Squares of Oppositions, Commutative Diagrams, and Galois Connections for Topological Spaces and Similarity Structures

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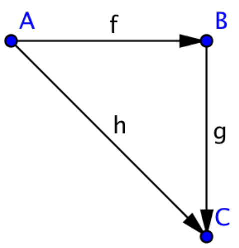
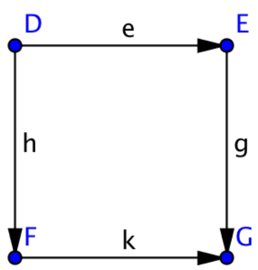
1.Introduction. The aim of this paper is to elucidate the relationship between Aristotelian conceptual oppositions, commutative diagrams of relational structures, and Galois connections. This is done by investigating in detail some examples of Aristotelian conceptual oppositions arising from topological spaces and similarity structures. The main technical device for this endeavor is the concept of Galois connections. In more detail this may be explained as follows.

Let A, B, C, ... a collection of relational structures such as sets, topological spaces, order structures, groups, or categories; let A——f—>B, B——g—>C, ... be a collection of structure-preserving maps between those relational structures. The concatenation of maps A——f—>B and B——g—>C is denoted by A——g•f—>C. A commutative diagram is a family of maps X——h—>Y for which all map compositions that start at the same relational structure X and end at the same structure Z are identical as maps. The simplest commutative diagrams are triangles and squares of the following kind:

(1.1)



g • f = h



g • e = k • h

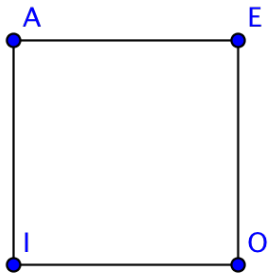
The commutative diagrams (1.1) are to be interpreted as functional equations, i.e., they are to be read as the propositions $\forall x(x \in A)$ $(g(f(x)) = h(x))$, and $\forall y$ $(y \in D)$ $(g(e(y)) = k(h(y))$, respectively. More complex commutative diagrams are to be interpreted analogously. A wealth of examples of commutative diagrams may be found in (Adámek, Herrlich and Strecker (1990)), (Goldblatt (1979)), or in any other textbook on category theory.

In the last decades, commutative diagrams have become an ever more important tool for presenting arguments and conceptual constructions in a neat and perspicuous way that otherwise could hardly be expressed in a palatable way. Commutative diagrams are not just helpful illustrations, often they are indispensable conceptual tools.

The aim of this paper is to show that commutative diagrams are useful for shedding new light on the logico-geometrical theory of conceptual oppositions cast in the framework of Aristotelian squares, hexagons, and similar figures. I want to show that the theory of conceptual oppositions can be elucidated in fruitful way with the help of commutative diagrams that naturally arise from certain Galois connections resulting from topological structures and similarity relations.

Beginning with Blanché's *Sur l'opposition des concepts* (1953) and Sesmat's Logique (1951) the issue of conceptual oppositions has gained considerable interest in recent decades (see Béziau (2012), Moretti (2006) and the literature mentioned there). Various kinds of conceptual oppositions that occur in thus different fields such as linguistics, modal logic, psychology and many others have been studied from a logico-geometrical perspective for which figures such as triangles, squares, hexagons, cubes, and more complex figures occupy centre stage.

(1.2) Definition (Propositional Square of Oppositions). Let A, I, E, and O be four propositions. Then the quadruple



is an Aristotelian square of opposition (A, E, I, O) iff the following four conditions are satisfied that are analogues to (1.2)(i) – (iv) (cf. Béziau (2012, 3)).

(i) A and E are contrary, i.e., they cannot both be true, but both may be false.

(ii) I and O are subcontrary, i.e., I and O cannot both be false, but both may be true.

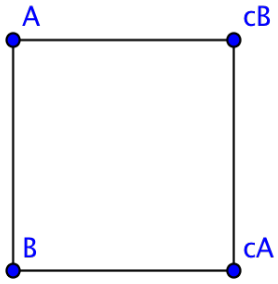
(iii) I and A and O and E, are in subalternation, i.e., A implies I, but not vice versa, and E implies O, but not vice versa.

(iv) A and O, and E and I are contradictory.♦

To be clear, the considerations of this paper are based on the so-called classical interpretation of the Aristotelian square and not on the modern one (cf. Westerstahl (2012, 195f)).

Interpreting propositions as sets of possible worlds a propositional Aristotelian square may be considered as a set-theoretical diagram:

<u>(1.3) Definition (Set-Theoretical Squares of Oppositions)</u>. Let A and B be two subsets of a „universal" set U. For $D \subset U$ denote the set-theoretical complement $U - D := \{x; x \in U \text{ and } x \notin D\}$ of D with respect to U by **C**D. For every proper inclusion $A \subset B$ the quadruple (A, **C**B, B, **C**A) defines a set-theoretical Aristotelian square that satisfies the set-theoretical analogues of (1.2)(i) – (iv):



Proof: Assume A ⊂ B. It is an elementary exercise in informal set theory to prove that the following four assertions are valid:

(i) A and **C**B are contrary, i.e., A ∩ **C**B = Ø and A ∪ **C**B ≠ U.

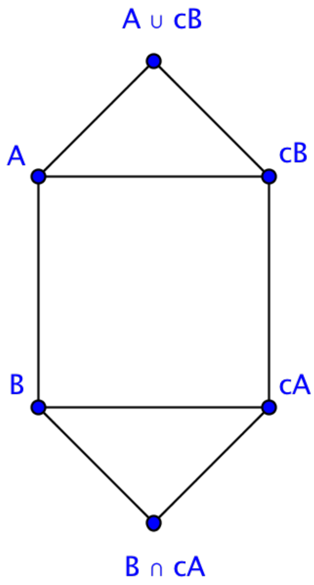
(ii) B and **C**A are subcontrary, i.e., B ∩ **C**A ≠ Ø and B ∪ **C**A = U.

(iii) A and B, and **C**B and **C**A are in subalternation, i.e., A ⊂ B and **C**B ⊂ **C**A.

(iv) A and **C**A, and B and **C**B are contradictory, i.e. A ∩ **C**A = B ∩ **C**B = Ø, and A ∪ **C**A = B ∪ **C**B = U.♦

A square such as (1.3) suggests a „Boolean closure“ (cf. Smessaert (2012, 176)) defined by adding two more vertices - one above the top horizontal edge <A, **C**B> and another one below the bottom edge <B, **C**A>. They are defined as the disjunction (union) of the two vertices A and **C**B, and the conjunction (intersection) of the two bottom vertices B and **C**A. This Boolean closure of (1.3) is geometrically illustrated by the following Aristotelian hexagon:

(1.4)



Due to the fact that de Morgam's law that is valid for Boolean lattices the top and the bottom of this hexagon are contradictory:

(1.5) $\mathbf{C}(A \cup CB) = \mathbf{C}A \cap \mathbf{CC}B = \mathbf{C}A \cap B = B \cap \mathbf{C}A$

For a detailed discussion of many examples of this kind of hexagons the reader may consult Béziau (2012, 34ff).

