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Kant and the object of determinate experience

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An influential view has it that the paradigm application of Kant's categories is Newton's dynamics. Though cogent, the claim makes the categories too narrow, because Newton's laws had explanatory limits known well before the 1780s. I show here that the categories are broad enough to avoid that problem: I prove that Kant can ground basic laws for all classical mechanics, which is demonstrably more general than Newton's theory.

To make my case, I survey three brands of Enlightenment dynamics, based respectively in the Principle of Least Action, the Principle of Virtual Work, and Euler's Laws of Motion. I obtain two results. Kant's Analogies of Experience support neither the Principle of Least Action nor the Principle of Virtual Work. But, they could well ground Euler's two laws of motion. However, to do so Kant must tweak his theory of matter: from a physical continuum to discrete mass-points.

And, I show that my reading makes Kant relevant to current foundations of mechanics. This makes his categories explanatorily sufficient *and* timely.

0. Introduction[1]

Soon after the First Critique came out, Kant became alarmed that readers may suspect his twelve categories are too abstract to apply to any objects of actual experience. To preempt that fear, in 1786 he wrote *Metaphysical Foundations of Natural Science*.[2] There, he assured them, was a set of "examples *in concreto*" for his pure transcendental concepts. Michael Friedman has long argued cogently that Kant meant his metaphysical foundations to ground primarily Newtonian science: the theory in Newton's *Principia* is the "paradigmatic instantiation" of Kant's constitutive

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[2] I refer to Kant's *Metaphysical Foundations of Natural Science* (1786) simply as '*Foundations*.' Unless otherwise noted, all translations are mine. I follow convention and cite Kant's works by volume and page number in the Academy Edition, as follows: '1' is Kant 1992, '4' is Kant 1911, and '14' is Kant 1925. 'A' and 'B' are the two editions of the First Critique, translated as Kant 1998.

apparatus, and his categories apply primarily to the objects and processes that Newton treated (Friedman 1992: 137; 2013: 410).

Friedman's result and sophisticated case—the apex of the venerable 'Marburg' approach to Kant—is so spectacular that it has obscured from us a worrisome fact we must no longer ignore. Let us grant, as Friedman implies, that Kant's categories apply primarily to *determinate* experience—the exact science of nature, that is—rather than to everyday acquaintance with common objects. Still, we must worry that his categories are too narrow even for that task. I do not mean that Riemann and Einstein proved Newtonian physics to hold just in a limit case. I mean that, even in a classical world—of objects at subluminal speeds in Euclidean space—Kant's categories are not broad enough to ground *all* rigorous experience as we had it until 1905 or so. That is because, not long after the *Principia* but well before the First Critique, it became apparent that Newton's fundamental notions had serious limitations. To wit, mechanics is explanatorily basic. Yet there were *basic* mechanical phenomena and causal processes that Newtonian theory could not explain in principle, not just provisionally. And so, if Newton's key concepts are too narrow, and yet they prove the "real possibility" of Kant's categories, we must infer that the categories are not general either. Newton's limitations, already visible in the 1740s, become Kant's too, so his categories are DOA—dated on arrival, as it were.

In this paper, I show a way to dispel that fear. Kant's categories, I argue, *are* general in the sense above. I make my case from a particular but uniquely powerful vantage point. Namely, I show that Kant's Analogies of Experience ground causal laws that explain the basic behavior of all 'classical' objects generally. My vantage point is privileged, in that it seeks to recover *ultimate* principles for causal reasoning within rigorous experience. To make my case, I marshal history and philosophy. I survey three Enlightenment theories—early versions of our mechanics—and ask if Kant's categories can ground any of them. For brevity's sake, I call these versions 'variational,' 'analytic' and 'Eulerian.' They are distinct in that they rest on different *basic* laws. My verdict is dual. Kant, it turns out, can ground neither variational nor analytic mechanics. And yet, he has a good chance of grounding Euler dynamics. But, he must adjust his mature theory of matter: namely, he must switch to a mass-point ontology. In historical terms, he must adopt a variant of his old physical monadology—though suitably modified in light of Transcendental Idealism.

First, a rationale: I show that Kant's contemporaries found Newton's laws too narrow to ground objecthood for mechanics (§ 1). Next, I explain three programs—variational, analytic and Eulerian—pursued in the Age of Reason to overcome Newton's limitations (§ 2). Then I establish that Kant has no good way of grounding the first two programs. Variational mechanics is not grounded in efficient causes, which Kant's Second Analogy demands of all physical theory. And, analytic mechanics introduces a type of fictive forces that violate Kant's three Analogies. In contrast, Kant has very good resources for grounding Euler dynamics (§ 3). However, to complete that task, Kant must adopt a version of his old view that matter is made of mass points, a phenomenal species of 'physical monads.' Thereby, he can derive the two fundamental laws of Eulerian mechanics (§ 4). Lastly, I invoke some modern results to show that Kant's discrete 'phenomenal monads' can support a theory of continuum phenomena as well (§ 5). So, his categories are more general *and* timely than we have thought.

Cui prodest? One benefit is that my study extends non-trivially Friedman's thesis that the *Principia* is the paradigmatic instantiation of Kant's categories. A critic might ask if Newton's theory is the paradigm—an exemplary case—or just the *sole* application of Kant's abstract categories to rigorous experience. That would lessen their value in proportion as the *Principia* turns out to have limits. I establish that Friedman has a point after all: Kant's categorical inventory *can* ground exact science beyond Newton. Another benefit is that it brings Kant's foundations into the present. For all its towering genius, the *Principia* is a work in early modern science. In contrast, classical mechanics is a live area of physics and philosophy (Sklar 2013; Batterman 2013). Thus, I show, Kant remains relevant.

Now, I must confess to two limitations. My account is not conclusive; I just uncover some initial barriers to fitting post-Newtonian dynamics into Kant's framework. But, a final verdict requires a fuller inquiry, for which I lack the space here. It is enough for now if I produce some first results and outline a research program. Second, I do not inspect here the grounding relation between Kant's foundations and basic dynamical laws. That relation is notoriously debated. It has been read variously as transcendental explanation of possibility, presuppositional analysis of dynamical theory, even confirmation (Friedman 1992; Watkins 1998a; DiSalle 2006).

A key caveat is in order, before I begin. I will argue below for a theory of matter in which the unit body is the mass-point. At times, I call those units 'phenomenal monads,' to signal their vicinity to Kant's thought. In the 1750s he held a doctrine, "physical monadology," that was a mass-point ontology in all but name. That is, a mass point and a physical monad largely overlap in their respective geometric makeup, kinematic behavior, and dynamical powers.[3] However, the metaphysics of those monads was transcendental realism. That standpoint becomes unacceptable to Kant after 1781, and I comply with his stricture.[4] Fortunately, physical monads can be made safe for transcendental idealism, as 'phenomenal monads.' They have the kinematic properties and causal powers of mass points, viz. three degrees of freedom, inertial mass, and action-at-a-distance forces. But, they are cis-noumenal, or a species of appearances. Specifically, they are the *phenomenal* substratum of all mechanical appearances in a classical regime. As to their epistemology, phenomenal monads are inferable by a systematic use of Kant's Postulates of Empirical Thought, not by armchair conceptual analysis. So, we need not fear the specter of the Second Antinomy as we adopt them on Kant's behalf.[5]

1. The limits of Newton's *Principia*

Mach in 1883 originated a myth that ensnared many in the Twentieth Century, namely, that the laws in the *Principia* are a complete basis for classical mechanics: "The principles of Newton suffice by themselves, *without* the introduction of any *new* laws, to explore thoroughly every mechanical phenomenon practically occurring" (1960: 342; my italics). His stature then helped turn it into conventional wisdom. Mach should have known better. Already in his time Hamel had proved that, to handle rigid bodies and deformable continua, Newton's three laws are not enough (1909). Rather, they need supplementation with other, equally *fundamental* laws. *Vox clamantis in deserto*—the myth reigned on for an-

[3] This is not true without qualification. In *Physical Monadology*, Kant really mixes two theories of matter, though the mass-point picture remains his dominant and official account. See the admirable Smith 2013.
[4] A thorough discussion of the Critical Kant's opposition to (his old) physical monadology is Pollok (2001: 251-73).
[5] I thank Rae Langton, Angela Breitenbach, and Konstantin Pollok for pressing me on this issue.

other half-century. Thankfully, historians began to look at 18th-century mechanics closely, and their findings awoke us from our dogmatic slumber. The Enlightenment doubted Newton's laws, altered them drastically or even replaced them outright (Caparrini & Fraser 2013).

