# Contractions of noncontractive consequence relations*

Rohan French and David Ripley
Monash University and the University of Connecticut
davewripley@gmail.com

November 6, 2014

**Abstract**

Some theorists have developed formal approaches to truth that depend on counterexamples to the structural rules of *contraction*. Here, we study such approaches, with an eye to helping them respond to a certain kind of objection. We define a contractive relative of each noncontractive relation, for use in responding to the objection in question, and we explore one example: the contractive relative of multiplicative-additive affine logic with transparent truth, or MAALT.

## 1 Introduction

A consequence relation is *contractive* iff it includes no counterexamples to either of Gentzen's [10] structural rules of contraction WL and WR:[1]

$$\text{WL: } \frac{\Gamma, A, A : \Delta}{\Gamma, A : \Delta} \qquad \text{WR: } \frac{\Gamma : A, A, \Delta}{\Gamma : A, \Delta}$$

According to some theorists, there is an important kind of consequence relation that is *noncontractive*. Indeed, some approaches to paradoxes of truth and validity (e.g. [25, 2, 33, 36]) turn on exactly this feature. According to these approaches, it is precisely the failure of some instances of contraction that defuses the apparent trouble that the paradoxes give rise to.

We agree with these authors that noncontractive consequence relations have much to teach us about paradoxes. However, there is a family of objections to

*Forthcoming as [7].

[1]Preliminaries: capital Roman letters are formulas; capital Greek letters (that are not also Roman letters) are finite multisets of formulas. We write $\Gamma : \Delta$ for an argument/sequent with premises $\Gamma$ and conclusions $\Delta$; we assume that premises and conclusions always come in finite multisets. Given a formula $A$ and closed term $t$, the formula $A[x/t]$ is the result of replacing all free occurrences of the variable $x$ in $A$ with $t$. We abbreviate in usual ways. Note that some authors prefer 'C' for contraction.

these noncontractive approaches that threatens to scupper the whole project. According to these objections, noncontractive relations can be interesting for many purposes, but they cannot capture at least some of the important things we want in a theory of consequence, since some of the things we want simply *must* obey contraction.

We are also sympathetic to these objections, but only to a point. When we are concerned with consequence, so we will argue, at least sometimes we are concerned with a relation that obeys contraction. But, we will also argue, this does not undermine the cited noncontractive approaches to paradox. The reason is this: every noncontractive consequence relation has a contractive next-door neighbour—we call this its *contraction*—that allows the noncontractivist to concede what's right about these objections without undermining any core part of the noncontractive story.

The noncontractivist can maintain that, while the underlying facts that ground consequence determine a noncontractive relation, nonetheless it is that relation's *contraction* that plays the roles the objector focuses on. Since the contraction is fully determined by its underlying noncontractive relation, this is not in any serious sense a retreat from a noncontractive position.

Interestingly, contractions of noncontractive relations tend to be *nontransitive*: to include counterexamples to Gentzen's structural rule of cut:

$$\text{Cut:} \quad \frac{\Gamma : A, \Delta \qquad \Gamma', A : \Delta'}{\Gamma, \Gamma' : \Delta, \Delta'}$$

Moreover, the particular counterexamples to contraction that occur in noncontractive solutions to the paradoxes of truth and validity become, in those solutions' contractions, exactly the counterexamples to cut put forward in the nontransitive approaches to those same paradoxes recommended in [28, 3].

So the machinery of contractions not only provides noncontractivists with a new response to familiar objections to their view, it also provides all of us with a new window on connections between contraction and transitivity, connections that may underly some of the similarities between noncontractive and nontransitive approaches to paradoxes of truth and validity.

With this application in mind, we use §2 to briefly rehearse some of the motivations for treating paradoxes in a substructural setting, whether noncontractive or nontransitive, and also to present one logical system of each type. §3 presents and explores one possible interpretation of noncontractive consequence, based on the notion of information extraction, and suggests a way for a noncontractivist favouring such an interpretation to meet the objection we mentioned above. §4 explores an example, and §5 reflects and concludes.

## 2 Two substructural approaches to paradox

In this section, we motivate and present two substructural approaches to paradoxes of truth and validity: one noncontractive and one nontransitive.

## 2.1 Liar paradox

Suppose there is a sentence $\lambda$ that is $\neg T\langle\lambda\rangle$, where $\langle\lambda\rangle$ is a name for $\lambda$ and $T$ is the truth predicate. That is, $\lambda$ is a sentence that says that it itself is not true. As is well known, such a sentence can cause some trouble in systems not prepared for it, at least if those systems are sophisticated enough to recognize the distinctive behaviour that a truth predicate ought to exhibit.

For example, consider the following rules for negation:

$$\neg\text{L:}\ \frac{\Gamma : A, \Delta}{\Gamma, \neg A : \Delta} \qquad \neg\text{R:}\ \frac{\Gamma, A : \Delta}{\Gamma : \neg A, \Delta}$$

These are usual rules for a negation exhibiting broadly classical 'flip-flop' behaviour. (Whether or not they determine full classical negation depends on the structural situation they find themselves in.) They are plausible enough, at least at first glance. Recall that multiple conclusions are interpreted disjunctively (in some sense), just as multiple premises are interpreted conjunctively (in some sense). If we have a bunch of conclusions among which $A$ is one option, then by helping ourselves to the additional premise $\neg A$ we can rule out that option, narrowing our conclusions down to whatever's left. This is captured by ¬L. Similarly, if we've relied on a premise $A$ to reach a bunch of conclusions, then we can open up an additional option by ditching that premise and considering the additional possibility $\neg A$. This is captured by ¬R. (All this is contentious, of course. We mean here merely to motivate.)

Similarly, consider the following rules governing the interaction between a truth predicate $T$ and a naming device $\langle\ \rangle$:

$$T\langle\ \rangle\text{L:}\ \frac{\Gamma, A : \Delta}{\Gamma, T\langle A\rangle : \Delta} \qquad T\langle\ \rangle\text{R:}\ \frac{\Gamma : A, \Delta}{\Gamma : T\langle A\rangle, \Delta}$$

These rules (in certain settings, including all those to be considered here—but see [29] for a counterexample) give us a *transparent* truth predicate, one such that $T\langle A\rangle$ is freely intersubstitutable with $A$, even when they occur as subsentences, without affecting the validity of any argument. These rules also make sense in their own right: if $A$ can be used as a premise to reach some conclusions, surely the claim that $A$ is true will do as well. Likewise, if $A$ can be concluded from some premises, then surely the claim that $A$ is true can be concluded as well. (Again, all this is contentious; but it should be clear this is at least tempting.)

Now, consider the following derivation:

$$\text{Cut:}\ \frac{\text{WR:}\ \dfrac{T\langle\ \rangle\text{R:}\ \dfrac{\neg\text{R:}\ \dfrac{T\langle\lambda\rangle : T\langle\lambda\rangle}{: T\langle\lambda\rangle, \lambda}}{: T\langle\lambda\rangle, T\langle\lambda\rangle}}{: T\langle\lambda\rangle} \qquad \text{WL:}\ \dfrac{T\langle\ \rangle\text{L:}\ \dfrac{\neg\text{L:}\ \dfrac{T\langle\lambda\rangle : T\langle\lambda\rangle}{T\langle\lambda\rangle, \lambda :}}{T\langle\lambda\rangle, T\langle\lambda\rangle :}}{T\langle\lambda\rangle :}}{:}$$

This derivation is one way to get yourself in Trouble with a liar sentence. (Deriving the empty sequent is Trouble, as far as we're concerned.) It relies on all the negation and truth rules we've seen so far, but it has a number of other dependencies: reflexivity, contraction, and cut. For the purposes of this paper, we won't question reflexivity.[2] If the above negation and truth rules are correct, then, we have two suspects left: contraction and cut.

## 2.2 Uniformity

But why should we assume that those negation and truth rules are correct? Almost all logical approaches to the paradoxes of truth, after all, reject at least one of these four rules.[3]

We cannot hope to consider the issue in any depth here, but we do want to point to what we take to be a compelling argument for holding the negation and truth rules innocent in the above derivation: an argument from *uniformity*.

The trouble with pinning the blame on rules for any particular vocabulary is this: there is no particular vocabulary required for paradox. While the liar paradox above uses truth and negation, it has relatives that use truth but not negation, like the curry paradox [4, 17], or that use neither truth nor negation, but a validity predicate instead [24, 22, 33, 2], or an inconsistency predicate, or an impossibility predicate; and this is still not to mention the closely-related set-theoretic versions of these paradoxes, and so on. It is implausible to think that all of these paradoxes have different sources. Yet there is no piece of vocabulary that they all share.

Arguments like this are common in the literature on paradox, and they have been responded to in various ways; we will not enter into that debate here. But the considerations they bring to bear are considerably more general than is often appreciated. For example, Priest [21] uses an argument like this to support rejection of ¬L, and Scharp [31] uses an argument like this to support doing without $T$ entirely. But by our lights, the push for uniformity undermines both of these approaches as well: it is a serious problem for all vocabulary-based treatments of these paradoxes, nonclassical and classical alike.

Instead of giving novel theories of particular pieces of vocabulary, we should focus instead on some ingredient that all these paradoxes have in common. There are three such ingredients: reflexivity, contraction, and cut. Without all three of these, it seems there are no problems forthcoming from any of these paradoxes. (Of course details matter; we are just skimming here, to provide some motivation.) For the purposes of this paper, we assume reflexivity. We are only concerned to treat the two families of substructural approaches best-explored in the literature: doing without contraction on the one hand, and doing without cut on the other.

