Scrutiny of Droste's Original Solution (1917)

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Scrutiny of Droste's Original Solution (1917)

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Abstract: In 1916, Johannes Droste independently found an exact (vacuum) solution to the Einstein's (gravitational) field equations in empty space. Droste's solution is quasi-comparable to Schwarzschild's one . Droste published his paper entitled *"The field of a single centre in Einstein's theory of gravitation, and the motion of a particle in that field"*. The paper communicated (in the meeting of May 27, 1916) by Prof. H.A. Lorentz, and published in 'Proceedings of the Royal Netherlands Academy of Arts and Science. **19** (1): 197-215 (1917)'. In the present article, the Droste's solution is scrutinized and proven to be invalid purely and simply because the procedure used by Droste is mathematically questionable since he had systematically, deliberately, and without any justification –removed the constant coefficient '2' from the differential term ($v'w'$) in Eq.(6) and added the differential term (wv'') to the same Eq.(6) in order to obtain Eq.(7) which *was* and *is* his principal objective, that is, the desired solution. Consequently, Eqs.(6,7) had clearly been falsified.

Keywords: Droste's original solution (1917), Einstein's (gravitational) field equations

1. Introduction

Historically, after the publication of Einstein's general relativity theory [1,2,3], and a little more than month, Schwarzschild found, for the first time, the exact (vacuum) solution in 1915 and published it in 1916 [4], and now it is usually called the 'Schwarzschild solution' also known as the 'Schwarzschild metric'. The explicit expression of this solution is

$$ds^2 = \left(1-\frac{\alpha}{R}\right)dt^2 - \left(1-\frac{\alpha}{R}\right)^{-1} dR^2 - R^2\left(d\vartheta^2 + \sin^2\vartheta d\phi^2\right), \quad R = \left(r^3+\alpha^3\right)^{1/3}. \tag{1}$$

However, it is our duty to draw readers' attention to the fact that many research articles, textbooks, historians and specialists of general relativity theory (GRT) have incorrectly attributed the following solution/metric

$$ds^2 = \left(1-\frac{r_S}{r}\right)c^2dt^2 - \left(1-\frac{r_S}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \qquad r_S = 2GM/c^2, \tag{2}$$

to Schwarzschild as *his* exact (vacuum) solution to the Einstein's field equations in empty space

$$R_{\mu\nu} = 0. \tag{3}$$

The solution (2) is supposed to be the correct description of the gravitational field outside a spherically symmetric mass. Also, the metric (2) is usually acknowledged as the conceptual basis for the investigation of GR-effects and leading to the concept of black hole. According to Birkhoff's

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theorem, the metric (2) is the most general spherically symmetric, vacuum solution of the Einstein's field equations (3).

In the Schwarzschild's original solution (1) there is only one singularity at $r=0$, however, the solution (2), which was wrongly accredited to Schwarzschild, appears to have two singularities at $r=0$ and at $r=r_S$ (the so-called Schwarzschild radius of the massive body, a scale factor which is related to its mass M by $r_S=2GM/c^2$). In reality, the metric (2) is Hilbert's solution [5,6] on which a more complete analysis of the singularity structure was given by Hilbert himself and he identified the two singularities. Although there was general consensus that the singularity at $r=0$ was a 'real physical' singularity, the exact nature of the singularity at $r=r_S$ remained unclear [7]. Consequently, the concept of black hole was originated from these two singularities. –But why did historians and experts of GRT wrongly attribute the metric (2) to Schwarzschild? Maybe because they did not read the Schwarzschild's original paper or maybe they ignored or neglected to do such a task in spite of the fact that the original paper has been translated from German to English.

Psychologically and physico-mathematically, the Schwarzschild original solution (1) had explicitly or implicitly stimulated and motivated many famous authors like, *e.g*., Droste, Hilbert, Lorentz, Painlevé, Gullstrand, Eddington and Lemaître to resolve Einstein's (gravitational) field equations (3) by means of other procedure, simple and different from that used by Schwarzschild. In this paper, we are particularly concerned with Droste's original solution (1917). More precisely, we should focus our attention on the Droste' procedure –published in the paper *"The field of a single centre in Einstein's theory of gravitation, and the motion of a particle in that field"* [8]–, which was behind his original solution:

$$ds^2=\left(1-\frac{\alpha}{r}\right)dt^2-\frac{dr^2}{1-\frac{\alpha}{r}}-r^2\left(d\vartheta^2+\sin^2\vartheta d\varphi^2\right). \qquad (4)$$

This solution numbered (7) on page 200 in Ref.[8]. Also, Droste added the following remark (footnote 1, page 200): *"After the communication to the Academy of my calculations, I discovered that also K. Schwarzschild has calculated the field. (...) Equation* (7) *agrees with* (14) *there, if R is read instead of r."* –Through this comment, Droste seemed as if he want to convince us of the fact that he was not familiar with Schwarzschild original solution (1).