Squares, hexagons, and other more complex (higher-dimensional) geometrical figures have been used in recent years as conceptual tools for elucidating various kinds of logical and conceptual oppositions that occur in logic, metaphysics, and other areas of knowledge.

Recently, Angot-Pellissier developed an account of „topological" Aristotelian squares and hexagons by constructing such squares and hexagons for appropriate regions of topological spaces. His account is interesting for at least two reasons. First, topology is one of the central theories of modern mathematics with many applications. Thus it is to be considered as a pleasing fact that the age-old Aristotelian theory of conceptual oppositions can be shown to be related to topology in a non-trivial way. Even more interesting is that modern topology comes along with its own diagrammatical devices which perhaps may be used to shed some new light on the classical theory of Aristotelian theory. Indeed, topology may be characterized as one of those theories of modern mathematics, in which diagrammatic reasoning plays a crucial role, similarly as in order theory, lattice theory, group theory, and category theory.

Up to now, the relationship between commutative diagrams and the „logical“ geometry of Aristotelian diagrams of oppositions has never been explicitly addressed. The aim of this paper is to show that there are interesting similarities between between these two kinds of diagrammatic representations of logical, conceptual, and mathematical relations. To put it in a nutshell, Aristotelian diagrams in topology and other areas can be realized as commutative functional diagrams. More precisely, appropriate commutative diagrams can be constructed that may serve as second-order diagrams, i.e., as operators operation on topological spaces, graphs (and possibly other structures) and taking as values first-order Aristotelian topological or graph-theoretical squares and hexagons.

The outline of this paper is as follows. In the next section 2 *Local Topological Squares and Hexagons* first we briefly recall the conceptual apparatus of topology and lattice theory necessary for the discussion of Aristotelian squares that arise from order theory, topology, and similar areas. This enables us in the section 3 *Global Commutative Aristotelian Diagrams in Topology* to reformulate Angot-Pellessier's account in such a way that his topological squares and hexagons can be characterized as „local values“ of certain „global“ commutative diagrams.

In section 4 *Local Aristotelian Diagrams for Similarity Structures* we show that Aristotelian squares and hexagons not only exist for topological spaces. Rather, one can show that every similarity structure (X, ~) i.e., a set X endowed with a binary reflexive and symmetric similarity relation ~, gives rise for a wealth of local Aristotelian squares defined on appropriate subsets of X. In the concluding section 5 *Commutative Diagrams and Galois Connections* it is shown that Galois connections can be used to define global commutative diagrams for topological spaces and graphs generate local Aristotelian squares and hexagons for topology and graph theory.

2. Local Topological Squares and Hexagons. To set the stage, let us first recall the definition of a topological space as it can be found in any textbook of topology (cf. Kuratowski and Mostowski 1976, Davey and Priestley, 1990):

(2.1) Definition. Let X be a set. Denote the power set of subsets of X by PX. A topological structure on X is defined as a subset OX ⊆ PX satisfying the following requirements:

(i) Ø, X ∈ OX.

(ii) If A, B ∈ OX, then A ∩ B ∈ OX.

(iii) If A_i ∈ OX then $\cup A_i$ ∈ OX.

The elements of OX are called open sets of the topological space (X, OX). OX is canonically ordered by set-theoretical inclusion ⊆. With respect to this order (OX, ⊆) is well-known to be a complete Heyting lattice.

The set-theoretical complement **C**A of an open set A is called a closed set of (X, OX). The set of closed sets is denoted by CX := {**C**A; A ∈ OX}. It is well known that CX is a complete co-Heyting algebra.

A subset A of X is clopen with respect to the topological structure (X, OX) if and only if A is open and closed. By definition Ø and X are clopen with respect to every topological structure. A topological space (X, OX) is connected if and only if Ø and X are the only clopen subsets of X. In this paper all topological spaces (X, OX) are assumed to be connected.♦

In the following sections we will often rely on a seemingly different but actually equivalent definition of topological spaces with (2.1). It may be traced back to Kuratowski (cf. Kuratowski and Muratowski 1976).

<u>(2.2) Definition</u>. A topological closure operator cl on a set X is a map PX—cl—>PX satisfying for all A, B ∈ PX the following axioms (cf. Kuratowski and Mostowski 1976, 27)

(i) $A \subseteq cl(A)$. (ii) $cl(cl(A)) = cl(A)$.

(iii) $cl(\emptyset) = \emptyset$. (iv) $cl(A \cup B) = cl(A) \cup cl(B)$.

A closure operator cl comes along with a kernel operator PX—int—>PX defined by int(A) := **C**cl**C**(A). Clearly, the two operators cl and int are interdefinable as cl = **C**int**C** and int = **C**cl**C**. The kernel operator int satisfies axioms dual to (i) – (iv):

(i)' $int(A) \subseteq A$. (ii)' $int(int(A)) = int(A)$.

(iii)' $int(\emptyset) = \emptyset$. (iv)' $int(A \cap B) = int(A) \cap int(B)$.

A set A ∈ PX is closed (with respect to cl) if and only if cl(A) = A. A set B ∈ PX is open if and only if B is the set-theoretical complement of a closed set A ∈ PX, i.e., B = **C**A and A = cl(A). Given a topological closure operator cl a uniquely determined topological space (X, OX) in the sense of (2.1) is defined by

$$OX := \{A;\ A = \mathbf{C}B \text{ and } B = cl(B)\}.$$

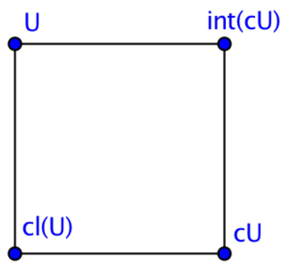
As is easily checked OX thus defined is indeed a topological structure in the sense of (2.1). On the other hand, given a topological structure (X, OX) in the sense of (2.1) a unique topological closure operator cl in the sense of (2.2) is defined by

$$cl(A) := \cap\{D;\ A \subseteq D,\ \mathbf{C}D \in OX\}.$$

A set A ∈ PX of X is regular open if and only if A = int(cl(A)). The set of regular open subsets of a topological space X is a subset O*X of OX. O*X has the structure of a Boolean algebra. Dually, a set B ∈ PX is regular closed if and only if cl(int(B)) = B. The set of regular closed subsets C*X is a subset of CX and has the structure of a Boolean algebra isomorphic to O*X. The map O*X——cl——>C*X defines a Boolean isomorphism between O*X and C*X.♦

After these preliminaries one can state Angot-Pellissier's „principal result" and „main discovery" on topological Aristotelian squares (cf. Angot-Pellissier (2012, 369/370, Theorem 2). Recall that it is assumed throughout that all topological spaces (X, OX) are connected. Moreover, open sets U, V for which Aristotelian squares are to be constructed, are assumed to be non-empty and different from X. The reasons for these technical restrictions are explained in (Angot-Pellissier 2012). They ensure that the resulting squares and hexagons are indeed Aristotelian in that they satisfy the required conditions of contrariness, contradiction, and subalternation.