Still, recounting the demise of false history is no argument. Then let me explain precisely why Newton was not enough. The key fact is that Newton applied his four laws merely to *free particles*. That is, their target objects are kinematically unconstrained; discrete, not continuous; point-sized, not extended; and single, not part of a system. But, that left Newton's original laws with three blind spots. Specifically, there were three kinds of processes for which his law $\mathbf{f} = m\mathbf{a}$ could not by itself yield equations of motion, the key task of dynamical principles.

One blind spot was constrained motion, in which the particle cannot move freely in space but is subject to kinematic limitations. E.g. a bead falling under gravity down a thin rigid rod swinging freely around a fixed pivot.[6] The Second Law predicts that the force will cause the particle to accelerate in the direction *of the force*. But, in constrained motion, the particle by definition cannot accelerate in that direction: the constraint prevents it.[7] And so, it is a mystery how $\mathbf{f} = m\mathbf{a}$ might be used to predict the particle's behavior, given that, for some regions and directions, there can be *no* actual **a**, no matter how great the actual **f**. Thus, to handle constrained motion, we can see in retrospect, the Second Law must be paired with an additional dynamical principle, viz. that constraint forces do zero mechanical work.

Another blind spot was the motion of rigid bodies and continuous media—the 'hard bodies' and 'infinitely divisible matter' of early modern science. Rigid bodies seemed intractable by Newton's laws because *every* part in them is constrained.[8] Moreover, they can turn or spin—unlike the *Principia*'s point-sized particles—thus forces might give them not just a

[6] Other setups studied intensely at the time: a ball sliding outward in a rotating tube; a set of weights individually attached to a hanging rope that sways; a fluid in laminar flow as its outer layer stays in contact with the container wall; a rigid body moving around a fixed internal point; and a body sliding down an inclined plane free to move on a rigid track under the body's weight; cf. Clairaut 1742 and d'Alembert 1743.

[7] Thus, in my example above, gravity pushes the bead vertically downward, but the oblique swinging rod on which it slides prevents it from going straight down—the bead must slide *along* the rod not *through* it.

[8] Namely, by the rigidity condition: every particle in a rigid body must move such that its distance to all other constituent particles stays constant.

linear impulse but also angular momentum. The Second Law cannot account for this, and so another basic principle was needed: the Torque Law, which I explain below. As to continuous media—fluids, elastic bodies, and plastic solids—the Second Law seemed outright inapplicable to them, as they were not made of discrete particles. Euler rose to the challenge by applying the Second Law to the least part of a continuous body, viz. the infinitesimal mass element *dM*, and integrating over the body. However, Euler soon discovered that, in general, Newton's principle was not enough. Mass elements in continua can twist under stresses, thus may acquire angular momentum, which the Second Law cannot handle. Again, a *new* principle is needed: the Torque Law.

Finally, Newton's laws had limits in respect to the very causal power they treated: gravity.[9] The *Principia* mathematized gravity mostly acting on particles.[10] To apply Newton's account to actual planets and satellites, these bodies had to be idealized away: for each planet, a *single point*—its mass center, a centroid—was chosen as representative. The *Principia*'s laws then let us track the motion of this special point when the external gravity is known. Now, this approach ignores the motion of the planet as an *extended body*: it leaves out of account how gravity might change its shape and volume, spin and orientation. But, actual planets do in fact experience such changes. And, most of these gravitational phenomena were found in the 1700s: equatorial bulging; terrestrial nutation; lunar libration; the precession of the equinoxes—all that the Enlightenment called *astronomia mechanica*. Newton's laws being insufficient in this wider realm, d'Alembert and Euler in the 1740s added novel *fundamental* principles to extend the reach of the *Principia* (Chapront-Touzé 2006; Verdun 2015).

Lest we think insight into the limits of Newton's laws is late-modern, hence anachronistic for Kant interpretation, listen to post-Newtonians express misgivings about their situation. As luck would have it, the fathers of Enlightenment dynamics were given to musings about their dis-

[9] The *Principia* is not a tract in general mechanics, just in gravitation theory. Newton did express the hope that a fully general mechanics may be developed from his notions of impressed force, action, and the Second Law. The fact is, however, that his book tackles just one kind of force—gravity.
[10] Some rare exceptions: the gravitational potential of a thin shell and a homogeneous sphere; the shape of the Earth.

cipline. By the 1740s they came to realize with dismay that the dynamical laws they had inherited were not general. Euler was appalled:

> These principles are of no use in the study of motion, unless the bodies are infinitely small, hence *the size of a point*—or at least we can *regard* them as such without much error: which happens when the direction of the soliciting power passes *through the center of gravity*.... But if it does not pass through that center, we *cannot* determine the entire effect of these powers. That is all the more so if the body to be moved is *not free*, viz. is constrained by some obstacle, depending on its structure. (1745, §17; my italics)

Daniel Bernoulli complained too: "the *general* laws of motion... are still hidden to us" (1746, 55; my italics). Their lament is precisely that the laws bequeathed by Newton, Huygens and Leibniz apply just to free particles or to a single point—a centroid, in our terms—in an extended free body. But clearly, bodies often behave in ways not even approximated by the motion of their centroid. A genuine mechanics ought to account for that.

In conclusion, Newton's laws had severe limitations: as the 18th century turned to new objects and processes, it found with consternation that the explanatory basis Newton *and others* had left it was too weak.[11]

2. Paradigms of mechanics in the Age of Reason

That started a long, arduous search for truly general principles. It was a collective effort that spanned the middle third of the century, and it slowly resulted in three different traditions of mechanics. I give below historical capsules of each, and explain their fundamental laws. Euler and Lagrange made the decisive contributions to all three.

Variational mechanics. Around 1740 in France Maupertuis saw a new way to look at mechanical processes. He found that, in bodies at rest on a balanced lever and in bodies in frontal collision, a certain quantity tends to a minimum value. He called that quantity 'action,' and claimed that it showed divine, economical wisdom at work. Euler, later his colleague at the Royal Academy in Berlin, moved to expand Maupertuis' insight. Euler showed that, if a unit-mass particle orbits in a plane under central forces, the time integral of its velocity multiplied by the differential arc element ($= \int v ds$) is a minimum (1744, 311f). Then in 1751 he wid-

[11] Variations on this theme are already in Truesdell 1968; Wilson 2013; Smith 2008.

ened the reach of Maupertuis' principle.[12] Euler let a particle M be attracted by central forces V, V', V'' proportional to some function of its distances z, z', z'' to the centers of force. For the sum of the integrals $\int V\mathrm{d}z$, $\int V'\mathrm{d}z'$, $\int V''\mathrm{d}z''$ Euler coined the term "effort," labeled Φ. (We recognize Φ in hindsight as the work function of the system.) Euler explained that, if the particle is in equilibrium, the "effort" will be an extremum, i.e. a maximum or minimum. [13] More importantly, he demonstrated that if the particle *moves*, the integral $\int\Phi\mathrm{d}t$ is also an extremum: "if the quantity Φ must be a minimum in case of equilibrium, the same laws of Nature seem to demand that in the case of motion the formula $\int\Phi\mathrm{d}t$ must be the smallest" (Euler 1753: 175).

