Post Completeness in Congruential Modal Logics

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Abstract

Well-known results due to David Makinson show that there are exactly two Post complete normal modal logics, that in both of them, the modal operator is truth-functional, and that every consistent normal modal logic can be extended to at least one of them. Lloyd Humberstone has recently shown that a natural analog of this result in congruential modal logics fails, by showing that not every congruential modal logic can be extended to one in which the modal operator is truth-functional. As Humberstone notes, the issue of Post completeness in congruential modal logics is not well understood. The present article shows that in contrast to normal modal logics, the extent of the property of Post completeness among congruential modal logics depends on the background set of logics. Some basic results on the corresponding properties of Post completeness are established, in particular that although a congruential modal logic is Post complete among all modal logics if and only if its modality is truth-functional, there are continuum many modal logics Post complete among congruential modal logics.

Keywords: Propositional Modal Logic, Post Completeness, Congruential Modal Logics, Classical Modal Logics

1 Introduction

The notion of Post completeness captures the intuitive idea of a logic being maximal, in the sense of it not being possible to strengthen the logic without collapsing it into inconsistency. This can be made precise in a very abstract setting: Let L be a set, informally understood as a set of formulas, and C a subset of the power set of L containing L itself, informally understood as the set of logics under consideration. A set $\Lambda \in C$ such that $\Lambda \neq L$ can then be defined to be *Post complete in* C if there is no $\Lambda' \in C$ such that $\Lambda \subset \Lambda' \subset L$. Post completeness in C is thus simply the property of being a coatom of C,

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partially ordered by $\subseteq$. Calling a member of C *consistent* if it is distinct from L, the coatoms of C can also be described as the maximal elements of the set of consistent members of C.

This abstract account of Post completeness makes clear that logics, understood as sets of formulas, are not Post complete *simpliciter*, but only Post complete *relative to* a given set of logics. Of course, if only one such set is considered, one may naturally talk about Post completeness simpliciter, leaving the relativity to this set implicit. Most of the literature on Post completeness in propositional modal logic operates on such an assumption, considering only Post completeness relative to the set of all *modal logics*, defined as sets of formulas containing all truth-functional tautologies and closed under the rules of modus ponens and uniform substitution. This is somewhat surprising, as the vast majority of research in modal logic focuses on the much more restrictive set of normal modal logics. The matter is partly explained by results due to Makinson [11], which show that among normal modal logics, the background set of logics is irrelevant, as a normal modal logic is Post complete in the set of all logics if and only if it is Post complete in the set of normal modal logics. (This section gives an informal introduction and overview; formal definitions and results will be stated more precisely below.)

The present article explores Post completeness beyond normal modal logics, in particular in the context of congruential modal logics. As will be shown, the background set of logics matters in this context, as most logics Post complete in the set of congruential modal logics fail to be Post complete in the set of all modal logics. First, however, some remarks on why Post completeness is an interesting notion.

Much of mathematical research in modal logic it is concerned with investigating various aspects of the set of normal modal logics, which, ordered by $\subseteq$, forms a complete lattice. Indeed, Rautenberg [13, p. 219] goes so far as to suggest that any investigation of normal modal logics is in effect aimed at improving our understanding of the lattice of normal modal logics. A natural part of such an enterprise is the investigation of the lattice's coatoms – the logics Post complete in it. What about its atoms? A version of Lindenbaum's lemma guarantees that every consistent normal modal logic can be extended to (i.e., is a subset of) a maximal consistent one (i.e., a coatom). But no such result guarantees that every normal modal logic distinct from the smallest normal modal logic **K** is an extension of an atom. In fact, Kracht [10, Theorem 7.7.2] notes, drawing on results due to Blok [2], that the lattice of normal modal logics is atomless.

A further reason for studying Post completeness arises from Makinson's results mentioned above, which also show that the lattice of normal modal logics has exactly two coatoms. This fact, and various specific details concerning these two logics, have proved extremely useful in a wide variety of applications (see, e.g., the appeals to "Makinson's Theorem" at various places in [10]). The intrinsic interest and evident usefulness of investigating Post completeness among normal modal logics therefore motivate studying this notion in wider classes of modal logics, and the present article makes a start, focusing in particular on congruential modal logics.

Section 2 briefly sets out the background theory of modal logics and corresponding models. Section 3 states the fundamental facts concerning Post completeness in normal modal logics which follow from Makinson's results, and some recent results due to Humberstone on Post completeness in congruential modal logics. Section 4 determines the number of logics Post complete in the lattice of congruential modal logics to be that of the continuum, and shows that infinitely many such logics are determined by a class of neighborhood frames. Section 5 shows that there are precisely four congruential modal logics which are Post complete in the lattice of all modal logics; with the first result of section 4, this shows that Post completeness among congruential modal logics is dependent on the background set of logics. Section 5 also shows that there is a continuum of extensions of congruential modal logics Post complete in the lattice of all modal logics which are not extensions of normal modal logics. Section 6 generalizes an observation of Humberstone's that the intersection of logics Post complete in lattice of all modal logics and closed under certain rules can be axiomatized using conditionals corresponding to these rules. Section 7 concludes, highlighting a number of open questions.