2. Profound difference between Mathematics and Physics

Before tackling the paper under discussion, it is judged important to recall the following pedagogical and epistemological considerations. Firstly, without entering into many details, let us begin by recalling the profound difference between mathematics and physics. Such a recall is indispensable because in the framework of GRT there is no clear and explicit distinction between a *physical equation* (an equation written in a purely physical context) and a *mathematical equation* (an equation written in a purely mathematical context). Secondly, the inhabitants of the mathematical world are purely abstract objects characterized by an absolute freedom. However, the inhabitants of

the physical world are purely concrete objects –in the theoretical sense and/or in the experimental/observational sense –and are characterized by very relative and restricted freedom.
Thirdly, when applied outside its original context, mathematics should play the role of an accurate language and useful tool, and gradually should lose its abstraction. However, when we are dealing with physical equations, abstraction and freedom together lose their absolutism and become very relative, and thus restricted, because each parameter contained in the physical equation has a well-defined role, fixed by its own physical dimensions.

3. In Mathematics, the Procedure and the Solutions are an essential component of the Reasoning

Mathematics is an exact science and has its own logic, terminology and convention. Mathematics is the language of Science itself. That's why *it* is the appropriate language of theoretical physics.
The *procedure* (method) is the *heart* of the mathematical reasoning. In mathematics, the pedagogical definition of the procedure is: 'procedure (method) is a set of rules and actions which is logically and rationally the accepted way of doing something, *e.g*., demonstrating a theorem or resolving an equation.'

Thus, methodically, pedagogically and mathematically speaking the correctness of the solution of any equation is strictly inseparable from the correctness of the procedure used for obtaining this solution because, in mathematics, the procedure and the solutions are an essential component of the reasoning. However, as we shall see soon, in his original paper [8], Droste found the solution (4) by means of mathematically questionable procedure thus, contextually, his solution is also mathematically unacceptable even if his solution may be correctly obtained *via* other method .

4. Proof of Falsification

Now, we arrive at our main subject namely the scrutiny of the Droste's original solution (4) on the way to prove its falsification purely and simply because the procedure used by Droste is mathematically questionable since in [8] he had systematically, deliberately, and without any justification –removed the constant coefficient '2' from the differential term ($v'w'$) in Eq.(6) and added the differential term (wv'') to the same equation Eq.(6). Thus he falsified Eq.(6) in order to get Eq.(7) which *was* and *is* his principal objective, that is, the desired solution.

In order to make our scrutiny more comprehensible, we are obliged to rewrite the author's central claims, word by word. We begin with page 199, on which the author wrote: " *The equations of the field being covariant for all transformations of the coordinates whatever, we are at liberty to choose instead of r a new variable which will be such a function of r, that in ds^2 the coefficient of the square of its differential becomes unity. That new variable we name r again and we put*

$$ds^2 = w^2 dt^2 - dr^2 - v^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right), \tag{4}$$

w and v only depending on r. We now find

$$G_{tr} = -\frac{4w''}{w},\ G_{\vartheta r} = G_{\varphi r} = -\frac{4v''}{v},\ G_{t\vartheta} = G_{t\varphi} = -\frac{4v'w'}{vw},\ G_{\vartheta\varphi} = \frac{4}{v^2} - \frac{4v'^2}{v^2}.$$

In these equations accents represent differentiations with respect to r*. So*

$$G=\frac{4}{v^2}-\frac{4v'^2}{v^2}-\frac{8v'w'}{vw}-\frac{8v''}{v}-\frac{4w''}{w}.$$

Now, as $\sqrt{-g}=v^2 w\sin\vartheta$*, the function to be integrated in the principle of variation becomes*

$$4\left(w-wv'^2-2vv'w'-2vwv''-v^2w''\right)\sin\vartheta.$$

We now apply the principle to the region $t_1\le t\le t_2,\quad r_1\le r\le r_2$.
By effecting the integrations with respect to t, ϑ *and* φ *we find the condition*

$$\delta\int_{r_1}^{r_2}\left(w-wv'^2-2vv'w'-2vwv''-v^2w''\right)dr=0. \qquad \text{(i)}$$

This gives us

$$2vv''+v'^2=1, \qquad (5)$$

and

$$vw''+v'w'+wv''=0. \qquad (6)$$

These are the equations of the field required.