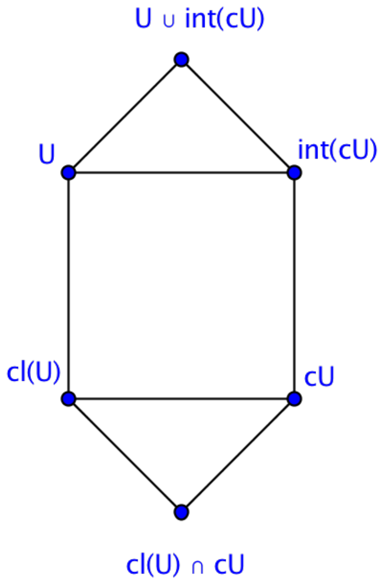
<u>(2.3) Theorem (Angot-Pellissier 2012)</u>. Let (X, OX) be a topological space with interior kernel operator int. For U ∈ OX and Ø ≠ U ≠ X the following square is an Aristotelian square, i.e., it satisfies the requirements (1.2)(i) – (iv):



This square is called the local Aristotelian topological square on U.

Proof: One has to show that the vertices of the edge <U, int(CU)> are contrary, the vertices of the edge <cl(U), CU> are subcontrary, and the two pairs <U, cl(U)> and <int(CU), CU> are in subalternation, respectively. Further, the diagonals <U, CU> and <int(CU), cl(U)> have to be contradictions. Using the definitions of the topological operators cl and int this is an elementary set-theoretical exercise.♦

Remark. Local Aristotelian topological squares are gregorious creatures. Given a connected topological space (X, OX) every non-trivial U ∈ OX comes along with a square defined on it.
This suggests to look for a kind of global Aristotelian square from which the local topological squares can be derived in one fell swoop. For this purpose it is necessary to exclude from our considerations the trivial sets Ø and X because for them the resulting squares in (2.3) are clearly not Aristotelian. Hence, from now on, the quantifier „for all U“ is to be understood as „for all U except Ø and X“.
The general recipe to go extend the realm of „logical figures“ beyond the classical square of oppositions is to introduce various kinds of intermediate propositions, sets, or concepts (see for instance (Béziau (2012)). An elementary example is given by the already mentioned set-theoretical example (1.3)(i). For the local topological square (2.1) the same recipe leads to a local topological hexagon for every U ∈ OX:
(2.4)



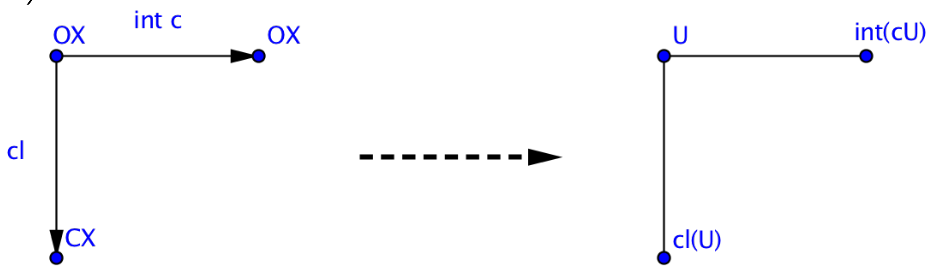
It is easily checked that the top U ∪ int(**C**U) and the bottom cl(U) ∩ **C**U of (2.4) are contradictions:

(2.5) **C**(U ∪ int(**C**U)) = **C**U ∩ **C**int(**C**U) = **C**U ∩ **CC**cl(U) = **C**U ∩ cl(U).

In sum, local topological squares and hexagons can be neatly derived from the elementary properties of the topological closure operator cl (or, equivalently, of course) the interior kernel operator int, of course). As will be shown in the next section, with some more effort one can construct global commutative diagrams from which the local squares and hexagons can be derived.

<u>3. Global commutative Aristotelian Diagrams in Topology</u>. The starting point for the construction of a global commutative diagram from which the local topological Aristotelian squares on U ∈ OX (2.3) is the following elementary observation: The upper left part of a local square on U is easily seen to be derivable from the global triangle that connects the lattices OX and CX by structure-preserving maps:

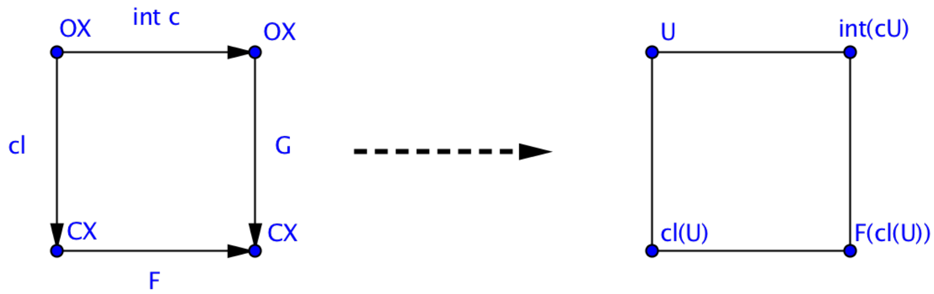
(3.1)



In other words, for U ∈ OX the corresponding half of the local topological square on U is obtained just by inserting U, intC(U), and cl(U) into the global lattice-theoretical diagram (3.1).

Then it is natural to ask whether the diagram (3.1) can be completed by functions CX——F——>CX and OX——G——>CX, respectively, so that (3.1) can be completed to a commutative diagram such that from this completed diagram

(3.2) (a) (3.2)(b)



with F(cl(U)) = G(int(**C**U)) = **C**U (cf. (1.1)). If this were the case the left commutative diagram (3.2)(a) could be characterized as a <u>global</u> topological square of the space (X, OX) from which all local topological squares (3.2)(b) on U ∈ OX are generated by inserting U.

It is easy to see that the global commutative diagram of (3.2) does not exist in general. Assume the contrary. Let us consider a well-behaved space, for instance a connected Hausdorff space such as the Euclidean space (E, OE) endowed with the familiar Euclidean topology OE.