2 Modal Logics and Algebraic Models

Let $\mathcal{L}$ be the set of formulas of a propositional modal language, built up as usual from a countably infinite set of proposition letters $p, q, \dots$ using the nullary operator $\top$ (trivial truth), the unary operator $\neg$ (negation), the binary operator $\wedge$ (conjunction) and the unary operator $\Box$ (the modality). Other operators, such as $\bot$, $\vee$ and $\rightarrow$ will be used as syntactic abbreviations as usual. Let a *substitution* be a function $\sigma : \mathcal{L} \rightarrow \mathcal{L}$ such that for all $\varphi, \psi \in \mathcal{L}$, $\sigma(\top) = \top$, $\sigma(\neg\varphi) = \neg\sigma(\varphi)$, $\sigma(\varphi \wedge \psi) = \sigma(\varphi) \wedge \sigma(\psi)$ and $\sigma(\Box\varphi) = \Box\sigma(\varphi)$. Let Σ be the set of substitutions. Let a *modal logic* be a set $\Lambda \subseteq \mathcal{L}$ such that Λ contains all propositional tautologies ($\varphi \in \mathcal{L}$ not containing $\Box$ true under every classical truth-value assignment) and is closed under modus ponens (if $\varphi, \varphi \rightarrow \psi \in \Lambda$ then $\psi \in \Lambda$) and uniform substitution (if $\varphi \in \Lambda$ then $\sigma(\varphi) \in \Lambda$ for any substitution σ). Let a modal logic be *consistent* if it is distinct from $\mathcal{L}$.

Usually, restrictions on modal logics are formulated in terms of containing certain axioms and being closed under certain rules. To provide an abstract framework for such restrictions, let a *rule* be a set of finite non-empty sequences of formulas. Let any $\Gamma \subseteq \mathcal{L}$ be *closed under* a rule R just in case for all $\langle \rho_0, \dots, \rho_n \rangle \in R$, if $\rho_i \in \Gamma$ for all $i < n$, then $\rho_n \in \Gamma$. The members of a rule will be called its *instances*. A rule R is *substitution-invariant* if for all $\langle \rho_0, \dots, \rho_n \rangle \in R$ and substitutions σ, $\langle \sigma(\rho_0), \dots, \sigma(\rho_n) \rangle \in R$. In this setting, the usual requirement of being closed under "necessitation" can be formulated as being closed under the substitution-invariant rule $\{\langle \sigma(p), \sigma(\Box p) \rangle : \sigma \in \Sigma\}$. The treatment of axioms can be subsumed under this account of rules, using sequences of formulas of length one. So, being a *normal* modal logic can be formulated as being a modal logic closed under the substitution-invariant rule

$N = \{\langle\sigma(K)\rangle, \langle\sigma(p), \sigma(\Box p)\rangle : \sigma \in \Sigma\}$, where K is the familiar distributivity axiom $\Box(p \to q) \to (\Box p \to \Box q)$.

For any substitution-invariant rule R, let $L(R)$ be the set of modal logics closed under R. It is routine to show that this is a complete lattice with top element $\mathcal{L}$. Let a modal logic Λ be *R-Post complete* just in case it is a coatom of $L(R)$ (a consistent member of $L(R)$ which is not a subset of any other consistent member of $L(R)$). Let R-Post be the set of $\Lambda \in L(R)$ which are R-Post complete, and, for any set $\Gamma \subseteq \mathcal{L}$, R-Post(Γ) the set of $\Lambda \in R$-Post which contain Γ. A routine version of Lindenbaum's lemma establishes that $L(R)$ is coatomic: every consistent modal logic closed under R can be extended to an R-Post complete modal logic.

Among the most important restrictions on modal logics are those of being normal, congruential, quasi-normal and quasi-congruential. (See [14] and [5] for general discussion of these classes and the models for them used below; the term "classical" is sometimes used instead of "congruential".) Normality was defined above as being closed under N; *congruentiality* can be defined analogously as being closed under the substitution-invariant rule $C = \{\langle\sigma(p \leftrightarrow q), \sigma(\Box p \leftrightarrow \Box q)\rangle : \sigma \in \Sigma\}$. A modal logic is *quasi-normal* if it is an extension of a normal modal logic, and *quasi-congruential* if it is an extension of a congruential modal logic. Note that quasi-normality and quasi-congruentiality can also be defined by appeal to substitution-invariant rules, *viz.* the rules whose instances are the singleton sequences of members of the smallest normal and congruential modal logics, respectively.

Although each of the four lattices of modal logics just defined gives rise to a distinct notion of Post completeness, not each of these notions gives rise to its own set of questions. To illustrate this, consider normal and congruential modal logics. It is routine to show that a congruential modal logic Λ (i.e., $\Lambda \in L(C)$) is normal (i.e., $\Lambda \in L(N)$) if and only if it contains both K and $\Box\top$. Thus $L(N)$ is simply the principal filter of $L(C)$ generated by the smallest congruential modal logic containing K and $\Box\top$, which is of course the familiar modal logic **K**. Thus it is clear that any $\Lambda \in L(N)$ is N-Post complete if and only if it is C-Post complete, which means that among normal modal logics, the notion of N-Post completeness coincides with that of C-Post completeness, and so the investigation of N-Post completeness is a special case of the investigation of C-Post completeness. A similar point applies to the constraints of being quasi-normal and quasi-congruential, as the notions of Post completeness to which they give rise are special cases of the notion of $\emptyset$-Post completeness.