Though important and novel, Euler's result had one serious limitation: it was provably true just for the dynamics of *one particle*. It was the young Lagrange who showed that the Principle of Least Action comes close to being truly general. He began by creating a new formalism: an analytic theory of δ, a variational operator that he let commute with d and $\int$, the respective basic operators in the differential and integral calculus (Lagrange 1762a). He then recast the Maupertuis-Euler law as follows:

> General principle: *Let there be an arbitrary number of bodies* M, M′, M″ *acting on each other in any manner. Let them be driven by forces proportional to some function of distance. Let* s, s′, s″*, etc. denote the spaces crossed by these bodies in time* t*, and let* u, u′, u″ *be their speeds at the end of this time. Then the formula* $M\int u\,ds + M'\int u'ds' + M''\int u''ds''$, *etc. will always be a minimum or a maximum.* (Lagrange 1762b: 198)

From this principle, Lagrange showed how to derive equations of motion for a vast range of mechanical systems: one particle, free *or* constrained, attracted by several forces; the three-body problem; three particles with constraints, rigid *and* deformable; the compound pendulum; the vibrating string; the rigid body, free *and* constrained; an incompressible inviscid fluid; and an elastic gas.[14]

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[12] Lucid analyses of these results are Goldstine 1980: 101-9 and Fraser 1983: 200-2.
[13] Here is an example of maximizing the action. (It comes from Euler.) Rest a cone of homogeneous mass on its vertex, on a flat rigid surface. It will stay in that equilibrium position, where its "effort" is a maximum. Any imbalance will make it move so as to come to rest on its base, where its "effort," or potential energy, is a minimum.
[14] In a nutshell, Lagrange proceeded as follows. First, he wrote the Principle of Least Action as $\delta\int M_i v_i ds_i = 0$. Second, he interchanged δ and $\int$, and expanded the variation

This brand of mechanics is variational because of its mathematics, the calculus of variations. Its basic law is the Principle of Least Action. In exact terms, it says that the variation of a certain integral—the "action"—is always null:[15]

[1] $$\delta \int Mv\,ds = 0$$

The physical insight behind this principle is: in every mechanical process, a system of bodies behaves so as to minimize or maximize collectively a certain quantity, namely the "action" defined above. At every point of its instantaneous location, the system has an infinite bundle of *possible* trajectories.[16] The *actual* trajectory will be that curve in the bundle for which the total action is an extremum, usually a minimum. To determine this curve more exactly—really, to infer equations of motion for the system—Lagrange had to rely on a subsidiary principle, the Conservation of Vis Viva.[17] Thereby (speaking modernly) he restricted the class of comparison curves, i.e. admissible variations, to paths of equal energy. In *this*, restricted class of possible trajectories, the system's actual path will be that which minimizes the total 'action.'

Analytic mechanics. Another path to generality began with Johann Bernoulli, who based it on an insight that goes back to Aristotle and Heron of Alexandria. First, in a 1713 letter, Johann introduced two novel concepts. One was "virtual velocity," explicated as "the mere disposition to move that forces have in a perfect equilibrium, i.e. when they do not move actually." Another was "energy," really his term for instantaneous virtual work in an infinitesimal displacement: "virtual velocity, multiplied by the absolute force, produces the momentum, or energy, of this

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of the action $\delta M_i v_i ds_i$. Thereby he got two terms, one in δds_i and another in δv_i, respectively. Third, he related the second term to forces $\mathbf{F}_i$ acting on the system, by means of Conservation of Vis Viva above. Thereby, he obtained a second expression, (E). Fourth, he formulated the first term in Cartesian coordinates, and called the resulting expression (V). Finally, he added E and V, and equated to zero their coefficients. This yielded for him the equations of motion for a general dynamical system, free or constrained. Cf. Lagrange 1762b: 205-9.

[15] M is mass, *v* is instantaneous velocity, and *s* is Cartesian distance.

[16] These curves are related to each other as the variations of the action functional.

[17] Consequently, he assumed that all these forces—even those responsible for the constraints—were induced by conservative potentials.

force."[18] Second, Bernoulli combined these notions into a new principle for statics. In another letter, he shared it with his disciple, Varignon, who made it public in his posthumous *Nouvelle Mécanique, ou Statique*: "In any equilibrium of forces whatsoever.... the sum of the positive Energies will equal the sum of the negative Energies with their signs reversed" (Varignon 1725: 176). In our terms, Bernoulli claims a sufficient condition for static equilibrium is that the virtual work of the applied forces vanishes. By 1740 his principle had gained wide acceptance (d'Alembert 1743: 182f).

The next, momentous step occurred in France, where the best minds were seeking a uniform method to tame constrained motion. D'Alembert had a brilliant though highly unintuitive idea. He posited that, in a constrained system, every part loses a velocity increment as it moves—because the constraints *prevent* it from acquiring the 'whole' motion that impressed forces would give it, were it free. D'Alembert called these increments "lost motions," and proposed a general heuristic based on them. Resolve the 'motion' impressed on every part into two motions, one lost and one acquired actually. D'Alembert asserted that in the system as a whole, the 'lost motions' are such as to induce equilibrium:[19] "if just the [lost motions] were impressed, the system would stay at rest" (1743: 51). This statical rule was meant to yield a general method for identifying, in a system, the 'motions' lost to constraints. Once found, these must be subtracted from the 'impressed' motions. The remainder would be the *actual* motions, i.e. the accelerations the system undergoes in fact. These would show up in the differential equations of motion.[20]

However, few could really follow d'Alembert's idea or his cumbersome implementation. Aware of this obstacle, he reformulated his insight. This time he left out of account the "lost" motions, and focused on the two other kinds, viz. "impressed" and the "acquired" motions. (Call them **I** and **A**, respectively. They are vectors, so I use boldface.) D'Alembert now claimed (1749: 35f) that:

[18] This letter (to Renau d'Elizagaray, a naval engineer) has not yet been printed. I translate here from an excerpt in Capecchi 2012: 434, which relies on an unpublished typescript by Patricia Radelet de Grave.

[19] That is to say: if the target system were in equilibrium, and a *different* set of forces acted on it, so as to impress *solely* the 'lost motions' as caused by the *previous* set, the system would remain in equilibrium.

[20] A lucid account of d'Alembert's original principle is Fraser 1985; Firode 2001 explains the large-scale structure of d'Alembert's mechanics.

[2] $$\sum \mathbf{I} = -\sum \mathbf{A}$$

In words: for each body or part of a system, if the impressed and the acquired motions—*the latter with their sign reversed*—were applied, the system as a whole would be in equilibrium.

D'Alembert's reformulation is important because Lagrange took it over, combined it with Bernoulli's law above, and made it the basis of his *second* general theory of mechanics. But, he made two key changes. First, he recast d'Alembert's principle in terms of forces, not 'motions.' Second, he rephrased Bernoulli's principle in terms of virtual *displacements*, not velocities: each is an "arbitrary small motion," whereby the particle "traverses an infinitely small space" (Lagrange 1878[1763]: 8-9).[21] These two changes to the insights of his predecessors yielded for him a novel method for deriving equations of motion. He first showcased it in a prize essay on lunar libration.[22] Lagrange lets the Moon—modeled as a set of mass elements m—be urged by two gravitational forces $\mathbf{F}_E$ and $\mathbf{F}_S$, exerted from the Earth and the Sun, respectively. Let $\mathbf{a}$ (= $d\mathbf{r}^2/d^2t$) be the net *actual* acceleration of m. Lagrange chooses to regard the products $m\mathbf{a}$ as *forces*. He claims that these 'forces,' *if reversed*, would balance exactly the gravities $\mathbf{F}_E$ and $\mathbf{F}_S$ above. In other words, these 'forces' are in equilibrium:

[3] $$\sum (\mathbf{F}_E + \mathbf{F}_S) = -\sum m\mathbf{a}$$

As justification, Lagrange claims the law "given by Mr. d'Alembert," which he calls the "general principle of Dynamics" (1878[1763]: 12, 8). Having turned the Moon into a system in equilibrium, Lagrange invokes a "principle universally true in Statics," namely his version of Bernoulli's law above. Specifically, the work done by all the (non-constraint) forces on a system in equilibrium is zero:[23]

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[21] Lagrange regrettably retained the term "virtual velocities." It is a dysphemism in his theory: by *vitesses virtuelles*, he means displacements—properly speaking, *timeless* variations of the coordinates.

