Non-deterministic algebras and algebraization of logics

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1. Introduction

Non-determinism was considered in Computer Science since its beginnings: from non-deterministic Turing machines to models of concurrency, event structures and Petri nets, as well as for variants of process languages and of l-calculus, the use of *multifunctions* instead of ordinary functions (asigning to each element of the domain a set of possible choices, instead of a single value) has revealed to be a extremely useful conceptual tool. Indeed, there is a need for abstraction when modelling computational procedures, by disregarding irrelevant information. Being so, instead of considering all the dependencies on all the possible parameters, they can be represented by (nondeterministic) choices.

In particular, the concept of *non-deterministic algebras* was introduced in Computer Science in order to deal with nondeterminism. Thus, for instance, non-deterministic algebras were proposed as an alternative to define Σ-X-tree-recognizers, which are designed to recognize terms from the free algebra generated by a signature Σ from a set X of generators (cf. [12]). An interesting monograph of non-de-

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terminism in Computer Science from an algebraic perspective can be found in [19].

In the realm of Logic, non-determinism was considered mainly as a tool for obtaining alternative semantics. Non-deterministic matrices constitute a good example of this alternative approach.

The non-deterministic matrices (Nmatrices, for short), introduced in [2], [3] and [1], are a generalization of the usual concept of logic matrix[1] and the main feature of this generalization is that the value that a valuation assigns to a complex formula can be chosen non-deterministically from a non-empty set options. That is, Nmatrices are based on non-deterministic algebras, in contrast with the usual logical matrices which are based on standard algebras.

Many propositional logic can be semantically characterized by the use of a single logic matrix (cf. [17]), but according to Avron and Lev [3], many of them have only infinite characteristic matrices and then such matrices do not provide a good decision procedure for these logics. The Nmatrices allow to replace, in many cases, an infinite characteristic matrix (for a given propositional logic) by a finite characteristic Nmatrix and thus obtain metaproperties such as, for example, decidability. Another problem that motivated Avron and his colaborators to introduce non-determinism (cf. [4]) is the fact that the principle of truth-funcionality[2], inherent to the matrix semantics in general and to classical logic in particular, conflicts with the information present in the "real world", which sometimes may be incomplete, inaccurate and/or inconsistent. Thus, Avron and his collaborators proposed the use of non-determinism (by means of Nmatrices) in order to weaken the principle of truth-funcionality as a solution to this problem.

Although Nmatrices have shown their usefulness in many examples, providing a finitary (and thus decidable) semantics for logic without a truth-functional semantics, such as some Logics of Formal Inconsistency - LFIs (cf. [4]) and certain modal logics (cf. [13]), a sistematic and rigorous study of the algebraic properties of Nmatrices is still missing in the literature. That is, the theory of Nmatrices has not yet been fully developed, from the point of view of its formal properties and expressive power.

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1 Additional information about logic matrices can be found at [18], [9], [17], [14] e [15] .

2 Principle in which the truth-value of a formula is determined functionally by the truth-value of its immediate sub-formulas.

Besides the applications to Computer Science mentioned above, there are few studies on non-deterministic algebras from the perspective of the discipline of Universal Algebra. The generalization of notions such as ultra products, reduced matrices and the Leibniz operator, among others, was not studied with full detail in the non-deterministic context. Thus, in this initial paper we propose the formal study of the theory of Nmatrices from the point of view of universal algebra, with the aim of establishing their potential applications in the realm of algebraic semantics.

In particular, we will focus our efforts in the methodology from Abstract Algebraic Logic (AAL, in short), inaugurated by W. Blok and D. Pigozzi (see [7], [8], [6]), extending techniques involving usual matrices for the more general context of Nmatrices. Thus, many of the known results in the literature on the application of the theory of logic matrices (most of these results can be found in [14] and [21]) can be applied to other logics that do not have a characterization by finite matrices.

This paper contains the initial notions and results developed in what we call *Non-deterministic universal algebra*, which is basically a theory designed to analyze from a very general perspective the usual concepts and results in universal algebra in order to adapt them to the non-deterministic context.

2. Elementary concepts in Universal Algebra

In this section we present some common results in universal algebra, required for the development of a theory of non-deterministic universal algebra.

Definition 1 (Signature): A *signature* Σ is a family $\{\Sigma_n : n \in \mathbb{N}\}$ where each Σ_n is a set (of *n-ary connectives*) such that, if $n \neq m$, then, $\Sigma_n \cap \Sigma_m = \emptyset$. The elements of Σ_0 are called *constants*. The *domain* of Σ is the set

$$|\Sigma| = \cup\Sigma = \{c : c \in \Sigma_n \text{ for some } n \geq 0\}.$$

Definition 2 (Algebra): Let Σ be a signature. An *algebra* A for Σ is a pair $\langle A, \bar{\sigma}\rangle$ where A is a non-empty set (the *domain* of A) and $\bar{\sigma}$ is a function that assigns, for every $n \geq 0$ and $c \in \Sigma_n$ an operation $\bar{\sigma}(c)$: $A^n \to A$ in A.

We will use, throughout the text, the expression $\wp(A)^+$ to denote the set $\wp(A) - \{\emptyset\}$ of all the non-empty subsets of the set A. Also we will often identify one signature Σ with its domain |Σ|, if the latter is finite and if the arity of the connectives are obvious in the context.

