Paraconsistent Belief Revision
based on a formal consistency operator*

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Abstract

In this paper two systems of AGM-like Paraconsistent Belief Revision are overviewed, both defined over Logics of Formal Inconsistency (**LFI**s) due to the possibility of defining a formal consistency operator within these logics. The AGM$\circ$ system is strongly based on this operator and internalize the notion of formal consistency in the explicit constructions and postulates. Alternatively, the AGMp system uses the AGM-compliance of **LFI**s and thus assumes a wider notion of paraconsistency – not necessarily related to the notion of formal consistency.

key-words Paraconsistent Belief Revision, paraconsistency, logics of formal inconsistency, contradiction, AGM-compliance.

1 Introduction

The presentation will be divided in four main parts:

- Present the Logics of Formal Inconsistency [3];
- Recall the notion of AGM-compliance [4];
- Present the AGMp system [9, 10];
- Present the AGM$\circ$ system [9, 10].

1.1 Rationality criteria of AGM system

Gärdenfors and Rott [5] adopt the following rationality criteria:

1. Where possible, epistemic states should remain consistent;[1]
2. Any sentence logically entailed by beliefs in an epistemic state should be included in the epistemic state;
3. When changing epistemic states, loss of information should be kept to a minimum;
4. Beliefs held in higher regard should be retained in favour of those held in lower regard.

1.1.1 Revision operation

Definition 1.1 (Internal Revision)**.** $K * \alpha = (K - \neg\alpha) + \alpha$

Definition 1.2 (External Revision)**.** $K * \alpha = (K + \alpha) - \neg\alpha$

Our main objective in constructing Paraconsistent Belief Revision systems is to allow the reasoning in contradictory epistemic states. Should the presence of contradictions make it impossible to derive anything sensible from a theory where such contradictions appear, as the classical logician would maintain? Or are there situations, like in the external revision, in which contradictions in theories are at least temporarily admissible?

2 On Paraconsistency

In classical logic, contradictoriness (the presence of contradictions in a theory) and triviality (the fact that such a theory entails all possible consequences) are assumed inseparable. This is an effect of a logical property known as *explosiveness* (*ex falso quodlibet* or *ex contradictione sequitur quodlibet*, that is, anything follows from a contradiction). According to it, from a contradiction everything is derivable. Therefore classical logic (as many other logics)

*The very first ideas of this paper was presented in [11]. The main final results are contained in the PhD thesis [9] (in portuguese). This preprint is a short English version of those technical main results. A final version, with substantial modifications and content incorporation, can be found in [10].

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[1]That is, they must be non-trivial.

equate consistency with freedom from contradictions. Thus such logics forcibly fail to distinguish between contradictoriness and other forms of inconsistency.

Paraconsistent logics are precisely the logics that challenge this assumption by rejecting the classical consistency presupposition.

2.1 The Logics of Formal Inconsistency

The Logics of Formal Inconsistency (**LFI**s) [3] constitute the class of paraconsistent logics which can internalize the meta-theoretical notions of consistency and inconsistency. As a consequence, despite constituting fragments of consistent logics, the **LFI**s can canonically be used to faithfully encode all consistent inferences.

Roughly, the idea in the **LFI**s is to express the meta-theoretical notions of consistency and inconsistency at the object language level, by adding to the language a new connective $\bullet$ with the intended meaning of "being inconsistent". However, it is the dual connective $\circ$ expressing "being consistent" that is used more frequently. Using the consistency operator, one can limit the applicability of the explosion principle to the case when α is consistent, that is, in any **LFI** it holds the following:

(1) Explosion Principle $\alpha, \neg\alpha \vdash \beta$ is not the case in general

(2) Gentle Explosion Principle $\alpha, \neg\alpha, \circ\alpha \vdash \beta$ is always the case.

The pragmatic point thus is not whether contradictory theories exist, but how to deal with them. In this work we present two systems of *Paraconsistent Belief Revision* – AGMp and AGM$\circ$ (see [9] for more details). Both systems are defined over Logics of Formal Inconsistency, but the constructions of the second are specially related to the formal consistency operator.[2]

Specifically, we define the constructions over a particular class of **LFI**s, developed by Carnielli, Coniglio and Marcos [3], in which the formal consistency is taken as a primitive operator. The most basic **LFI** considered there is the propositional logic **mbC** – which can be assumed as being the smallest logic that respects the above criteria, .

[2]Notably the terms consistency and inconsistency captures a more sensible definition in the **LFI**s. In order to avoid misunderstanding, in this presentation it will be used, for those logics, specifically the terms *formal consistency* and *formal inconsistency*. So the terms *consistency* and *inconsistency* will maintain the usual interpretation, namely *non-triviality* and *triviality*, respectively.

Definition 2.1 (**mbC**[3])**.** *The logic* **mbC** *is defined as follows:*
Axioms:

(A1) $\alpha \to (\beta \to \alpha)$

(A2) $(\alpha \to \beta) \to ((\alpha \to (\beta \to \delta)) \to (\alpha \to \delta))$

(A3) $\alpha \to (\beta \to (\alpha \wedge \beta))$

(A4) $(\alpha \wedge \beta) \to \alpha$

(A5) $(\alpha \wedge \beta) \to \beta$

(A6) $\alpha \to (\alpha \vee \beta)$

(A7) $\beta \to (\alpha \vee \beta)$

(A8) $(\alpha \to \delta) \to ((\beta \to \delta) \to ((\alpha \vee \beta) \to \delta))$

(A9) $\alpha \vee (\alpha \to \beta)$

(A10) $\alpha \vee \neg\alpha$

(bc1) $\circ\alpha \to (\alpha \to (\neg\alpha \to \beta))$

Inference Rule:

(Modus Ponens) $\alpha, \alpha \to \beta \vdash \beta$

It is worth noticing that (A1)-(A9) plus *Modus Ponens* constitutes an axiomatization for the classical positive logic $\mathbf{CPL}^+$.

Different **LFI**s entail distinct logical consequences and therefore substantially alter the rationality captured by the *principle of deductive closure.*

Definition 2.2 (Extensions of **mbC** [2])**.** *Consider the following axioms:*

(ciw) $\circ\alpha \vee (\alpha \wedge \neg\alpha)$

(ci) $\neg\circ\alpha \to (\alpha \wedge \neg\alpha)$

(cl) $\neg(\alpha \wedge \neg\alpha) \to \circ\alpha$

(cf) $\neg\neg\alpha \to \alpha$

Some relevant extensions of **mbC** *are the following:*

mbCciw = **mbC**+(ciw)

mbCci = **mbC**+(ci)

bC = **mbC**+(cf)

Ci = **mbC**+(ci)+(cf) = **mbCci**+(cf)

mbCcl = **mbC**+(cl)

Cil = **mbC**+(ci)+(cf)+(cl) = **mbCci**+(cf)+(cl) = **mbCcl**+ (cf) + (ci) = **Ci**+(cl)

The technical details of these logics as well as a taxonomy of **LFI** systems can be found in the references. Although these are fundamental to the AGM$\circ$ system, mainly for the understanding of the various theorems presented, the general facts outlined above are sufficient for this presentation.