[22] Perforce, I must greatly simplify Lagrange's complex reasoning in that paper, and also modernize his notation. An exquisite account is Fraser 1983; Capecchi 2003 is more explicit about the role of the two principles above in Lagrange's paper.

[23] $\delta\boldsymbol{e}$ and $\delta\boldsymbol{s}$ are virtual displacements in the directions of the forces $\mathbf{F}_E$ and $\mathbf{F}_S$, respectively, whereas $\delta\boldsymbol{r}$ is in the opposite direction of the actual acceleration $\mathbf{a}$.

[4] $$\sum (\mathbf{F}_{\mathrm{E}} \cdot \delta \boldsymbol{e} + \mathbf{F}_{\mathrm{S}} \cdot \delta \boldsymbol{s}) + \sum m\mathbf{a} \cdot \delta \boldsymbol{r} = 0$$

This early, special application was the gist of his later, comprehensive theory in the 1788 masterpiece *Méchanique analitique*.[24] There, from his basic dynamical law he derived equations of motion for a system of free particles; a particle under external holonomic constraints; a system of particles with internal constraints; a rigid body, free and constrained; an incompressible inviscid fluid in laminar flow; and an elastic gas (Barroso Filho 1994: 267-87). Lagrange's approach is exactly the same as in his 1763 *Recherches sur la libration de la Lune*. The sole conceptual innovation is a novel way of handling constraints.[25]

Officially, this brand of mechanics rested on a single basic law, which Lagrange called the General Principle of Virtual Velocities.[26] In physical terms, it says that in a mechanical system, the work of the non-constraint applied forces $\mathbf{F}_i$ in a virtual displacement vanishes:

[5] $$\sum \mathbf{F}_i \cdot \delta \boldsymbol{r}_i = 0$$

And yet, this statement obscures the insight behind it. Lagrange's general principle above is really a conjunction of *two principles*. One is a basic law for statics: in a system in equilibrium, the net virtual work by the applied external forces $\mathbf{S}_i$ is null:

[6] $$\sum \mathbf{S}_i \cdot \delta \boldsymbol{r}_i = 0$$

Another is a fundamental law for dynamics, the theory of systems in motion. Lagrange baptized it "d'Alembert's Principle," and it caught on. This law contains two ideas, deeply at odds with Newtonian intuitions

[24] A handwritten note by Lagrange claims d'Alembert had told him, in a letter, "I read your piece on lunar libration…. and I said, as John the Baptist said [of Jesus], *He must increase, but I must decrease*." (Lagrange 1882: 10)
[25] In the 1763 paper, Lagrange did not have to worry about constraint forces. A kinematic condition—his Moon is a rigid ellipsoid—supplied him with a purely geometric way to express the action of constraints. In the general theory of *Méchanique analitique*, he inserted in the Principle of Virtual Work a set of undefined, force-like coefficients λ, μ, ν, etc.—the 'Lagrange multipliers,' as we call them—and set them equal to zero, to express the insight that constraints are workless. Lagrange's mature dynamics of constrained systems is sadly understudied. Duhem 1903 remains useful.
[26] Some modern authors call it 'Lagrange's Principle,' e.g. Papastavridis 2002: 386.

about mechanics. First, regard the *accelerations* of masses as a species of *force*. Then change the sign of these 'forces,' and call the result 'reverse effective forces,' labeled $\mathbf{J}_i$.[27] Second, d'Alembert's Principle now claims that the actual, impressed forces $\mathbf{I}_i$ balance exactly the reverse effective forces $\mathbf{J}_i$:

$$[7] \qquad \sum \mathbf{I}_i = -\sum \mathbf{J}_i$$

In effect, d'Alembert's Principle reduces a dynamical system to a *static* system in equilibrium. So reduced, the system is acted on by a net applied force $\sum \mathbf{F}_i$ from two sources, viz. the impressed forces and the reverse effective forces:

$$[8] \qquad \sum \mathbf{F}_i = \sum \mathbf{I}_i + \sum \mathbf{J}_i$$

Thereby, the system comes under the authority of [5], the statical Principle of Virtual Work above. Lagrange then sought to express the 'variations' $\delta \boldsymbol{r}$ in terms of 'generalized coordinates' and their first derivatives. This led him in every case to the 'Lagrangian' equations of motion for that system.

Euler mechanics. The last program was largely due to Euler, but it too began with Johann Bernoulli. In the late 1730s he found a new way to quantify and predict the behavior of so-called Newtonian fluids. Earlier, his son Daniel had used a Leibnizian strategy—an energy method, in our parlance—for the same task. Daniel posited Conservation of Vis Viva for the extended fluid mass. Next, he sought to find the motion of *individual* fluid particles from this *global* condition. Dissatisfied with his son's approach, Johann pursued a *methodus directa*.[28] His genial idea was to

[27] I follow Routh 1905 in this terminology. Modern authors call them 'inertial forces,' 'kinetic reactions,' 'constraint reactions,' 'reaction forces.' All these terms mislead badly.

[28] 18th-century *mécaniciens* had a duality of method, "direct" vs. "indirect." The former amounted to applying some dynamical law directly to the least part or individual component of a mechanical system, so as to obtain the equation of motion. The latter would impose a global condition on the *system*—usually, one or more integrals of motion—then try to infer *indirectly* the (differential) equation of motion for single components. The global principle—e.g., energy conservation—functioned as a constraint on admissible motions for components. But, it did not *explain* them in causal terms. In contrast, the direct method *had* causal-explanatory import: it relied on dynamical laws about efficient causation. E.g., Newton's approach to free fall is *metho-*

treat the fluid as a physical continuum, and apply the Second Law not to particles—there *are* none there—but to every mass element *dM*, or infinitesimal bit of matter in that fluid. Johann shared his insight with Euler, who was elated: "You have given me now the greatest light in this matter, whereas previously I would approach it in a great fog, and was unable to determine it other than by the indirect method.[29] It was the breakthrough Euler needed so as to treat most of the problems and setups that Newton's *Principia* and his own *Mechanica* of 1736 had left untouched.

I said 'successfully,' but not 'easily.' It took Euler over three decades to fulfill his program. Through the 1740s, Euler showed how to apply the Second Law to mass-points constrained to move on mobile surfaces, and to systems of free particles. In 1750, he found a way to use the Law to obtain equations of motion for a rigid body under external forces. (At the time, the rigid sphere was becoming the key model for the celestial mechanics of *extended* bodies, which exhibit precession and nutation.) During that same decade, Euler systematized and generalized the dynamics of 'Newtonian,' or inviscid fluids, by grounding their behavior in the Second Law and his new-fangled notion of internal pressure. However, by the 1770s Euler came to see that, except for these fluids above, in general Newton's law was *not* enough by itself to predict all the possible motions of all possible bodies. Another basic principle is needed, to handle the "moments of forces," i.e. *torques*. In his later treatment of flexible and rigid bodies, he would start with two fundamental principles, known as 'Euler's Laws of Motion':[30]

[9] $$\mathbf{f} = d\mathbf{p}/dt$$

and also

[10] $$\mathbf{h} = d\mathbf{l}/dt$$

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dus directa, whereas Leibniz's treatment of 2-body collision is indirect. Terse accounts of this duality are in Euler's correspondence with Daniel Bernoulli; and in Anonymous 1751: 71f.

[29] Euler to Johann Bernoulli, 5 May 1739; Eneström 1905: 25.