Definition 3 (Formulas): Let Σ be a signature and let Ξ be a countable set $\{\xi_m : m \geq 0\}$ of symbols called *variables*. The algebra freely generated by Σ from Ξ will be denoted by L(Σ, Ξ). The elements of L(Σ, Ξ) are called *formulas* (or *schema formulas*) over Σ.

From now on, and given the set Ξ of variables, we only consider signatures Σ such that $\Xi \cap \Sigma_n = \emptyset$ for all $n \geq 0$. The set of variables occurring in a formula $\varphi \in L(\Sigma, \Xi)$ will be denoted by VAR(φ).

Definition 4: Let Σ be a signature, and $\Xi' \subset \Xi$. We denote by L(Σ, Ξ') the subset of L(Σ, Ξ) formed by the schema formulas φ such that $VAR(\varphi) \subseteq \Xi'$. In particular, if $\Xi_n = \{\xi_i : 0 \leq i \leq n\}$ for $n \geq 0$, then $L(\Sigma, \Xi_n)$ is the subset of L(Σ, Ξ) formed by the schema formulas φ such that $VAR(\varphi) \subseteq \{\xi_0, \ldots, \xi_n\}$.

Definition 5 (Total and partial multifunctions): Let A and B be two non-empty sets. A *total multifunction* g from B to A, denoted by g: $B \to_M A$, is a function g: $B \to \wp(A)^+$ in the usual sense. A function g: $B \to \wp(A)$, in turn, corresponds to what we call a *partial multifunction* g from B to A.

Throughout the rest of this text we only use the concept of total multifunction. Thus, a total multifunction will be referred to simply as a multifunction.

Definition 6 (Composition of multifunctions): Let A, B and C be not-empty sets, and let $g_1: C \to_M B$ and $g_2: B \to_M A$ be two multifunctions. The *composed multifunction* is the multifunction $g_2 \circ g_1: C \to_M A$ given by $(g_2 \circ g_1)(c) = \cup\{g_2(b) : b \in g_1(c)\}$, for every $c \in C$.

The proof of the following result is straightforward:

Proposition 7: The partial operation of composition between multifunctions is associative.

3. ND-algebras and ND-Homomorphisms

In this section we present the formal notions of non-deterministic algebras (or ND-algebras) and of homomorphisms between ND-algebras, which are fundamental for the development of non-deterministic universal algebra.

Definition 8 (ND-algebra): Let Σ be a given signature. A *ND--algebra A over Σ* is a pair ⟨A, σ⟩ where A is a non-empty set (the *domain* of A) and σ is a function that assigns to each $n \geq 0$ and $c \in \Sigma_n$, a multifunction $\sigma(c)$: $A^n \rightarrow A$ in A, such that σ (c) corresponds to an unitary set A, if $c \in \Sigma_0$.

We will write henceforward, for simplicity, c^A instead of σ(c). If $c \in \Sigma_0$, the only element of c^A will be denoted by c_A, that is, $c^A = \{c_A\}$. Throughout this text, we can write $\bar{a}$ to denote any n-tuple $a_1, \ldots a_n$ of elements in A. That is, $\bar{a}$ belongs to the Cartesian product A^n.

Avron in 5, p. 162 and p. 163] presents two non-deterministic matrices (or Nmatrices), M^B_3 and M^B_5, that semantically characterize the logical system B, which is known in literature as mbC, one of the Logics of Formal Inconsistency (LFI's)[3]. These Nmatrices will be presented in the following two examples, and subsequently analyzed in the light of the concepts introduced, along with other Nmatrices introduced in the literature.

Example 1: Let $\Sigma = \{\wedge, \vee, \rightarrow, \neg, \circ\}$ and let $M_5 = \langle A_5, D_5, O_5 \rangle$ be the Nmatrix over Σ such that

$A_5 = \{t, t_I, I, f, f_I\}$;

$D_5 = \{t, t_I, I\}$;

[3] Introduced by W. Carnielli and J. Marcos in [11], and thereafter studied in detail in [10].

For each connective c, the multifunction $O_5(c) = c^{A5}$ is defined by the following tables (here F = {f, f_I}).

$\vee^{A5}$	t	t_I	I	f	f_I
t	D_5	D_5	D_5	D_5	D_5
t_I	D_5	D_5	D_5	D_5	D_5
I	D_5	D_5	D_5	D_5	D_5
f	D_5	D_5	D_5	F	F
f_I	D_5	D_5	D_5	F	F

$\wedge^{A5}$	t	t_I	I	F	f_I
t	D_5	D_5	D_5	F	F
t_I	D_5	D_5	D_5	F	F
I	D_5	D_5	D_5	F	F
f	F	F	F	F	F
f_I	F	F	F	F	F

$\rightarrow^{A5}$	t	t_I	I	f	f_I
t	D_5	D_5	D_5	F	F
t_I	D_5	D_5	D_5	F	F
I	D_5	D_5	D_5	F	F
f	D_5	D_5	D_5	D_5	D_5
f_I	D_5	D_5	D_5	D_5	D_5

	$\neg^{A5}$
T	F
t_I	F
I	D5
F	D5
f_I	D5

	$\circ^{A5}$
T	D_5
t_I	F
I	F
F	D_5
f_I	F

Clearly M_5 induces a ND-algebra $A_5 = \langle A_5\ \sigma_5\rangle$ over Σ such that $\sigma_5 = O_5$.

Example 2: Let $\Sigma = \{\wedge, \vee, \rightarrow, \neg, \circ\}$ and let $M_3 = \langle A_3, D_3, O_3\rangle$ be the Nmatrix such that

$A_3 = \{t', I', f'\}$;

$D_3 = \{t', I'\}$;

For each connective c, the multifunction $O_3(c) = c^{A3}$ is defined by the following tables.