3 The AGMp system

3.1 Formal Preliminaries

Let us assume an **LFI**, namely **L**, such that **L** is an extension of **mbC**. The deductively closed theories of **L** are called belief sets (or epistemic states) of **L**. The set of belief sets of **L** is denoted by $Th(\mathbf{L})$, and $Cn_{\mathbf{L}}(X)$ is the set of logical consequences (in **L**) of the set of formulas X. The language $\mathbb{L}$ of **L** is generated by the connectives $\wedge, \vee, \rightarrow, \neg, \circ$ and the constant f (falsum). The classical negation (or strong negation) is defined by $\sim\alpha =_{def} (\alpha \rightarrow f)$, and $\alpha \leftrightarrow \beta$ is an abbreviation for $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$. The consequence relation of **L** will be denoted by $\vdash_{\mathbf{L}}$ or simply $\vdash$, when **L** is obvious from the context. Similarly, we will write $Cn(X)$ when **L** is obvious.

The following property of **L** is important in order to prove the representation theorems since it guarantees proof by cases. The full proof of this result can be found in the appendix together with the proofs of the main original results presented.

Lemma 3.1 (α-local non-contravention). *Let $X \cup \{\alpha\} \subseteq \mathbb{L}$. Then,*

$$X, \alpha \vdash \neg\alpha \ \ \textit{implies} \ \ X \vdash \neg\alpha.$$

3.2 AGM-compliance

An AGM-compliant logic is simply one in which is possible to completely characterize the contraction operation via the classical postulates. Formally we have the following:

Definition 3.2 (AGM-compliance[3]). *A logic **L** is AGM-compliant if it admits at least one operation $- : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ on **L** which satisfies the postulates for contraction.*

Such compatibility is related to the fact that logic is decomposable. The intuition is that the result K' of a contraction $K - \alpha$ should "fill the gap" between K and α, i.e., it should be possible the decomposition of K with respect to α into two sets, namely $Cn(\alpha)$ and K', such that they both contain less information than K when taken separately but they have the same informational power of K when combined – they are equivalent to K. Thus the resulting theory $K' = K - \alpha$ can be seen as a kind of complement of K relative to α.

A logic is called in [4] *decomposable* if, for every K and every α, there is at least one complement of Krelative to α. Formally:

Definition 3.3 (Decomposability). *A logic $\langle\mathbb{L}, Cn\rangle$ is decomposable if, for every $K \subseteq \mathbb{L}$ and every $\alpha \notin Cn(\emptyset)$, there is $K' \subseteq \mathbb{L}$ such that:*

1. $Cn(K') \subset Cn(K)$
2. $K' + \alpha = Cn(K)$

[3]The definitions and the theorem in this section are adaptations of the results in [4].

Given the definitions presented above, the following theorem asserts wich logics are AGM-compliant.

Theorem 3.4 (AGM-compliance – Flouris [4]). *A logic $\langle\mathbb{L}, Cn\rangle$ is AGM-compliant iff is decomposable.*

Compact and supra-classical logics such as the **LFI**s considered here are decomposable and, hence, AGM-compliant. Furthermore, in this kind of logic recovery ($K \subseteq (K - \alpha) + \alpha$) and relevance (if $\beta \in K \setminus K - \alpha$ then there exists K' such that $K - \alpha \subseteq K' \subseteq K$, $\alpha \notin K'$ and $\alpha \in K' + \beta$) are equivalent. Hence, although this is not valid in general (see [7, 8]), relevance and recovery can be used indistinguishably for the logics considered here.

3.3 Expansion

Expansion is defined as in the classical AGM way:

Definition 3.5 (expansion). *An expansion over **L** is a function $+ : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ defined by $K + \alpha = Cn(K \cup \{\alpha\})$, for all K and α.*

3.4 Contraction

3.4.1 Postulates

Definition 3.6 (Postulates for AGMp contraction). *A contraction over **L** is a function $- : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ satisfying the following postulates:*

(closure) $K - \alpha = Cn(K - \alpha)$.[4]

(success) *If $\alpha \notin Cn(\emptyset)$ then $\alpha \notin K - \alpha$.*

(inclusion) $K - \alpha \subseteq K$.

(relevance) *If $\beta \in K \setminus K - \alpha$ then there exists K' such that $K - \alpha \subseteq K' \subseteq K$, $\alpha \notin K'$ and $\alpha \in K' + \beta$*

3.4.2 Partial meet contraction

Definition 3.7 (**Remainder [1]**). *A set $K' \subseteq \mathbb{L}$ is a maximal subset of K that does not entail α if and only if:*

(i) $K' \subseteq K$.

(ii) $\alpha \notin Cn(K')$.

(iii) *If $K' \subset K'' \subseteq K$ then $\alpha \in Cn(K'')$.*

[4]Rigorously speaking, this postulate is redundant since by definition the co-domain of the function $-$ is Th(**L**). However, in order to keep closer to the classical AGM presentation, we decide to maintain this postulate in all the operations presented here.

The set of all the maximal subsets of K that do not entail α is called the remainder set of (K, α)*, and is denoted by $K \perp \alpha$.*

Lemma 3.8. *If $K' \in K \perp \alpha$, then $K' \in Th(\mathbf{L})$.*

Lemma 3.9 (Upper-bound)**.** *Let K be a belief set in $\mathbf{L}$ and $\alpha \in \mathbb{L}$. If $X \subseteq K$ is such that $\alpha \notin Cn(X)$, then there is a set $X' \in K \perp \alpha$ such that $X \subseteq X'$.*

Definition 3.10 (selection function). *A selection function in $\mathbf{L}$ is a function $\gamma : Th(\mathbf{L}) \times \mathbf{L} \longrightarrow \wp(Th(\mathbf{L})) \setminus \{\emptyset\}$ such that, for every K and α:*

1. *$\gamma(K, \alpha) \subseteq K \perp \alpha$ if $\alpha \notin Cn(\emptyset)$.*
2. *$\gamma(K, \alpha) = \{K\}$ otherwise.*

The partial meet contraction is the intersection of the sets selected by the choice function:

$$K -_\gamma \alpha = \bigcap \gamma(K, \alpha).$$

Theorem 3.11 (Representation for AGMp contraction). *An operation $- : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ satisfies the postulates of Definition 3.6 iff there exists a selection function γ in $\mathbf{L}$ such that $K - \alpha = \bigcap \gamma(K, \alpha)$, for every K and α.*

3.5 Revision

Definition 3.12 (AGMp external revision)**.** *An AGMp external revision over $\mathbf{L}$ is an operation $* : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ satisfying the following postulates:*

(closure) $K * \alpha = Cn(K * \alpha)$

(success) $\alpha \in K * \alpha$

(inclusion) $K * \alpha \subseteq K + \alpha$

(vacuity) *if $\neg\alpha \notin K$ then $K + \alpha \subseteq K * \alpha$*

(non-contradiction) *if $\neg\alpha \in K * \alpha$ then $\vdash \neg\alpha$*

(relevance) *if $\beta \in K \backslash (K * \alpha)$ then there exists X such that $K * \alpha \subseteq X \subseteq K + \alpha$, $\neg\alpha \notin Cn(X)$ and $\neg\alpha \in Cn(X) + \beta$*

(pre-expansion) $(K + \alpha) * \alpha = K * \alpha$

By reverse Levi identity we use the partial meet AGMp contraction to define a construction for an external revision operator defined over belief sets:

$$K *_\gamma \alpha = (K + \alpha) -_\gamma \neg\alpha = \bigcap \gamma(Cn(K \cup \{\alpha\}), \neg\alpha).$$

As expected, external partial meet revision is fully characterized by the postulates of Definition 3.12.