Several of the following results will appeal to standard algebraic models for congruential modal logics. Let a *modal matrix* be a structure $\mathfrak{A} = \langle A, 1, -, \sqcap, *, D\rangle$ such that $\langle A, 1, -, \sqcap\rangle$ is a Boolean algebra, $* : A \to A$ and $D \subseteq A$ is a filter of the algebra. For such a modal matrix $\mathfrak{A}$, let an *interpretation* be a function ι mapping proposition letters to elements of A. Extend such functions implicitly to $\mathcal{L}$, by letting $\iota(\top) = 1$, $\iota(\neg\varphi) = -\iota(\varphi)$, $\iota(\varphi \wedge \psi) = \iota(\varphi) \sqcap \iota(\psi)$ and $\iota(\Box\varphi) = *\iota(\varphi)$. Define $\Lambda(\mathfrak{A})$, the *logic of* $\mathfrak{A}$, to be the set of formulas φ such that $\iota(\varphi) \in D$ for all interpretations ι. Using Lindenbaum-Tarski algebras, it is routine to show that a modal logic is quasi-congruential if and only if it is the logic of a modal matrix, and congruential if and only if it is the logic of a modal matrix with singleton filter (see, e.g., Hansson and Gärdenfors [8]).

3 Makinson's and Humberstone's Results

It follows from the results of Makinson [11] that among normal modal logics, $\emptyset$-Post completeness coincides with N-Post completeness, and that these properties are had by exactly two logics, both of which interpret $\Box$ truth-functionally. These logics are most naturally described using algebraic models:

Treating T and F as the usual truth-values, every function $* : \{T, F\} \to \{T, F\}$ gives rise to a modal matrix $\mathfrak{T}^* = \langle\{T, F\}, T, -, \sqcap, *, \{T\}\rangle$ in which $-$ and $\sqcap$ are the usual truth-functional operations of negation and conjunction. Let t and f be the constant one-place functions to T and F, respectively, i the identity function and n the function mapping T and F to each other. Each such function $*$ thus gives rise to a logic $\Lambda^* = \Lambda(\mathfrak{T}^*)$ in which the modality $\Box$ behaves according to the truth-function $*$. Call these the four *truth-functional modal logics*. It is easy to see that no two truth-functions give rise to the same truth-functional modal logic. The two normal modal logics which are $\emptyset/N$-Post complete are Λ^t and Λ^i, and thus every consistent normal modal logic can be extended to at least one of these.

Can similar results be obtained for congruential modal logics? Humberstone [9] gives a negative answer, by showing that some congruential modal logics cannot be extended to any truth-functional modal logic. Since every consistent congruential modal logic can be extended to a C-Post complete one, it follows that some C-Post complete modal logics are not truth-functional. So, what is the extent of the property of Post completeness among congruential and quasi-congruential modal logics, and how do these sets of modal logics relate to each other and to the set of truth-functional modal logics? The remainder of this paper gives some basic answers to these and closely related questions.

4 Continuum Many C-Post Complete Logics

The first result to be established shows that the number of modal logics Post complete in the lattice of congruential modal logics is $\beth_1$ $(= 2^{\aleph_0})$:

Theorem 4.1 *The number of C-Post complete modal logics is $\beth_1$.*

Proof. Let $\langle A, \omega, -, \cap\rangle$ be the countable Boolean algebra of finite and cofinite sets of natural numbers. Let B be the set of finite non-empty sets of natural numbers, and $\langle b_n : n \in \omega\rangle$ an enumeration of B. For each set of natural numbers $S \subseteq \omega$, define a modal matrix $\mathfrak{A}_S = \langle A, \omega, -, \cap, *, \{\omega\}\rangle$, where $*$ is defined as follows:

$*(\omega) = b_0$

$*(b_n) = b_{n+1}$ for all $n \in \omega$

$*(-b_n) = \omega$ for all $n \in S$

$*(-b_n) = \emptyset$ for all $n \in \omega\backslash S$

$*(\emptyset) = \omega$

Since every modal matrix with a singleton filter determines a congruential modal logic, $\Lambda(\mathfrak{A}_S)$ is a congruential modal logic. Furthermore, for all $S \subseteq \omega$ and $n \in \omega$,

$\Box\neg\Box^n\Box\top \in \Lambda(\mathfrak{A}_S)$ iff $n \in S$

$\neg\Box\neg\Box^n\Box\top \in \Lambda(\mathfrak{A}_S)$ iff $n \notin S$

where $\Box^n$ is a string of n $\Box$ operators. Thus for any distinct $S, S' \subseteq \omega$, $\Lambda(\mathfrak{A}_S)$ and $\Lambda(\mathfrak{A}_{S'})$ cannot be extended to the same consistent modal logic. As every consistent congruential modal logic can be extended to a C-Post complete modal logic, there are $\beth_1$ such logics. $\Box$

Theoremhood in $\Lambda(\mathfrak{A}_S)$ depends not only on S, but also on the choice of the enumeration of B. E.g., consider a set S containing 2 but not 3. Then $\Box(\Box\top \wedge \Box\Box\top) \in \Lambda(\mathfrak{A}_S)$ if $b_0 \cap b_1 = b_2$, but not if $b_0 \cap b_1 = b_3$.