[2]This should not be as uncontroversial as it is! But one battle at a time.

[3]For examples, see [21, 6, 1, 8, 12, 31].

## 2.3 MAALT and MASTT

Moreover, we will only look at a single example of each approach, although we think the examples we've chosen are representative of the families they come from.

Our noncontractive logic is MAALT: Multiplicative-Additive Affine Logic with Truth. Our nontransitive logic is MASTT: Multiplicative-Additive Strict-Tolerant logic with Truth.[4] We will mainly be concerned with the (multiple conclusion) consequence relations of these logics, which we understand as the set of valid sequents they determine. This set of sequents will be determined via a sequent calculus presentation. When we write 'MAALT' or 'MASTT', then, we might mean a set of sequents, or we might mean a sequent calculus; we trust context to disambiguate.

The two calculi have a very large common part: we give it in Figure 2.3. (In this figure, when two formulas in parentheses figure in the premise-sequent of a rule, the rule can be validly applied with *either* (but just one!) of them. $t$ can be any term, and $a$ must be an *eigenvariable*: a variable not occurring in the conclusion-sequent of the rule application.) Conjunction, disjunction, and implication all bifurcate in the present setting, coming in an *additive* flavour ($\sqcap, \sqcup, \mapsto$, respectively) and a *multiplicative* one ($\otimes, \oplus, \twoheadrightarrow$, respectively).[5] We suppose that there is some $\langle\ \rangle$-name for each formula; this can be achieved, for example, by the method outlined in [27, p.355].

Note that the negation rules and the truth rules considered above are all in the common fragment of these calculi, together with the rule Id, which guarantees the reflexivity of the consequence relation. If the calculi are to avoid deriving the empty sequent (and, via K, every sequent), then, neither can include both contraction and cut. Indeed, this is precisely the difference between them. MASTT is the common fragment plus the rules WL and WR of contraction. As it happens, the common fragment alone *admits* cut: adding just the rule of cut to the common fragment would not change which sequents are derivable.[6] So our formulation of MAALT is simply the common fragment itself; however, you should remember that it is closed under cut, and so fully transitive. Thus, every MAALT derivation is a MASTT derivation, but not vice versa.

Note as well that, minus the truth rules, MASTT determines the usual consequence relation of first-order classical logic (with two copies of each binary connective). It is only in the presence of applications of these truth rules, then, that MASTT becomes nontransitive.

[4]MASTT is very closely related to the STTT of [3], the G1cT of [29], and the ST of [28], but it is not identical to any of these, as MASTT includes both additive and multiplicative vocabulary, to make for easier comparisons with MAALT.

[5]Notation is not yet standardised in this area; please note ours! We stick to *additive* quantifiers only; while we agree with [19] that there is some need for a theory of multiplicative quantifiers as well, there is a briarpatch over there that it's better for our purposes here just to dodge. See [36] for one interesting approach to multiplicative quantifiers.

[6]This can be shown following the techniques of [36]; but the situation here is in some ways simpler, since Zardini's multiplicative quantifiers (and so infinitely-wide derivations) are not needed.

$$\text{Id: } \frac{}{A : A} \qquad \text{K: } \frac{\Gamma : \Delta}{\Gamma, \Gamma' : \Delta, \Delta'}$$

$$\neg\text{L: } \frac{\Gamma : A, \Delta}{\Gamma, \neg A : \Delta} \qquad \neg\text{R: } \frac{\Gamma, A : \Delta}{\Gamma : \neg A, \Delta}$$

$$\otimes\text{L: } \frac{\Gamma, A, B : \Delta}{\Gamma, A \otimes B : \Delta} \qquad \otimes\text{R: } \frac{\Gamma : A, \Delta \qquad \Gamma' : B, \Delta'}{\Gamma, \Gamma' : A \otimes B, \Delta, \Delta'}$$

$$\sqcap\text{L: } \frac{\Gamma, (A), (B) : \Delta}{\Gamma, A \sqcap B : \Delta} \qquad \sqcap\text{R: } \frac{\Gamma : A, \Delta \qquad \Gamma : B, \Delta}{\Gamma : A \sqcap B, \Delta}$$

$$\oplus\text{L: } \frac{\Gamma, A : \Delta \qquad \Gamma', B : \Delta'}{\Gamma, \Gamma', A \oplus B : \Delta, \Delta'} \qquad \oplus\text{R: } \frac{\Gamma : A, B, \Delta}{\Gamma : A \oplus B, \Delta}$$

$$\sqcup\text{L: } \frac{\Gamma, A : \Delta \qquad \Gamma, B : \Delta}{\Gamma, A \sqcup B : \Delta} \qquad \sqcup\text{R: } \frac{\Gamma : (A), (B), \Delta}{\Gamma : A \sqcup B, \Delta}$$

$$\twoheadrightarrow\text{L: } \frac{\Gamma : A, \Delta \qquad \Gamma', B : \Delta'}{\Gamma, \Gamma', A \twoheadrightarrow B : \Delta, \Delta'} \qquad \twoheadrightarrow\text{R: } \frac{\Gamma, A : B, \Delta}{\Gamma : A \twoheadrightarrow B, \Delta}$$

$$\mapsto\text{L: } \frac{\Gamma : A, \Delta \qquad \Gamma, B : \Delta}{\Gamma, A \mapsto B : \Delta} \qquad \mapsto\text{R: } \frac{\Gamma, (A) : (B), \Delta}{\Gamma : A \mapsto B, \Delta}$$

$$\forall\text{L: } \frac{\Gamma, A[x/t] : \Delta}{\Gamma, \forall x A : \Delta} \qquad \forall\text{R: } \frac{\Gamma : A[x/a], \Delta}{\Gamma : \forall x A, \Delta}$$

$$\exists\text{L: } \frac{\Gamma, A[x/a] : \Delta}{\Gamma, \exists x A : \Delta} \qquad \exists\text{R: } \frac{\Gamma : A[x/t], \Delta}{\Gamma : \exists x A, \Delta}$$

$$T\langle\ \rangle\text{L: } \frac{\Gamma, A : \Delta}{\Gamma, T\langle A \rangle : \Delta} \qquad T\langle\ \rangle\text{R: } \frac{\Gamma : A, \Delta}{\Gamma : T\langle A \rangle, \Delta}$$

**Figure 1:** Rules common to MAALT and MASTT

## 2.4 Sparser formulations

There are a lot of different connectives in play in MAALT and MASTT, and we can in fact get by with a lot fewer. In the presence of negation, any one additive connective allows us to define the others, and similarly for any one multiplicative connective. For example, note that $\neg(A \otimes \neg B)$ obeys the rules given for $A \twoheadrightarrow B$:

$$\neg\text{L: } \dfrac{\otimes\text{R: } \dfrac{\Gamma : A, \Delta \qquad \neg\text{R: } \dfrac{\Gamma', B : \Delta'}{\Gamma' : \neg B, \Delta'}}{\Gamma, \Gamma' : A \otimes \neg B, \Delta, \Delta'}}{\Gamma, \Gamma', \neg(A \otimes \neg B) : \Delta, \Delta'}$$

$$\neg\text{R: } \dfrac{\otimes\text{L: } \dfrac{\neg\text{L: } \dfrac{\Gamma, A : B, \Delta}{\Gamma, A, \neg B : \Delta}}{\Gamma, A \otimes \neg B : \Delta}}{\Gamma : \neg(A \otimes \neg B), \Delta}$$

Similarly, $\neg(\neg A \otimes \neg B)$ can stand in for $A \oplus B$, and corresponding definitions will work for the additives as well. Also, $\exists$ can be defined as $\neg\forall\neg$ in the usual way, as is quick to verify.

## 2.5 Additive/multiplicative collapse

The distinction between additive and multiplicative connectives will loom large in what follows. But it is a difference that disappears in MASTT. In the presence of both weakening and contraction, additive connectives can be shown to obey their multiplicative twins' rules, and vice versa. For example:[7]

$$\text{WL: } \dfrac{\sqcap\text{L: } \dfrac{\sqcap\text{L: } \dfrac{\Gamma, A, B : \Delta}{\Gamma, A \sqcap B, B : \Delta}}{\Gamma, A \sqcap B, A \sqcap B : \Delta}}{\Gamma, A \sqcap B : \Delta}$$

$$\sqcap\text{R: } \dfrac{\text{K: } \dfrac{\Gamma : A, \Delta}{\Gamma, \Gamma' : A, \Delta, \Delta'} \qquad \text{K: } \dfrac{\Gamma' : B, \Delta'}{\Gamma, \Gamma' : B, \Delta, \Delta'}}{\Gamma, \Gamma' : A \sqcap B, \Delta, \Delta'}$$

$$\otimes\text{L: } \dfrac{\text{K: } \dfrac{\Gamma, (A), (B) : \Delta}{\Gamma, A, B : \Delta}}{\Gamma, A \otimes B : \Delta} \qquad \text{WR}^\star\text{: } \dfrac{\text{WL}^\star\text{: } \dfrac{\otimes\text{R: } \dfrac{\Gamma : A, \Delta \qquad \Gamma : B, \Delta}{\Gamma, \Gamma : A \otimes B, \Delta, \Delta}}{\Gamma : A \otimes B, \Delta, \Delta}}{\Gamma : A \otimes B, \Delta}$$

[7] In annotating our derivations, we add $\star$ to the name of a rule to indicate that it might have to be iterated some number of times (including 0). For instance, in showing that $A \otimes B$ obeys the rule for $\sqcap$, it might take many applications of WL to contract $\Gamma, \Gamma$ to $\Gamma$, since $\Gamma$ might have many members. Alternately, $\Gamma$ might be empty, in which case no applications at all of WL are needed. We will also (later) write things like $WL^n$, to indicate $n$ applications of the rule WL, or $A \otimes B^n$, to indicate $n$ occurrences of $A \otimes B$.