Theorem 3.13. *An operation $* : Th(\mathbf{L}) \times \mathbb{L} \rightarrow Th(\mathbf{L})$ is an AGMp external revision over $\mathbf{L}$ iff it is an external partial meet revision operator over $\mathbf{L}$, that is: there is a selection function γ for AGMp in $\mathbf{L}$ such that $K * \alpha = \bigcap \gamma(K + \alpha, \neg\alpha)$, for every K and α.*

4 The AGM∘ system

4.1 Expansion

Let K be a belief set in $\mathbf{L}$ and $\alpha \in \mathbb{L}$. The expansion of K by a sentence α, i.e. the operation that just adds α and removes nothing, denoted by $K + \alpha$, is defined as in the classical AGM way:

Definition 4.1 (expansion)**.** *An expansion over $\mathbf{L}$ is a function $+ : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ defined by $K + \alpha = Cn(K \cup \{\alpha\})$, for all K and α.*

4.2 Contraction

Definition 4.2 (Postulates for AGM∘ contraction)**.** *A contraction over $\mathbf{L}$ is a function $- : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ satisfying the following postulates:*

(closure) *$K - \alpha = Cn(K - \alpha)$.*

(success) *If $\alpha \notin Cn(\emptyset)$ and $\circ\alpha \notin K$ then $\alpha \notin K - \alpha$.*

(inclusion) *$K - \alpha \subseteq K$.*

(failure) *If $\circ\alpha \in K$ then $K - \alpha = K$.*

(relevance) *If $\beta \in K \backslash K - \alpha$ then there exists K' such that $K - \alpha \subseteq K' \subseteq K$, $\alpha \notin K'$ and $\alpha \in K' + \beta$.*

Our system, in particular, incorporates the idea of non-revisibility in the selection function. This strategy proves to be quite natural when we consider that, in fact, the consistent beliefs are not an option in the retraction – even if they were retracted as the last option such as the more entrenched beliefs. Rather, the consistent belief remains in the epistemic state in any situation, unless the agent retract the own fact that such belief is consistent.

Definition 4.3 (selection function for AGM∘ contraction). *A selection function in $\mathbf{L}$ is a function $\gamma : Th(\mathbf{L}) \times \mathbf{L} \longrightarrow \wp(Th(\mathbf{L})) \setminus \{\emptyset\}$ such that, for every K and α:*

1. *$\gamma(K, \alpha) \subseteq K \perp \alpha$ if $\alpha \notin Cn(\emptyset)$ and $\circ\alpha \notin K$.*
2. *$\gamma(K, \alpha) = \{K\}$ otherwise.*

The partial meet contraction is the intersection of the sets selected by the choice function:

$$K -_\gamma \alpha = \bigcap \gamma(K, \alpha).$$

Theorem 4.4 (Representation for AGM∘ contraction). *An operation $- : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ satisfies the postulates of Definition 4.2 iff there exists a selection function γ in $\mathbf{L}$ such that $K - \alpha = \bigcap \gamma(K, \alpha)$, for every K and α.*

The main objective of the AGM∘ system is to allow modelling contradictory theories. Punctually, the focus is to ensure the possibility of modelling external revision, in which there is an intermediate contradictory epistemic state as perceived by the definition of reverse Levi identity.

4.3 Internal Revision

Definition 4.5 (Postulates for internal AGM∘ revision). *An internal AGM∘ revision over* $\mathbf{L}$ *is an operation* $* : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *satisfying the following:*

(closure) $K * \alpha = Cn(K * \alpha)$.

(success) $\alpha \in K * \alpha$.

(inclusion) $K * \alpha \subseteq K + \alpha$.

(non-contradiction) *If* $\neg\alpha \notin Cn(\emptyset)$ *and* $\circ\neg\alpha \notin K$ *then* $\neg\alpha \notin K * \alpha$.

(failure) *If* $\circ\neg\alpha \in K$ *then* $K * \alpha = K + \alpha$

(relevance) *If* $\beta \in K \setminus K * \alpha$ *then there exists* K' *such that* $K \cap K * \alpha \subseteq K' \subseteq K$ *and* $\neg\alpha \notin K'$, *but* $\neg\alpha \in K' + \beta$.

It is worth noticing that the *failure* of this operation illustrates the case in which the negation of the sentence to be incorporated is consistent in K and thus the prior removal (as is shown below) is not possible due to the *failure* of contraction.

By Levi identity, as in the classical model, we use the partial meet contraction to define a construction for internal revision:

$$K *_\gamma \alpha = (K -_\gamma \neg\alpha) + \alpha = Cn((\bigcap \gamma(K, \neg\alpha)) \cup \{\alpha\}).$$

Theorem 4.6 (Representation for internal AGM∘ partial meet revision). *An operation* $* : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *over* $\mathbf{L}$ *satisfies the postulates of Definition 4.5 if and only if there exists a selection function* γ *in* $\mathbf{L}$ *such that* $K * \alpha = (\bigcap \gamma(K, \neg\alpha)) + \alpha$, *for every* K *and* α.

4.4 External Revision

Definition 4.7 (Postulates for external AGM∘ revision). *An external revision over* $\mathbf{L}$ *is a function* $* : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *satisfying the following postulates:*

(closure) $K * \alpha = Cn(K * \alpha)$.

(success) $\alpha \in K * \alpha$.

(inclusion) $K * \alpha \subseteq K + \alpha$.

(non-contradiction) *if* $\neg\alpha \notin Cn(\emptyset)$ *and* $\sim\alpha \notin K$ *then* $\neg\alpha \notin K * \alpha$.

(failure) *If* $\sim\alpha \in K$ *then* $K * \alpha = \mathbb{L}$

(relevance) *If* $\beta \in K \setminus K * \alpha$ *then there exists* K' *such that* $K * \alpha \subseteq K' \subseteq K + \alpha$ *and* $\neg\alpha \notin K'$, *but* $\neg\alpha \in K' + \beta$.

(pre-expansion) $(K + \alpha) * \alpha = K * \alpha$.

The *pre-expansion* highlights the main feature of a external revision. Moreover, as in the case of contraction, this operation fails – in this case, by *failure* when trying to revise K by a sentence α strongly rejected.