Similar to relational frames for normal modal logics, so-called neighborhood frames are naturally used to provide possible world models for congruential modal logics. A neighborhood frame is a pair $\langle W, N\rangle$ such that W is a set (the "worlds") and $N : \mathcal{P}(W) \to \mathcal{P}(W)$, from which a model can be obtained by adding a valuation function V which maps every proposition letter to a set of worlds. Truth of a formula at a world is defined as in relational frames, with the following clause for the modal operator:

$\langle W, N, V\rangle, w \vDash \Box\varphi$ iff $\{v \in W : \langle W, N, V\rangle, v \vDash \varphi\} \in N(w)$

The logic of a class of neighborhood frames is the set of formulas true at every world in every model based on a frame in the class.

For every set W, the powerset $\mathcal{P}(W)$ forms a Boolean algebra, and it is easy to see that the modal matrices with singleton filter based on $\mathcal{P}(W)$ correspond uniquely to the neighborhood frames on W. Neighborhood frames can therefore be seen as modal matrices with singleton filters based on powerset algebras. Not every Boolean algebra is isomorphic to a powerset algebra, however, which opens up the possibility that some congruential modal logic is not the logic of any class of neighborhood frames. That there are such logics was shown by Gerson [6]. With Theorem 4.1, this raises the question how widespread such incompleteness is among logics Post complete in the set of congruential modal logics. The following result gives a partial answer, by showing that there are infinitely many C-Post complete logics which are the logic of a class of neighborhood frames. What their precise number is will be left open, as well as the question whether there are any C-Post complete logics which fail to be the logic of a class of neighborhood frames, and if so, how many such logics there are.

Theorem 4.2 *There are at least $\aleph_0$ C-Post complete modal logics each of which is the logic of a class of neighborhood frames.*

Proof. For every natural number n, let $\mathfrak{A}_n$ be a matrix based on the powerset algebra A on $n = \{0, \ldots, n-1\}$ with filter n and function $*$ to be defined. Let

$B = A \backslash \{n\}$ and $\langle b_i : i < 2^n - 1\rangle$ an enumeration of B such that $b_0 = \emptyset$. Define $*$ as follows:

$*(n) = n$

$*(b_i) = b_{i+1}$ for all $i < 2^n - 2$

$*(b_{2^n-2}) = b_0$

Let $\Lambda_n = \Lambda(\mathfrak{A}_n)$, which is congruential by construction. For any n, the smallest $l > 0$ such that $\neg\Box^l\bot \in \Lambda_n$ is $2^n - 1$. So if $n \neq n'$, $\Lambda_n \neq \Lambda_{n'}$. Since $\mathfrak{A}_n$ is based on a power set algebra, Λ_n is the logic of the corresponding neighborhood frame. It only remains to argue that Λ_n is C-Post complete.

Consider any congruential modal logic Λ properly extending Λ_n. Then there is a $\varphi \in \Lambda$ which is not in Λ_n. Since $\varphi \notin \Lambda_n$, there is an interpretation ι such that $\iota(\varphi) \neq n$. For each element $x \in A$, there is a formula $\delta(x)$ containing no proposition letters such that $\kappa(\delta(x)) = x$ for every interpretation κ: let $\delta(n) = \top$ and $\delta(b_i) = \Box^i\bot$. Let σ be the substitution mapping each proposition letter p to $\delta(\iota(p))$. A routine induction on the complexity of formulas shows that for every interpretation κ, $\kappa(\sigma(\varphi))) = \iota(\varphi)$, and so $\kappa(\sigma(\varphi))) \neq n$.

One the one hand, by construction of $\mathfrak{A}_n$, there is a number k such that $\kappa(\Box^k\sigma(\varphi)) = \emptyset$ for all interpretations κ. So $\neg\Box^k\sigma(\varphi)$ is a member of Λ_n and thus a member of Λ. On the other hand, since $\varphi \in \Lambda$, by uniform substitution, $\sigma(\varphi) \in \Lambda$. So $\top \leftrightarrow \sigma(\varphi) \in \Lambda$, and therefore by k applications of the congruentiality rule, $\Box^k\top \leftrightarrow \Box^k\sigma(\varphi)$. But $\Box^k\top$ is a member of Λ_n and so a member of Λ, hence $\Box^k\sigma(\varphi) \in \Lambda$. It follows that Λ is inconsistent. $\Box$

5 Truth-Functionality and Quasi-Congruentiality

From Theorem 4.1, it follows immediately that there are $\beth_1$ congruential modal logics which cannot be extended to a truth-functional modal logic. In this sense, C-Post completeness differs markedly from N-Post completeness. But it turns out that one consequence of Makinson's results does extend to congruential modal logics: $\emptyset$-Post completeness coincides with truth-functionality also among congruential modal logics. The proof relies on a lemma of Segerberg [15, p. 712, Lemma A]; to state it, let $\mathcal{L}_0$ be the set of formulas containing no proposition letters (i.e., built up entirely from $\top$).