$\vee^{A3}$	t'	I'	f'
t'	D_3	D_3	D_3
I'	D_3	D_3	D_3
f'	D_3	D_3	{f'}

$\wedge^{A3}$	t'	I'	f'
t'	D_3	D_3	{f'}
I'	D_3	D_3	{f'}
f'	{f'}	{f'}	{f'}

$\rightarrow^{A3}$	t'	I'	f'
t'	D_3	D_3	{f'}
I'	D_3	D_3	{f'}
f'	D_3	D_3	D_3

	$\neg^{A3}$
t'	{f'}
I'	D_3
f'	D_3

	$\circ^{A3}$
t'	A_3
I'	{f'}
f'	A_3

Let A_3 = <A3, σ3> such that $\sigma_3 = O_3$. Thus, A_3 is an ND-algebra over Σ.

Example 3: Let Σ = {∧, ∨, →, ¬, ∘} and let $M'_3 = \langle A'_3, D'_3, O'_3 \rangle$ be the Nmatrix such that

$A'_3 = \{t', t'_I, I', f', f'_I\}$;

$D'_3 = \{t', I'\}$;

For each connective c, the multifunction $O'_3(c) = c^{A'3}$ is defined by the following tables (here F ' = {f '}).

$\vee^{A'3}$	t'	t'_I	I'	f'	f'_I
t'	D'_3	D'_3	D'_3	D'_3	D'_3
t'_I	D'_3	D'_3	D'_3	D'_3	D'_3
I'	D'_3	D'_3	D'_3	D'_3	D'_3
f'	D'_3	D'_3	D'_3	F'	F'
f'_I	D'_3	D'_3	D'_3	F'	F'

$\wedge^{A'3}$	t'	t'_I	I'	f'	f'_I
t'	D'_3	D'_3	D'_3	F'	F'
t'_I	D'_3	D'_3	D'_3	F'	F'
I'	D'_3	D'_3	D'_3	F'	F'
f'	F'	F'	F'	F'	F'
f'_I	F'	F'	F'	F'	F'

$\rightarrow^{A'3}$	t'	t'_I	I'	f'	f'_I
t'	D'_3	D'_3	D'_3	F'	F'
t'_I	D'_3	D'_3	D'_3	F'	F'
I'	D'_3	D'_3	D'_3	F'	F'
f'	D'_3	D'_3	D'_3	D'_3	D'_3
f'_I	D'_3	D'_3	D'_3	D'_3	D'_3

	$\neg^{A'3}$
t'	F'
t'_I	F'
I'	D'_3
f'	D'_3
f'_I	D'_3

	$\circ^{A'3}$
t’	{t’, I’, f ‘}
t'_I	{t’, I’, f ‘}
I’	F ‘
f ‘	{t’, I’, f ‘}
f'_I	{t’, I’, f ‘}

Clearly M'_3 induces a ND-algebra $A'_3 = \langle A'_3, \sigma'_3\rangle$ over Σ such that $\sigma'_3 = O'_3$.

Definition 9 (Homomorphism of ND-algebras): Let $A = \langle A, \sigma\rangle$ and $B = \langle B, \sigma'\rangle$ be two ND-algebras over a signature Σ. A *homomorphism* h: A → B of ND-algebras over Σ is a function h: A ⟶ B such that for all $n \geq 0$, $c \in \Sigma_n$ and $a_1, \ldots, a_n \in A$,

$h[c^A(a_1, \ldots, a_n)] \subseteq c^B(h(a_1), \ldots, h(a_n))$.

In particular, $h(c_A) = c_B$, if $c \in \Sigma_0$.[4]

Notation: We will use the brackets: ‘’[‘’ and ‘’]”, to differentiate when a function is applied on a set of when it is applied to an element of its domain.

Example 4: Let $A_5 = \langle A_5\ \sigma_5\rangle$ and $A_3 = \langle A_3, \sigma_3\rangle$ be the ND-algebras introduced in examples 1 and 2, respectively. Let h: $A_5 \rightarrow A_3$ be a function such that $h(t) = h(t_I) = t'$; $h(t_I) = I'$ and h(f) = h(f ‘) = f ‘. Clearly

$h[D_5] = D_3$ and h(F) = {f ‘}.

h defines a homomorphism h: $A_3 \rightarrow A_5$.

On the other hand, the function h’: $A_3 \rightarrow A_5$ such that h’ (t’) = I; h’ (I’) = f_I and h’ (f ‘) = h’ (f ‘) = t_I does not define a homomorphism h’: $A_3 \rightarrow A_5$.

Henceforward, and when there is no chance of confusion, we assume that the ND-algebras are defined over a fixed signature Σ.

Proposition 10: Let Σ be a signature. The ND-algebras over Σ, together with their homomorphisms form a category, which will be called ND (Σ).

4 Remember that , if $c \in \Sigma_0$, we write $\sigma(c) = c^A = \{c_A\}$.

The proof of this fact is easy: it is enough to show that the usual composition of functions produces a homomorphism and that the identity maps produce the identity homomorphisms.

