Imprecise Probabilities in Quantum Mechanics

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1 Introduction

In his entry on "Quantum Logic and Probability Theory" in the *Stanford Encyclopedia of Philosophy*, Alexander Wilce (2012) writes that "it is uncontroversial (though remarkable) that the formal apparatus of quantum mechanics reduces neatly to a generalization of *classical probability* in which the role played by a Boolean algebra of events in the latter is taken over by the *'quantum logic'* of projection operators on a Hilbert space." For a long time, Patrick Suppes has opposed this view (see, for example, the papers collected in Suppes and Zanotti 1996). Instead of changing the logic and moving from a Boolean algebra to a non-Boolean algebra, one can also 'save the phenomena' by weakening the axioms of probability theory and work instead with *upper and lower probabilities*. However, it is fair to say that despite Suppes' efforts upper and lower probabilities are not particularly popular in physics as well as in the foundations of physics, at least so far. Instead, quantum logics is booming again, especially since quantum information and computation became hot topics. Interestingly, however, imprecise probabilities are becoming more and more popular in formal epistemology as recent work by authors such as James Joyce (2010) and Roger White (2010) demonstrates.
In this essay I would like to give *one more reason* for the use of upper and lower probabilities in quantum mechanics and outline the research program that they inspire. The remainder of this essay is organized as follows. Sec. 2 introduces upper and lower probabilities. Sec. 3 turns to quantum mechanics and presents the CHSH inequality. We show that there is not always a joint probability distribution that reproduces observed quantum correlations. Sec. 4 argues that imprecise probabilities can be defined in these cases, and Sec. 5 concludes with a number of open questions.

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2 Imprecise Probabilities

Imprecise probabilities are well known from the theory of uncertain reasoning (Halpern 2005, Walley 1991). The starting point of the formal developments is the question of how to represent one's ignorance about a probability value. One way to do this is to introduce a lower probability measure P_* and an upper probability measure P^*, where the difference between the two is an agent's measure of her uncertainty about a probability assignment. To illustrate this, consider a coin tossing experiment and start with $P_*(\text{Heads}) = 0$ and $P^*(\text{Heads}) = 1$, which means that the agent is in a state of full uncertainty about the outcomes of the coin tossings. Then collect evidence and update $P_*(\text{Heads})$ and $P^*(\text{Heads})$ accordingly. If the coin is fair, then both measures will eventually converge to $1/2$, i.e. the probability of a fair coin to land heads. Note that the use of uppers and lowers is compatible with the existence of a probability value. The uppers and lowers only express our uncertainty about the probability value.
Upper and lower probability measures are defined as follows (Suppes and Zanotti 1996).

Definition 1 (Upper Probability) *Let Ω be a nonempty set, $\mathcal{B}$ a Boolean algebra on Ω, and P^* a real-valued function on $\mathcal{B}$. Then $\Omega = (\Omega, F, P^*)$ is an upper probability space if and only if for every A and B in $\mathcal{B}$, (i) $0 \leq P^*(A) \leq 1$, (ii) $P^*(\emptyset) = 0$ and $P^*(\Omega) = 1$, (iii) if $A \cap B = \emptyset$, then $P^*(A \cup B) \leq P^*(A) + P^*(B)$.*

Definition 2 (Lower Probability) *Let Ω be a nonempty set, $\mathcal{B}$ a Boolean algebra on Ω, and P_* a real-valued function on $\mathcal{B}$. Then $\Omega = (\Omega, F, P_*)$ is a lower probability space if and only if for every A and B in $\mathcal{B}$, (i) $0 \leq P^*(A) \leq 1$, (ii) $P_*(\emptyset) = 0$ and $P_*(\Omega) = 1$, (iii) if $A \cap B = \emptyset$, then $P_*(A \cup B) \geq P_*(A) + P_*(B)$.*

We also note the following definition:

Definition 3 (Upper-Lower Pair) *We call a pair (P_*, P^*) an upper-lower probability pair Ω, if for every A in $\mathcal{B}$ we have $P_*(A) \leq P^*(A)$.*