Lemma 5.1 *A consistent modal logic Λ has exactly one $\emptyset$-Post complete extension if and only if for all $\varphi \in \mathcal{L}_0$, $\varphi \in \Lambda$ or $\neg\varphi \in \Lambda$.*

Theorem 5.2 *A congruential modal logic is $\emptyset$-Post complete if and only if it is truth-functional.*

Proof. The *if* direction is routine, so consider a congruential modal logic $\Lambda \in \emptyset$-Post. It follows by Lemma 5.1 that for each of $\Box\top$ and $\Box\bot$, Λ contains either it or its negation. The rest of the argument follows along the lines of [11]: Since Λ is congruential, $\Lambda = \Lambda(\mathfrak{A})$ for some modal matrix $\mathfrak{A} = \langle A, 1, -, \sqcap, \dagger, \{1\}\rangle$. Let $h : \{T, F\} \to A$ map T to 1 and F to 0 $(= -1)$. Since $\{\dagger(1), \dagger(0)\} \subseteq \{1, 0\}$, a truth-function $*$ can be defined as $h^{-1} \circ \dagger \circ h$, and it is easily seen that h is a homomorphism from $\mathfrak{T}^*$ to $\mathfrak{A}$. Consequently, Λ is a sublogic of Λ^*, and so as $\Lambda \in \emptyset$-Post, $\Lambda^* = \Lambda$. □

Among congruential modal logics, C-Post completeness and $\emptyset$-Post completeness therefore come wide apart. The results established so far thus emphasize the remarkableness of the fact that these two properties coincide among normal modal logics.

Consider now quasi-congruential modal logics, which can be characterized as the modal logics closed under the substitution-invariant rule $QC = \{\langle\varphi\rangle : \varphi \in \mathbf{E}\}$, where $\mathbf{E}$ is the smallest congruential modal logic. As noted above, $\emptyset$-Post completeness and QC-Post completeness trivially coincide on $L(QC)$. Moreover, the number of such logics has already been determined to be $\beth_1$ by Segerberg [15, p. 713], who shows that there are $\beth_1$ $\emptyset$-Post complete quasi-normal modal logics.[2] Given Theorem 5.2, there must be $\beth_1$ of them which are not congruential. This observation leaves open the possibility that all of them are quasi-normal, but the proof of Theorem 4.1 can be adapted to rule this out:

Theorem 5.3 *There are $\beth_1$ $\emptyset$-Post complete quasi-congruential modal logics which are not quasi-normal.*

Proof. Since the modal matrices used in the proof of Theorem 4.1 are based on an algebra generated by the single element 1, it follows with [12, Theorem 1] that extending the filter of any such modal matrix $\mathfrak{A}_S$ to an ultrafilter produces a matrix which determines a quasi-congruential modal logic which is $\emptyset$-Post complete. Let $\mathfrak{A}_S(U)$ be such a matrix with ultrafilter U such that $-b_n \in U$ for some $n \in \omega$. Then $\neg\Box^{n+1}\top \in \Lambda(\mathfrak{A}_S(U))$, which entails that this logic is not quasi-normal. As in Theorem 4.1, $\Lambda(\mathfrak{A}_S(U)) \neq \Lambda(\mathfrak{A}_{S'}(U))$ for any distinct S and S', from which the claim to be proven follows. □

6 Characterizing Intersections of Post-Complete Extensions

For any substitution-invariant rule R and set of formulas Γ, let $\Lambda_R(\Gamma)$ be the smallest modal logic closed under R which contains Γ; since $L(R)$ is a complete lattice, this is well-defined. Call $\Lambda_R(\Gamma)$ the R-logic *axiomatized by* Γ. Humberstone [9] notes that $\bigcap(\emptyset\text{-Post} \cap L(N))$, the intersection of the two $\emptyset$-Post complete normal modal logics, is the normal modal logic axiomatized by the formula $NC = p \to \Box p$, i.e., $\Lambda_N(\{NC\})$. He also notes that the intersection of the four truth-functional modal logics is the congruential modal logic axiomatized by the "extensionality conditional" $EC = (p \leftrightarrow q) \to (\Box p \leftrightarrow \Box q)$. With Theorem 5.2, it follows that this is the intersection of the $\emptyset$-Post complete congruential modal logics.

The axioms appealed to in these observations strongly suggest a general connection between $\emptyset$-Post complete logics closed under a given substitution-

[2] The claim in [7, pp. 133, 136 & 142] that there are only two such logics is therefore incorrect; this also seems to affect the discussion in [9, Coda].

invariant rule and the conditionals corresponding to the instances of this rule. This section establishes such a connection. The result to be proven shows, for any set of formulas Γ, how to characterize the intersection of the $\emptyset$-Post complete logics closed under a substitution-invariant rule R which contain Γ using the conditionals corresponding to the instances of R. The natural conjecture is that this intersection is simply the modal logic axiomatized by the union of Γ and the set of these conditionals. It turns out that this is incorrect, but that applying a natural operation to the logic so axiomatized produces the desired intersection.

To motivate the required operation, consider the case of $R = \emptyset$. The natural conjecture just mentioned says that $\bigcap \emptyset\text{-Post}(\Gamma)$ is $\Lambda_\emptyset(\Gamma)$, the modal logic axiomatized by Γ. This is not the case: As shown by Segerberg [16], $\emptyset\text{-Post}(\Gamma) = \emptyset\text{-Post}(\Delta)$ whenever $\Gamma \cap \mathcal{L}_0 = \Delta \cap \mathcal{L}_0$. Thus, as long as no $\mathcal{L}_0$ formulas are added, $\Lambda_\emptyset(\Gamma)$ can be expanded without adding formulas not in $\bigcap \emptyset\text{-Post}(\Gamma)$. (That there are logics which can be so expanded will follow from lemmas to be established presently.) This problem can be solved by expanding $\Lambda_\emptyset(\Gamma)$, adding all formulas whose substitution instances in $\mathcal{L}_0$ are already contained in $\Lambda_\emptyset(\Gamma)$. This turns out to be the required operation. It will now be defined formally, and some lemmas will be established, with which the desired result can be established.