Definition 11 (Full homomorphism of ND-algebras): Let A = $\langle A, \sigma \rangle$ and B = $\langle B, \sigma' \rangle$ be two ND-algebras over a signature Σ. A *full homomorphism* h: A → B of ND-algebras over Σ is a function h: A → B such that h is a homomorphism and for all n > 0, $c \in \Sigma_n$ and $a_1, \ldots, a_n \in A$,

$$c^B(h(a_1), \ldots, h(a_n)) \subseteq h[c^A(a_1, \ldots, a_n)].$$

That is, h is full homomorphism if, and only if

$$h[c^A(a_1, \ldots, a_n)] = c^B(h(a_1), \ldots, h(a_n)).$$

for all $c \in \Sigma_n$ and $a_1, \ldots, a_n \in A$, with n > 0.

4. Sub-ND-algebras and Sub-ND-Universes.

Now we analyze the notion of sub-ND-algebra, fundamental to our overall study of ND-algebras.

Definition 12 (Sub-ND-algebra): Let A = $\langle A, \sigma \rangle$ and B = $\langle B, \sigma' \rangle$ be two ND-algebras over Σ such that $B \subseteq A$. We say that B is a *sub--ND-algebra* of A over Σ, denoted by B ⊆ A, if for every $n \geq 0$, $c \in \Sigma_n$ and $b_1, \ldots, b_n \in B$, $c^B(b_1, \ldots, b_n) = c^A(b_1, \ldots, b_n)$.

As with the usual algebras, a non-empty subset of the domain of a ND-algebra generates a single sub-ND-algebra.

Example 5: Let $A_3 = \langle A_3, \sigma_3 \rangle$ and $A'_3 = \langle A'_3, \sigma'_3 \rangle$ be the ND-algebras introduced in examples 2 and 3, respectively, such that $A_3 \subseteq A'_3$.

By the definition of σ_3 and σ'_3, is immediate that A_3 is sub-ND-algebra A'_3, that is, $A_3 \subseteq A'_3$.

Definition 13 (Sub-ND-universe): Let A = $\langle A, \sigma \rangle$ be a ND-algebra over Σ. A *sub-ND-universe* of A over Σ is a non-empty subset B of A that is closed under the operations of A. That is, for any $n \geq 0$, $c \in \Sigma_n$ and $b_1, \ldots, b_n \in B$, $c^A(b_1, \ldots, b_n) \subseteq B$.

Example 6: Let $\mathbf{A}_3 = \langle A_3, \sigma_3\rangle$ and $\mathbf{A'}_3 = \langle A'_3, \sigma'_3\rangle$ be the ND-algebras introduced in examples 2 and 3, respectively, such that $A_3 \subseteq A'_3$. By the definition of σ'_3, is immediate that A_3 is sub-universe of $\mathbf{A'}_3$.

Definition 14 (Generated sub-ND-universe): Let $\mathbf{A} = \langle A, \sigma\rangle$ be a ND-algebra over Σ and $\emptyset \neq X \subseteq A$. The *sub-universe of* $\mathbf{A}$ *generated by* X *over* Σ, denoted by $sg^{\mathbf{A}}_{\Sigma}(X)$ (or simply $sg(X)$) is defined as follows:

$sg(X) = \bigcap\{B : B \text{ is a sub-ND-universe of } \mathbf{A} \text{ over } \Sigma, \text{ and } X \subseteq B\}$.

Note that A is a always a sub-ND-universe of $\mathbf{A}$ over Σ containing X, then $\{B : B \text{ is a sub-ND-universe of } \mathbf{A} \text{ over } \Sigma, \text{ and } X \subseteq B\} \neq \emptyset$. Thus, $sg(X)$ is well defined.

Proposition 15: The set $sg(X)$ is a sub-ND-universe of $\mathbf{A}$ over Σ.

Proof: Note that $sg(x) \subseteq A$ and $sg(X) \neq \emptyset$, because $\emptyset \neq X \subseteq sg(X)$. Let $n \geq 0$, $c \in \Sigma_n$ and $b_1, ..., b_n \in sg(X)$. Let B be a sub-ND-universe of $\mathbf{A}$ such that $X \subseteq B$. Since $b_1, ..., b_n \in B$ then $c^{\mathbf{A}}(b_1, ..., b_n) \subseteq B$. Hence, $c^{\mathbf{A}}(b_1, ..., b_n) \subseteq sg(X)$, and then $sg(X)$ is a sub-ND-universe of $\mathbf{A}$. ■

As in the case of the usual algebras, it is possible to give a constructive definition of $sg(X)$:

Proposition 16: Let $\mathbf{A} = \langle A, \sigma\rangle$ be a ND-algebra on Σ and $\emptyset \neq X \subseteq A$. Consider the family $\{E^n(X): n \geq 0\}$ of subsets defined by induction as follows:

$E^0(X) = X$;

$E^{n+1}(X) = E^n(X) \cup \bigcup\{ c^{\mathbf{A}}(a_1, ..., a_k): k \geq 0; c \in \Sigma_k \text{ and } a_1, ..., a_k \in E^n(X)\}$.

So, $sg(X) = \bigcup\{E^n(X): n \geq 0\}$.

The proof is obtained by showing separately that $sg(X) \subseteq \bigcup\{E^n(X): n \geq 0\}$ and that $\bigcup\{E^n(X): n \geq 0\} \subseteq sg(X)$. The first half is easily done by definition, and the second half can be easily proved by induction on n.

Definition 17 (Sub-ND-algebra generated): Let $\mathbf{A} = \langle A, \sigma\rangle$ be a ND-algebra over Σ and $\emptyset \neq X \subseteq A$. We say that $\langle A, \sigma\rangle$ is *generated* by X, if $sg(X) = A$.