Note that lower probabilities are super-additive and upper probabilities are sub-additive, which has several consequences: First, the sum over all atoms of the algebra may lead to a value greater than 1 for uppers and smaller than 1 for lowers. Second, while for a probability measure $P(A) = \sum_{A',B,B'} P(A, A', B, B')$ holds, the following inequalities hold for uppers and lowers:

$$\begin{aligned} P^*(A) &\leq \sum_{A',B,B'} P^*(A, A', B, B') \\ P_*(A) &\geq \sum_{A',B,B'} P_*(A, A', B, B'). \end{aligned}$$

Interestingly, if *monotonicity* holds, then uppers and lowers are related in the following way: $P_*(A) = 1 - P^*(\overline{A})$, where $\overline{A}$ is the complement of A in $\mathcal{B}$. (We will see later that this relation does not hold in quantum mechanics.) For an interpretation of upper and lower probabilities in terms of betting odds, see Walley (1991).

3 Quantum Mechanics and the CHSH Inequality

Let us consider four binary random variables A, A', B and B' that can take the values $a_i, a'_i, b_i, b'_i = \pm 1$ for $i = 1, 2$. We assume *symmetry*, i.e. we only consider situations where $E(A) = E(A') = E(B) = E(B') = 0$ with the expectation value E defined in the usual way, i.e. $E(A) := \sum_{i=1}^{2} a_i \, p(a_i)$. Next, we define the quantity

$$\mathcal{F} := |E(AB) + E(AB') + E(A'B) - E(A'B')|, \tag{1}$$

where the expectation value

$$E(AB) := \sum_{i,k=1}^{2} a_i \, b_k \, P(a_i, b_k) = \sum_{i,j,k,l=1}^{2} a_i \, b_k \, P(a_i, a'_j, b_k, b'_l) \tag{2}$$

measures the *correlation* between the random variables A and B. P is a probability measure. Note that $E(AB)$ takes values in the interval $[-1, 1]$ and that these correlations can be measured. Generalizing Bell's theorem, Clauser, Horn, Shimony and Holt (1969) effectively showed the following.

Theorem 1 *If there is a joint probability distribution $P(A, A', B, B')$, then $\mathcal{F} \leq 2$ ("CHSH inequality").*

The proof is in the appendix.

As is generally known, the CHSH inequality does not always hold. There are experimental setups that exhibit (quantum) correlations which violate the CHSH inequality. In experiments with correlated photons, for example, one can measure values of $\mathcal{F}$ up to $2\sqrt{2}$. These experiments starts with an EPR state of correlated photons, i.e. with the state $|\text{EPR}>= 1/\sqrt{2} \cdot (|10> - |01>)$ where $|0>$ and $|1>$ represent the photon polarizations of the two subsystems **A** and **B**. One can then find measurement angles α and α' (at **A**) and β and β' (at **B**) such that the CHSH inequality is violated. Hence, there is not always a joint probability distribution over A, A', B and B' that reproduces the expectation values $E(AB)$ etc. Note that these expectation values can be calculated from quantum mechanics and that the experiments confirm the theory.

Let us now study the CHSH inequality for atoms. Experiments similar to the just-mentioned photon experiments can be performed with an EPR state of two 2-level atoms that are trapped in a cavity. Here $|0>$ and $|1>$ represent the states of a single 2-level atom being in the ground state or the excited state, respectively. Let $A := X_1$, $A' := Z_1$, $B := X_2 + Z_2$ and $B' := X_2 - Z_2$, where X_1 denotes the Pauli matrix σ_x applied to the state of subsystem 1. Z_1, X_2 etc. are defined accordingly. Note that *symmetry* holds i.e. $E(A) = E(A') = E(B) = E(B') = 0$. Next, we calculate $E(AB) = E(AB') = E(A'B) = -1/2\sqrt{2}$ and $E(A'B') = 1/2\sqrt{2}$. Hence $\mathcal{F} = 2\sqrt{2}$, i.e. the CHSH inequality is maximally violated.

Next, we examine what happens if the quantum state under consideration decays under the influence of decoherence (Schlosshauer 2007). Clearly, how fast the state decays will depend on the experimental context. It is known, for example, that the decay is slower in a cavity than in free space. What is important to us is that if the EPR state decoheres, then the correlations in the system also decay and the CHSH inequality will eventually be satisfied after some time τ_0. Once the CHSH inequality is satisfied, the correlations can be explained classically, i.e. by a non-contextual local hidden variables model. Moreover, these correlations can then be accounted for by a joint probability distribution.