[30] Cf. Euler 1766, on celestial bodies in rotation; Euler 1771, on the elastica; Euler 1776a, on the vibrating string; and Euler 1776b, on the rigid body; **f** is force, **h** is torque; **p** is linear momentum, **l** is angular momentum.

One is the Force Law, which asserts that the net impressed force equals the time-rate of change in linear momentum. The other is the Torque Law: the net torque equals the rate of change in angular momentum.

I call this tradition Eulerian, as more than half of it comes from Euler. And yet, though handy, the label is glib. More properly called, it is Newton-Euler dynamics, because it took its cue from Newton's *Lex Secunda*. Its explanatory core is *two* logically *independent* principles relating dynamical efficient causes and kinematic effects.[31] The causes come in two kinds: forces and torques. So do their respective effects: linear and angular accelerations. All are vector magnitudes. And, the two principles are differential laws: they start with *infinitesimal* changes in the *least* part of a mechanical system (i.e., the particle or the mass element, respectively). In other words, the reference of these laws is the part not the whole. To find out how the whole moves, one must integrate over it. This is wholly unlike the other two programs, whose basic laws—the Principle of Least Action and the Principle of Virtual Work—are about scalar quantities, not vectors, and make assertions about the whole (system), not the part—which does *not* necessarily obey the basic law.

Against this backdrop, let us see now what chances Kant has to expand his foundations.

3. Kant, the Analogies, and the three paradigms

For some time now, Michael Friedman has made a broad, sophisticated and robust case that Kant aimed to secure apodictic foundations for mechanics from *constitutive* resources.[32] Namely, Kant's categories, applied to the empirical concept <matter>, guarantee that mathematics can be used in mechanics, thus lending its certainty and exactness to the latter. Likewise, the Analogies of Experience, if applied to matter yield synthetic a priori dynamical laws. Lastly, the Postulates of Empirical Thought constrain the future addition of new empirical laws to dynamics. Thus, the structures and activities of the *understanding*—listed in the First Cri-

[31] Some call them, respectively, Euler's First Law and Second Law; e.g., Truesdell 1991 and Wilson 2013. Handbooks of modern continuum mechanics also call them the Balance Law of Linear Momentum and of Angular Momentum, respectively.
[32] See Friedman 1992, 2012, 2013. His approach is in self-conscious opposition to an older construal, epitomized by Buchdahl 1992, on which it is primarily Kant's *regulative* principles that frame mechanical theory.

tique, applied to matter in *Foundations*—suffice to ground mechanical theory.

Friedman relies explicitly and heavily on Newton as the test case for his reading of Kant. Sensibly, he limits his focus to the four laws in the *Principia*. Now, I have explained that these laws are not general. Then let us ask: can we use constitutive resources alone, as Friedman has done for Newton, to show that Kant could have truly general laws of mechanics? More broadly put, can we prove that Kant's Analytic and *Foundations* support a *truly general* notion of determinate object of experience? I submit that, without some qualification, we cannot: there are serious conceptual obstacles in accommodating any of the three programs to Kant's official foundations in the 1780s. Let us take them in turn.

Variational mechanics. Kant thought highly of Maupertuis, and very favorably of his research (Ferrari 1999). His regard for Euler needs no proof. Sadly, Kant never learned of Lagrange's first unification of mechanics from the Principle of Least Action, PLA, though it would not have surprised him.

Still, Kant would be hard pressed to ground it in *constitutive* resources. As Friedman and also Watkins (1997, 1998b) have shown amply, Kant always expects his Analogies of Experience to ground the basic dynamical laws, be they Newtonian, Leibnizian or otherwise.[33] The Second Analogy, in particular, imposes a severe constraint on mechanical theory. It posits that all substantial change—ergo, all change of mechanical state—has an efficient cause that precedes it in time. The two are related as "cause and effect, the former of which determines the latter in time, as its consequence" (B 234). A fortiori, the basic mechanical principles must be laws of efficient causation. And, presumably, all explaining must terminate at last in explanations from efficient causes and their basic laws of operation. To be sure, these causes need not be just forces. They may be torques, stresses, body couples—any kind of ultimate efficient agency in mechanics.

However, the PLA is deeply at odds with Kant's constraint. For one, it is not at all a statement about forces, its cognates above or any efficient causes whatsoever. In Kant's time, it was about *non-causal* features of

[33] Watkins 1998a made a conclusive case that Kant's grounding does not amount to substitution—replacing, as it were, category-terms with physical notions. Still, I do believe that Kant means to ground his basic laws by *deductive* argument [*Beweis*], in which at least one Analogy of Experience appears as an indispensable premise.

body, viz. mass, velocity, and infinitesimal displacement. Moreover, the PLA is an integral law, as I explained. It does not link instantaneous causes and their (directly successive) effects. Rather, it asserts a fact about a global feature of the system throughout a finite time.

There is a second, grave difficulty. The PLA does not underwrite explanations from efficient causes. In this mechanics, the system's position and velocities at a particular instant are explained by a fact about a time integral—of its action functional—which extends into the future, and includes *subsequent* instants. In turn, that integral picks out the system's actual path (in configuration space) from a set of infinitely close possible paths, viz. the variations of the action. And, to explain why *this*, rather than any of its variational siblings, is the actual path, we must appeal in part to the *final* configuration of the system. Thereby, we rely on facts about a future state to explain mechanical behavior at previous instants. This rests Kant's Second Analogy squarely on its head: in PLA, the future explains the past, instead of the past explaining the present.

Nor is it clear that we can anchor the PLA in some other Kantian constitutive principle. Take the First Analogy: with it, Kant posits the permanence of phenomenal substance. In 1787, he modified it to assert a global conservation principle: the amount of (phenomenal) substance "is neither increased nor diminished in nature" (B 224). Friedman has shown ingeniously how Kant linked this rephrasing to his aim in *Foundations* of using the First Analogy to derive Conservation of Mass in mechanics (2013: 315f). Can we not do some creative interpretation, and try to imitate Kant and Friedman by yoking the PLA to his updated First Analogy somehow? It does not seem possible. That principle supports conservation laws alone, and the PLA is not one. It asserts that, in every process, the 'action' takes an *extremal* value—a minimum or maximum—not that it is conserved.

Analytic mechanics. Regrettably, Kant was unaware of Lagrange's second mechanics, though it gestated just south from him, in Berlin. We must again resort to interpretive conjecture about how we might ground it in his principles. It seems that we must start with the official basic law in it, or the Principle of Virtual Work, PVW. However, there is a formidable problem nearby. The PVW is useless without d'Alembert's Principle, the linchpin between statics and dynamics in Lagrange's theory. And, that law seems fundamentally unable to rest on Kant's foundations, for two reasons.

One is that d'Alembert's Principle requires us to add *in thought* a set of wholly *fictive* forces to the given system. These *imaginary* forces are meant to balance the actual, real ones, so as to convert the system to static equilibrium, the necessary condition for the PVW to take over and yield equations of motion. In effect, d'Alembert's Principle claims: if a *different* set of forces were acting on the moving system, it would be a system at static rest. Many modern authors fail to declare the wholly fictive existence of their so-called 'constraint reactions,' 'inertial forces,' 'reaction forces' and such synonyms. But Lagrange knew it:

> If we *imagine* impressing in each body, in the opposite direction, the motion it is to have, clearly the system will be reduced to rest.... Thereby we can reduce the entire Dynamics to a single general formula. For, to apply the formula of equilibrium to the motion of a system, it is enough to *introduce* forces that come from the [actual] change in the motion of each body, which motions must be destroyed. (Lagrange 1811: 240; my emphasis)

And so did Euler, who first explained clearly the virtual work approach to mechanical problems (1862: 46f). Some modern authors still remembered it (Routh 1905: 46). Even more perceptive writers are overt about it. Hamel on d'Alembert's Principle: "Add to the impressed forces the negative accelerations of masses, as fictive forces [*Scheinkräfte*]. Then treat the system as one in equilibrium" (1949: 218).