Let a substitution σ be a 0-*substitution* if $\sigma(\varphi) \in \mathcal{L}_0$ for all formulas φ. For any set of formulas Γ, define

$$\varepsilon_0(\Gamma) = \{\varphi \in \mathcal{L} : \sigma(\varphi) \in \Gamma \text{ for every 0-substitution } \sigma\}.$$

Call this the 0-*expansion of* Γ.

Lemma 6.1 *For any modal logics* Λ, Λ'*:*

(i) $\Lambda \subseteq \varepsilon_0(\Lambda)$

(ii) $\varepsilon_0(\Lambda)$ *is a modal logic.*

(iii) *If* Λ *is closed under a given substitution-invariant rule, so is* $\varepsilon_0(\Lambda)$*.*

(iv) *If* Λ *is consistent, so is* $\varepsilon_0(\Lambda)$*.*

(v) *If* $\Lambda \cap \mathcal{L}_0 \subseteq \Lambda'$ *then* $\varepsilon_0(\Lambda) \subseteq \varepsilon_0(\Lambda')$*.*

Proof. Routine. □

With this lemma, it is easy to see that for every substitution-invariant rule R, ε_0 is a closure operator on $L(R)\backslash\mathcal{L}$, the consistent modal logics closed under R, ordered by $\subseteq$.

Lemma 6.2 *For every substitution-invariant rule* R *and* R*-Post complete logic* Λ*,* $\varepsilon_0(\Lambda) = \Lambda$*.*

Proof. Let $\Lambda \in L(R)$ such that $\varepsilon_0(\Lambda) \neq \Lambda$. Then by Lemma 6.1 (i), $\Lambda \subset \varepsilon_0(\Lambda)$. By Lemma 6.1 (ii) and (iii), $\varepsilon_0(\Lambda) \in L(R)$. Since $\Lambda \subset \varepsilon_0(\Lambda)$, Λ is consistent, and so with Lemma 6.1 (iv), $\varepsilon_0(\Lambda)$ is consistent. So Λ is not R-Post complete. □

Before applying this lemma to establish the main theorem of this section, it is worth relating the operation of 0-expansion to the closely related notions of 0-reducibility and general Post completeness, which Chagrov and Zakharyaschev [4, chapter 13] discuss in detail. A modal logic Λ is 0-*reducible* just in case for every formula $\varphi \notin \Lambda$, there is a 0-substitution σ such that $\sigma(\varphi) \notin \Lambda$. It is easy to see that Λ is 0-reducible just in case $\varepsilon_0(\Lambda) = \Lambda$, and that $\varepsilon_0(\Lambda)$ is the smallest 0-reducible modal logic containing Λ. Λ is *generally Post complete* if Λ is R-Post complete, where R is the union of substitution-invariant rules under which Λ closed. Chagrov and Zakharyaschev [4, Theorem 13.11] show that a consistent modal logic Λ is 0-reducible if and only if it is generally Post complete.[3] Since there are consistent modal logics which are not generally Post complete (see, e.g., their Theorem 13.2), it follows, as claimed above, that there are modal logics Λ such that $\varepsilon_0(\Lambda) \neq \Lambda$. The connections just drawn also show that fittingly, a modal logic is generally Post complete just in case it is R-Post complete for some substitution-invariant rule R; this observation provides an alternative route to establishing Lemma 6.2.

Returning to the theorem to be established, define, for any rule R, $\overrightarrow{R} = \{\bigwedge_{i<n} \rho_i \to \rho_n : \langle \rho_0, \ldots, \rho_n \rangle \in R\}$. A final lemma leads to the desired result:

Lemma 6.3 *For any $\emptyset$-Post complete modal logic Λ closed under a substitution-invariant rule R, $\overrightarrow{R} \subseteq \Lambda$.*

Proof. Let $\Lambda \in \emptyset\text{-Post} \cap L(R)$ and consider any $\langle \rho_0, \ldots, \rho_n \rangle \in R$. Since by Lemma 6.2, $\varepsilon_0(\Lambda) = \Lambda$, it suffices to show, for an arbitrary 0-substitution σ, that $\bigwedge_{i<n} \sigma(\rho_i) \to \sigma_n(\rho) \in \Lambda$. This can be done by a case distinction, using Lemma 5.1: If $\bigwedge_{i<n} \sigma(\rho_i) \notin \Lambda$, then $\neg \bigwedge_{i<n} \sigma(\rho_i) \in \Lambda$, and so $\bigwedge_{i<n} \sigma(\rho_i) \to \sigma(\rho_n) \in \Lambda$. If $\bigwedge_{i<n} \sigma(\rho_i) \in \Lambda$, then as Λ is closed under the substitution-invariant rule R, $\sigma(\rho_n) \in \Lambda$, and therefore $\bigwedge_{i<n} \sigma(\rho_i) \to \sigma(\rho_n) \in \Lambda$. □

Theorem 6.4 *For any set of formulas Γ and substitution-invariant rule R,*

$$\bigcap(\emptyset\text{-Post}(\Gamma) \cap L(R)) = \varepsilon_0(\Lambda_\emptyset(\Gamma \cup \overrightarrow{R})).$$