Despite this collapse, we include the full range of additive and multiplicative vocabulary in MASTT, to ease comparison with MAALT.

In MAALT, where there is no rule of contraction, these collapsing derivations are not always available. Since MAALT includes K, however, *some* of the derivations are. (As in the example derivations, one direction of each collapse depends just on contraction, and the other direction just on weakening.) As a result, in MAALT, a multiplicative conjunction is strictly *stronger* than its additive twin, and multiplicative disjunctions and implications are strictly *weaker* than theirs.

### 2.6 The paradoxes

Both MAALT and MASTT block the disastrous derivation of §2.1. Of course, that alone is not enough to guarantee that they are free of other trivializing derivations. But, as it happens, they are.

We do not prove this here, or consider full approaches to the paradoxes based on MAALT, MASTT, or their relatives; that would take us too far afield. For more on noncontractive approaches to paradoxes, see [25, 11, 20, 36, 2, 33, 15]. For more on nontransitive approaches, see [34, 35, 28, 3, 32]. Both approaches are susceptible to a cut-elimination style non-triviality proof (see [20, 36] for the noncontractive approaches, and [28] for the nontransitive approaches). In addition, the non transitive approach can also be shown to be non-trivial using the argument given in [14] (cf. [27]).

## 3 Interpreting noncontractive consequence

In this section, we briefly lay out a representative theory of noncontractive consequence, and offer its proponents a novel way to grapple with a familiar objection. We start from the familiar notion of 'resource sensitivity', and develop it in ways suggested in [15], a recent and, we think, representative noncontractive approach to paradox.

### 3.1 Resource sensitivity

The metaphor of *resource sensitivity* is a commonly-deployed one where noncontractive logics are involved.[8] With this metaphor in hand, it is clear how failures of contraction can arise. For example, think of money: two dollars will get you a candy bar, but one will not. There is nothing to distinguish one dollar from another, however; it takes two of the very same thing to perform the transaction (see also [18, p.16]).

Now think of validity as a transaction, and premises and conclusions as like money; then we ought to expect that two occurrences of some premise or

[8]Since we're assuming full weakening throughout this paper, the appropriate way to deploy this metaphor for our purposes is as concern over whether we have *enough* resources or not. (Without weakening, we could also worry about whether we have *too many* resources; in the presence of weakening this question is obscured.)

conclusion might be needed for validity, where one occurrence wouldn't yet be enough. For example, we might need to appeal twice to a particular premise on the way to a conclusion; it can happen that one appeal alone isn't enough.

The analogy works to explain failures of contraction because when it comes to money, *amount* is crucial. But it goes a bit too far: for money, amount is *all* that's crucial. Different types of money are convertible into each other, so long as you have enough. Premises and conclusions are not like this: even if two $p$s 'buy' you more than one $p$ does, no amount of $p$s will 'buy' you a single $q$. For premises and conclusions, *type* is also crucial. This is not to deny that amount might matter, but only to point out that amount cannot replace type. We must attend to type as well.[9]

Suppose that the resource analogy works; suppose the real facts about entailment, at their core, relate premises to conclusions with attention to both amount and type. We might be interested in these real facts for any number of reasons. But we might also be interested in just those aspects that depend *only on type*: for example, given just the types of premises and conclusions, is there *any* way of setting up amounts to achieve validity?[10] This is the question we will pursue in a moment. First, though, we want to present one way to make the idea of resouce sensitivity more concrete.

## 3.2 Informational extraction

In [15], Mares and Paoli give a theory of a certain kind of consequence that, they argue, should not be expected to obey contraction. (They call it 'internal consequence'; they also consider a notion of 'external consequence', which they argue should be expected to contract. We do not discuss external consequence here.)

As they cast it, the pertinent notion of consequence is a matter of 'information extraction'; a conclusion follows from a bunch of premises iff the information in the conclusion can be extracted from the information in the premises. Unfortunately, they do not consider cases of multiple conclusions (in their philosophical gloss: their technical material happily accommodates multiple conclusions), nor do they give an account of what 'the information in the premises' is, even granting them the notion of the information in a single premise.

Nonetheless, we can fill the details in for them, at least in an abstract way, by supposing that there are two operations of combination—call them *premise combination* $\circ$ and *conclusion combination* $\bullet$. These are binary operations (as-

[9] A tighter analogy comes from [18, p. 26]: three testimonies of the form 'I saw the defendant at the scene of the crime' might be enough to convict, even if one is not. In an ordinary case, this is because there is some chance that any particular testimony is false—whether mistakenly or deliberately so—but it is supposed that the more agreement there is, the smaller the chance of falsehood. Here, both type and amount are clearly at issue: no number of testimonies of the form 'The defendant is an egg' is enough to convict, in most cases. This is still only an analogy, however, since noncontractive consequence of the sort we're interested in is not meant to be understood as a matter of probability.

[10] For Paoli's testimony example, this becomes: given just these claims, is there any degree of certainty we could have in them that would result in a conviction?

sumed to be commutative and associative) on pieces of information, and are not to be confused with connectives. Let $\circ(\Gamma)$ be the result of combining all the members of $\Gamma$ via $\circ$, and similarly for $\bullet(\Delta)$. (Since $\Gamma$ and $\Delta$ are assumed to be *finite* multisets, and since $\circ$ and $\bullet$ are both commutative and associative, this is well-defined.) Then, given a multiset $\Gamma$ of premises, each with its own piece of information, we can instead focus on $\circ(\Gamma)$.[11] Similarly, given a multiset $\Delta$ of conclusions, we can focus on $\bullet(\Delta)$. Now we need only to know when one piece of information can be extracted from another, and we have our account: $\Gamma : \Delta$ is valid iff $\bullet(\Delta)$ can be extracted from $\circ(\Gamma)$.

Although Mares and Paoli work without weakening, we are more interested in systems with it. We do not think this blocks application of their ideas, however: even if they are correct that there are notions of extraction and combination that do not obey weakening, surely there are notions that do. For example, if 'can be extracted from' is transitive and we can always extract $x$ from $\circ(x, y)$, then weakening on the left will preserve validity. (Similarly if we can always extract $\bullet(x, y)$ from $x$, for weakening on the right.)

But there are any number of ways to see the situation that will obey these constraints: so long as extraction does not have to use all the information it is given, the constraints are more or less to be expected. And in many reasoning situations, this is just the case: we collect a whole lot of information to see what it can show us, without any thought that in the end we *must* use it all to reach a conclusion; or we pile up a variety of conclusions we'd be happy to reach, without any thought that in the end narrowing things down further would be a *mistake*. For some purposes the question 'is this exactly what's needed?' is the appropriate one; for the ones we mean to focus on, the question 'is this enough?' is more appropriate.

## 3.3 Informational extraction and cut

One possible drawback of this kind of approach to consequence is that it makes cut into a remarkably unintuitive condition. Although this is a bit of a side issue, the situation is interesting, so we linger for a moment. Let $x \sqsubseteq y$ mean that information $y$ can be extracted from information $x$; then cut amounts to the constraint that whenever $y \sqsubseteq \bullet(x, a)$ and $\circ(w, a) \sqsubseteq z$, then $\circ(w, y) \sqsubseteq \bullet(x, z)$. At the very least, such a constraint needs explanation. Concrete theories of premise combination, conclusion combination, and extraction may help here; but it looks difficult to motivate cut from the mere shape of the information-extraction account. (Note that simple transitivity is much easier to motivate: the transitivity of extraction alone is enough, and there is no need for this complex dance of side-information and modes of combination.)[12]

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[11] We treat premises and conclusions as interchangeable with the information they carry; this sloppiness should cause no trouble.

[12] This is related to a strange phenomenon in lattice-theoretic treatments of consequence. Suppose $\circ$ and $\bullet$ are the meet and join, respectively, of some lattice, and that 'can be extracted from' is the lattice order. (In this case, full contraction will be in force, since meets and joins are idempotent.) Even in this much stronger setting, cut is still not guaranteed; it preserves

An anonymous referee points out that the situation changes if $\circ$ is residuated; that is, if there is a binary operation $>$ on pieces of information such that $z \sqsubseteq x > y$ iff $z \circ x \sqsubseteq y$. Indeed, the referee claims that this suffices for cut. But if the referee means what we do by 'cut', this is incorrect; residuating $\circ$ is not enough for cut, but only a single-conclusion restriction of it, which we now give. Suppose that $x \sqsubseteq y$, and $z \circ y \sqsubseteq w$. Then (implicitly appealing to the commutativity of $\circ$, which we will not further comment on), $y \sqsubseteq z > w$. By simple transitivity, then, $x \sqsubseteq z > w$; and so $x \circ z \sqsubseteq w$. (We suspect, of course, that this weaker property is what the referee means by 'cut'.) But there remain counterexamples to full cut.