These 'forces' are fictive or imaginary in two senses. They are not actual or present in the system—not even latently so, like the normal component of gravity on an inclined plane. (Again, recall that the system is in motion, not at static rest.) And, they do not originate in actual bodies, whether present within the system or outside it. For that reason, d'Alembert's Principle is irreconcilable with Kant's Second Analogy. The latter requires that all mechanical agencies be species of the causal powers of actual substances (Watkins 2006: 256ff). The reverse effective 'forces' in d'Alembert's Principle are neither: they are sourceless, non-actual forces.

Nor is the Third Analogy of much help here. Kant's *Grundsatz* entails that all mutual forces are balanced—which seems to put d'Alembert's Principle within its purview. But that is a vain hope. The real consequence of the Third Analogy for mechanics is that all *actual* impressed forces are interactions: they come in pairs, not singly; and act on *differ-*

ent, actual bodies. D'Alembert's Principle fails this demand too: it says that two kinds of force—one actual and one fictive—*could* be impressed on the *same* body.

In conclusion, so far the outlook is gloomy. There is some sunshine coming, I will argue, but it is still in the distance.

Euler dynamics. This would seem the inevitable choice for Kant, given his strong commitments to the theory in the *Principia*, which Euler made fully general. So, I must show that he is in a position to derive its two basic causal principles, the Force Law and the Torque Law. Now, Newton's Second Law is notably absent from Kant's foundations. A fortiori, so is its Eulerian generalization, the Force Law. Still, this obstacle is just temporary. Kant does have enough resources for a theoretical equivalent of *Lex Secunda*, namely the Parallelogram of Forces.[34] In his foundations of kinematics, Kant proves the Parallelogram of *Velocities*. For him, it is synthetic a priori, and holds for accelerations as well.[35] Kant also claims that forces generate (linear) accelerations along their line of action. And, he has an indirect proof that single forces are proportional to the accelerations they induce. (I expand on this argument in § 4.) So, accelerations add like vectors; forces are commensurate and colinear with their accelerations; thus forces add like vectors. Then Kant has a substitute for Newton's Second Law, hence also for Euler's Force Law.[36]

What about the Torque Law? Do Kant's foundations entail it? In modern accounts of mechanics, that law is sometimes derived as a theorem, from mechanical axioms. There are really two ways to do so. **(1)** We may start with the Force Law and the so-called Strong Third Law, viz. the principle that inter-particle forces equal, opposite, and *central*, i.e. acting on the line between the particles.[37] From these two premises we can prove that the impressed torque equals the rate of change in angular momentum. **(2)** Or, we may start with the Force Law and another,

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[34] The Parallelogram of Forces is Newton's Corollary II, in the *Principia*. The law of inertia and the *Lex Secunda* entail it jointly. Thus, the Second Law and the Parallelogram of Forces are equivalent: given (translational) inertia, each entails the other.

[35] Or so he thinks. But, it deserves scrutiny whether Kant's official proof can be extended to accelerations. I thank David Hyder for pressing me on this point.

[36] Friedman agrees that Kant has the resources for it (2013: 373-5).

[37] The Strong Third Law differs from the *Weak* Third Law, viz. the principle that all forces are merely pairwise equilibrated—namely, equal and opposite—with no assumption about the particular *direction* in which they act.

quite different premise: the symmetry of the stress tensor. Both entail the Torque Law.[38]

There is a subtle but deep difference between the two avenues I outlined above. Proof strategy (1) requires that matter be made of mass points or rigid atoms, whereas (2) assumes that it is a physical continuum.[39] Kant in *Foundations* clearly asserts bodies to be continuous, and so he has no choice but to pursue strategy (2). Unfortunately, Kant weakens his chances to implement it. The physical meaning of the premise that the stress tensor be symmetric is: in every *contact* interaction, certain shear forces on bodies—namely, the conjugate shear stresses—are equal. This fact is entirely out of Kant's reach: not because 'shear stress' may be anachronistic for him, but because he denies that there can be any shear forces *at all*. Kant is adamant that *all* forces between "matters" are central, i.e. they act *along* the straight line between them: "all motion that one matter can impress on another, since in this regard each of them is considered only as a point, must always be viewed as imparted in the straight line between the two points" (4:498). However, shear stresses act *tangentially* to any two matters in contact, and so Kant's claim entail that shear force has *no* place in mechanics.

That is unfortunate, for two reasons. Kant's strong conceivability claim rules out a workable strategy to secure for his mechanical foundations the generality they deserve. Moreover, his thesis puts him at odds with his age. Euler had just calculated the shear force on a one-dimensional elastic continuum. In *True Principles of the Equilibrium and Motion of Flexible and Elastic Bodies*, he had determined, from the Force Law, the normal and *tangential* component [*vis tangentialis*] on an element *ds* in the elastica (Euler 1771). But, the 'tangential' force-component above is just the shear between two contiguous elements—precisely the sort of force Kant banishes from mechanics.[40]

To sum up, Kant's constitutive foundations can support neither analytic nor variational mechanics. And, prima facie the path to Newton-

[38] For details and discussion, see Truesdell 1968b.
[39] Briefly, it is because the Strong Third Law holds *solely* for discrete particles interacting by forces at a distance—while the concept of stress, and the condition that its tensor be symmetric, is meaningful *solely* for continuous bodies not particles; Truesdell 1968b.
[40] There is a reason for Kant's decision to make all forces central. He needed it so as to prove the Parallelogram of Forces, which plays the role of Newton's Second Law in Kant's system, as I explained above. For details, see Friedman 2013: 373-5.

Euler dynamics is closed to Kant. However, the case is far from hopeless. I revisit the latter issue below, to see if my conclusion is really inevitable. It turns out that it is not: Kant has a way out. What keeps him from grounding Eulerian mechanics is just one facet of his mature theory of matter. But, that facet is dispensable, and can be removed without much damage. Then Kant will have all he needs to solve the problem that motivates my paper.

4. Categories, phenomenal monads, determinate objects

I move to make a positive argument now: I establish that Kant after all did have the conceptual resources to ground objecthood for a general mechanics, the explanatory basis of all determinate experience. My case has two parts. First I present a theory of matter, "physical monadology," that Kant used to hold in the 1750s; and show that, adjusted to his strictures, the theory entails the two laws of Euler dynamics. Second, I argue that Kant's foundation—*reworked* into 'phenomenal monads' as I explain—is a constitution theory for a general object of exact knowledge, or determinate experience. Physical monads are discrete particles, which leads us to doubt they can ground continuous bodies and their behavior. And yet, I show, there are modern techniques for grounding continuum mechanics in discrete matter, as Kant should have had it.

In *Physical Monadology*, Kant set out to reconcile two tenets in conflict: geometry entails that material substance is divisible because it is extended; however, metaphysics demands that basic substance be partless, thus *in*divisible. His solution was to dissociate extension from divisibility. There is a type of substance, Kant showed astutely, that takes up space and yet is not divisible; he called it a "physical monad." In his official account, a monad has inert mass and exerts two forces, attraction and repulsion. Crucially, it has two key features: its mass is concentrated at a point, not distributed over a volume; and its forces are actions-at-a-distance.[41] In effect, his monad is a mass-point, and so, in modern terms, it has no true size, just an effective volume. In Kant's words, a monad has a "determinate volume" that is however not the "diameter of the

[41] Kant declared, without argument, that repulsion was an inverse-cube and attraction an inverse-square force. Still, for my argument it is irrelevant what particular force laws these monads might obey. It is enough that they be ruled by some (inverse)-power law.

monad itself" (1:484, 481). These premises entail an elegant solution to Kant's initial problem. The monad is extended: it exerts repulsive force over a finite volume, so it "fills space" (1:480). It is impenetrable in that no two monads can be superimposed. And, it is indivisible: its mass takes up a point, which cannot be divided; and neither can its sphere of activity, which is just an acceleration field, not a volume filled with mass.