We can now to define the sub-ND-algebra generated by a non--empty subset of its domain:

Proposition 18: Let A = $\langle A, \sigma \rangle$ be a ND-algebra over Σ, and $\emptyset \neq X \subseteq A$. Then $SG(X) = \langle sg(X), \sigma^X \rangle$ such that $c^{SG(x)}(a_1, \ldots, a_n) = c^A(a_1, \ldots, a_n)$ for any $n \geq 0$, $c \in \Sigma_n$ and $a_1, \ldots, a_n \in sg(X)$, is the only sub-ND--algebra of A generated by X.

Proof: (Existence) Clearly, $SG(X) \subseteq A$, then, by definition, $sg(X) \subseteq A$ and by definition of $SG(X)$, $c^{SG(X)}(a_1, \ldots, a_n) = c^A(a_1, \ldots, a_n)$ for any $n \geq 0$, $c \in \Sigma_n$ and $a_1, \ldots, a_n \in sg(X)$.

(Uniqueness) Let X and Y be two sets, such that $X \neq Y$, $\emptyset \neq X \subseteq A$ and $\emptyset \neq Y \subseteq A$.

Assume that $SG(X) = \langle sg(X), \sigma^X \rangle$ where $c^{SG(X)}(a_1, \ldots, a_n) = c^A(a_1, \ldots, a_n)$ for any $n \geq 0$ and $c \in \Sigma_n$ is a sub-ND-algebra of A generated by X, and $SG(Y) = \langle sg(Y), \sigma^Y \rangle$ such that $c^{SG(Y)}(b_1, \ldots, b_n) = c^A(b_1, \ldots, b_n)$ for any $n \geq 0$, $c \in \Sigma_n$ and $b_1, \ldots, b_n \in sg(Y)$ is a sub-ND-algebra of A generated by Y. Clearly $sg(X) = sg(Y)$ and $c^{SG(X)}(a_1, \ldots, a_n) = c^A(a_1, \ldots, a_n) = c^A(b_1, \ldots, b_n) = c^{SG(Y)}(b_1, \ldots, b_n)$, thus $\langle sg(X), \sigma^X \rangle = \langle sg(Y), \sigma^Y \rangle$.

Now we will prove that, if h: A → B is a homomorphism of ND--algebras, then the image by h(X) of $SG(X)$ is contained in $SG(h[X])$.

Lemma 19: Let A = $\langle A, \sigma \rangle$ and B = $\langle B, \sigma' \rangle$ be two ND-algebras over Σ, $\emptyset \neq X \subseteq A$, and let h: A → B be a homomorphism of ND-algebras. If $E^n(X)$ and $E^n(h[X])$ are defined inductively as in Proposition 16, then $h[En(X)] \subseteq E^n(h[X])$.

Proof: The proof is by induction on n, for $n \geq 0$. If n = 0, $h[E^0(X)] = h[X] = E^0(h[X])$. Suppose that $h[E^n(X)] \subseteq E^n(h[X])$, then

$h[E^{n+1}(X)] =$

$h[E^n(X) \cup \bigcup\{ c^A(a_1, \ldots, a_k) : k \geq 0, c \in \Sigma_k \text{ and } a_1, \ldots, a_k \in E^n(X) \}] =$

$h[E^n(X)] \cup h[\bigcup\{c^A(a_1, \ldots, a_k) : k \geq 0, c \in \Sigma_k \text{ and } a_1, \ldots, a_k \in E^n(X) \}] =$

$E^n(h[X]) \cup \bigcup\{h[c^A(a_1, \ldots, a_k)] : k \geq 0, c \in \Sigma_k \text{ and } a_1, \ldots, a_k \in E^n(X) \} =$

$E^n(h[X]) \cup \bigcup\{c^B(h(a_1), \ldots, h(a_k))] : k \geq 0, c \in \Sigma_k \text{ and } h(a_1), \ldots, h(a_k) \in E^n(h(X)) \} =$

$E^{n+1}(h[X])$.

Theorem 20: Let A = $\langle A, \sigma\rangle$ and B = $\langle B, \sigma'\rangle$ be two ND-algebras, $\emptyset \neq X \subseteq A$, and let h: A → B be a homomorphism of ND-algebras. Then

$h[sg(X)] \subseteq sg(h[X])$.

Proof: By Proposition 16 we have

$h[sg(X)] = h[\cup\{E^n(X) : n \geq 0\}] = \cup\{h[E^n(X)] : n \geq 0\}$.

Using the previous lemma and Proposition 16 we have again

$\cup\{h[E^n(X)] : n \geq 0\} \subseteq \cup\{E^n(h[X]) : n \geq 0\} = sg(h[X])$.

Therefore, $h[sg(X)] \subseteq sg(h[X])$.

Definition 21: Let A = $\langle A, \sigma\rangle$ and B = $\langle B, \sigma'\rangle$ be two ND-algebras over a signature Σ, h: A → B is a full homomorphism of ND-algebras over Σ, and let A = $\langle A', \sigma''\rangle$ be a sub-ND-algebra of A. Then h(A') = $\langle h[A'], \sigma^{h(A')}\rangle$ is the sub-ND-algebra such that, for all $n \geq 0$, $c \in \Sigma_n$ and $b_1, ..., b_n \in h[A']$,

$c^{h(A')}(b_1, ..., b_n) = \cup\{h[c^{A'}(a'_1, ..., a'_n)] : h(a'_i) = b_i \text{ for } 1 \leq i \leq n\}$.