The existence of the map CE——F——>CE is disproved as follows: If such a map F existed it would have to satisfy the equation F(cl(U)) = **C**U for all U $\in$ OE. Since E is connected and Hausdorff, the open set U has infinitely many points. Hence, choosing a point $\alpha \in$ U the set U_1 := U – {α} is still open and non-empty. Due to the fact that E is Hausdorff, this entails that cl(U_1) = cl(U). Therefore, one would have F(cl(U)) = F(cl(U_1)) = **C**U = **C**U_1. This entails U = U_1. This is a contradiction since by definition U $\neq$ U_1.

In a similar vein, one proves that a right vertical edge map G in (3.2)(b) cannot exist due to the fact that there exist different open sets V, $V_1 \in$ OX with int(CV) = int(CV_1) so that one would obtain G(int(CV)) = G(intC(V_1)) = **C**V = **C**V_1. This entails that V = V_1 which is a contradiction. Thus, there is no global commutative diagram from which the local topological squares (2.1) could be derived. This means that in general a global commutative diagram (3.2)(a), which could generate the local topological squares for U, does not exist.

Fortunately, this disappointing result can be sidestepped in several ways. In order to carry out this endeavor it is expedient to recall some elementary lattice-theoretical facts concerning the structures OX, CX, O*X, and C*X. For a more detailed representation of the and lattice-theoretical aspects of topology the reader may consult (Davey and Priestley 1990), (Gierz et al. 2003), or (Johnstone 1986). For the following we only need:

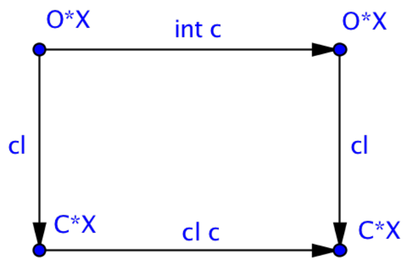
(3.3) On Lattice-theoretical Aspects of Topological Spaces.

(i) The set OX of open sets of a topological space X is a complete Heyting algebra with respect to set-theoretical inclusion $\subseteq$. This entails, in particular, that every a $\in$ OX has a Heyting inverse a* with $a \wedge a^* = 0$, $a^{***} = a^*$, and $0^* = 1$, $1^* = 0$.

(ii) The set CX of closed sets of a topological space X is a complete co-Heyting algebra with respect to the set-theoretical inclusion $\subseteq$. This entails, in particular, that every b $\in$ CX has a Co-Heyting inverse $b^\#$ with $b \vee b^\# = 1$, $b^{\#\#\#} = b^\#$, and $0^\# = 1$, $1^\# = 0$.

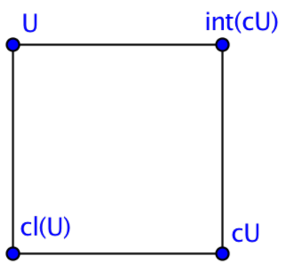
(iii) The set of regular open subsets O*X and the set of regular closed subsets C*X, are Boolean algebras.♦

(3.4) Theorem. Let (X, OX) be a connected topological space.Then for all regular open U ∈ O*X the diagram



is a commutative diagram that generates local topological Aristotelian squares:

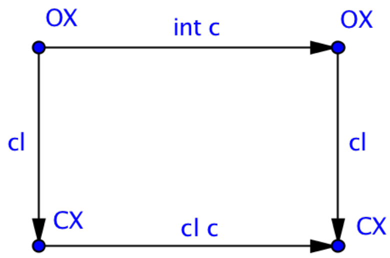
(3.5)



Proof. The commutativity of the diagram (3.4) follows directly from the definition of the operators cl and int (see (2.2)). To prove the contrariety of the top edge <U, int(CU)> one argues as follows: (i) U ∩ int(**C**U) ⊆ U ∩ **C**U = Ø. Assume U ∪ int(CU) = X. This entails that X is the disjoint union of two clopen subsets U and int(**C**U). This is impossible because X is assumed to be connected. That the vertical edges are in subalternation is proved similarly by also invoking the connectedness of X. The remaining assertions are proved similarly.♦

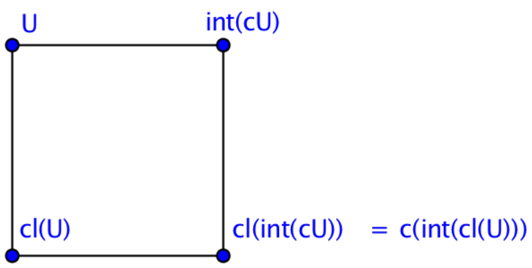
Instead of restricting the domain of regions for which Aristotelian squares are to be constructed another way to cast local topological squares in the framework of global commutative diagrams is to weaken the requirements for an Aristotelian square. More precisely, one observes that a global commutative diagram exists

(3.6)



which yields „almost Aristotelian“ local squares for all U ∈ OX:

(3.7)



The only difference of an „almost Aristotelian“ to a „fully Aristotelian“ local square is that the diagonal <U, cl(int(CU))> may not be a contradiction but only a contrariety. For U ∈ O*X the local square on U is fully Aristotelian, because in this case one has **C**int(cl(**C**U))) = U.

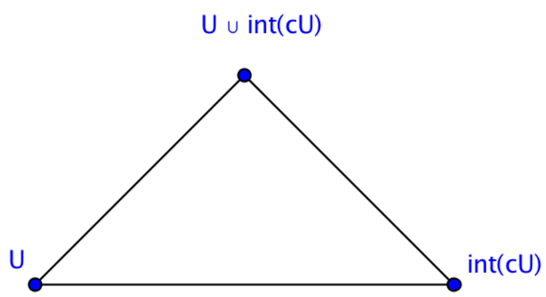
<u>(3.8) Example</u>. The diagonal <U, cl(int(**C**U)> in (3.7) is a contrariety but may not be contradiction.