Granted, Kant in *Physical Monadology* did not always adhere to the clean view of matter I attributed to him above, though all my ascriptions *are* explicit in his account of monads.[42] And, Kant subsequently abandoned it; by 1786 he had come to assert that matter is a deformable continuum. Still, a (transcendental-idealist) variant of his monads is a better ontology than his later doctrine in *Foundations*, for it can ground the causal principles of Euler dynamics. Here is how.[43]

First, physical monads obey the Force Law, viz. Euler's generalization of Newton's *Lex Secunda*. This is shown indirectly.[44] They obey it because they entail an equivalent principle, namely the Parallelogram of Forces. Admittedly, Kant did not carry out an overt derivation of the Parallelogram Rule for forces. But, he was confident that it *could* be derived from his a priori resources. He was right: three premises established independently in *Foundations* do entail the Rule jointly. The first is the Parallelogram of Velocities, PV, about which Kant is explicit. In Phoronomy, the kinematic part of *Foundations*, Kant gives a geometric proof of PV (4:487-95; cf. also Friedman 2013: § 4). Inter alia, PV asserts that if a material point acquires two velocity increments, it will move with the vector-resultant of the increments.[45] The second premise is also explicit: all forces induce accelerations (i.e. velocity increments) in the direction of their action, namely the line between any two interacting particles: "all motion that one matter can impress on another…. must always be viewed as *imparted* in the straight line between the two points" (4:498-9; my italics). The third premise is an unstated corollary of Kant's

[42] Smith 2013 argues conclusively that Kant in his paper oscillates between two views of matter: mass points, as I claimed; and deformable continua, a distinct picture.
[43] I offer my derivations here as strong plausibility arguments, not as formal proofs as Hilbert and modern axiomatic mechanics requires. My sense was the prevalent notion of 'proof' in Enlightenment mechanics. As illustration, a Hilbertian reconstruction of Newton-Euler dynamics for continua (*sans* constitutive relations) is Noll 1973.
[44] I reprise here an argument from § 3, where I confronted Kant with Euler.
[45] Namely, the point acquires an acceleration equal to, and in the direction of, the diagonal of the parallelogram formed by the two initial velocity increments.

indirect proof of the Weak Third Law (see below). Specifically, it is the thesis that single forces induce accelerations proportional to them. In sum, (i) accelerations add like vectors; forces are (ii) collinear with their accelerations, and (iii) directly proportional to them. Ergo, forces add like vectors, i.e. according to the Parallelogram Rule. In addition, monadic mass—whereby a monad resists accelerations—is *officially* located at a point, not distributed over a volume. So, intermonadic forces act on point masses, not on volumes or surfaces. All this entails that the Parallelogram of Forces governs Kant's monads. *Eo ipso*, the Force Law applies to them.[46]

Second, physical monads obey the Strong Third Law. This may be shown in two parts. **(1)** All intermonadic forces are equal and opposite. Kant in *Foundations* has a justification for it. It is an argument by *reductio*, as follows. Deny the Weak Third Law. That is, suppose interactions are *not* equal and opposite. Then the resultant force has a non-zero projection on an arbitrary straight line passing through the mass center of the "world-edifice," or bulk of all matter [*Weltganze*]. In consequence, this mass center will be accelerated. But, this acceleration is not a possible object of experience; it is "absolute motion," and that is "simply impossible."[47] Consequently, any physical principle that entails it must be false. For that reason, "any proof for a law of motion that argues the law's opposite would entail the rectilinear motion of the whole world-edifice is an apodictic proof of that law" (4:562). Hence, denying the Weak Third Law—the "law of *antagonism*," as Kant calls it—entails an epistemic impossibility. Ergo, the law must be true. **(2)** All intermonadic forces are central. Kant's support for it is a conceivability claim: "Only [forces acting in a straight line between two material points] can be thought" (4:489). If this comes too close to an *ipse dixit*, I offer a better argument. Recall that Kant's monads are essentially endowed with two forces [*vires*], attraction and repulsion. These 'forces' are really irrotational potentials induced by a point-sized source. Then the 'Newtonian'

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[46] I thank Michael Friedman and David Hyder for very illuminating discussions on this difficult topic.

[47] Kant is a relationist about true motion: any true acceleration consists in a change of relation between "matters," be they bodies or their constituents. By 'absolute motion,' he means a displacement that is no kinematic change relative to some matter. In *Foundations*, he makes a long case that absolute motion—in contrast to true motion—is not an object of scientific experience. Friedman seems to concur with my argument for point (1) above; cf. (2013: 497-99).

force, viz. the generic force-term **f** in **f** = m**a**, is just the negative gradient $-\nabla$ of each Kant-monadic 'force' at a point away from the monad itself.[48] And, the gradient always points toward the point source of the 'force,' in virtue of the mathematical features of these monadic agencies. So, whenever a phenomenal monad C sits at point *B*, the force exerted on it by another monad A is in the straight line between *B* and *A*. Hence, all intermonadic forces are central. But (1) and (2) together are just the Strong Third Law. These two laws, provably implied by Kant's foundations as I re-read them, imply the second basic principle of Euler dynamics, i.e. the Torque Law.[49] Its real content is that the *external* torque on a body equals the rate of change in angular momentum. To derive it from Kant's premises, start with the idea that a body is a lattice of phenomenal monads, or mass-points. Write the Force Law for the *j*-th monad in the body:

$$[11] \qquad {}^{E}\mathbf{f}_j + \sum_k \mathbf{f}_{jk} = \mathrm{m}_j\, \mathbf{a}_j$$

Here, ${}^{E}\mathbf{f}$ is the net impressed force external to the *body*, whatever its sources. In contrast, $\mathbf{f}_{jk}$ are *internal* forces, exerted on *j* by every other monad *k* in the body. Now, take the cross product of each member in [11] above with $\mathbf{r}_O$, the distance vector to an arbitrary point *O*. The result says: the net torque about *O*, internal *and* external, equals the rate of change in angular momentum relative to *O*. Finally, invoke the Strong Third Law. Because all monadic forces are central, it follows that the net torque of all the internal forces is zero. So, the total external torque on a poly-monadic body equals the increment of angular momentum. Ergo, phenomenal monads, in conjunction with Kant's conceptual inventory in *Foundations*—which in turn relies on his categories in the First Critique—entail the two basic laws of Euler dynamics.

As I explained in the introduction, physical monads were young Kant's account of matter as thing-in-itself, established by a priori reason-

[48] The reader need not worry that this is anachronistic. Already in Kant's age, Lagrange had grasped this idea, though not with our full clarity. He introduced for gravitational systems a function Ω such that the impressed force on a body at point *P* is the negative gradient of Ω; cf. Lagrange 1777. (So, Ω is the potential energy function of that system.) Physically, Lagrange's Ω is a species of Kant's monadic force of attraction. Mathematically, they are alike, viz. zero-curl scalar potential functions.
[49] I give here just a verbal proof sketch. Some modern textbooks give more rigorous proofs; e.g. Joos 1934 and Spivak 2010: 199f.

ing, viz. reflection on the concept of material substance *an sich*. The Critical Kant forbids such moves resolutely, and I follow him in that respect. The mass points I sought to ascribe to him are better called 'phenomenal monads.' Because they have the same material constitution (or geometric, kinematic and dynamical attributes) as young Kant's *unitates physicae*—and so are partless and indivisible—they count as monads. However, they are not matter *als Noumenon*, being in fact an ontology of matter as it appears to us, though mediated by mechanical theory. Thus, they are phenomenal. That gives me license to call them 'mass points' and 'phenomenal monads' interchangeably.

5. Kant for our times

Objection: I won a Pyrrhic victory for Kant. I showed he could have general basic laws—and so a general notion of objecthood for exact science—but the price is to adopt discrete matter. Then ultimately he will be unable to account for vast areas of mechanics—like elasticity, fluid dynamics and plasticity theory—where matter must be modeled as a physical continuum, not discrete points. So, Kant's laws are not truly general. Then it seems that my proposed Kantian foundation cannot ground *all* determinate experience.