Corollary 22: Let A = $\langle A, \sigma\rangle$ and B = $\langle B, \sigma'\rangle$ be two ND-algebras over a signature Σ, h: A → B is a full homomorphism of ND-algebras, and $\emptyset \neq X \subseteq A$. Then, the image by h of *SG*(X) is a sub-ND-algebra of *SG*(h[X]).

Proof: It is clear that Theorem 20 is still valid when h is a full homomorphism, thus $[sg(X)] \subseteq sg(h[X])$ and for any $n \geq 0$, $c \in \Sigma_n$ and $b_1, \ldots, b_n \in h[sg(X)]$, we have that

$c^{h(SG(X))}(b_1, ..., b_n) = \cup\{h[c^{SG(X)}(a_1, ..., a_n)] : h(a_i) = b_i \text{ for } 1 \leq i \leq n\} =$
$\cup\{h[c^{A}(a_1, ..., a_n)] : h(a_i) = b_i \text{ for } 1 \leq i \leq n\} =$
$\cup\{c^{B}(h(a_1), ..., h(a_n)) : h(a_i) = b_i \text{ for } 1 \leq i \leq n\} =$
$c^{B}(b_1, ..., b_n) = c^{SG(h[X])}(b_1, ..., b_n)$.

5. Products of ND-Algebras.

Now, we analyze the definition of products in the category of ND-algebras, adapting the classic definition of products given in universal algebra. Thus, it will be shown that the category of ND-algebra over a given signature is closed by arbitrary products.

Definition 23 (ND-Products): Let $A_1 = \langle A_1, \sigma_1\rangle$ and $A_2 = \langle A_2, \sigma_2\rangle$ be two ND-algebras over Σ. The (direct) ND-product $A_1 \times A_2$ is the ND--algebra $\langle A_1 \times A_2, \sigma^P\rangle$ over Σ such that $c^{A1\times A2}(\langle a_{11}, a_{21}\rangle, \ldots, \langle a_{1n}, a_{2n}\rangle) = c^{A}(a_{11}, \ldots, a_{1n}) \times c^{A}(a_{21}, \ldots, a_{2n})$ for any $a_{1j} \in A_1$ and $a_{2j} \in A_2$, with $1 \leq j \leq n$. In particular, if $c \in \Sigma_0$, $c_{A1\times A2} = \langle c_{A1}, c_{A2}\rangle$.

Definition 24 (canonical projections): Let A_1 and A_2 be sets. The function π_i: $A_1 \times A_2 \longrightarrow A_i$ defined by $\pi_i(\langle a_1, a_2\rangle) = a_i$ for every $a_1 \in A_1$ and $a_2 \in A_2$, is called the *ith- canonical projection* of $A_1 \times A_2$, for i = 1,2.

Proposition 25: Let $A_1 = \langle A_1, \sigma_1\rangle$ and $A_2 = \langle A_2, \sigma_2\rangle$ be two ND--algebras on Σ. Then, the canonical projections π_i: $A_1 \times A_2 \rightarrow A_i$ (i = 1,2) are full homomorphisms.

The proof is immediate from the definitions.

Proposition 26: Let $A_1 = \langle A_1, \sigma_1\rangle$ and $A_2 = \langle A_2, \sigma_2\rangle$ be two ND--algebras over Σ, and let π_i: $A_1 \times A_2 \rightarrow A_i$ (i = 1,2) be the canonical projections of $A_1 \times A_2$. Then $\langle A_1 \times A_2, \langle \pi_1, \pi_2\rangle\rangle$ is the product of A_1 and A_2 in the category ND(Σ).

Proof: We have to show that $\langle A_1 \times A_2, \langle \pi_1, \pi_2\rangle\rangle$ satisfies the following universal property: if $B = \langle B, \sigma'\rangle$ is a ND-algebra and f_i: $B \longrightarrow A_i$, for i = 1, 2, are homomorphisms, then there is a unique homomorphism h: $B \rightarrow A_1 \times A_2$, such that $f_i = \pi_i \circ h$ for i = 1, 2. Thus, consider the function h: $B \rightarrow A_1 \times A_2$ such that $h(b) = \langle f_1(b), f_2(b)\rangle$, for all $b \in B$.

I) $h[c^B(b_1, \ldots, b_n)] = \{h(b) : b \in c^B(b_1, \ldots, b_n)\} = \{\langle f_1(b), f_2(b)\rangle : b \in c^B(b_1, \ldots, b_n)\} \subseteq$

$f_1[c^B(b_1, \ldots, b_n)] \times f_2[c^B(b_1, \ldots, b_n)]$. As by hypothesis f_i: $B \rightarrow A_i$, for i = 1, 2 are homomorphisms, then $f_i[c^B(b_1, \ldots, b_n)] \subseteq c^{Ai}(f_i(b_1), \ldots, f_i(b_n))$ and, thus $h[c^B(b_1, \ldots, b_n)] \subseteq c^{A1}(f_1(b_1), \ldots, f_1(b_n)) \times c^{A2}(f_2(b_1), \ldots, f_2(b_n)) = c^{Ai\times Ai}(h(b_1), \ldots, h(b_n))$. Therefore h: $B \rightarrow A_1 \times A_2$ is a homomorphism.

II) $f_i(b) = \pi_i(\langle f_1(b), f_2(b)\rangle) = \pi_i(h(b))$, for i = 1,2, by the definition of h.