In fact, even assuming that $\circ$ is residuated *and* that $\bullet$ is coresiduated (that is, that there is a binary connective $<$ on pieces of information such that $z < y \sqsubseteq x$ iff $z \sqsubseteq y + x$) *and* that both $\circ$ and $\bullet$ support weakening (that is, that $x \circ y \sqsubseteq x$ and $x \sqsubseteq x \bullet y$) is *still* not enough for cut. To see this, consider the structure $\mathcal{A} = \langle A, \circ, >, \bullet, <, \sqsubseteq \rangle$ determined as follows: $A = \{0, 1, 2\}$, $x \sqsubseteq y$ iff $x \geq y$, and operations given by the tables:

| $\circ$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 2 | 2 |
| 1 | 2 | 2 | 2 |
| 2 | 2 | 2 | 2 |

| $>$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 |

| $\bullet$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 |
| 2 | 0 | 1 | 2 |

| $<$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 2 | 1 | 0 |
| 1 | 2 | 2 | 1 |
| 2 | 2 | 2 | 2 |

It is immediate that $\sqsubseteq$ is a partial order, and it can be verified from the tables that $\circ$ and $\bullet$ are commutative, associative, and residuated and coresiduated by $>$ and $<$, respectively. Moreover, both $\circ$ and $\bullet$ support weakening. But we still have a counterexample to cut: $0 \sqsubseteq 1 \bullet 1 = 0$, and $2 = 0 \circ 1 \sqsubseteq 2$, but $0 = 0 \circ 0 \not\sqsubseteq 2 \bullet 1 = 1$. (This counterexample was found by Mace4; see [16].)

In the absence of lattice properties, then, even the combination of residuation, coresiduation, and weakening is not enough for full cut.[13] Of course, just as residuation on its own suffices for the single-conclusion restriction of cut considered above, so too does coresiduation on its own suffice for a single-premise restriction. There are any number of other properties related to cut that we do not consider here; suffice it to say the situation is rich.

We close this discussion by noting a set of conditions in the area that *is* sufficient for cut. Let $\circ$ be *inner residuated* iff there is a binary operation $>$ on pieces of information such that $w \circ x \sqsubseteq y \bullet z$ iff $w \sqsubseteq (x > y) \bullet z$. This is 'residuation with side conclusions'. If there is some piece $0_\bullet$ of information with $x \bullet 0_\bullet = x$ for all $x$, then inner residuation implies residuation; but if not, it need not. (Define *inner coresiduation* in the obvious 'side-premisey' way.)

extractability (if and) *only if* the lattice in question is distributive. (See [30] for proof.) Cut thus contrasts notably with weaker transitivity properties (which tend to come automatically) in a range of algebraic settings.

[13] A lattice whose meet is residuated must be distributive, and so cut would indeed follow from residuation alone in combination with lattice properties.

Now, if $\circ$ is residuated and $\bullet$ is coresiduated, *and* either $\circ$ is inner residuated or $\bullet$ is inner coresiduated, *then* $\sqsubseteq$ supports cut.[14] Take the case where $\circ$ is inner residuated, and suppose that $a \sqsubseteq b \bullet c$ and $d \circ b \sqsubseteq e$. Then we can reason as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{a \sqsubseteq b \bullet c}{a < c \sqsubseteq b}\ \text{Coresiduation} \qquad \dfrac{d \circ b \sqsubseteq e}{b \sqsubseteq d > e}\ \text{Residuation}}{a < c \sqsubseteq d > e}\ \text{Transitivity}}{a \sqsubseteq (d > e) \bullet c}\ \text{Coresiduation}}{d \circ a \sqsubseteq e \bullet c}\ \text{Inner residuation}$$

In the case where $\bullet$ is inner coresiduated instead, we can switch the order of the last two steps in the above reasoning (mutatis mutandis), reaching the same result.

Many discussions of substructural logics take place in settings where residuation of $\circ$ is taken for granted but $\bullet$ is entirely absent (e.g. [26, 9]). Others take place in settings where $\bullet$ is present but not coresiduated (or inner coresiduated), even though $\circ$ is residuated and inner residuated (e.g. [18]). In settings like these, full cut cannot be taken for granted; it is a substantive additional constraint on the interaction between premise-combination, conclusion-combination, and consequence itself, and it does not follow from the transitivity of the order in question, even in the presence of implication-like operations. Care is definitely called for.

## 3.4 Objection and response

A natural objection to this information-extraction theory of consequence proceeds as follows:

> That's all well and good as a theory of information extraction (whatever that is), but what we wanted was a theory of *valid argumentation*, of when conclusions *follow from* premises. And this simply *must* obey contraction, for the following reasons.
>
> When an argument proceeds from premises, it is perfectly legitimate for it to make multiple appeals to the same premise. If the premises are known, they do not become unknown after we appeal to them. If they are merely supposed, we are free to suppose them again. And so on: whatever status our premises have is one that is *easy to duplicate*. So reasoning from $\Gamma, A, A$ is nothing more (and nothing less) than reasoning from $\Gamma, A$. This gives us WL.
>
> Similarly, when an argument narrows things down to a range of conclusions, multiplicity cannot matter. If particular premises entail $A, A, \Delta$, then we know (given those premises) where the truth must lie: with $A$ or with something in $\Delta$. Listing the first of these

[14] All these conditions are needed, as Mace4 will quickly confirm. Adding weakening does not affect the situation.

possibilities twice does not change the situation: it is still just one possibility. So validly concluding $A, A, \Delta$ is nothing more (and nothing less) than validly concluding $A, \Delta$. This gives us WR.

We do not here endorse this objection. It can certainly be resisted, but our present goal is not to resist it either. Instead, we want to show just how far an advocate of the information-extraction conception we've outlined of noncontractive consequence can go towards meeting the demands of the objection. (If the demands are not in the end legitimate, then things are all the better for such an advocate.)

Suppose that, as the argument has it, valid argumentation must obey contraction, because there is no difference for argumentative purposes between having a premise once and having it multiple times, or between having a conclusion once and having it multiple times. Still, we think, valid argumentation might *reduce* to information extraction in a particularly simple way.

What we have in mind is this: we can continue to use information extraction as the underlying core of the consequence relation, but simply choose to ignore the multiplicity of our premises and conclusions. Recall that the *underlying set* (sometimes called the 'root set') of a multiset $\Gamma$ is the set whose members are exactly those things that occur at least once in $\Gamma$. Then let an argument $\Gamma : \Delta$ count as valid (in the objector's sense) iff there are multisets $\Gamma', \Delta'$ with the same underlying sets as $\Gamma, \Delta$ respectively such that $\bullet(\Delta')$ can be extracted from $\circ(\Gamma')$.

This gives the objector exactly what they want: having a premise even once in $\Gamma$ counts the same as having it arbitrarily many times, and having a conclusion even once in $\Delta$ counts the same as having it arbitrarily many times. But it allows us to see the objector's notion of validity, important as it may be, as holding (when it does) only in virtue of the *prior* facts about information extraction. The objector's notion of validity, on this suggestion, is just what you get when you take information extraction and 'forget' about multiplicities. Seen from this angle, the objection is not much of an objection at all. The objector simply wants to focus on one aspect of information extraction, and they are free to do so.[15]

Although we have presented the situation by focusing on [15]'s notion of information extraction, we reckon this is a valuable and underexplored avenue for noncontractivists in general. Whatever the notion of noncontractive consequence in play, it has a contractive relative that can be found in just this way.[16]

## 3.5 Contractions

Here, we turn to the formal situation. We define the *contraction* of a consequence relation. While MAALT will return as our running example in §4, the notion of

[15] Note that this response differs from the one considered in [15]; their notion of 'external consequence', which plays a similar role, is quite different.

[16] For example, on the instability understanding of noncontractive consequence sketched in [36], the 'contractive relative' will be some variety of stable entailment.

a contraction should be of wider use for answering the objection of §3.4, along the lines we suggested there. As such, we here give the general idea.

Recall that a multiset $\Gamma$ is a *submultiset* of a multiset $\Gamma'$ iff they have the same underlying set, and everything that occurs in $\Gamma$ occurs at least as many times in $\Gamma'$. (Example: $[A, B, B, C]$ is a submultiset of $[A, A, B, B, C, C]$, but not of $[A, B, C]$.)

**Definition 1.** A multiset $\Gamma$ is a *contraction* of a multiset $\Gamma'$ iff: $\Gamma$ and $\Gamma'$ have the same underlying set and $\Gamma$ is a submultiset of $\Gamma'$. We also say $\Gamma'$ is an *expansion* of $\Gamma$.

A sequent $\Gamma : \Delta$ is a *contraction* of a sequent $\Gamma' : \Delta'$ iff: $\Gamma$ is a contraction of $\Gamma'$ and $\Delta$ is a contraction of $\Delta'$. We also say $\Gamma' : \Delta'$ is an *expansion* of $\Gamma : \Delta$.

Note that $\Gamma : \Delta$ is a contraction of $\Gamma' : \Delta'$ iff it can be reached from $\Gamma' : \Delta'$ by some number of applications of the rules WL and WR. We write $\widehat{\Gamma}, \widehat{\Gamma}_1, \widehat{\Gamma}_2$ for (possibly distinct) expansions of a multiset $\Gamma$.

**Definition 2.** For a set $C$ of sequents, its *contraction* $\mathrm{W}C$ is the set of contractions of sequents in $C$.