By reverse Levi identity we use the partial meet AGM∘ contraction to define a construction for an external revision operator defined over belief sets:

$$K *_\gamma \alpha = (K + \alpha) -_\gamma \neg\alpha = \bigcap \gamma(Cn(K \cup \{\alpha\}), \neg\alpha).$$

Theorem 4.8 (Representation for external AGM∘ partial meet revision). *An operation* $* : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *over* $\mathbf{L}$ *satisfies the postulates for external partial meet AGM∘ revision (see Definition 4.7) iff there is a selection function* γ *in* $\mathbf{L}$ *such that* $K * \alpha = \bigcap \gamma(K + \alpha, \neg\alpha)$, *for every* K *and* α.

Remark 4.9. *The logical possibility of defining an external revision operator over* $\mathbf{L}$ *challenges the need of a prior contraction, as in the internal revision. Thus, it is possible to interpret the contraction underlying an internal revision as an unnecessary retraction and therefore as a violation of the principle of minimality. On the other hand, if we consider the non-contradiction principle as a priority, then the internal revision remains to be the only rational option. This illustrates the clear opposition between the principle of non-contradiction and that of minimality. Such opposition deserves further attention in future works.*

By capturing two different principles of rationality, both revisions differ both intuitively and logically.

4.5 Consolidation and Semi-revision

Definition 4.10 (Postulates for AGM∘ consolidation). *An AGM∘ consolidation over* $\mathbf{L}$ *is an operation* $! : Th(\mathbf{L}) \longrightarrow Th(\mathbf{L})$ *satisfying the following postulates:*

(closure) $K! = Cn(K!)$.

(inclusion) $K! \subseteq K$.

(non-contradiction) *If* $K \neq \mathbb{L}$, *then* $K!$ *is not contradictory.*

(failure) *If* $K = \mathbb{L}$, *then* $K! = \mathbb{L}$.

(relevance) *If* $\beta \in K \setminus K!$ *then there exists* K' *such that* $K! \subseteq K' \subseteq K$ *and* K' *is not contradictory, but* $K' + \beta$ *is contradictory.*

It can be noted that consolidation is a particular case of contraction, so it is natural that many of its postulates and the explicit construction follow that operation.

As in the case of contraction, a choice function over a remainder set will be used for each consolidation operator. The particularity of the definition of remainder sets is that, in the case of consolidation, these sets are defined over collections of belief sets.

Definition 4.11 (Remainder for sets). *Let K be a belief set in $\mathbf{L}$ and $A \subset \mathbb{L}$. The set $K \perp_P A \subseteq \wp(\mathbb{L})$ is such that for all $X \subseteq \mathbb{L}$, $X \in K \perp_P A$ iff the following is the case:*

1. $X \subseteq K$
2. $A \cap Cn(X) = \emptyset$
3. *If* $X \subset X' \subseteq K$ *then* $A \cap Cn(X') \neq \emptyset$.

Consolidation considers a specific subset A, that is, the one that represents the totality of contradictory sentences in K, defined as follows:

Definition 4.12 (Contradictory set). *Let K be a belief set in $\mathbf{L}$. The set Ω_K of contradictory sentences of K. is defined as follows:*

$$\Omega_K = \{\alpha \in K : \text{ exists } \beta \in \mathbb{L} \text{ such that } \alpha = \beta \wedge \neg\beta\}.$$

Definition 4.13 (Consolidation function). *A* consolidation function *in $\mathbf{L}$ is a function $\gamma : Th(\mathbf{L}) \longrightarrow \wp(Th(\mathbf{L})) \setminus \{\emptyset\}$ such that, for every belief set K in $\mathbf{L}$:*

1. *If* $K \neq \mathbb{L}$ *then* $\gamma(K) \subseteq K \perp_P \Omega_K$
2. *If* $K = \mathbb{L}$ *then* $\gamma(K) = \{K\}$

The consolidation operator defined by a consolidation function γ is then defined as follows: for every belief set K in $\mathbf{L}$,

$$K!_\gamma = \bigcap \gamma(K)$$

Theorem 4.14 (Representation of consolidation). *An operation $! : Th(\mathbf{L}) \longrightarrow Th(\mathbf{L})$ over $\mathbf{L}$ satisfies the postulates of definition 4.10 iff there exists a consolidation function γ in $\mathbf{L}$ such that $K! = \bigcap \gamma(K)$ for every belief set K in $\mathbf{L}$.*

From consolidation for belief sets, it is now possible to define semi-revision for belief sets.

As stated previously, both revisions require effective integration of the new belief. On the other hand, from the definition of external revision, it is possible to define a revision in which the *principle of primacy of new information*, tacitly accepted in internal and external revisions, is challenged. In the context of belief bases it is called *semi-revision* by Hansson (see [6]), which is characterized by the expansion-consolidation scheme.

The semi-revision for belief sets can be defined as a generalization of external-revision, in which the choice for the removal is left to the selection function.

$$K?_\gamma\alpha = (K + \alpha)!_\gamma$$

In short, the AGM∘ system of Paraconsistent Belief Revision captures the dynamics of contradictory theories, particularly represented by the operators of external revision and semi-revision. Diagonally, this system provides to the Logics of Formal Inconsistency an intuitive interpretation for the formal consistency connective, and raises an interesting contrast between the principles of *minimality* and *non-contradiction*. Moreover, the important distinction between *consistency* and *coherence* is deepened, which certainly puts new perspectives to the coherence interpretation of epistemic justification.

5 Final Remarks

The AGMp system can be seeing, in a sense, as a complementary theory of classical AGM since it permits, taken as primitive the classical contraction, to define external revision and also semi-revision, by an expansion-consolidation schema (like AGM∘). As previously stated, the main difference between internal and external revision is the primacy of the consistency criterion in the former, and the minimality in the latter. The semi-revision can also be understood as a generalization of the latter in which the primacy of the new information is not valid.

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Appendix: Proofs of the main results

Lemma 3.1 *Let $X \cup \{\alpha\} \subseteq \mathbb{L}$. Then,*

$$X, \alpha \vdash \neg\alpha \ \text{ implies } \ X \vdash \neg\alpha.$$

Proof: Suppose that $X, \alpha \vdash \neg\alpha$. It is always the case that $X, \neg\alpha \vdash \neg\alpha$, so $X, \alpha \vee \neg\alpha \vdash \neg\alpha$. Here we are assuming that $\mathbf{L}$ have a classical disjunction $\vee$, as it happens with every extension of **mbC**. But $\vdash \alpha \vee \neg\alpha$ and then $X \vdash \neg\alpha$. ■

Theorem 3.13 *An operation $* : Th(\mathbf{L}) \times \mathbb{L} \rightarrow Th(\mathbf{L})$ is an AGMp external revision over $\mathbf{L}$ iff it is an external partial meet revision operator over $\mathbf{L}$, that is: there is a selection function γ for AGMp in $\mathbf{L}$ such that $K * \alpha = \bigcap \gamma(K + \alpha, \neg\alpha)$, for every K and α.*