2. To solve (6)*, we introduce instead of* r *the quantity* $x=v$ *as an independent variable by which, on taking account of* (5), (6) *changes into*

$$\left(1-x^2\right)\frac{d^2w}{dx^2}-2x\frac{dw}{dx}+2w=0. \qquad \text{(ii)}$$

This equation is satisfied by $w=x$*. The other particular solution is now also easily found, viz.*

$$w=1-\tfrac{1}{2}x\log\frac{1-x}{1+x}. \qquad \text{(iii)}$$

But we want w *to be a finite constant if* $v'=1$ (*for* $r=\infty$).
Then w must be equal to x, if we take the constant to be 1 (*the speed of light then approaches to* 1 *at large distances from the centre*)*."*

On page 200, he wrote: *" The introduction of* x *in* (5) *gives*

$$\frac{dv}{dx}=\frac{2xv}{1-x^2},$$

From which we immediately find

$$v=\frac{\alpha}{1-x^2},$$

α *being a constant of integration.*

Differentiating this relation with respect to r*, we get*

$$v' = \frac{2\alpha x}{\left(1-x^2\right)^2}\frac{dx}{dr},$$

or, v' being equal to x,

$$dr = \frac{2\alpha dx}{\left(1-x^2\right)^2}\ .$$

So (4) *changes into*

$$ds^2 = x^2dt^2 - \frac{4\alpha^2}{\left(1-x^2\right)^4}dx^2 - \frac{\alpha^2}{\left(1-x^2\right)^2}\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)\ . \qquad \text{(iv)}$$

So we have now been led again to introduce another variable instead of r, viz. x. The form obtained leads us to introducing the variable $\xi = 1 - x^2$*.Then*

$$ds^2 = \left(1-\xi\right)dt^2 - \frac{4\alpha^2}{\left(1-\xi\right)\xi^4}d\xi^2 - \frac{\alpha^2}{\xi^2}\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right).$$

Lastly we put

$$\xi = \frac{\alpha}{r}.$$

This r is not the same as occurs in (4). *We obtain*

$$ds^2 = \left(1-\frac{\alpha}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{\alpha}{r}\right)} - r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right). \qquad (7)$$

We have chosen the coordinates in a particular manner; it is now of course also very easy to introduce for r another variable, which is a function of r."

It is clear from the above considerations, Eqs.(5,6) are not only the equations of the required field, but also they are the 'cornerstone' of the author's procedure. At first glance, these equations seemed valid, but on closer inspection we shall find that the Eq.(6) is not only incorrect but also falsified. Our scrutiny reveals the intentional omission of the constant coefficient '2' from the differential term (*v'w'*) in Eq.(6) and the deliberate addition of the differential term (*wv''*) to the same Eq.(6). Therefore, Eq.(6) cannot be deduced from (i). In fact, we can deduce two possible systems of differential equations from (i).

–The first system may be deduced as follows: Let us rewrite (i) in the following form

$$\delta\int_{r_1}^{r_2}\left(w\left[1 - v'^2 - 2vv''\right] - v\left[2v'w' + vw''\right]\right)dr = 0\,. \qquad \text{(v)}$$

This provides us

$$\begin{cases} 2vv'' + v'^2 = 1, \\ vw'' + 2v'w' = 0. \end{cases} \tag{vi}$$

By comparing Eqs.(5,6) with the system (vi), we can see that Eq.(5) is identical to the first equation of system (vi), but Eq.(6) is completely different from the second equation of system (vi). Consequently, contrary to the author's claim, the second equation, *i.e.*, $(vw'' + 2v'w' = 0)$ cannot change into Eq.(ii) even when *we introduce instead of* r *the quantity* $x = v$ *as an independent variable.*

–The second system may be deduced like this: Let us rewrite (i) in the subsequent form

$$\delta\int_{r_1}^{r_2}\left(w\left[1 - v'^2 - vv''\right] - v\left[2v'w' + wv'' + vw''\right]\right)dr = 0\,. \tag{vii}$$

This gives us

$$\begin{cases} vv'' + v'^2 = 1, \\ vw'' + 2v'w' + wv'' = 0. \end{cases} \tag{viii}$$