It happens that $\mathrm{W}$, so defined, is a closure operation; for all $C, C'$:

- $C \subseteq \mathrm{W}C$,
- If $C \subseteq C'$, then $\mathrm{W}C \subseteq \mathrm{W}C'$, and
- $\mathrm{WW}C = \mathrm{W}C$.

Moreover, a set of sequents is $\mathrm{W}$-closed iff it is closed under contraction; so $\mathrm{W}C$ is the closure of $C$ under contraction.

We are concerned with the case where $C$ is the set of sequents valid in some logical system. If the system is contractive, then $C$ is $\mathrm{W}$-closed already—$\mathrm{W}C = C$—and the idea is not of much use. So we are particularly interested in the case where $C$ is the set of sequents valid in some *noncontractive* logical system. In such a case, $C$ can be what you like, and $\mathrm{W}C$ then emerges as the contractive relative of $C$ demanded by the objector of §3.4.

An anonymous referee objects here, saying that the objector should demand that: '*at any stage* of a [derivation], reasoning from assumptions $\Gamma, A, A$ is nothing more than reasoning from $\Gamma, A$, and *at any stage* of a [derivation], validly concluding $A, A, \Delta$ is nothing more (and nothing less) than concluding $A, \Delta$'. In other words, it is not sufficient to contract the *output* of a calculus; we must instead add the rule of contraction to the calculus itself. (As will emerge presently, these produce different results.)

But this is to mistake the situation. We are supposing that a noncontractive system $C$ gives the *correct* story about some underlying phenomenon: in our example, information extraction. The objector demands a contractive story about *something else*, say reasoning from assumptions. In response we offer such a story built from noncontractive resources, via a bridge principle like the following: conclusions $\Delta$ follow validly from assumptions $\Gamma$ iff the information in some expansion of $\Delta$ can be extracted from the information in some expansion of $\Gamma$. (See [23] for a similar idea in a different setting.)

Contraction in the *course* of a derivation cannot be justified by such a bridge principle. We would first need to know that the other rules of the calculus, which we are assuming are correct for information extraction, are *also* correct for reasoning from assumptions. But this cannot be justified on the basis of the above discussion.

## 4 WMAALT

Here, we turn to MAALT, to characterize its contraction, WMAALT. To begin, we just give two examples, to give the flavour of the idea.

**Example 1.** The sequent $: p \sqcup \neg p, p \sqcup \neg p$ is MAALT-valid. Therefore, the sequent $: p \sqcup \neg p$ is WMAALT-valid, even though it is not MAALT-valid.

**Example 2.** The sequent $p \sqcap (q \sqcup r), p \sqcap (q \sqcup r) : (p \sqcap q) \sqcup (p \sqcap r)$ is MAALT-valid. Therefore, the sequent $p \sqcap (q \sqcup r) : (p \sqcap q) \sqcup (p \sqcap r)$ is WMAALT-valid, even though it is not MAALT-valid.

These examples show that while additive excluded middle and additive distribution are not valid in MAALT, they are not as badly off as, say, the sequent $p : q$ is—they at least have MAALT-valid expansions, where $p : q$ has none. WMAALT captures an interesting 'intermediate' status for sequents to have: it's not nothing to have this status, but it's still easier to have the status than it is to be fully MAALT-valid.

Moreover, this intermediate status is just what we have appealed to on behalf of the MAALT theorist in answering the objection considered in §3.4. WMAALT is the relation that holds between premises and conclusions when *enough copies* of those premises would allow us to extract the information in *enough copies* of those conclusions. Additive distribution is not MAALT-valid, but it does have this status. For example, if we know $p \sqcap (q \sqcup r)$, we may appeal to this knowledge (twice, as it happens) to conclude $(p \sqcap q) \sqcup (p \sqcap r)$. On the other hand, $p : q$ does not have this status. Even if we know $p$, we cannot appeal to this knowledge any number of times to conclude any number of repetitions of $q$.

What is WMAALT, then? Since it is the closure of MAALT (as a set of sequents!) under contraction, a natural first hypothesis is that it is the same as MASTT, which results from adding a rule of contraction to (cut-free) MAALT. This hypothesis receives some confirmation from noting that additive excluded middle and additive distribution are both WMAALT-valid. It receives still further confirmation from noting that $: \lambda$ and $\lambda :$ are both WMAALT-valid. (As we saw in §2.1, $: \lambda, \lambda$ and $\lambda, \lambda :$ are both MAALT-valid.) And of course, the empty sequent is not WMAALT-valid, since it is not MAALT-valid. So WMAALT handles the liar paradox in just the same way as MASTT does: by allowing cut to fail.

## 4.1 WMAALT is not MASTT

These appearances are not (very) deceiving; WMAALT and MASTT are indeed related. But they are *not* identical. For example, consider the following MASTT derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p : p}\,\text{Id}}{: p, \neg p}\,\neg\text{R}}{: p, p \sqcup \neg p}\,\sqcup\text{R}}{: p \sqcup \neg p, p \sqcup \neg p}\,\sqcup\text{R}}{: p \sqcup \neg p}\,\text{WR} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p : p}\,\text{Id}}{: p, \neg p}\,\neg\text{R}}{: p, p \sqcup \neg p}\,\sqcup\text{R}}{: p \sqcup \neg p, p \sqcup \neg p}\,\sqcup\text{R}}{: p \sqcup \neg p}\,\text{WR}}{: (p \sqcup \neg p) \otimes (p \sqcup \neg p)}\,\otimes\text{R}$$

The sequent $: (p \sqcup \neg p) \otimes (p \sqcup \neg p)$, then, is MASTT-valid. But it is not WMAALT-valid.

**Fact 1.** *No expansion of* $: (p \sqcup \neg p) \otimes (p \sqcup \neg p)$ *is derivable in* MAALT.

*Proof.* Abbreviate $p \sqcup \neg p$ as $D$. Any expansion of the sequent has the form $: (D \otimes D)^n$, for some $n$ (exponents here indicate multiple occurrences). Note that if there is a MAALT derivation of $: (D \otimes D)^n$, for some $n$, then there is a MAALT derivation of $: (D \otimes D)^n, D$ as well; and there is likewise a derivation of $: (D \otimes D)^n, p$; and one of $: (D \otimes D)^n, \neg p$; these all follow by K. We show by induction on $n$, however, that for no $n$ is there a MAALT derivation of any of the following sequents:

$$(i_n) \quad : (D \otimes D)^n, p \quad (ii_n) \quad : (D \otimes D)^n, \neg p \quad (iii_n) \quad : (D \otimes D)^n, D.$$

The target claim follows immediately. In fact, it follows from any one of the nonderivability claims; but the induction depends on their conjunction. (For the remainder of this proof, 'derivation' means MAALT derivation.)

Note that the only rules in MAALT are operational rules, weakening, and $T\langle\,\rangle$ rules (recall that cut is admissible in MAALT, but is not amongst its structural rules). These facts make an exhaustive analysis of possible proofs feasible: any given sequent is only of the right form to be the conclusion of so many possible rule applications.

To demonstrate the above underivability claim we proceed by induction on $n$. For the base case ($n = 0$), there is no derivation of $: D$, or of $: p$, or of $: \neg p$, as is quick to verify.

Now, suppose there is no derivation of $(i_m)$, $(ii_m)$, or $(iii_m)$, for any $m < n$, with an eye to showing that there is no derivation of $(i_n)$, $(ii_n)$, or $(iii_n)$.

First, consider the case of $(i_n)$. How could $: (D \otimes D)^n, p$ potentially be derived? It could come:

1. by weakening from $: (D \otimes D)^{n-1}, p$, or
2. by $\otimes$R from $: (D \otimes D)^j, D, p$ and $: (D \otimes D)^k, D$, for $j + k = n - 1$ (and so $k < n$), or

3. by weakening from $: (D \otimes D)^n$, which itself could have come:
    (a) by weakening from $: (D \otimes D)^{n-1}$, or
    (b) by $\otimes$R from $: (D \otimes D)^j, D$ and $: (D \otimes D)^k, D$, for $j + k = n - 1$ (and so $j, k < n$).

But all of these possibilities contradict the inductive hypothesis (case 1 contradicts $(i_{n-1})$, case 2 contradicts $(iii_k)$, case 3a contradicts $(i_{n-1})$ by a step of $K$, and 3b contradicts $(iii_j)$ and $(iii_k)$); so there is no derivation of $(i_n)$. By similar reasoning (complicated only slightly by the presence of $\neg$), there is no derivation of $(ii_n)$ either. It remains only to show that there is no derivation of $(iii_n)$.

How could $: (D \otimes D)^n, D$ potentially be derived? It could come:

1. by weakening from $: (D \otimes D)^{n-1}, D$, or
2. by weakening from $: (D \otimes D)^n$, or
3. by $\sqcup$R from $: (D \otimes D)^n, p$, or
4. by $\sqcup$R from $: (D \otimes D)^n, \neg p$, or
5. by $\otimes$R from $: (D \otimes D)^j, D, D$ and $: (D \otimes D)^k, D$, for $j + k = n - 1$ (and so $k < n$).

These are the only options. Options 1 and 5 contradict the inductive hypothesis. And in considering $(i_n)$ and $(ii_n)$, we already ruled out options 2–4 as well. So there is no derivation of $(iii_n)$. This completes the inductive step, and the result follows.