Proof: (construction $\Rightarrow$ postulates)

closure: By the definition of $*$.

success: Let $X \in (K + \alpha)\bot(\neg\alpha)$ and suppose that $\alpha \notin X$. Consider $X' = X \cup \{\alpha\}$. Since $X \subset X' \subseteq K+\alpha$ then $\neg\alpha \in Cn(X')$, by property iii. of definition 3.7 (it is maximal), that is, $X, \alpha \vdash \neg\alpha$. Hence $X \vdash \neg\alpha$ by lemma 3.1. But that contradict the fact that $\neg\alpha \notin Cn(X)$, by item ii. of definition 3.7. Hence $\alpha \in X$ for all $X \in (K+\alpha)\bot(\neg\alpha)$. If $(K+\alpha)\bot(\neg\alpha) \neq \emptyset$ then $\alpha \in \bigcap \gamma((K + \alpha)\bot(\neg\alpha)) = K * \alpha$. In the case that $(K + \alpha)\bot(\neg\alpha) = \emptyset$ then it is the case that $\alpha \in \bigcap \gamma((K + \alpha)\bot(\neg\alpha)) = K*\alpha$, since in this case $\gamma((K+\alpha)\bot(\neg\alpha)) = \{K + \alpha\}$, by definition of 4.3 (and obviously $\alpha \in K + \alpha$).

inclusion: Clearly $K * \alpha = (K + \alpha)-(\neg\alpha) \subseteq K + \alpha$, by the contraction postulates.

vacuity: Suppose that $\neg\alpha \notin K$. Hence $\neg\alpha \notin (K + \alpha)$, by lemma 3.1. Then $K * \alpha = (K + \alpha)-(\neg\alpha) = (K + \alpha)$ by contraction postulates.

non-contradiction: Supose that $\neg\alpha \in K * \alpha = (K+\alpha)-(\neg\alpha)$. By contradction postulates $\vdash \neg\alpha$.

relevance: Let $\beta \in K \setminus ((K + \alpha)-(\neg\alpha))$. Hence $(K + \alpha)\bot(\neg\alpha) \neq \emptyset$ (otherwise $(K + \alpha)-(\neg\alpha) = K + \alpha$ and then $K \setminus ((K + \alpha)-(\neg\alpha)) = \emptyset$, a contradiction). Then there exists $X \in \Upsilon(K + \alpha, \neg\alpha) \subseteq (K + \alpha)\bot(\neg\alpha)$ such that $\beta \notin X$. By definition of $*$, $K * \alpha \subseteq X \subseteq K + \alpha$. Let $X' = X \cup \{\beta\}$. Hence $X \subset X' \subseteq K + \alpha$ since $\beta \in K$. By definition 3.7, $X' \vdash \neg\alpha$, that is, $X, \beta \vdash \neg\alpha$.

pre-expansion: $(K + \alpha) * \alpha = ((K + \alpha) + \alpha)-(\neg\alpha) = (K + \alpha)-(\neg\alpha) = K * \alpha$.

(postulates $\Rightarrow$ construction)
Let $*$ be an operator satisfying the postulates and let γ be the following function:

$$\gamma(K, \neg\alpha) = \{X \in K\bot\neg\alpha : K * \alpha \subseteq X\}$$

We will prove that 1) it is a selection function for AGMp (recall Definition 3.10), and 2) $K * \alpha = \bigcap \gamma(K + \alpha, \neg\alpha)$.

1. It is obvious that $\gamma(K + \alpha, \neg\alpha) \subseteq (K + \alpha)\bot(\neg\alpha)$ when $(K + \alpha)\bot(\neg\alpha) \neq \emptyset$. In order to consider γ as a selection function for AGMp we must prove that $\gamma(K+\alpha, \neg\alpha) \neq \emptyset$ if $(K+\alpha)\bot(\neg\alpha) \neq \emptyset$. Then suppose that $(K+\alpha)\bot(\neg\alpha) \neq \emptyset$. Hence $\nvdash \neg\alpha$ by item ii of Definition 3.7. By *non-contradiction* it is the case that $\neg\alpha \notin K*\alpha$. By *closure* and *inclusion* $\neg\alpha \notin K * \alpha = Cn(K * \alpha) \subseteq K + \alpha$. Hence, by the upper bound property, there exists $X \in (K + \alpha)\bot(\neg\alpha)$ such that $K * \alpha \subseteq X$. Then $X \in \gamma(K + \alpha, \neg\alpha)$ and so $\gamma(K+\alpha, \neg\alpha) \neq \emptyset$ if $(K+\alpha)\bot(\neg\alpha) \neq \emptyset$.

2. Now let us prove that $K * \alpha = (K + \alpha) - (\neg\alpha) = \bigcap \gamma(K + \alpha, \neg\alpha)$.

1. Suppose that $(K + \alpha)\bot(\neg\alpha) \neq \emptyset$.
Clearly $K * \alpha \subseteq \bigcap \gamma(K + \alpha, \neg\alpha)$ by definition of γ.
Let $\beta \notin K * \alpha$. We have to prove that there exists $X \in \gamma(K + \alpha, \neg\alpha)$ such that $\beta \notin X$. If $\beta \notin K + \alpha$ then $\beta \notin X$ for all $X \in \gamma(K+\alpha, \neg\alpha)$ (since all $X \in \gamma(K + \alpha, \neg\alpha)$ is in $K + \alpha$). Suppose that $\beta \in K + \alpha$. By *pre-expansion* $\beta \notin (K + \alpha) * \alpha$ and then by *relevance*, there exists Z such that $K * \alpha = (K+\alpha) * \alpha \subseteq Z \subseteq (K+\alpha) + \alpha = K + \alpha$, $\neg\alpha \notin Cn(Z)$ and $\neg\alpha \in Cn(Z)+\beta$. By upper bound property there exists $X \in (K + \alpha)\bot(\neg\alpha)$ such that $K * \alpha \subseteq Z \subseteq X$. Hence $X \in \gamma(K+\alpha, \neg\alpha)$. Since $\neg\alpha \in Cn(Z)+\beta$, then $X, \beta \vdash \neg\alpha$ and hence $X \nvdash \beta$ (otherwise $X \vdash \neg\alpha$). Then $\beta \notin X$ and $\beta \notin \bigcap \gamma(K + \alpha, \neg\alpha)$. It proves that $K * \alpha = \bigcap \gamma(K + \alpha, \neg\alpha)$ if $(K + \alpha)\bot(\neg\alpha) \neq \emptyset$.

2. Finally suppose that $(K + \alpha)\bot(\neg\alpha) = \emptyset$. Then $\bigcap \gamma(K+\alpha, \neg\alpha) = K + \alpha$, by definition of γ. On the other hand, if there exists $\beta \in (K+\alpha)\setminus(K*\alpha)$ then, by the same way it was proved above, $(K + \alpha)\bot(\neg\alpha) \neq \emptyset$, a contradiction.