Proof. Clearly U ∩ cl(int(**C**U)) ⊆ int(cl(U)) ∩ cl(int(**C**U)) = int(cl(U)) ∩ **C**intcl(U) = Ø. On the other hand, it may happen that U ∪ cl(int(**C**U) ≠ X. An elementary example is the following one: Take X to be the Euclidean plane and U the punctured open unit disk U := D – {(0,0)}, D := {x; 0 ≤ |x| < 1}. Then

U ∪ cl(int(**C**U) = D – {(0,0)} ∪ cl**C**D) = X - {(0,0)} ≠ X

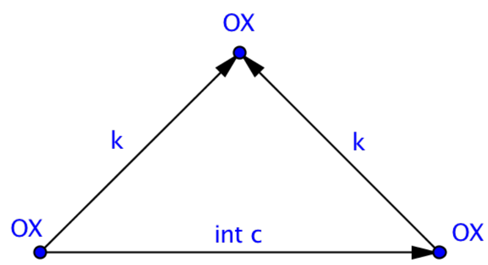
Thus, <U, cl(int(**C**U)> is contrariety but not necessarily a contradiction for all U.♦

After having constructed global diagrams for local squares it is natural to proceed to the task of constructing global diagrams for local topological hexagons. That is to say, to cast local triangles of contrariety (Béziau 2012) to cast into the framework of commutative diagrams. This task has naturally two parts: the construction of a global triangle for the top and for the bottom of the hexagon. The format of a local top triangle



suggests the following format of a global commutative triangle:

(3.9)



with k the set-theoretical map OX———k———>OX defined by k(U) := U ∪ int(**C**U)). This works perfectly well for regular open U:

$$k(\mathrm{int}\mathbf{C}\,(U)) = \mathrm{int}(\mathbf{C}U)) \cup \mathrm{int}(\mathbf{C}\mathrm{int}(\mathbf{C}U)) = U \cup \mathrm{int}(\mathbf{C}U)) = k(U)$$

because in this case U = int(**C** (int(**C**U))). Hence, for regular open U ∈ O*X the triangle (3.9) is a commutative diagram. Moreover, U and int(**C**U) define a contrariety because U ∩ int(**C**U) = O and U ∪ int(**C**U) ≠ X due to the fact that X is assumed to be connected. For U open but not regular open (3.9), however, does not commute:

$$k(\mathrm{int}\mathbf{C}\,(U)) := \mathrm{int}(\mathbf{C}U) \cup \mathrm{int}(\mathbf{C}\,(\mathrm{int}(\mathbf{C}U)) \neq k(U) := U \cup \mathrm{int}(\mathbf{C}U)$$

This slight lack of commutativity can be overcome by replacing the map k by k* defined as

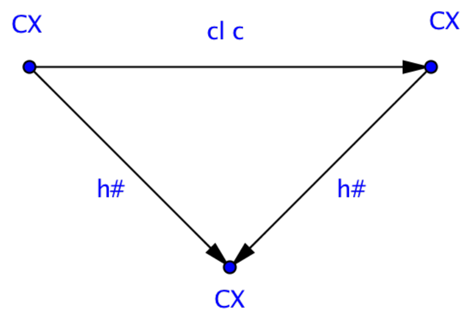
$$k^*(U) := \mathrm{int}(\mathbf{C}\mathrm{int}(\mathbf{C}U) \cup \mathrm{int}(\mathbf{C}U)$$

because OX is a Heyting algebra for which the threefold negation equals simple negation. Thus one obtains k*(U) = k*(int(**C**U). Evidently, k* renders the triangle commutative for all U ∈ OX.

Dually, the bottom triangle of the local squares on U can be generated by the commutative global diagram

CX———clC————>CX

(3.10)



defined by $h^{\#}(V) := cl(CV) \cap cl(\mathbf{C}(cl(\mathbf{C}V))$. Assembling the global top triangle, the global square (3.6), and the global bottom triangle then yields a global commutative Aristotelian hexagon because the top and bottom elements are contradictory as is shown by the following calculation:

$$\mathbf{C}(U^{*} \cup U^{**}) = \mathbf{C}U^{*} \cap \mathbf{C}U^{**}$$

By definition of * one obtains for the first factor of this intersection

$$CU^{*} = CintCCclU = clCclU = cl(U)^{\#}$$

The second factor of this intersection is calculated as

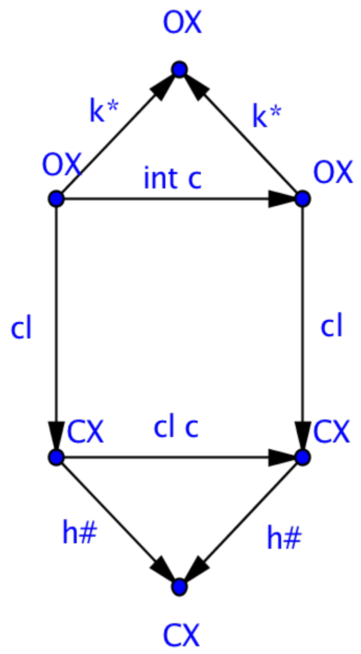
$$Cint(cl(U)) = clCclCclU = clCCintCCclU = clintclU = clU = clU^{\#\#}$$

Thus we finally obtain

$$C(int(CU) \cup int(cl(U)) = Cint(CU)) \cap Cint(cl(U)) = cl(U)^{\#} \cap cl(U)^{\#\#}$$

In sum, given a connected topological space (X, OX) we obtain a global commutative diagram that generates local Aristotelian topological hexagons for all $U \in OX$:

(3.11)



In sum, the construction of global commutative diagrams for local squares and hexagons works smoothly for U ∈ O*X. For non-regular open sets U ∈ OX one has to be content with the construction of weakly Aristotelian commutative diagrams, however. This is due to the fact that one has to iron out the effects of the non-Booleaness of the Heyting algebra OX and the co-Heyting algebra CX, respectively. This is not the case for O*X, since it is Boolean.

4. Similarity Structures and Galois Connections. The aim of this section is to present a new class of local Aristotelian squares generated by global commutative diagrams that do not arise from topological spaces but from similarity structures.

(4.1) Definition. A similarity structure (X, ~) is defined as a set X endowed with a reflexive and symmetric (not necessarily transivitive) binary relation ~ ⊆ X × X. The relation ~ is called a similarity relation defined on X. If (x, y) ∈ ~ the elements x and y are said to be similar to each other. As usual this is denoted by x ~ y. For x ∈ X the similarity neighborhood co(x) of x is defined as co(x) := {y: x ~ y}. Since ~ is reflexive the similarity neighborhood co(x) is never empty but contains at least x.♦

<u>(4.2) Definition</u>. Let (X, ~) a similarity structure. The similarity relation ~ defines two operators PX——h——>PX and PX——s——>PX by:

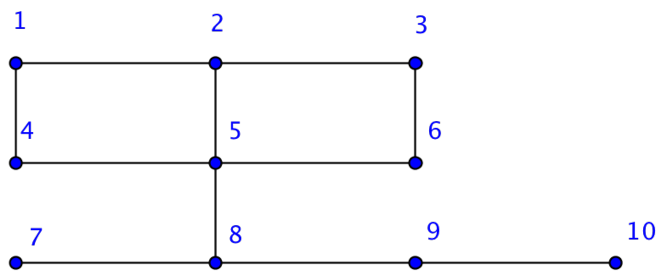
$$h(A) := \{x: co(x) \subseteq A\} \qquad s(A) := \{x: \exists y\ (y \in A \text{ and } x \sim y\}$$

The operators h and s are interdefinable, namely, h = **C**s**C** and s = **C**h**C**. Clearly, h(A) ⊆ A ⊆ s(A), but neither h nor s are closure operators in the sense of (2.2). As will be shown later the concatenation hs is a closure operator (although not a topological one). Hence a set A ∈ PX is called closed if and only if hs(A) = A, and B ∈ PX is called open if and only if sh(B) = B.♦

For a thorough-going calculatory investigation of the operators h and s the reader may consult the paper of Breysse and de Glas (2007). Similarity structures (X, ~) with few elements can be nicely illustrated by simple undirected graphs: Every vertex of the graph corresponds to an element of X, and two non-identical elements that are similar to each other define a uniquely determined edge of the graph.