□

This shows that WMAALT is not MASTT, despite extending MAALT and being closed under contraction. Along the way, it also shows one *reason* that WMAALT is not MASTT: WMAALT is not closed under the rule $\otimes$R. After all, $: p \sqcup \neg p$ is certainly WMAALT-valid, but we just saw that $: (p \sqcup \neg p) \otimes (p \sqcup \neg p)$ is not.

Easy variations on the above will show that WMAALT is not closed under $\oplus$L or $\twoheadrightarrow$L. It can also similarly be shown that WMAALT is not closed under $\forall$R or $\exists$L.[17]

Note that this also shows that, even in the absence of the rules for $T\langle\ \rangle$, WMAALT is not closed under cut. If it were then, given that both $: p \sqcup \neg p$ and $p \sqcup \neg p : (p \sqcup \neg p) \otimes (p \sqcup \neg p)$ are WMAALT-valid (in the latter case because $p \sqcup \neg p, p \sqcup \neg p : (p \sqcup \neg p) \otimes (p \sqcup \neg p)$ is MAALT-valid), a single application of cut would deliver us $: (p \sqcup \neg p) \otimes (p \sqcup \neg p)$, contradicting Fact 1.[18]

[17] For $\forall$R: the sequent $: Pa \sqcup \neg Pa$ is WMAALT-valid, but $: \forall x(Px \sqcup \neg Px)$ is not.
[18] Thanks to Francesco Paoli for both raising and resolving this question.

## 4.2 Characterizing WMAALT

In this section, we give a set of derivations that is sound and complete for WMAALT validity. We do this by focusing in on a particular kind of MASTT derivation. As we've seen, not just any MASTT derivation ends in a WMAALT-valid sequent. But there is a simple characterization of those that do; in this section, we will work our way to it.

**Definition 3.** A MASTT derivation is *Y-shaped* iff no application of a contraction rule (WL, WR) occurs in the derivation above any application of a rule other than a contraction rule. (A *Y-shaped derivation* is a Y-shaped MASTT derivation.)

Y-shaped derivations are thus the MASTT derivations with all their contractions at the bottom.

**Lemma 1.** *A sequent is* WMAALT*-valid iff it has a Y-shaped derivation.*

*Proof.* LTR: if a sequent $S$ is WMAALT-valid, then it is a contraction of some MAALT-derivable sequent $S'$. Consider a MAALT derivation of $S'$. Like all MAALT derivations, this is a MASTT derivation, and includes no applications of either contraction rule. Now extend the derivation with enough applications of WL and WR to reach $S$: the result is a Y-shaped derivation of $S$.

RTL: take a Y-shaped derivation of a sequent $S$, and chop it in half at the premise sequent $S'$ of the first application of a contraction rule in the derivation. (If the derivation has no applications of either contraction rule, let $S'$ be $S$—so the bottom 'half' is empty.) Above the chop, we are left with a MAALT derivation of $S'$; below the chop, a demonstration that $S'$ is an expansion of $S$. So $S$ has a MAALT-derivable expansion: it is WMAALT-valid. □

This already gives a characterization of WMAALT via derivations. But there is a much wider class of MASTT derivations we can use; the rest of this section is devoted to this liberalisation.

**Definition 4.** A MASTT derivation meets *the multiplicative restriction* iff: every application of a two-premise multiplicative rule ($\otimes$R, $\oplus$L, $\twoheadrightarrow$L) in the derivation has at least one of its premise sequents derived without any use of contraction.

**Definition 5.** A MASTT derivation meets *the eigenvariable restriction* iff: every application of an eigenvariable rule ($\forall$R, $\exists$L) in the derivation has its premise sequent derived without any use of contraction.

**Definition 6.** A *restricted* MASTT derivation is a MASTT derivation that meets both the multiplicative restriction and the eigenvariable restriction. (A *restricted derivation* is a restricted MASTT derivation.)

Now, the theorem:

**Theorem 1.** *A sequent is* WMAALT*-valid iff it has a restricted derivation.*

*Proof.* LTR: This is the 'completeness' direction, showing that there are enough restricted derivations. But every Y-shaped derivation *is* a restricted derivation, so Lemma 1 is all we need.

RTL: This is the 'soundness' direction, showing that every restricted derivation derives a WMAALT-valid sequent. In an inversion of the usual situation, this is where all the work is. We proceed by showing how to transform every restricted derivation of a sequent $S$ step by step into a MAALT derivation of *some expansion of* $S$.[19]

We proceed inductively on the restricted derivation. The base case is quick: suppose our restricted derivation ends like so:

$$\text{Id:}\quad \frac{}{A : A}$$

In this case, that application of Id is the entire derivation, which is already a MAALT derivation of $A : A$ (which is an expansion of itself); no transformation is needed.

For the inductive step, we need to consider each other rule of MASTT. Here, we consider only representative cases; each omitted case is similar to one of these, and we have indicated these similarities.

- For K: suppose a restricted derivation ending with:

  $$\text{K:}\quad \frac{\Gamma : \Delta}{\Gamma, \Gamma' : \Delta, \Delta'}$$

  By the inductive hypothesis, there is some MAALT derivation $\delta$ of $\widehat{\Gamma} : \widehat{\Delta}$, for some expansions $\widehat{\Gamma}, \widehat{\Delta}$ of $\Gamma, \Delta$ respectively. Add to $\delta$ as follows:

  $$\text{K:}\quad \frac{\widehat{\Gamma} : \widehat{\Delta}}{\widehat{\Gamma}, \Gamma' : \widehat{\Delta}, \Delta'}$$

  Here, $\widehat{\Gamma}, \Gamma' : \widehat{\Delta}, \Delta'$ is an expansion of $\Gamma, \Gamma' : \Delta, \Delta'$, so we are set.

- For WL (WR is similar): suppose a restricted derivation ending with:

  $$\text{WL:}\quad \frac{\Gamma, A, A : \Delta}{\Gamma, A : \Delta}$$

  By the inductive hypothesis, there is some MAALT derivation $\delta$ of $\widehat{\Gamma}, A^m, A^n : \widehat{\Delta}$ (with $m, n \geq 1$). But $\widehat{\Gamma}, A^m, A^n : \widehat{\Delta}$ is *already* an expansion of $\Gamma, A : \Delta$, so we are set; $\delta$ itself is our target derivation.

[19] One way to look at this transformation is as 'pushing contractions down' and off the root of the derivation. The general strategy of transforming one derivation into another, including by permuting inferences, is well-explored (see e.g. [13, 5]); the innovation here, if any, is allowing the endsequent to change in the course of these transformations. The notion of an *expansion* is what allows us to keep control of these changes.

- For ¬L (¬R, $T\langle\,\rangle$L, and $T\langle\,\rangle$R are similar): suppose a restricted derivation ending with:

$$\neg\text{L:}\quad \frac{\Gamma : A, \Delta}{\Gamma, \neg A : \Delta}$$

By the inductive hypothesis, there is some MAALT derivation $\delta$ of $\widehat{\Gamma} : A^n, \widehat{\Delta}$ (with $n \geq 1$). Add to $\delta$ as follows:

$$\neg\text{L}^n\text{:}\quad \frac{\widehat{\Gamma} : A^n, \widehat{\Delta}}{\widehat{\Gamma}, \neg A^n : \widehat{\Delta}}$$

- For ⊗L (⊕R and ↠R are similar): suppose a restricted derivation ending with:

$$\otimes\text{L:}\quad \frac{\Gamma, A, B : \Delta}{\Gamma, A \otimes B : \Delta}$$

By the inductive hypothesis, there is some MAALT derivation $\delta$ of $\widehat{\Gamma}, A^n, B^m : \widehat{\Delta}$ (with $m, n \geq 1$). Add to $\delta$ as follows:

$$\begin{array}{rc} \text{K:} & \dfrac{\widehat{\Gamma}, A^n, B^m : \widehat{\Delta}}{\widehat{\Gamma}, A^{\max(n,m)}, B^{\max(n,m)} : \widehat{\Delta}} \\ \otimes\text{L}^{\max(n,m)}\text{:} & \overline{\widehat{\Gamma}, A \otimes B^{\max(n,m)} : \widehat{\Delta}} \end{array}$$

- For ⊗R (⊕L and ↠L are similar): suppose a restricted derivation ending with:

$$\otimes\text{R:}\quad \frac{\Gamma : A, \Delta \qquad \Gamma' : B, \Delta'}{\Gamma, \Gamma' : A \otimes B, \Delta, \Delta'}$$

Since this is a *restricted* derivation, it can't be that both $\Gamma : A, \Delta$ and $\Gamma' : B, \Delta'$ used contraction in their derivations. One of them, then, already has a MAALT derivation $\delta_1$. Wlog, suppose it's $\Gamma' : B, \Delta'$. By the inductive hypothesis, there is some MAALT derivation $\delta_2$ of $\widehat{\Gamma} : A^n, \widehat{\Delta}$ (with $n \geq 1$).