Hence, $K * \alpha = K + \alpha = \bigcap \gamma(K + \alpha, \neg\alpha)$.

■

Lemma 3.8 *If $K' \in K\bot\alpha$, then $K' \in Th(\mathbf{L})$.*

Proof: If $\beta \in Cn(X') \setminus X'$ then $\alpha \in Cn(X' \cup \{\beta\})$. Since $\mathbf{L}$ is Tarskian, this implies that $\alpha \in Cn(X')$, a contradiction. Then $X' = Cn(X')$ and so $X' \in K\bot\alpha$. ■

Lemma 3.9 *Let K be a belief set in $\mathbf{L}$ and $\alpha \in \mathbb{L}$. If $X \subseteq K$ is such that $\alpha \notin Cn(X)$, then there is a set $X' \in K\bot\alpha$ such that $X \subseteq X'$.*

Proof: First, assuming that the language $\mathbb{L}$ is denumerable, let us arrange the sentences of K into a sequence $\beta_1, \beta_2, \ldots$ (if $\mathbb{L}$ is not denumerable, the proof above must be extended in order to use transfinite induction). Let $X = X_0$ and for each $i \geq 1$ we define X_i as follows:

$$X_i = \begin{cases} X_{i-1} & \text{if } \alpha \in Cn(X_{i-1} \cup \{\beta_i\}) \\ X_{i-1} \cup \{\beta_i\} & \text{otherwise} \end{cases}$$

By construction, for every i, $\alpha \notin Cn(X_i)$. Let $X' = \bigcup_i X_i$. It is easy to verify that $X \subseteq X' \subseteq K$. By compactness, if $\alpha \in Cn(X')$ then $\alpha \in Cn(X'')$ for some finite $X'' \subseteq K$. It follows that $\alpha \in Cn(X_j)$ for some j, a contradiction. Then $\alpha \notin Cn(X')$. Moreover, if $\beta \in K$ and $\beta \notin X'$ then, in particular, $\beta \notin X_i$ where i is such that $\beta = \beta_i$. This means that $\alpha \in Cn(X_{i-1} \cup \{\beta\})$, by construction, and so $\alpha \in Cn(X' \cup \{\beta\})$, by monotonicity. ■

Theorem 4.4 *An operation* $-: Th(\mathbf{L}) \times \mathbf{L} \longrightarrow Th(\mathbf{L})$ *satisfies the postulates of Definition 4.2 iff there exists a selection function* γ *in* $\mathbf{L}$ *such that* $K - \alpha = \bigcap \gamma(K, \alpha)$*, for every* K *and* α*.*

Proof: (construction ⇒ postulates)

closure: Let $X \in K \bot \alpha$ and $\beta \in Cn(X)$ then $\alpha \notin Cn(X \cup \{\beta\})$ and, since X is maximal, $\beta \in X$. So for all $X \in K \bot \alpha$ it is the case that $X = Cn(X)$. So $K -_\gamma \alpha = \bigcap \gamma(K, \alpha)$ and the elements of $\gamma(K, \alpha)$ are closed sets and, since the intersection of closed sets are also closed, it is the case that $K -_\gamma \alpha$ is closed.

sucess: If $\alpha \notin Cn(\emptyset)$ then by the upper bound lemma $K \bot \alpha \neq \emptyset$.

inclusion: Follows directly from the construction.

failure: Follows directly from the construction.

relevance: If $\beta \in K \setminus K - \alpha$ than exists a $X \in \gamma(K, \alpha)$ such that $\beta \notin X$. By definition, $K -_\gamma \alpha \subseteq X \subseteq K$, $\alpha \notin Cn(X)$ and $\alpha \in Cn(X \cup \{\beta\})$.

(postulates ⇒ construction)
Let $-$ be an operator satisfying the postulates for contraction and let γ be the following function:

$$\begin{aligned} \gamma(K, \alpha) &= \{X \in K \bot \alpha : K - \alpha \subseteq X\} \text{ if} \\ & \quad \alpha \notin Cn(\emptyset) \text{ or } \circ\, \alpha \notin K \\ &= \{K\} \text{ otherwise.} \end{aligned}$$

We have to prove that 1) γ is a selection function and 2) $K - \alpha = \bigcap \gamma(K, \alpha)$.

1. The fact thats $\gamma(K, \alpha) \subseteq K$ folows directly from construction. If $\alpha \notin Cn(\emptyset)$ e $\circ\alpha \notin K$ then the *sucess* and *inclusion* guarantees that $\alpha \notin K - \alpha \subseteq K$. By the upper bound lemma, exists X such that $K - \alpha \subseteq X \in K \bot \alpha$ and, hence, $\gamma(K, \alpha) \neq \emptyset$.

2. If $\alpha \in Cn(\emptyset)$ then *relevance* and *inclusion* guarantees that $K - \alpha = K$. Similarly $\circ\alpha \in K$ and *failure* guarantees that $K - \alpha = K$. In both cases $\bigcap \gamma(K, \alpha) = K$, since $\gamma(K, \alpha) = \{K\}$. If $\alpha \notin Cn(\emptyset)$ then $K - \alpha \subseteq K -_\gamma \alpha$ by construction. Now we have to show that $K -_\gamma \alpha \subseteq K - \alpha$. Let $\beta \notin K - \alpha$ and suppose that $\beta \in K$ (otherwise $\beta \notin \bigcap \gamma(K, \alpha)$ trivially). By *relevance*, exists K' such that $K - \alpha \subseteq K' \subseteq K$, $\alpha \notin Cn(K')$ and $\alpha \in Cn(K' \cup \{\beta\})$. By the upper bound lemma exists X such that $K' \subseteq X \in K \bot \alpha$. Since $K' \subseteq X$, $\alpha \in Cn(K' \cup \{\beta\})$ e $\alpha \notin Cn(X)$, it is the case that $\beta \notin X$. Hence, $\beta \notin \bigcap \gamma(K, \alpha)$.

■

Theorem 4.6 *An operation* $*: Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *over* $\mathbf{L}$ *satisfies the postulates of Definition 4.5 if and only if there exists a selection function* γ *in* $\mathbf{L}$ *such that* $K * \alpha = (\bigcap \gamma(K, \neg\alpha)) + \alpha$*, for every* K *and* α*.*

Proof:
(construction ⇒ postulates)
Let γ be a selection function, and define $K * \alpha = (\bigcap \gamma(K, \neg\alpha)) + \alpha$. We have to prove that $*$ satisfies the postulates for internal AGM∘ partial meet revision.

The postulates of *closure, success, inclusion and non-contradiction* follows like the previous theorem.

relevance: Let $\beta \in K \setminus K * \alpha$ then $\beta \notin \bigcap \gamma(K, \neg\alpha) + \alpha$ hence there exists X such that $\beta \notin X \in K \bot \neg\alpha$. Besides $K \bigcap K *_\gamma \alpha \subseteq X + \alpha$. From the fact that $X \in K \bot \neg\alpha$, then $X \subseteq K$, $\neg\alpha \notin X$ and, by the fact that $\beta \in K \setminus X$, $\neg\alpha \in X + \beta$.

failure: If $\circ\neg\alpha \in K$ then $K - \neg\alpha = K$ by definition of selection function, hence $(K - \neg\alpha) + \alpha$ is $K + \alpha$.