The following example offers an illustration of how the operators h, s, and their concatenations work for a small graph (X, ~):

<u>(4.3) Example (Graph-theoretical Illustration of small similarity structures and the operations of h and s).</u> Let (X, ~) be the similarity structure given by the following graph:



According to this graph the elements 1 and 2, 2 and 3, 3 and 6, ... are similar to each other, while 1 and 3, 4 and 6, 6 and 10 etc ... are not similar to each other.

For a subgraph Y of (X, ~) defined as Y := 7.——8.——9. the operations of the operators h, s, and their concatenations can be calculated as follows:

h(Y) = (7) hh(Y) = Ø, s(Y) = (5, 7, 8, 9, 10)

sss(Y) = X, sh(Y) = (7, 8) hs(Y) = (7, 8, 9, 10).

The subgraph Y is neither closed (hs(Y) = Y) nor open (sh(Y) = Y). The subgraph hs(Y) is, of course, closed ♦

One observes that neither h is an interior kernel operator nor s is a closure operator. One has to invest some more work to squeeze out from h and s of some interesting structure. For technical reasons one has to restrict one's attention to connected similarity structures in the following sense:

<u>(4.4) Definition</u>. A similarity structure (X, ~) is connected if and only if for all A ⊆ X the following holds

$$(A)(A \subseteq X \text{ and } A = sA \Rightarrow A = \emptyset \text{ or } A = X).$$

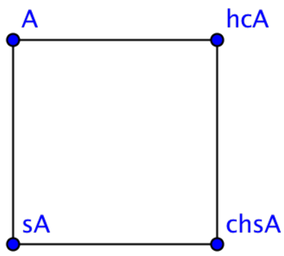
Equivalently, (X, ~) is connected if and only if

(B) (B ⊆ X and B = hB entails B = Ø or B = X).♦

From now on, all similarity structures are assumed to be connected. Since A ≠ Ø, X is assumed throughout this entails that in the following A and sA are always in subalternation.♦

The natural first step for constructing global commutative diagrams of Aristotelian opposition for similarity structures is to begin with local squares:

<u>(4.5) Proposition</u>. Let (X, ~) be a connected similarity structure and A ⊆ X with Ø ≠ A ⊂ X and A = hsA. Then the following diagram is an Aristotelian square:



<u>Proof</u>. One has to show that the edges <A, sA>, <sA, ChsA>, ... of this square satisfy the required Aristotelian conditions. The proofs are routine:

(i) The top edge <A, hCA> is a contrariety: By definition (4.2) of the operator h one has A ∩ hCA ⊆ A ∩ CA = O. Further, A ∪ hCA ≠ X because otherwise X would not be connected. On the other hand, <A, hCA> is not a contradiction. Assume the contrary, i.e., A ∪ hCA = X. This entails that Ø = C(A ∪ hCA) = CA ∩ ChCA = CA ∩ sCCA = CA ∩ sA. This entails that sA = A. In other words, X would not be connected. This is contradicts the assumption that X is connected.

(ii) The bottom edge <sA, shCA> is a subcontrariety: by definition (4.2) $sA \cup ChsA \supseteq A \cup CA = X$ and $sA \cap ChsA \neq \emptyset$, because otherwise X would not be connected.

(iii) The vertical edges <A, sA> and <hCA, shCA> are in subalternation by definition of s and the assumption that X is connected.

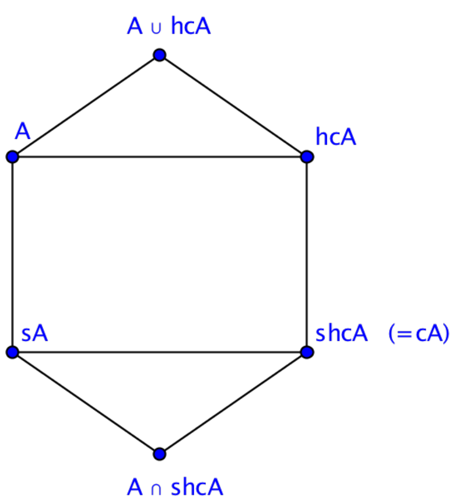
(iv) The diagonals are contradictory: For <A, ChsA> one calculates <A, ChsA> = <A, CA)>, because A is closed, i.e., hsA = A. For <sA, hCA> one obtains <sA, hCA> = <sA, CsA>. Both pairs are clearly contradictory.♦

Further, squares of type (4.5) can be extended in a canonical way to commutative hexagons with contradictory tops $A \cup hCA$ and bottoms $sA \cap ChsA$, respectively:

$$C(A \cup hCA) = CA \cap ChCA = ChsA \cap sA.$$

In sum, for connected similarity structures (X, ~) one obtains an analogue of Angot-Pellessier's theorem:

<u>(4.6) Theorem</u>. Let (X, ~) be a connected similarity structure. Then for all $A \in PX$ with $A = hsA$ and $\emptyset \neq A \neq X$ there is a local Aristotelian hexagon on A:



As will be shown in the next section, the operator hs is indeed a (non-topological) closure operator, i.e., an operator that satisfies the conditions (2.2)(i) – (iii) (but usually not (2.2)(iv)). Hence, a subset A of X with A = hsA may be called a closed set (with respect to hs). According to (4.6) every such closed set defines a local Aristotelian hexagon. The aim of the next section is to construct a global commutative diagram from which all local hexagons can be derived. As it will turn out this can be carried out in an even neater way than for global topological diagrams.