I have two ways to blunt the force of this criticism. First, my proposal is historically sufficient. In Kant's time, an initially successful program began in France for grounding all of physics in discrete particles—called 'molecules' by Cauchy, Poisson, Navier and Saint-Venant—endowed with short-range, action-at-a-distance forces. In that respect, those particles were *just like* Kant's phenomenal monads. Laplacian physics, as we named that program, dominated the first third of the 19th century (Fox 1974; Arnold 1978). Cauchy endorsed it as equivalent to the physical-continuum approach:

> In investigating the equations that express the equilibrium conditions or the laws of interior motion for solid or fluid bodies, we may regard these bodies as continuous masses whose density varies from one point to another by insensible degrees; or as a system of material points distinct yet separated by very small distances. (Cauchy 1828: 160)

And, in elasticity, discrete matter competed with continuum models well into the twilight of classical physics (Capecchi, Ruta & Trovalusci 2010).

So, Kant's foundation as I read it above is sufficient for the 19th century.[50]

Second, it turns out that mass points *can* ground a mechanics of continua after all. Admittedly, this fact is late-modern, so at most we may retrofit it to Kant's system. Still, it does fit seamlessly with Kant's foundations—his 'monadological' theory of matter as I read it above, and his basic dynamical laws above. Ever since the 1950s, a theoretical effort has been underway to give continuum-mechanical fields a physical interpretation in terms of discrete particles. The key conceptual move is: a *continuum* field value (at a point) is physically interpreted as the localized, space-and-time average of a *discrete* set of analogous values. We use two tools to bridge the discrete and the continuous. The formal instrument is a family of weighting functions w that yield weighted averages of discrete quantities. These yields are scale-dependent: the averaging kernel can be so chosen as to give values for the desired length scale. The modeling tool is *mass-points*.[51] Crucially for my argument, these mass-points are just like Kant's monads: they have the same kinematic structure, and obey the *same* dynamical laws.

As illustration, here is how we ground continuum kinematics in phenomenal monads. Start with a discrete set of points having mass, position and velocity. Next, use a weighting function w to pass from discreta to continua.[52] Using w enables us to define a notion of density *at a point*: a localized space average, or weighted sum over a volume. In turn, this notion allows us to define continuum *fields*. The two primary fields so defined are mass density ρ and momentum density $\mathbf{p}$. From them, all the

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[50] Granted, this is not quite enough. While in the heyday of Laplacian physics 'molecules' often designated mass points—thus were equivalent to 'phenomenal monads'—Poisson in the 1830s, challenged by new experimental evidence in elasticity, made his molecules ellipsoidal (Duhem 1903: 84f). This breaks their similarity with Kant's monads, whose true size is zero (hence, shapeless) and volume is spherical, not ellipsoidal.

[51] In this modern project, mass-points seem to be an ontological extra layer—introduced chiefly for the sake of computation, not explanation—between molecules and continuous matter. It aims to model molecular interactions, so for each mass point the three parameters stand for a molecule's mass, and the position and velocity of its mass-center.

[52] w is a function of displacements, and it assigns a greater (relative) weight to molecules closer to a point $\mathbf{x}$ than to those further away. Since it is scale dependent, one may always choose w such that it gives weight zero to molecules beyond a distance ε from point $\mathbf{x}$. Lastly, w is constrained by the normalization condition $\int_{\text{all displacements}} w(\mathbf{u})d\mathbf{u} = 1$. See details in Murdoch 2011, § 2.

core kinematic concepts for continua are obtained: velocity, material point, placement, and motion.[53]

Moreover, this program can derive not just kinematic foundations but also dynamical laws for continua. When we start with discrete monads, the greatest conceptual difficulty is grounding a notion of *contact* force. (Physical monads are point-sized, thus all their forces are actions at a *distance*.) And yet, the program I am presenting has found a way to overcome it. Namely, it has a way to define a Cauchy stress tensor, i.e. a notion of internal contact force within a continuous body. In physical terms, this tensor measures the (weighted) density of inter-particle forces at a point (see below). With these ideas in place, dynamical principles are then established: the Continuity Equation, or conservation of mass; and the balance laws of momentum (linear and angular) and energy. For instance, Cauchy's First Law is derived thus. Start with the Force Law for systems of mass-points:[54]

[12] $$\mathbf{b}_i + \sum_{i \neq j} \mathbf{f}_{ij} = m_i \mathbf{a}_i$$

Weight each member by means of an appropriately chosen function w. Next, define an interaction tensor $\mathbf{T}^*_w$ based on the weighted *internal* forces acting at a point, thus: $\mathrm{div}\mathbf{T}^*_w = \mathbf{f}_w$. Algebraic manipulation of the right-hand side of [12] yields a weighted rate of change in linear momentum $\rho\mathbf{a}_w$ and a thermal quantity $\boldsymbol{D}_w$.[55] Now we define the *Cauchy* stress tensor $\mathbf{T}_w$ as a derived concept, thus: $\mathbf{T}_w = \mathbf{T}^*_w - \boldsymbol{D}_w$. At this point, we are in a position to write:

[13] $$\nabla\mathbf{T}_w + \mathbf{b}_w = \rho\mathbf{a}_w$$

This is the Force Law for *continua* (i.e. Cauchy's First Law) in terms whose reference is *discrete*, viz. 'phenomenal monads.' The other basic principle, viz. the Torque Law, is likewise derivable from resources so far

[53] E.g. velocity $\mathbf{v}$ at a point is defined as the ratio $\boldsymbol{\rho}$ over $\mathbf{p}$. This way is the *reverse* of standard continuum mechanics, in which 'body,' 'placement' and 'motion' are primitive notions, and 'velocity,' 'mass density' and 'momentum' are derived *from* them.

[54] In [12] below, $\mathbf{b}_{ij}$ is the force exerted by a monad j located within the body on the i-th monad. In contrast, $\mathbf{f}_i$ is a force that originates in some agency external to the body, e.g. gravity.

[55] $\boldsymbol{D}_w$ is a tensor value associated with heat. The trace of $\boldsymbol{D}_w$ is twice the heat energy density in the volume averaged by the function $\boldsymbol{w}$; cf. Murdoch 2012: 125.

available (Murdoch 2012: 115-21). Mass points, therefore, suffice to ground a continuum theory. And so, modern mechanics implicitly vindicates Kant's constitution theory as I reformulated it here.

6. Conclusions

I have examined if Kant's Analogies of Experience support a sufficient notion of object of determinate (exact) experience. Specifically, I asked if the Analogies support basic causal laws that avoid the known limitations of Newton's science—Kant's paradigmatic application of his categories to determinate objects. In response, I investigated three fundamental principles of physical action. It turns out that two candidates, the basic laws of variational and analytic mechanics, seem beyond the reach of Kant's inventory of categories. And yet, Kant is in a position to ground Newton-Euler dynamics, a provably *general* theory. But, to do that Kant must abandon a tenet of his mature doctrine: the infinite divisibility of matter. Fortunately, his ensuing picture is still 'dynamistic,' as he always desired: matter is endowed with essential forces. In effect, I have advocated restoring a purified, updated version of his early matter theory. I called that version 'phenomenal monadology.' In modern terms, I have argued that grounding determinate experience requires Kant to swap continuous matter for mass points.

And so, Friedman's program to read Kant's categories as constitutive of scientific objecthood is vindicated and extended here. To be sure, much remains to be done. For instance, we need to inspect more closely the reach of Kant's categories of quantity, to see how much *kinematic* structure they support; and his categories of modality, to determine the notion of objective mechanical behavior that they ground. Still, there are good reasons to stay on the path Friedman broke, instead of following Buchdahl. My paper has thus outlined a research program that continues—in respect to post-Newtonian classical theory—Friedman's account of Kant's constitution of determinate objects. Equally, I have shown a way to make Kant an interlocutor in current dialogue on the foundations of classical science.

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