Proof. $\subseteq$: Consider any formula $\varphi \notin \varepsilon_0(\Lambda_\emptyset(\Gamma \cup \overrightarrow{R}))$. Thus there is a 0-substitution σ such that $\sigma(\varphi) \notin \Lambda_\emptyset(\Gamma \cup \overrightarrow{R})$. A routine argument shows that then, $\Lambda_\emptyset(\Gamma \cup \overrightarrow{R} \cup \{\neg\sigma(\varphi)\})$ is consistent, which can therefore be extended to a $\emptyset$-Post complete modal logic Λ. Since Λ contains the conditionals in $\overrightarrow{R}$, it is closed under R. Thus $\Lambda \in \emptyset\text{-Post}(\Gamma) \cap L(R)$. As Λ is consistent, $\varphi \notin \Lambda$, and therefore $\varphi \notin \bigcap(\emptyset\text{-Post}(\Gamma) \cap L(R))$.

$\supseteq$: Consider any $\Lambda \in \emptyset\text{-Post}(\Gamma) \cap L(R)$ (if there is no such element, this direction is trivial). It suffices to show that $\varepsilon_0(\Lambda_\emptyset(\Gamma \cup \overrightarrow{R})) \subseteq \Lambda$. Since Λ is $\emptyset$-Post complete, it follows with Lemma 6.2 that $\varepsilon_0(\Lambda) = \Lambda$. So by Lemma 6.1 (v), it suffices to show that $\Lambda_\emptyset(\Gamma \cup \overrightarrow{R}) \subseteq \Lambda$, which is immediate using Lemma 6.3.□

[3] The treatment of rules in [4] is slightly different from the present treatment, which affects the definition of general Post completeness. For present purposes, the difference is merely a matter of presentation.

The case of $R = \emptyset$ showed that the operation of 0-expansion appealed to in this result is essential, but Humberstone's observation did not appeal to it. However, the observation falls out as a corollary of Theorem 6.4 with the following lemma:

Lemma 6.5 *For any modal logic* Λ *containing EC,* $\varepsilon_0(\Lambda) = \Lambda$.

Proof. Assume $EC \in \Lambda$; note that this means that Λ is congruential. It follows by a routine induction that for any substitution σ and formula φ built up from proposition letters $p_0, \dots, p_{n-1}$, $\bigwedge_{i<n}(p_i \leftrightarrow \sigma(p_i)) \to (\varphi \leftrightarrow \sigma(\varphi)) \in \Lambda$. By Lemma 6.1 (i), it suffices to show that $\varepsilon_0(\Lambda) \subseteq \Lambda$, so consider any $\varphi \in \varepsilon_0(\Lambda)$ built up from proposition letters $p_0, \dots, p_{n-1}$. Let Σ' be a finite set of 0-substitutions such that for every truth-value assignment among $p_0, \dots, p_{n-1}$, there is $\sigma \in \Sigma'$ mapping each proposition letter among $p_0, \dots, p_{n-1}$ correspondingly to $\top$ or $\bot$. Then $\bigvee_{\sigma \in \Sigma'} \bigwedge_{i<n}(p_i \leftrightarrow \sigma(p_i))$ is a tautology. With the schema derived earlier, it follows that $\bigvee_{\sigma \in \Sigma'}(\varphi \leftrightarrow \sigma(\varphi)) \in \Lambda$. Since $\varphi \in \varepsilon_0(\Lambda)$, $\sigma(\varphi) \in \Lambda$ for all $\sigma \in \Sigma'$, and therefore $\varphi \in \Lambda$. □

The desired corollary follows immediately from Theorem 6.4 and Lemma 6.5:

Corollary 6.6 *For any set of formulas* Γ,

$$\bigcap(\emptyset\text{-Post}(\Gamma) \cap L(C)) = \Lambda_\emptyset(\Gamma \cup \{EC\}).$$

As a second corollary of Theorem 6.4, another characterization of general Post completeness can be obtained:[4]

Corollary 6.7 *A modal logic is generally Post complete if and only if it is the intersection of a non-empty set of* $\emptyset$*-Post complete modal logics.*

Proof. The claim is immediate for inconsistent modal logics, so let Λ be a consistent modal logic. As noted above, Λ is generally Post complete if and only if $\varepsilon_0(\Lambda) = \Lambda$. Letting $R = \emptyset$, it follows from Theorem 6.4 that $\bigcap \emptyset\text{-Post}(\Lambda) = \varepsilon_0(\Lambda)$. If $\varepsilon_0(\Lambda) = \Lambda$, then $\bigcap \emptyset\text{-Post}(\Lambda) = \Lambda$. If $\Lambda = \bigcap S$ for some nonempty set $S \subseteq \emptyset\text{-Post}$, then $S \subseteq \emptyset\text{-Post}(\Lambda)$, so $\Lambda = \bigcap \emptyset\text{-Post}(\Lambda)$, and thus $\varepsilon_0(\Lambda) = \Lambda$. □

Having characterized the intersection of the modal logics extending a given set which are closed under a substitution-invariant rule R and $\emptyset$-Post complete, it is natural to ask for characterizations of similar intersections. First, one might ask for a characterization of the intersection of the modal logics extending Γ which are closed under R and *generally* Post complete. Second, one might ask for a characterization of the intersection of the modal logics extending Γ which are R-Post complete (and thus closed under R). Third, given a substitution-invariant rule $R' \subseteq R$, one might ask for a characterization of the intersection of the modal logics closed under R extending Γ which are R'-Post complete.