Let us now calculate the time τ_0 when this is the case. One way of modeling decoherence is by coupling the quantum system to a reservoir. One can then write down the Schrödinger equation for the system plus the reservoir (environment), make the Born-Markov approximation, trace out the environment and obtain a quantum master equation for the reduced state ρ of the system. ρ then satisfies the following quantum master equation, which is of the Lindblad form (Breuer and Petruccione 2002):

$$\frac{d\rho}{dt} = -\frac{B}{2} \sum_{i=1}^{2} [\sigma_+^{(i)}\sigma_-^{(i)}\,\rho + \rho\,\sigma_+^{(i)}\sigma_-^{(i)} - 2\sigma_-^{(i)}\,\rho\,\sigma_+^{(i)}], \qquad (3)$$

with the decay constant B. Using the theory of Generalized Dicke States (Hartmann 2012), this equation can be solved analytically. We then obtain for the time evolution of the initial state $\rho(0) = |\mathrm{EPR}><\mathrm{EPR}|$:

$$\rho(\tau) = e^{-\tau}\,\rho(0) + (1 - e^{-\tau})\,|00><00|, \qquad (4)$$

with $\tau := B\,t$.

Next, we calculate the expectation values of A, A', B and B' as defined above for a system in the state $\rho(\tau)$ and obtain:

$$<A>= 0, \quad <A'>=<B>= -<B'>= e^{-\tau} - 1 \qquad (5)$$

To make sure that *symmetry* holds for all times τ, we replace $A \to \tilde{A} := A - <A>$ etc. Clearly, we then have $E(\tilde{A}) = E(\tilde{A}') = E(\tilde{B}) = E(\tilde{B}') = 0$. For the correlations, we obtain:

$$
\begin{array}{lll}
< \tilde{A}\tilde{B} > = < AB > & , & < \tilde{A}'\tilde{B} > = < A'B > -(e^{-\tau} - 1)^2 \\
< \tilde{A}\tilde{B}' > = < AB' > & , & < \tilde{A}'\tilde{B}' > = < A'B' > +(e^{-\tau} - 1)^2
\end{array} \tag{6}
$$

Next, we calculate $\tilde{\mathcal{F}}$ as a function of τ (see eq. (1)). It is easy to see that a joint probability distribution over $\tilde{A}, \tilde{A}', \tilde{B}$ and $\tilde{B}'$ exists if $\tau > \tau_0 := 245$, i.e. after a relatively short period of time after the quantum state starts to decay (in units of the inverse decay constant B). Figure 1 shows $\tilde{\mathcal{F}}$ and, for comparison, also $\mathcal{F}$ as a function of τ, where $\mathcal{F}$ is calculated using the original operators A, A', B and B'.

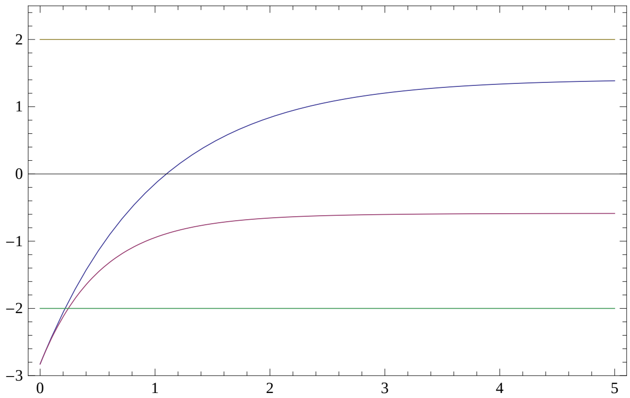

Figure 1: $\mathcal{F}$ (blue) and $\tilde{\mathcal{F}}$ (violet) as a function of τ.