(postulates ⇒ construction)
Let $*$ be an operator satisfying the postulates and let γ be the following function:

$$\begin{aligned} \gamma(K, \neg\alpha) &= \{X \in K \bot \neg\alpha : K \cap K * \alpha \subseteq X\} \\ & \quad \text{if } K \bot \neg\alpha \neq \emptyset \\ &= \{K\} \text{ otherwise.} \end{aligned}$$

Like the previous theorem, γ is well defined and we will prove that 1) γ is a selection function and 2) $K * \alpha = \bigcap \gamma(K, \neg\alpha) + \alpha$

1. $\gamma(K, \neg\alpha) \subseteq K\bot\neg\alpha$ by definition. If $\neg\alpha \notin Cn(\emptyset)$ and $\circ\neg\alpha \notin K$ then by non-contradiction $\neg\alpha \notin K * \alpha$ and by upper bound there exists X' such that $K \cap K * \alpha \subseteq X' \in K\bot\neg\alpha$, hence $X' \in \gamma(K, \neg\alpha)$ and therefore $\gamma(K, \neg\alpha) \neq \emptyset$

2. First we must prove that $K * \alpha \subseteq \bigcap\gamma(K, \neg\alpha) + \alpha$. By construction, $K \cap K * \alpha \subseteq \bigcap\gamma(K\neg\alpha)$. Hence $(K \cap K * \alpha) + \alpha \subseteq \bigcap\gamma(K\neg\alpha) + \alpha$ and therefore $K + \alpha \cap (K * \alpha + \alpha) \subseteq \bigcap\gamma(K\neg\alpha) + \alpha$ by distributivity. Besides, by success, inclusion and closure, $K * \alpha \subseteq \bigcap\gamma(K, \neg\alpha) + \alpha$. To prove the other side, we have two cases.:

   1. if $\circ\neg\alpha \in K$, in this case by failure, $K * \alpha = K + \alpha$ and since $\bigcap\gamma(K, \neg\alpha) \subseteq K$ it follows, by closure and siccess that $\bigcap\gamma(K, \neg\alpha) + \alpha \subseteq K * \alpha$.

   2. If $\circ\neg\alpha \notin K$, then we have two cases:

      1. if $\neg\alpha \in Cn(\emptyset)$, then in this case, by relevance, it follows that $K \subseteq K * \alpha$. In that way, since there can not exists $\beta \in K \setminus K * \alpha$, then $\bigcap\gamma(K, \neg\alpha) \subseteq K * \alpha$.

      2. Let $\neg\alpha \notin Cn(\emptyset)$. In this case, suppose by absurd that $\beta \in \bigcap\gamma(K, \neg\alpha) \setminus K * \alpha$. Since $\beta \in \bigcap\gamma(K, \neg\alpha)$ then $\beta \in K$ and hence $\beta \in K \setminus K * \alpha$. By relevance, there exists K' such that $K \cap K * \alpha \subseteq K'$, $K' \subseteq K$, $\neg\alpha \notin K'$ and $\neg\alpha \in K' + \beta$. By upeer bound, there exists K'' such that $K' \subseteq K'' \in K\bot\neg\alpha$. Since $\circ\neg\alpha \notin K$ and $\neg\alpha \notin Cn(\emptyset)$ then $\bigcap\gamma(K, \neg\alpha) \subseteq K''$ and therefore $\beta \in K''$.
      Since $\neg\alpha \in K' + \beta$ e $K' \subseteq K''$ then if $\beta \in K''$ it is the case that $\neg\alpha \in Cn(K'')$. Therefore $\beta \notin K''$, by the previous cases 1 and 2. Hence $\bigcap\gamma(K, \neg\alpha) \subseteq K * \alpha$.

   In both cases since $\bigcap\gamma(K, \neg\alpha) \subseteq K * \alpha$, $\bigcap\gamma(K, \neg\alpha) + \alpha \subseteq K * \alpha + \alpha$ and by success and closure, $\bigcap\gamma(K, \neg\alpha) + \alpha \subseteq K * \alpha$.

■

Theorem 4.8 *An operation* $* : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *over* $\mathbf{L}$ *satisfies the postulates for external partial meet AGM*$\circ$ *revision (see Definition 4.7) iff there is a selection function* γ *in* $\mathbf{L}$ *such that* $K * \alpha = \bigcap\gamma(K + \alpha, \neg\alpha)$, *for every* K *and* α.

Proof: (construction ⇒ postulates)

closure: Follows as the previous theorem.

sucess: In the cases that $\neg\alpha \in Cn(\emptyset)$ or $\circ\alpha \in K$ by definition it is the case that $K *_\gamma \alpha = K + \alpha$ and sucess follows trivially.

Let $X \in (K + \alpha)\bot\neg\alpha$, and suppose (by absurd) that $\alpha \notin X$. Let $X' = X \cup \{\alpha\}$. Since $X \subset X' \subseteq K + \alpha$, it is the case that $\neg\alpha \in Cn(X')$ by the maximality of $\bot$. Therefore $\neg\alpha \in Cn(X \cup \{\alpha\})$ and by the lemma 3.1 it is the case that $\alpha \in Cn(X)$. That contradicts the fact that $\neg\alpha \notin Cn(X)$. Therefore $\alpha \in X$ for all $X \in (K + \alpha)\bot\neg\alpha$. Therefore $\alpha \in K *_\gamma \alpha$.

inclusion: Follows by construction.

non-contradiction: Suppose that $\neg\alpha \in K * \alpha = (K + \alpha) - \neg\alpha$. By success and contraction $\neg\alpha \in Cn(\emptyset)$ or $\circ\alpha \in K$.

failure: If $\sim\alpha \in K$ than $K + \alpha = \mathbb{L}$ and hence $\circ\neg\alpha \in K + \alpha$. By failure (of contraction) $K + \alpha - \neg\alpha = \mathbb{L}$.

relevance: Let $\beta \in K \setminus ((K + \alpha) - \neg\alpha)$.

Therefore $(K + \alpha)\bot\neg\alpha \neq \emptyset$ (otherwise $(K + \alpha) - \neg\alpha = K + \alpha$ and $K \setminus ((K + \alpha) - \neg\alpha) = \emptyset$, a contradiction). Hence exists $X \in \gamma(K + \alpha, \neg\alpha) \subseteq (K + \alpha)\bot\neg\alpha$ such that $\beta \notin X$. By construction $K * \alpha \subseteq X \subseteq K + \alpha$. Let $X' = X \cup \{\beta\}$. Therefore $X \subset X' \subseteq K + \alpha$ by the fact that $\beta \in K$. By definition $\neg\alpha \in Cn(X')$ and hence $\neg\alpha \in X + \beta$.

pre-expansion: $(K + \alpha) * \alpha = ((K + \alpha) + \alpha) - \neg\alpha = (K + \alpha) - \neg\alpha = K * \alpha$.