[4] The result is due to David Makinson (p.c.), who provided a more direct proof, appealing to the observation that ε_0 commutes with intersection: $\varepsilon_0(\bigcap\{\Gamma_i : i \in I\}) = \bigcap\{\varepsilon_0(\Gamma_i) : i \in I\}$.

The first question can be answered relatively easily; the others will be left open. Writing g-Post(Γ) for the set of generally Post complete modal logics extending Γ, a natural characterization can be given as follows:

Theorem 6.8 *For any set of formulas* Γ *and substitution-invariant rule* R*,*

$$\bigcap(g\text{-Post}(\Gamma) \cap L(R)) = \varepsilon_0(\Lambda_R(\Gamma)).$$

Proof. The claim is immediate if $\Lambda_R(\Gamma)$ is inconsistent, so assume otherwise.

$\subseteq$: It suffices to show that $\varepsilon_0(\Lambda_R(\Gamma)) \in g\text{-Post}(\Gamma) \cap L(R)$. As noted earlier, $\varepsilon_0(\Lambda_R(\Gamma))$ is 0-reducible and therefore generally Post complete; by Lemma 6.1 (iii), it is a member of $L(R)$.

$\supseteq$: For any $\Lambda \in g\text{-Post}(\Gamma) \cap L(R)$, $\Lambda_R(\Gamma) \subseteq \Lambda$. As noted earlier, since Λ is generally Post complete, $\varepsilon_0(\Lambda) = \Lambda$, and therefore with Lemma 6.1 (v), $\varepsilon_0(\Lambda_R(\Gamma)) \subseteq \Lambda$. □

7 Conclusion

This paper made a start at investigating Post completeness among congruential modal logics. Both similarities and differences to the case of normal modal logics were established, most importantly that while $\emptyset$-Post completeness coincides with truth-functionality in both settings, there are $\beth_1$ modal logics Post complete in the set of congruential modal logics, in contrast to the two modal logics Post complete in the set of normal modal logics. The few elementary results established here bring out many open questions.

Two clusters of questions were already mentioned above: First, how many modal logics Post complete in the set of congruential modal logics are the logic of a class of neighborhood frames, and how many (if any) are not? An analogous question arises for quasi-congruential modal logics and neighborhood frames with distinguished elements (and quasi-normal modal logics and relational frames with distinguished elements, a question which seems not to have been considered). Second, for any substitution-invariant rules $R' \subseteq R$ and set of formulas Γ, how can one characterize the intersection of modal logics extending Γ which are R-Post complete, and the intersection of modal logics extending Γ which are R'-Post complete and closed under R?

Similar to the questions considered in Bellissima [1], one could also investigate, for each cardinal $\kappa \leq \beth_1$, the number of congruential modal logics Λ such that $|C\text{-Post}(\Lambda)| = \kappa$.[5] It would also be interesting to know whether what Segerberg [16] calls "Halldén's Theorem" holds among congruential modal logics, in the sense that for all congruential modal logics Λ and Λ', $C\text{-Post}(\Lambda \cap \Lambda') = C\text{-Post}(\Lambda) \cup C\text{-Post}(\Lambda')$. More generally, the question could be asked for any substitution-invariant rule.

Further interesting questions arise from the intersection of modal logics whose only Post complete extension is a specific logic. As Blok and Köhler [3,

[5] [1] is concerned with an analogous question for $\emptyset$-Post completeness, focusing on normal modal logics. Surprisingly, it seems to presuppose the truth of of the continuum hypothesis without mentioning it; see the four-fold case distinction in the proof of Theorem 3.1, p. 133.

p. 952–954] note, one can show with Lemma 5.1 that for any $\emptyset$-Post complete modal logic Λ, the set of modal logics whose only $\emptyset$-Post complete extension is Λ contains its intersection. This, as they note, does not carry over to normal modal logics: while the set of normal modal logics whose only N-Post complete extension is Λ^i contains its intersection, the set of normal modal logics whose only N-Post complete extension is Λ^t does not. The former intersection is $\mathbf{D}$, the smallest normal modal logic containing the axiom $D = \Diamond\top$, and the latter intersection is $\mathbf{K}$. Furthermore, $\mathbf{D}$ and Λ^t give rise to a so-called *splitting* of the lattice of normal modal logic, since for any normal modal logic Λ, $\mathbf{D} \subseteq \Lambda$ or $\Lambda \subseteq \Lambda^t$ but not both. (See [10, section 7.2] for more on splittings.) Since there are far more C-Post complete modal logics than N-Post complete ones, it is an interesting question to ask for which C-Post complete modal logics Λ the set of congruential modal logics whose only C-Post complete extension is Λ contains its intersection, and for cases in which the answer is affirmative, whether the relevant intersection gives rise to a splitting of the lattice of congruential modal logics. One might also ask which C-Post complete modal logics give rise to a splitting of this lattice, and investigate whether there are cases in which the intersection of congruential modal logics whose only C-Post complete extension is a given logic and another C-Post complete modal logic form a splitting pair. More generally, one could consider an arbitrary set S of C-Post complete modal logics, and ask similar questions concerning the set of congruential modal logics whose C-Post complete extensions are precisely the members of S.

Many more questions could be asked concerning Post completeness in congruential modal logics, but the ones mentioned so far should suffice to indicate that much remains to be done in this area.

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