<u>5. Aristotelian Squares and Global Commutative Diagrams for Similarity Structures</u>. The aim of this section is to introduce the fundamental concept of a Galois connection. More precisely, we are going to show that the operators h and s of a similarity structure (X, ~) define such a connection on PX. This entails the existence of a commutative diagram that defines a global Aristotelian square of oppositions for appropriate subgraphs on U ∈ SCX ⊆ PX. Moreover, it turns out that this construction of global squares for similarity structures (X, ~) is fully analogous to the construction of global squares for topological spaces (X, OX) in previous sections. To set the stage, let as first briefly recall the general definition of a Galois connetion (cf. Gierz et al. 2003, Denecke et al. 2004):

<u>(5.1) Definition</u>. Let (X, ≤) and (Y, ≤) be partially ordered sets. The order-preserving maps $X \xrightarrow{h} Y$ and $Y \xrightarrow{s} X$ define a Galois connection (h, s) between X and Y iff for all (a, b) ∈ X × Y one has ha ≥ b ⇔ a ≥ sb.♦

<u>Remark</u>: It should be noted that the concept of a Galois connection usually is an asymmetric concept, i.e., if (h, s) is a Galois connection between X and Y, in general there is no reason to expect that (s, h) is a Galois connection too. In the literature several concepts of Galois

connections can be found, some inconsistent with each other. Often, the map h is called the „upper" (or the „right") adjoint and s is called the „lower" (or the „left") adjoint of the Galois connection (h, s). This paper adopts the definition of (cf. (Gierz et al. (2003, 22-23)).

<u>(5.2) Example</u>. Let (X, OX) be a topological space, OX and CX the lattices of open and closed subsets, and int and cl the closure and the interior operators, respectively. Then CX——int——>OX and OX——cl——>CX define a are a Galois connection (int, cl) between CX and OX, i.e. for $A \in CX$ and $B \in OX$ one has

$$\mathrm{int}(A) \supseteq B \Leftrightarrow A \supseteq \mathrm{cl}(B).$$

<u>Proof</u>: Assume $B \subseteq \mathrm{int}(A)$. Then $B \subseteq \mathrm{int}(A) \subseteq A$ and $\mathrm{cl}(B) \subseteq \mathrm{cl}(A) = A$. On the other hand, assume $\mathrm{cl}(B) \subseteq A$. Then $\mathrm{int}(\mathrm{cl}(B)) \subseteq \mathrm{int}(A)$. Since for $B \in OX$ one has $B \subseteq \mathrm{int}(\mathrm{cl}(B))$, one obtains $B \subseteq \mathrm{int}(\mathrm{cl}(B)) \subseteq \mathrm{int}(A)$.♦

<u>Remark</u>. Although the topological operators cl and int are defined on all of PX, they do not define a Galois connection on PX as is shown by elementary examples.

Galois connections abound in mathematics, logic, and elsewhere (cf. Deneke et al. (2004), Gierz et al. (2003)). Actually, Galois connections are special cases of adjoint situations which are considered as the fundamental concept category theory *überhaupt* (cf. Adámek, Herlich, and Strecker (1990), Awoday (2010), In the following we will discuss a type of Galois connections that provides a rich source for Aristotelian squares and hexagons. Indeed, one may contend that these Aristotelian squares and hexagons are more easily constructed than those of topological origin.

<u>(5.3) Proposition</u>. Let (X, ~) be a similarity structure, and h and s maps from PX onto itself as defined in (4.2). Assume PX to be ordered by set-theoretical inclusion ⊆. With respect to this

relation the maps h and s are order-preserving and define a Galois connection (h, s) on (PX, ⊆), i.e., for A, B ∈ PX one has

$$B \subseteq hA \quad \Leftrightarrow \quad sB \subseteq A$$

Proof. Let A, B ∈ PX and assume B ⊆ hA and x ∈ sB. One has to show that x ∈ A. From x ∈ sB one infers that there this a y ∈ B such that x ~ y. This means that x ∈ co(y). From B ⊆ hA one obtains co(y) ⊆ A. Hence a ∈ A.

To prove the other implication, assume sB ⊆ A and y ∈ B. One has to show that y ∈ hA. From sB ⊆ A one infers that co(y) ⊆ A. This is just to assert that y ∈ hA.♦

From the technical result (5.3) a lot of useful results can be derived (cf. Gierz et al. (2003), Denecke et al. (2004)).

(5.4) Proposition. Let (h, s) be a Galois connection between order structures (X, ≤) and (Y, ≤). Denote the identity maps of X and Y by id_X and id_Y, respectively. Then the following holds:

(i) $id_X \leq hs$ and $sh \leq id_Y$.

(ii) hsh = h and shs = s.

(iii) hs is a closure operator (satisfying (2.2)(i) – (iii)) and sh is a kernel operator (satisfying (2.2)(j) – (jjj)) on Y and X, respectively.

(iv) For X = Y h and s induce a chain of inclusions: .. ⊆ hA ⊆ shA ⊆ A ⊆ hsA ⊆ sA ⊆ ....

Proof. (i) Use the fact that (h, s) is a Galois connection and apply the definition of a Galois connection to (sB, B) and (A, hA). Then one obtains sB ⊆ sB ⇔ B ⊆ hsB, and hA ⊆ hA ⇔ shA ⊆ A. This proves (5.4)(i). The assertions (5.4)(ii) – (iv) immediately follow from (i).♦

(5.5) Definition. Let (X, ~) be a similarity structure with Galois connection (h, s).

(i) The set SCX := {A; A = hsA, A ∈ PX} (SCX for „similarity closed“) is called the set of (similarity-)closed subsets of PX. Due to (5.2)(ii) one has hs(hA) = hshA = hA. Hence SCX may be equivalently defined as hPX = {hA: A ∈ PX}. SCX is a complete inf semi-lattice with respect to set-theoretical intersection ∩. Hence it can be rendered a complete lattice (SCX, ∧, ∨) by defining A ∧ B := A ∩ B and A ∨ B := hs(A ∪ B).