4 Imprecise Probabilities in Quantum Mechanics

We have seen that there is a joint probability distribution P for $\tau \geq \tau_0$ that reproduces the experimentally measurable correlations in the decaying EPR state. But how can we account for the correlations before that time? Hartmann and Suppes (2010) have explicitly constructed an upper probability distribution P^* that accounts for the correlations of a decaying EPR state at all times, i.e. before, at, and after τ_0. We therefore have unified account, which allows us to stick to a Boolean algebra of events. It is not necessary to work with a non-Boolean algebra in the quantum domain and a Boolean algebra in the classical domain, as quantum logicians do. All correlations can be accounted for by an upper probability distribution. This measure is explicitly sub-additive for times $\tau < \tau_0$ and turns into an additive probability measure for $\tau \geq \tau_0$. I take this to be a main advantage of the proposed approach to work with imprecise probabilities in quantum mechanics compared to the alternative quantum logical account, which do not allow for such a unified treatment.

It is interesting to note that the situation discussed here is similar to the learning situation discussed in Sec. 2. In the learning case, the upper probability distribution approximates the proper joint probability distribution more and more as the number of coin tosses increases. They coincide in the limit of an infinite number of coin tosses. In the quantum mechanical case, the upper probability distribution approximates the proper joint probability distribution more and more as the state decays. It coincides with the joint probability distribution once the CHSH inequality is satisfied (after a finite decay time). The joint probability distribution emerges from the interaction of the quantum state with its environment.

For the decaying EPR state, there is also a lower probability measure. This measure also converges into a probability measure which is defined for times $\tau \geq \tau_0$. However, the lower and the upper probability distributions are not related via $P_*(A) = 1 - P^*(\overline{A})$, i.e. they do not form an upper-lower pair. This is in line with the fact that there is no joint distribution for times $\tau < \tau_0$. Consequently, the *monotonicity* condition is violated in quantum mechanics, and upper and lower probability distributions have to be calculated independently by fitting them to the quantum mechanical expectation values. It is interesting to further explore the implications of the failure of monotonicity in quantum mechanics.

5 Open Questions

In future work, we plan to address the following four questions. *First*, how do our results generalize? Is it always possible, i.e. for all quantum states and corresponding sets of measurement operators, to fit an upper and a lower probability distribution? It would be nice to have a general proof that this is always possible, or a counter example showing that it is not. Our evidence so far is only episodic as we focused on the EPR state. *Second*, what is the proper interpretation of upper and lower probabilities in quantum mechanics? To address this question, the failure of *monotonicity* in quantum mechanics has to be understood. It will also be interesting to relate the discussion of upper and lower probabilities in quantum mechanics to the recent work on Quantum Bayesianism (Caves et al. 2007) which may shed some light on interpretational questions regarding upper and lower probabilities in quantum mechanics. *Third*, to further explore the relation between logic and probability in quantum mechanics, Gleason's Theorem has to be analyzed (Hughes 1989). Here special attention has to be paid to the additivity assumption, which shows up in the proof of the theorem. We ask: What follows if one allows for sub- and super additive measures? *Fourth and finally*, what is the advantage of upper and lower probabilities compared to negative probabilities for which our decoherence story can be told as well? Negative probabilities where famously discussed by Feynman (1987) and have recently attracted the interest of Patrick Suppes. It will be worth to compare negative probabilities with imprecise probabilities.

Acknowledgments

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Appendix: Proof of Theorem 1

To prove Theorem 1, we first simplify the notation and denote the value -1 by 0. Next, we introduce the following abbreviations:

$$\begin{array}{lcl}
P(1111) = P(0000) := x_1 & , & P(1110) = P(0001) = x_2 \\
P(1101) = P(0010) := x_3 & , & P(1100) = P(0011) = x_4 \\
P(1011) = P(0100) := x_5 & , & P(1010) = P(0101) = x_6 \\
P(1001) = P(0110) := x_7 & , & P(1000) = P(0111) = x_8,
\end{array}$$

where we have made use of the *symmetry* requirement. Note that $0 \leq x_i \leq 1$ for $i = 1, \ldots, 8$ and that $\sum_{i=1}^{8} x_i = 1/2$. We then obtain by using eq. (2) and similar equations for the other expectation values:

$$\begin{aligned}
\mathcal{F} &= 4|x_1 + x_2 - x_3 - x_4 + x_5 - x_6 + x_7 - x_8| \\
&\leq 4|x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8| \\
&\leq 4 \times 1/2 = 2,
\end{aligned}$$

which completes the proof.