Obviously, the first and second equation of system (viii) are different from Eqs.(5,6). Thus, in contrast to the author's claim, the second equation of system (viii), *i.e.*, $(vw'' + 2v'w' + wv'' = 0)$ cannot change into Eq.(ii) even when *we introduce instead of* r *the quantity* $x = v$ *as an independent variable.*

It seems that, from the beginning, Droste knew perfectly well the Schwarzschild's original solution (1) and he used *it* as a central target. However, to arrive at his wanted aim, he falsified Eq.(6) by removing the constant coefficient ‘2’ from the differential term ($v'w'$) in Eq.(6) and added the differential term (wv'') to the same Eq.(6) in order to obtain Eq.(7) which *was* and *is* his principal objective, that is, the desired solution. He knew, in advance, that if he had *correctly* used the system (vi or viii), he cannot find the solution (7). For that reason, the solution (7) is mathematically meaningless.

Historically, the Droste's original paper (1917) had been communicated (in the meeting of May 27, 1916) by Lorentz himself to the ‘Proceedings of the Royal Netherlands Academy of Arts and Science’ and published in 1917, so the paper should be peer-reviewed, at least, by two reviewers. However, possibly the reviewers focused their attention only on the solution itself without concerning themselves with the procedure behind this solution particularly the falsified Eq.(6) and its application as an equation of the required field.

4.1. Solutions of system (vi)

Now, let us investigate and resolve the system (vi) in order to show more conclusively that its solutions cannot be used to get the solution (7). Let us rewrite the first equation of system (vi):

$$2vv'' + v'^2 = 1, \tag{vi.1}$$

and recalling that the functions v and w are only dependent on r, *i.e.*, $v \equiv v(r)$ and $w \equiv w(r)$. Eq.(vi.1) has a *singular solution*

$$v = r.$$

If we multiply Eq.(vi.1) by v' and taking into account the fact that $(v'^2)' = 2v'v''$, we get, after some manipulation, the *second solution*

$$r = \sqrt{v(v-c_1)} + c_1 \ln\left(\sqrt{v} + \sqrt{(v-c_1)}\right) + c_2, \qquad c_1,\, c_2 \in IR$$

If we take into consideration the singular solution $v = r$ of Eq.(vi.1), the second equation of system (vi) may be rewritten as

$$r w'' + 2w' = 0, \qquad \text{(vi.2)}$$

which has two particular solutions, namely, $w_1 = 1$ and $w_2 = r^{-1}$. The general solution is

$$w = k_1 w_1 + k_2 w_2, \qquad k_1,\, k_2 \in IR.$$

It is clear, from the above solutions, we cannot find the solution (7). This proves us more conclusively that Droste had really fabricated *his* solution.

4.2. Solutions of system (viii)

Like before, let us investigate and resolve the system (viii) with the purpose of showing that its solutions cannot be utilized to obtain the solution (7). Rewriting the first equation of system (viii):

$$vv'' + v'^2 = 1, \qquad \text{(viii.1)}$$

Eq.(viii.1) has a *particular solution* $v = r$, and if we multiply Eq.(viiii.1) by v' and taking into account the fact that $\frac{1}{2}(v'^2)' = \frac{1}{2}(2v'v'')$, we get, after some manipulation, the *second solution*

$$v = \pm\sqrt{(r^2 - \alpha_2) + \alpha_1}, \qquad \alpha_1,\, \alpha_2 \in IR,$$

which may be also considered as a general solution of Eq.(viii.1). Finally, the second equation of system (viii) may be rewritten as

$$vw'' + 2v'w' + wv'' = [(vw)']' = \frac{d}{dr}[(vw)'] = 0. \qquad \text{(viii.2)}$$

A direct integration gives us $vw = a_1 r + a_2$ with $a_1,\, a_2 \in IR$. Let us now determine the explicit expression of w. We have for the case when $v = r$, the first expression for the second solution of Eq.(viii.2):

$$w = a_1 + a_2 r^{-1},$$

and for the other case, *viz.*, $v = \pm\sqrt{(r^2 - \alpha_2) + \alpha_1}$, we get the second expression

$$w = \frac{a_1 r + a_2}{\pm\sqrt{(r^2 - \alpha_2) + \alpha_1}} .$$

Obviously, the above solutions, cannot be used to obtain the solution (7). Again, this shows us more convincingly that Droste had really fabricated *his* solution.

5. Conclusion

The so-called Droste exact solution (1917) to the Einstein's (gravitational) field equations in empty space is scrutinized and proven to be not only wrong but mathematically meaningless basically because the cornerstone of the procedure used by Droste to obtain this solution is completely falsified.

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