Continue like so:

$$\begin{array}{l} \otimes\text{R:}\quad \dfrac{\overset{\delta_2}{\widehat{\Gamma} : A^n, \widehat{\Delta}} \qquad \overset{\delta_1}{\Gamma' : B, \Delta'}}{\widehat{\Gamma}, \Gamma' : A^{n-1}, A \otimes B, \widehat{\Delta}, \Delta'} \qquad \overset{\delta_1}{\Gamma' : B, \Delta'} \\ \otimes\text{R:}\quad \overline{\widehat{\Gamma}, \Gamma'^2 : A^{n-2}, A \otimes B^2, \widehat{\Delta}, \Delta'^2} \qquad \overset{\delta_1}{\Gamma' : B, \Delta'} \\ \otimes\text{R:}\quad \overline{\widehat{\Gamma}, \Gamma'^3 : A^{n-3}, A \otimes B^3, \widehat{\Delta}, \Delta'^3} \\ \qquad\qquad \vdots \end{array}$$

After $i$ applications of $\otimes$R (for $i \leq n$), this pattern gives a MAALT derivation of $\widehat{\Gamma}, \Gamma'^i : A^{n-i}, A \otimes B^i, \widehat{\Delta}, \Delta'^i$. Thus, by applying $\otimes$R $n$ times in this way, we get a MAALT derivation of $\widehat{\Gamma}, \Gamma'^n : A \otimes B^n, \widehat{\Delta}, \Delta'^n$. Since $n \geq 1$, this is an expansion of $\Gamma, \Gamma' : A \otimes B, \Delta, \Delta'$, so we are set.

- For $\sqcap$L ($\sqcup$R and $\mapsto$R are similar): suppose a restricted derivation ending with:

$$\sqcap\text{L:} \quad \frac{\Gamma, (A), (B) : \Delta}{\Gamma, A \sqcap B : \Delta}$$

By the inductive hypothesis, there is some MAALT derivation $\delta$ of $\widehat{\Gamma}, (A^n), (B^m) : \widehat{\Delta}$. Note that because only one of $A, B$ appears outside $\Gamma$ and $\Delta$ in $\Gamma, (A), (B) : \Delta$, only one of them appears outside $\widehat{\Gamma}$ and $\widehat{\Delta}$ in $\widehat{\Gamma}, (A^n), (B^m) : \widehat{\Delta}$. Wlog, suppose it's $A$. So we have a MAALT derivation $\delta$ of $\widehat{\Gamma}, A^n : \widehat{\Delta}$. Add to $\delta$ as follows:

$$\sqcap\text{L}^n\text{:} \quad \frac{\widehat{\Gamma}, A^n : \widehat{\Delta}}{\widehat{\Gamma}, A \sqcap B^n : \widehat{\Delta}}$$

- For $\sqcap$R ($\sqcup$L and $\mapsto$L are similar): this is the tricky one. Suppose a restricted derivation ending with:

$$\sqcap\text{R:} \quad \frac{\Gamma : A, \Delta \qquad \Gamma : B, \Delta}{\Gamma : A \sqcap B, \Delta}$$

By the inductive hypothesis, there is a MAALT derivation $\delta_1$ of $\widehat{\Gamma}_1 : A^m, \widehat{\Delta}_1$, and a MAALT derivation $\delta_2$ of $\widehat{\Gamma}_2 : B^n, \widehat{\Delta}_2$ (with $m, n \geq 1$). Let $\widehat{\Gamma}$ be the least common expansion of $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_2$, and similarly for $\widehat{\Delta}$. Extend $\delta_1$ and $\delta_2$ with an application of K each to become MAALT derivations of $\widehat{\Gamma} : A^n, \widehat{\Delta}$ and $\widehat{\Gamma} : B^m, \widehat{\Delta}$, respectively. From here, let $\delta_1$ and $\delta_2$ be these extended derivations.[20]

We begin with two substeps; the point of these is to get one of $m, n$ down to 1. (If one of them already is 1, we can skip these substeps.)

  - First substep:
    Adding a further step of K to $\delta_1$ can bring us to $\widehat{\Gamma} : A^m, B^{n-1}, \widehat{\Delta}$. Now, suppose that $\widehat{\Gamma} : A \sqcap B^i, A^j, B^{n-1}, \widehat{\Delta}$ is MAALT derivable (note that we already have this for $i = 0, j = m$). Then we have the following:

[20] All that really matters is that $\widehat{\Gamma}$ be *a* common expansion of $\widehat{\Gamma}_1$ and $\widehat{\Gamma}_2$, and similarly for $\widehat{\Delta}$; leastness isn't really important. But: waste not, want not.

$$
\sqcap\mathrm{R}:\ \dfrac{\widehat{\Gamma} : A \sqcap B^i, A^j, B^{n-1}, \widehat{\Delta} \qquad \mathrm{K}:\ \dfrac{\overset{\delta_2}{\widehat{\Gamma} : B^n, \widehat{\Delta}}}{\widehat{\Gamma} : A \sqcap B^i, A^{j-1}, B^n, \widehat{\Delta}}}{\widehat{\Gamma} : A \sqcap B^{i+1}, A^{j-1}, B^{n-1}, \widehat{\Delta}}
$$

  By putting this strategy's output back as an input to the same strategy and repeating $j$ times, we reach $\widehat{\Gamma} : A \sqcap B^{i+j}, B^{n-1}, \widehat{\Delta}$. Starting as we do from $i = 0, j = m$, in this way we build a MAALT derivation of $\widehat{\Gamma} : A \sqcap B^m, B^{n-1}, \widehat{\Delta}$. By a parallel method, we build a MAALT derivation of $\widehat{\Gamma} : A \sqcap B^n, A^{m-1}, \widehat{\Delta}$.

  - Second substep:
    Now, let $\Theta_1$ be $\widehat{\Delta}, A \sqcap B^{\max(m,n)}$. With appropriate weakenings (in one case; we already have the other), we can use the first substep to derive both $\widehat{\Gamma} : A^{m-1}, \Theta_1$ and $\widehat{\Gamma} : B^{n-1}, \Theta_1$, still in MAALT.

    Here's what's happened: from the sequents $\widehat{\Gamma} : A^m, \widehat{\Delta}$ and $\widehat{\Gamma} : B^n, \widehat{\Delta}$, we've derived sequents with one fewer occurrence of $A$ (in one case) and $B$ (in the other), by adding $\max(m, n)$ occurrences of $A \sqcap B$ to the side conclusions. Moreover, we can repeat this as much as we like; just take these new results and do to them what we did to these sequents, and we'll get down to $m - 2$, $n - 2$, and so on. We'll add a whole mess of $A \sqcap B$s to the side conclusions as we go (determining $\Theta_2 = \Theta_1, A \sqcap B^{\max(m,n)-1}$, then $\Theta_3 = \Theta_2, A \sqcap B^{\max(m,n)-2}$, and so on), but that's all we add.

    Repeat until we have a MAALT derivation $\delta_\star$ of one of $\widehat{\Gamma} : A, \Theta_x$ or $\widehat{\Gamma} : B, \Theta_x$. (This will take $\min(m, n) - 1$ rounds, so $x = \min(m, n) - 1$.)

Wlog, assume $\delta_\star$ is a MAALT derivation of $\widehat{\Gamma} : B, \Theta_x$. Now, finish as follows:

$$
\sqcap\mathrm{R}:\ \dfrac{\sqcap\mathrm{R}:\ \dfrac{\mathrm{K}:\ \dfrac{\overset{\delta_1}{\widehat{\Gamma} : A^n, \widehat{\Delta}}}{\widehat{\Gamma} : A^n, \Theta_x} \qquad \mathrm{K}:\ \dfrac{\overset{\delta_\star}{\widehat{\Gamma} : B, \Theta_x}}{\widehat{\Gamma} : B, A^{n-1}, \Theta_x}}{\widehat{\Gamma} : A \sqcap B, A^{n-1}, \Theta_x} \qquad \mathrm{K}:\ \dfrac{\overset{\delta_\star}{\widehat{\Gamma} : B, \Theta_x}}{\widehat{\Gamma} : B, A^{n-2}, \Theta_x}}{\widehat{\Gamma} : A \sqcap B^2, A^{n-2}, \Theta_x}
$$

$$
\vdots
$$

$$
\widehat{\Gamma} : A \sqcap B^n, \Theta_x
$$

Recall that $\Theta_x$ is just a bunch of copies of $A \sqcap B$ together with $\widehat{\Delta}$. So this is indeed an expansion of $\Gamma : A \sqcap B, \Delta$, derived in MAALT.

- For $\forall$L ($\exists$R is similar): suppose a restricted derivation ending with:

$$
\forall\mathrm{L}:\ \dfrac{\Gamma, A[x/t] : \Delta}{\Gamma, \forall x A : \Delta}
$$

By the inductive hypothesis, there is some MAALT derivation $\delta$ of $\widehat{\Gamma}, A[x/t]^n : \widehat{\Delta}$. Add to $\delta$ as follows:

$$\forall\text{L}^n: \quad \frac{\widehat{\Gamma}, A[x/t]^n : \widehat{\Delta}}{\widehat{\Gamma}, \forall x A^n : \widehat{\Delta}}$$

- For $\forall$R ($\exists$L is similar): suppose a restricted derivation ending with:

$$\forall\text{R}: \quad \frac{\Gamma : A[x/a], \Delta}{\Gamma : \forall x A, \Delta}$$

Since this is a *restricted* derivation, $\Gamma : A[x/a], \Delta$ has been derived without the use of contraction. The restricted derivation, then, is already a MAALT derivation of $\Gamma : \forall x A, \Delta$, which is an expansion of itself, so we are set.

□

## 5 Conclusion

We close with a series of comments on Theorem 1 and its proof.