(ii) The set SOX := {B; B = shB, B ∈ PX} (SOX for „similarity open“) is called the set of (similarity-)open subsets of PX. Due to (5.2)(ii) one has sh(sB) = shsB = sB. Hence SOX may be equivalently defined as SOX := {hB: B ∈ PX}. SOX is a complete sup semi-lattice with respect to set-theoretical union ∪. Hence it can be rendered a complete lattice SOX, (SCX, ∧, ∨) by defining A ∧ B := sh(A ∩ B) and A ∨ B := A ∪ B.♦

The lattices SCX and SOX turn out to have important extra structures that are essential for the construction of Aristotelian squares of oppositions for A ∈ SCX:

<u>(5.6) Definition (cf. (Beran (1984, II.1, p. 29)</u>. A lattice (L, ∧, ∨) is an ortholattice if and only if there is a map („negation“) L——§——>L satisfying the following requirements:

(i) D ∧ §(D) = 0 (iii) §(§(D)) = D,

(ii) D ∨ §(D) = 1 (iv) D ≤ E ⇒ §(E) ≤ §(D).♦

<u>(5.7) Proposition</u>. Let (X, ~) be a similarity structure with the lattices (SCX, ∧, ∨) and (SOX, ∧, ∨) of closed and open subsets, respectively. Then

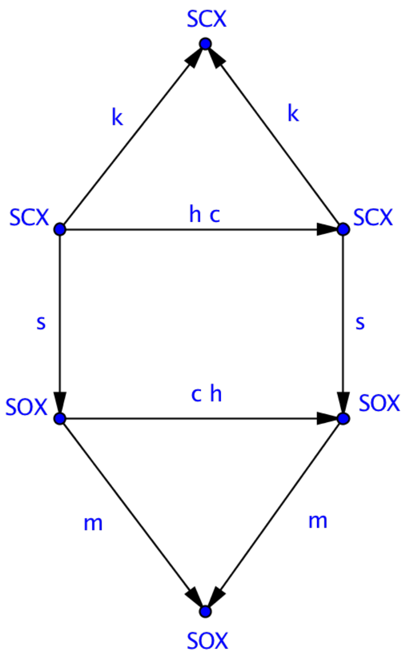
(i) SCX is an ortholattice with respect to the negation§ defined by §(A) := h(CA).

(ii) SOX is an ortholattice with respect to the negation § defined by §(B) := s(CB).

<u>Proof:</u> A direct proof without using the fact that (h, s) define a Galois connection can be found in (Breysse and De Glas (2007)). A quicker proof is provided by (5.4).♦

Examples show that the lattices SCX and SOX in general are not distributive. Hence, these lattices need not to be Boolean lattices. The Galois connection (h, s) defined on PX by the similarity structure (X, ~) can be restricted to a Galois connection (h, s) SOX——h——>SCX and SCX——s——>SOX. Thereby we obtain:

(5.8) Theorem. Let (X, ~) be a connected similarity structure with Galois connection (h, s). Then there exists the following global commutative diagram:



with k and m defined for $A \in SCX$ and $B \in SOX$, respectively, by $k(A) := A \cup h(CA)$, and $m(B) := B \cap Ch(s(B))$, respectively. The upper and the lower triangle are well-defined as commutative diagrams because

(5.9) $k(A) := A \cup hCA = hsA \cup hCA = hCCsA \cup hCA = k(hCA)$

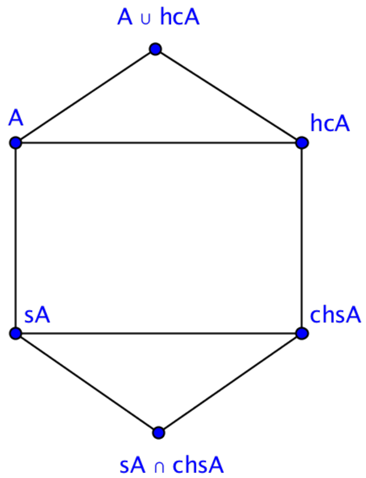
Analogously for the lower triangle one obtains for $B \in SOX$:

(5.9)' $m(B) := B \cap ChB = shB \cap ChB = sCChB \cap hB = m(ChB)$

Thereby, from (5.9) and (5.9)’ one calculates show that the top vertex and the bottom vertex of a local hexagon generated by (5.8) on A ∈ SCX are contradictory.♦

For non-trivial A ∈ SCX the global commutative diagram (5.8) generates local Aristotelian hexagons:

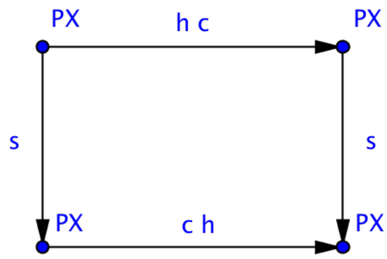
(5.10)



In contrast to topological spaces (X, OX), for similarity structures (X, ~) one need not restrict the domain of generating commutative diagrams to regular subsets. This is due to the fact that for similarity structures (X, ~) the „closed“ and the „open“ lattices SCX and SOX are ortholattices. In contrast, for topological spaces (X, OX) the lattices OX and CX are not ortholattices, but only Heyting and co-Heytings ones, respectively.

Theorem (5.8) may be generalized for arbitrary A ∈ PX provided one accepts a slight weakening of what is to be understood by an Aristotelian square (and hexagon) (cf. (4.5)):

<u>(5.11) Theorem</u>. Let (X, ~) be a connected similarity structure with Galois connection (h, s) defined on PX. Then the global commutative diagram



induces for all A ∈ PX, A ≠ Ø, X, squares and hexagons that are almost Aristotelian in the sense that they satisfy all requirements that an Aristotelian square has to satisfy with the possible exception that the diagonal <A, **C**hsA> may not be a contradiction but only a contrariety. This is the case exactly if A is not a closed subset of X with respect to the closure operator hs.

Proof. Analogous to the proof of (4.5).♦

6. Concluding Remarks. In this paper the first steps have been taken to explore the relationship between two important types of „logical diagrams", viz. Aristotelian squares and hexagons on the one side, and commutative diagrams of relational structures such as topological spaces and similarity structures on the other.

Commutative diagrams and Aristotelian squares are not the only types of diagrams. Also Euler diagrams, Venn diagrams, and Peircean graphs deserved to be mentioned. Recently, Demey and Smessaert pointed out that Aristotelian diagrams and Hasse diagrams (= diagrams of partially ordered structures) are related in interesting ways (cf. Demey and Smessaert (2014)). The co-existence of many types of different diagrams suggests that it would not be a promising strategy to look for one "best" type of diagram for all purposes. Rather, one should look for appropriate combinations of diagrammatical presentations of conceptual relations for specific purposes.

The global commutative diagrams (3.11), (5.8), and (5.10), based on the lattices OX, CX, SOX, SCX, PX etc., give rise to a family of Aristotelian figures of oppositions, indexed by the elements of OX, SOX, and PX, respectively. Further, conceiving OX itself as an order structure with respect to set-theoretical inclusion, Aristotelian squares and hexagons defined on U, V ∈ OX with U ⊂ V give rise to 3-dimensional figures (cubes and others) in a natural way. In this way the combination of various types of diagrams offers richer infor–mation than relying on any one type of diagram alone.

Another issue worth to be explored further is the general relationship between Aristotelian conceptual oppositions and Galois connections. In this paper it has been shown that Galois connections that arise from topological structures and similarity structures give rise in a natural way to various types of Aristotelian conceptual oppositions. It is not clear, however, how Galois connections and Aristotelian conceptual oppositions are related in general.

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