(postulates ⇒ constructions)

Let $*$ be an operator satisfying the postulates and let γ be the following function:

$$\begin{aligned}\gamma(K, \neg\alpha) &= \{X \in K\bot\neg\alpha : K * \alpha \subseteq X\} \\ &\quad \text{if } \circ\,\alpha \notin K \text{ and } \neg\alpha \notin Cn(\emptyset) \\ &= \{K\} \text{ otherwise.}\end{aligned}$$

We have to prove that 1) γ is a selection function and 2) $K * \alpha = (K + \alpha) - \neg\alpha = \bigcap\gamma(K + \alpha, \neg\alpha)$

1. It follows direct by construction that $\gamma(K + \alpha, \neg\alpha) \subseteq (K + \alpha)\bot\neg\alpha$ in the case that $\circ\alpha \notin K$ e $\neg\alpha \notin Cn(\emptyset)$.

   If $\circ\alpha \in K$ or $\neg\alpha \in Cn(\emptyset)$ then $\gamma(K + \alpha, \neg\alpha) = \{K\}$ by definition. Otherwise we have to show that $\gamma(K + \alpha, \neg\alpha) \neq \emptyset$. By *non-contradiction* we have that $\neg\alpha \notin K * \alpha$. By *closure* and *inclusion*, $\neg\alpha \notin K * \alpha = Cn(K * \alpha) \subseteq K + \alpha$. Therefore by the upper bound lemma exists $X \in (K + \alpha)\bot\neg\alpha$ such that $K * \alpha \subseteq X$. It follows that $X \in \gamma(K + \alpha, \neg\alpha)$ and then $\gamma(K + \alpha, \neg\alpha) \neq \emptyset$.

2. Let $\circ\alpha \notin K$ e $\neg\alpha \notin Cn(\emptyset)$. In this case, $K * \alpha \subseteq \bigcap \gamma(K + \alpha, \neg\alpha)$ by construction.

   Let $\beta \notin K * \alpha$. We have to show that $X \in \gamma(K+\alpha, \neg\alpha)$ such that $\beta \notin X$. If $\beta \notin K+\alpha$ then $\beta \notin X$ for all $X \in \gamma(K+\alpha, \neg\alpha)$ (since all $X \in \gamma(K+\alpha, \neg\alpha)$ are in $K+\alpha$).

   Let $\beta \in K + \alpha$. By *pre-expansion*, $\beta \notin (K + \alpha) * \alpha$ and then, by *relevance*, exists Z such that $K * \alpha = (K + \alpha) * \alpha \subseteq Z \subseteq (K + \alpha) + \alpha = K + \alpha$, $\neg\alpha \notin Cn(Z)$ and $\neg\alpha \in Z + \beta$. By upper bound lemma, exists $X \in (K+\alpha)\perp\neg\alpha$ such that $K*\alpha \subseteq Z \subseteq X$. Hence $X \in \gamma(K + \alpha, \neg\alpha)$. Since $\neg\alpha \in Z+\beta$, then $\neg\alpha \in X+\beta$ and therefore $\beta \in Cn(X)$ (otherwise, $\neg\alpha \in Cn(X)$). It follows that $\beta \notin X$ and then $\beta \notin \bigcap \gamma(K+\alpha, \neg\alpha)$. We conclude that $K*\alpha = \bigcap \gamma(K+\alpha, \neg\alpha)$.

   Now, if $\circ\alpha \in K$ or $\neg\alpha \in Cn(\emptyset)$ we have, by construction, that $\bigcap \gamma(K + \alpha, \neg\alpha) = K + \alpha$. In the other hand, if exists $\beta \in (K+\alpha) \setminus (K*\alpha)$ then $(K+\alpha)\perp\neg\alpha \neq \emptyset$, a contradiction. We conclude that $K * \alpha = K + \alpha = \bigcap \gamma(K + \alpha, \neg\alpha)$.

■

Theorem 4.14 *An operation* $! : Th(\mathbf{L}) \longrightarrow Th(\mathbf{L})$ *over* $\mathbf{L}$ *satisfies the postulates of definition 4.10 iff there exists a consolidation function* γ *in* $\mathbf{L}$ *such that* $K! = \bigcap \gamma(K)$ *for every belief set* K *in* $\mathbf{L}$.

Proof:
(construction ⇒ postulates)

closure: It follows as the previous theorem

inclusion: It follows from construction.

non-contradiction: By upper bound $K\perp_P\Omega_K \neq \emptyset$. Then by definition $\bigcap \gamma(K) \cap \Omega_K = \emptyset$.

failure: Follows from definition of γ.

relevance: Let $\beta \in K \setminus K!$. There exists $X \in \gamma(K) \subseteq K\perp_P\Omega_K$ such that $\beta \notin X$. By construction, $K! \subseteq X \subseteq K$. Let $X' = X \cup \{\beta\}$. Then $X \subset X' \subseteq K$ by the fact that $\beta \in K$. By definition, $\Omega_K \cap Cn(X') \neq \emptyset$, that is, $\Omega_K \cap (X + \beta) \neq \emptyset$.

(postulates ⇒ construction)
Consider the following function:

$$\gamma(K) = \{X \in K\perp_P\Omega_K : K! \subseteq X\} \text{ if } K \neq \mathbb{L}$$

$$\gamma(K) = \{K\} \text{ otherwise}$$

We must prove that 1) γ is a consolidation function 2) $K! = \bigcap \gamma(K)$

1. It follows from construction that $\gamma(K) \subseteq K\perp_P\Omega_K$. We need to show that $\gamma(K) \neq \emptyset$. By *non-contradiction* it follows that $\Omega_K \cap K! = \emptyset$. By *inclusion*, $K! \subseteq K$. By upper bound there exists $X \in K\perp_P\Omega_K$ such that $K! \subseteq X$. It follows that $X \in \gamma(K)$ and then $\gamma(K) \neq \emptyset$.

2. It follows by construction that $K! \subseteq \gamma(K)$. We must show that $\gamma(K) \subseteq K!$. It is sufficient to show that there exists $\beta \notin K!$ such that $\beta \notin \bigcap \gamma(K)$. Let $\beta \notin K!$ and suppose that $\beta \in K$ (otherwise $\beta \notin \gamma(K)$ trivially). By *relevance* there exists K' such that $K! \subseteq K' \subseteq K$, $K' \cap \Omega_K = \emptyset$, but $K' + \beta \cap \Omega_K \neq \emptyset$. By upper bound $X \in K\perp_P\Omega_K$ such that $K! \subseteq K' \subseteq X$. Hence $X \in \gamma(K)$. Since $\Omega_K \cap K' + \beta \neq \emptyset$ it follows that $\beta \notin Cn(X)$ (otherwise $\Omega_K \cap X \neq \emptyset$). Hence $\beta \notin \bigcap \gamma(K)$.

■

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