III) Suppose that there are two homomorphisms h_1: B → $A_1 \times A_2$ and h_2: B → $A_1 \times A_2$ such that $f_i = \pi_i \circ h_j$ for i, j = 1,2. So $\pi_i(h_1(b)) = f_i(b) = \pi_i(h_2(b))$ for i = 1,2 and b ∈ B. Therefore, $h_1 = h = h_2$ and so the homomorphism h: B → $A_1 \times A_2$ is unique.

Definition 27 (general canonical projections): Let I be a set and let $(A_i)_{i\in I}$ be a family of ND-algebras over Σ. The function π_j: $\prod_{i\in I} A_i \to A_j$ defined by $\pi_j(a) = a(j)$, is called the *j-th canonical projection* of $\prod_{i\in I} A_i$.

Definition 29 (general products): Let I a set, such that i ∈ I and $(A_i)_{i\in I}$ is a family of ND-algebras on Σ. The product (direct) A = $\prod_{i\in I} A_i$ is the ND-algebra $\langle\prod_{i\in I} A_i, \sigma^{P\Pi}\rangle$ on Σ such that $c^{A}(a_1, \ldots , a_n) = \prod_{i\in I} c^{Ai}(a_{i1}, \ldots , a_{in})$, for all c ∈ Σ_n and $a_1, \ldots, a_n \in \langle\prod_{i\in I} A_i\rangle$.

Proposition 28: The canonical projections π_j: $\prod_{i\in I} A_i \to A_j$ are full homomorphisms.

Proposition 30: Let I be a set, $(A_i)_{i\in I}$ is a family of ND-algebras over Σ, and let π_j: $\prod_{i\in I} A_i \to A_j$ be the j-th canonical projection of $\prod_{i\in I} A_i$. Then $\langle\prod_{i\in I} A_i, (\pi_i)_{i\in I}\rangle$ is the product of the family $(A_i)_{i\in I}$ in the category ND(Σ).

6. Interpretation of formulas in ND-algebras.

In this section we define the concept of interpretation of formulas (over a signature Σ) in an ND-algebra (over Σ). To do this, we must use assignments, which will interpret the schema variables occurring in the formula.

Definition 31 (Selector): Let A and B be non-empty sets, g: B $\to_M$ A is a multi-function, and A^B is the set of all functions from B to A. A *selector* of g is a function λ: B → A such that λ (b) ∈ g(b) for all b ∈ B. Let SEL(g) = {λ ∈ A^B : λ is a selector of g}.

Definition 32 (ND-assignment): Let A = ⟨A, σ⟩ be a ND-algebra. A *ND-assigment* in A is a function ρ: Ξ ⟶ A.

Note that, as well as the constants assume a single value in ND--algebras (instead of a multiplicity of values), we will define, in a coherent way, that the variables are instantiated in individual values of the algebra, rather than being instantiated in non-empty sets of elements of the algebra. So, from our perspective, the non-determinism in the ND-algebras only appears in the complex level, that is, when operators (different of the constants) are effectively applied to the elements of the algebra.

Definition 33 (interpretation of formulas in a ND-algebra): Let $A = \langle A, \sigma\rangle$ be a ND-algebra and let ρ be a ND-assigment in A. The multifunction $(\cdot)^{A\rho}$: $L(\Sigma, \Xi) \to_M A$ is the *interpretation* of φ in A by ρ is the non-empty subset $\varphi^{A\rho}$ of A defined by induction on the complexity of the formula φ as follows:

$\xi^{A\rho} = \{\rho(\xi)\}$, if $\xi \in \Xi$;

$c^{A\rho} = \{c_A\}$, if $c \in \Sigma_0$;

$c(\varphi_1 ..., \varphi_n)^{A\rho} = \cup\{c^A(a_1, \ldots, a_n) : a_i \in \varphi_i^{A\rho}$ for $1 \leq i \leq n\}$, if $n > 0$, $c \in \Sigma_n$ and $\varphi_i \in L(\Sigma, \Xi)$, for $1 \leq i \leq n$.

Notation: If ρ is an ND-assignment in a ND-algebra $A = \langle A, \sigma\rangle$, $\varphi \in L(\Sigma, \Xi_n)$ and $\rho(\xi_i) = a_i$, with $1 \leq i \leq n$, we will write $\varphi^A(a_1, ..., a_n)$ instead of $\varphi^{A\rho}$.

7. ND-congruences and ND-Quotient Algebras.

The concepts of congruence and quotient algebra are essential tools in Blok and Pigozzi's theory of algebraization of logical systems. Aiming for possible applications of ND-algebra theory within the algebraic semantics of logical systems, this section will discuss the definition of congruence and quotient algebra in the context of ND-algebras.

Definition 34 (ND-Congruence): Let $A = \langle A, \sigma\rangle$ be a ND-algebra over a signature Σ and let $\theta \subseteq A \times A$ be a relation in A. We say that θ is a *congruence* in A if, and only if:

θ is an equivalence relation;

for all $n > 0$, $c \in \Sigma_n$ and $a_1, ..., a_n, b_1, ..., b_n \in A$, if $a_i\theta b_i$ for all $1 \leq I \leq n$, then:

for all $a \in c^{A}(a_1 ..., a_n)$ there is $b \in c^{A}(b_1 ..., b_n)$ such that $a\theta b$;
for all $b \in c^{A}(b_1 ..., b_n)$ there is $a \in c^{A}(a_1 ..., a_n)$ such that $b\theta a$.