Since WMAALT is not closed under $\otimes$R ($\oplus$L, $\twoheadrightarrow$L) or $\forall$R ($\exists$L), we cannot completely remove the restrictions on these rules; the resulting system would be unsound. However, we *could* loosen the restrictions a bit further without breaking the above theorem. The restrictions we gave focused on the entire derivation above the tricky rules, but in the proof of the theorem, the side formulas of these rules turn out not to matter; it is only the principal formulas that must be prevented from multiplying. This would allow us to loosen the restrictions slightly, at the cost of making them more complex to state.

Taking advantage of this loosening on the restriction points the way towards a *local* sequent calculus which directly determines WMAALT.[21] This sequent calculus makes use of 'signed formulas' where, approximately speaking, the sign of a formula occurrence registers whether any of the ancestors of that occurrence have been contracted. More formally, a a *signed formula* is an ordered pair of the form $(A, \alpha)$ where $A$ is a WMAALT-formula and $\alpha$ is either 0 or 1.[22] The first coordinate of a signed formula is called its *formula* and the second its *sign*. We take signs to be ordered such that $0 < 1$. $X, Y, \ldots$ are finite, possibly empty, multisets of signed formulas. $(\Gamma, \overline{\alpha})$ is short for the multiset $[(A_1, \alpha), \ldots, (A_n, \alpha)]$ if $\Gamma = [A_1, \ldots, A_n]$. Given a signed formula $S = (A, \alpha)$ let $Form(S) = A$ and

[21] An earlier version of this paper left the existence of a local sequent calculus as an open question. We would like to thank an anonymous referee for suggesting the following calculus.

[22] For the remainder of this discussion we will drop the convention adopted in Section 2.3 regarding how to interpret a formula appearing in parentheses in the statement of a rule so as to avoid both confusion and an unsightly proliferation of parentheses.

$Sign(S) = \alpha$, with $Form(X)$ and $Sign(X)$ defined as expected when $X$ is a multiset of signed formulas.

The system WMAALT$^{\pm}$is defined in Figure 5, where it is given a 'sparse formulation' (c.f. §2.4). Let us write $\vdash_{\text{WMAALT}^{\pm}} \Gamma : \Delta$ iff there exist multisets of signed formulas $X$ and $Y$ s.t. $Form(X) = \Gamma$, $Form(Y) = \Delta$ and $X : Y$ is derivable in WMAALT$^{\pm}$—i.e. if there is some way of giving signs to all the formulas in $\Gamma$ and $\Delta$ so that the resulting sequent of signed formulas is derivable in WMAALT$^{\pm}$. When this is the case we will say that the sequent $\Gamma : \Delta$ is WMAALT$^{\pm}$-derivable. Let us, briefly, sketch an argument demonstrating that WMAALT consists of precisely those sequents which are WMAALT$^{\pm}$-derivable.

- **If** $\vdash_{\text{WMAALT}} \Gamma : \Delta$ **then** $\vdash_{\text{WMAALT}^{\pm}} \Gamma : \Delta$ To demonstrate this all we need to note is that every sequent which has a WMAALT-derivation has a *restricted* derivation (Theorem 1), and that we can always assign signs to formula occurrences in a restricted derivation to transform it into a signed derivation.

- **If** $\vdash_{\text{WMAALT}^{\pm}} \Gamma : \Delta$ **then** $\vdash_{\text{WMAALT}} \Gamma : \Delta$ This direction is, perhaps unsurprisingly, where most of the work is. The main notion which is required here is that of what we might call a *principally-restricted derivation*, which (approximately speaking) is a MASTT derivation in which one of the principal formula occurrences in every instance $\otimes$R and $\forall$R (and $\oplus$L, $\twoheadrightarrow$L and $\exists$L in any calculus with those connectives primitive) has been derived without the sub-formula occurrences required to derive it having been contracted. This is not a precise definition of the target class of derivations, but we trust that the reader can see how this should be made precise. In particular it is worth noting that all this requires is that there be some chain of formula occurrences in a proof which are not the result of an application of contraction, even if there are other occurrences of the same formula in the same position in the same sequent which have been contracted. For example, in the following MASTT derivation, while formula occurrences of $p$ and $q$ have been contracted in both premise sequents to $\otimes$R, there is still a chain of formula occurrences in the proof which are not the result of contraction (those listed in boldface), so the contracted formula occurrences do not need to be used to derive the $p \otimes q$ in the endsequent.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p : \mathbf{p}}\ \text{Id}}{p : \mathbf{p}, p, p}\ \text{K}}{p : \mathbf{p}, p}\ \text{WR} \qquad \dfrac{\dfrac{\dfrac{}{q : \mathbf{q}}\ \text{Id}}{q : \mathbf{q}, q, q}\ \text{K}}{q : \mathbf{q}, q}\ \text{WR}}{p, q : p \otimes q, p, q}\ \otimes\text{R}$$

Thus, the above proof is an example of one which is principally-restricted. It is easy to see that any sequent which has a WMAALT$^{\pm}$-derivation has a principally-restricted derivation (which is just the result of removing all signs from its WMAALT$^{\pm}$-derivation, this being a MAALT-derivation). We can then prove that any sequent which has a principally-restricted

$$\text{Id: } \frac{}{(A,1):(A,1)} \qquad \text{K: } \frac{X:Y}{(\Gamma,\overline{1}),X:Y,(\Delta,\overline{1})}$$

$$\text{WL: } \frac{(A,\alpha),(A,\beta),X:Y}{(A,0),X:Y} \qquad \text{WR: } \frac{X:Y,(A,\alpha),(A,\beta)}{X:Y,(A,0)}$$

$$\neg\text{L: } \frac{\Gamma:(A,\alpha),Y}{X,(\neg A,\alpha):Y} \qquad \neg\text{R: } \frac{X,(A,\alpha):Y}{X:(\neg A,\alpha),Y}$$

$$\otimes\text{L: } \frac{X,(A,\alpha),(B,\beta):Y}{X,(A\otimes B,\min(\alpha,\beta)):Y}$$

$$\otimes\text{R1: } \frac{X:(A,1),Y \qquad X':(B,\alpha),Y'}{X,X':(A\otimes B,\alpha),Y,Y'}$$

$$\otimes\text{R2: } \frac{X:(A,\alpha),Y \qquad X':(B,1),Y'}{X,X':(A\otimes B,\alpha),Y,Y'}$$

$$\sqcap\text{L: } \frac{X,(A,\alpha):Y}{X,(A\sqcap B,\alpha):Y} \qquad \sqcap\text{L: } \frac{X,(B,\alpha):Y}{X,(A\sqcap B,\alpha):Y}$$

$$\sqcap\text{R: } \frac{X:(A,\alpha),Y \qquad X:(B,\beta),Y}{X:(A\sqcap B,\min(\alpha,\beta)),Y}$$

$$\forall\text{L: } \frac{X,(A[x/t],\alpha):Y}{X,(\forall xA,\alpha):Y} \qquad \forall\text{R: } \frac{X:(A[x/a],1),Y}{X:(\forall xA,1),Y}$$

$$T\langle\,\rangle\text{L: } \frac{X,(A,\alpha):Y}{X,(T\langle A\rangle,\alpha):Y} \qquad T\langle\,\rangle\text{R: } \frac{X:(A,\alpha),Y}{X:(T\langle A\rangle,\alpha),Y}$$

**Figure 2:** A 'sparse' formulation calculus of signed formulas WMAALT$^{\pm}$

derivation is WMAALT-valid by the same proof as was used for Theorem 1, mutatis mutandis.

So, from the argument sketched above, we can see that a sequent is WMAALT-valid iff it is WMAALT$^{\pm}$-derivable. This signed calculus has the advantage of being local—whether we can apply a rule just depends on the form of the premise sequents—but the disadvantage of working with signed formulas.

It is worth noting the vital role that *weakening* plays in the proof of Theorem 1; it is needed for both the one-premise multiplicative rules and the two-premise additive rules. Our techniques, then, will need modification if they are to be able to treat logics without weakening. But such logics are important contenders in this area. (Indeed, [15], our main example, defends a logic without weakening, and most of the logics explored in [25] also lack weakening.) Philosophically, our suggested response to the objection in §3.4 should adapt to these logics without difficulty. But our logical tools may not adapt so cleanly. More work is needed to describe the contractions of systems without weakening.[23]

We can also use Theorem 1 to give a tight connection between *fragments* of our two systems. Let the *restricted language* be the language without quantifiers or multiplicative connectives, and with names only for formulas in the restricted language. Note that if a sequent built from the restricted language is derivable (in either MAALT or MASTT), then it has a derivation entirely within the restricted language. (The only way a formula can disappear moving down the tree is via an application of a $T\langle\ \rangle$ rule, but even here it leaves a trace: a name for itself.) Within the restricted language, though, *every* MASTT derivation is a restricted derivation. So within the restricted language (or any sublanguage of it), our two systems exactly coincide.

In sum, we have here outlined a potential understanding of noncontractive consequence, inspired by [15], offered a strategy for an advocate of such an understanding to answer an expected objection, and explored the technical situation surrounding our response for the case of MAALT. This response creates a bridge between noncontractive and nontransitive theories of consequence and truth, highlighting the fertile interplay between these two approaches.[24]

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[24] For discussion of this material, thanks to the Melbourne Logic Group, the UConn Logic Group, and the audience at the MCMP workshop 'Truth and Paradox'. Thanks as well to comments from two anonymous referees. This research was partially supported by the Government of Spain via the grant *Non-Transitive Logics*, number FFI2013-46451-P.

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