Example 7: Let $\Sigma = \{\wedge, \vee, \rightarrow, \neg, \circ\}$, A'_3 the ND-algebra introduced in Example 3, and let $\theta = \{\langle t'_I, t' \rangle, \langle t', t'_I \rangle, \langle f'_I, f' \rangle, \langle f', f'_I \rangle\} \cup \{\langle a, a \rangle \in A'_3\} \subseteq A'_3 \times A'_3$. Then θ is a congruence in A'_3.

Proposition 35: Let $A = \langle A, \sigma \rangle$ be a ND-algebra over a signature Σ and let $\theta \subseteq A \times A$ be a congruence on A. Then, for all $\varphi \in L(\Sigma, \Xi_n)$ (with $n > 0$) and for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A^n$ such that $a_i\theta b_i$ (for $1 \leq I \leq n$), the following holds:
for all $a \in \varphi^{A}(a_1, \ldots, a_n)$ there is $b \in \varphi^{A}(b_1, \ldots, b_n)$ such that $a\theta b$;
for all $b \in \varphi^{A}(b_1, \ldots, b_n)$ there is $a \in \varphi^{A}(a_1, \ldots, a_n)$ such that $b\theta a$.

The proof can be easily done by induction on the complexity of φ.

Definition 36: Let $A = \langle A, \sigma \rangle$ be a ND-algebra over a signature Σ and let $\theta \subseteq A \times A$ be a congruence on A. The *ND-algebra quotient* of A by θ, denoted by A/θ, is the ND-algebra of universe A/θ with operations $c^{A/\theta}(a_1/\theta, \ldots, a_n/\theta) = \{a/\theta : a \in c^{A}(a_1, \ldots, a_n)\}$, where a/θ is the equivalence class of a, also called the *congruence class* of a.

Proposition 37: If $A = \langle A, \sigma \rangle$ is a ND-algebra over a signature Σ and $\theta \subseteq A \times A$ is a congruence on A, then A/θ is indeed a ND-algebra whose operations are well defined.

The proof is straightforward.

8. Filters, ultrafilters and ultraproducts.

In this section we will show, using our definition of ND-algebra quotient, that it is possible to define the ultraproduct (this is, the reduced product with respect to an ultrafilter) of any family of ND-algebras.

Proposition 38: Let I be a set, $U \subseteq \wp(I)$ an ultrafilter on I, $(A_i)_{i\in I}$ a family of ND-algebras over Σ and $\theta_U \subseteq (\prod_{i\in I} A_i)^2$ defined as follows:

$a\theta_U b$ if and only if $\{i \in I : a(i) = b(i)\} \in U$. Then, θ_U is a congruence on the ND-algebra $A = \prod_{i\in I} A_i$.

Proof: Clearly θ_U is an equivalence relation. Now, we show that θ_U satisfies the definition of ND-congruence. Let $n > 0$, $c \in \Sigma_n$ and $a_1, \ldots, a_n, b_1, \ldots, b_n \in A = \prod_{i\in I} A_i$, such that $a_j\theta_U b_j$ for $1 \leq j \leq n$. Then, by definition of θ_U and by the properties of U we have that $R = \{i : a_1(i) = b_1(i), ..., a_n(i) = b_n(i)\} \in U$.

Now, let $S = \{i : c^{A_i}(a_1(i), \ldots, a_n(i)) = c^{A_i}(b_1(i), \ldots, b_n(i))\} \in U$. Clearly $R \subseteq S$, therefore $S \in U$.

Let $x \in c^A(a_1, \ldots, a_n) = \prod_{i\in I} c^{A_i}(a_1(i), \ldots, a_n(i))$, and define $y \in A$ such that $y(i) = x(i)$, for $i \in S$, and $y(i) \in c^{A_i}(b_1(i), \ldots, b_n(i))$, if $i \notin S$. Since $c^A(b_1, \ldots, b_n) = \prod_{i\in I} c^{A_i}(b_1(i), \ldots, b_n(i))$, then $y \in c^A(b_1, \ldots, b_n)$. Moreover, $S \subseteq \{i : x(i) = y(i)\} = T$ and then $T \in U$. Therefore $x\theta_U y$.

Analogously we can prove that, if $y \in c^A(b_1, \ldots, b_n)$, there is $x \in c^A(a_1, \ldots, a_n)$ such that $y\theta_U x$.

This shows that θ_U is a congruence on the ND-algebra A.

Definition 39 (Ultraproduct): Let I be a set, $U \subseteq \wp(I)$ an ultrafilter on I, $(A_i)_{i\in I}$ a family of ND-algebras on Σ and $\theta_U \subseteq (\prod_{i\in I} A_i)^2$. The *ultraproduct* $\prod_{i\in I} A_i/U$ is the ND-algebra quotient $\prod_{i\in I} A_i/\theta_U$.

Final considerations

The study of the usual logical matrices and Nmatrices, but mainly the fundamental tools of universal algebra, enabled the development of the first original results in what we call *non-deterministic universal algebra*.

In this theory, non-deterministic algebraic structures called *ND--algebras* were introduced, whose non-deterministic operations produce non-empty sets of values, rather than individual values. Several notions and basic constructions from universal algebra were adapted to the non-deterministic framework.

Concerning the next steps, we will focus our efforts in the methodology from Abstract Algebraic Logic (AAL, in short), inaugurated by W.

Blok and D. Pigozzi (see [7], [8], [6]), extending techniques involving usual matrices for the more general context of Nmatrices. Thus, many of the known results in the literature on the application of the theory of logic matrices (most of these results can be found in [14] and [20]) could be applied to other logics that do not have a characterization by finite matrices.

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