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Euler, Newton, and Foundations for Mechanics

Marius Stan

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Abstract and Keywords

This chapter looks at Euler's relation to Newton, and at his role in the rise of Newtonian mechanics. It aims to give a sense of Newton's complicated legacy for Enlightenment science and to point out that key "Newtonian" results are really due to Euler. The chapter begins with a historiographical overview of Euler's complicated relation to Newton. Then it presents an instance of Euler extending broadly Newtonian notions to a field he created: rigid-body dynamics. Finally, it outlines three open questions about Euler's mechanical foundations and their proximity to Newtonianism: Is his mechanics a monolithic account? What theory of matter is compatible with his dynamical laws? What were his dynamical laws, really?

Keywords: Newton, Euler, Newtonian, Enlightenment science, rigid-body dynamics, classical mechanics, Second Law, matter theory

Though the *Principia* was an immense breakthrough, it was also an unfinished work in many ways. Newton in Book III had outlined a number of programs that were left for posterity to perfect, correct, and complete. And, after his treatise reached the shores of Europe, it was not clear to anyone how its concepts and laws might apply beyond gravity, which Newton had treated with great success. In fact, no one was sure *that* they extend to all mechanical phenomena. Moreover, as the eighteenth century turned, the *Principia* was not its sole agenda for mechanics. Two other sets of tools and problems had been handed down to it. One came from Huygens, a genius inspired by Torricelli and Mersenne. Another came from Leibniz and his followers, Jakob Hermann and the elder Bernoullis.

Thus, on the eve of the Enlightenment, it was unclear how or whether Newton's gravitation theory might be extended to all mechanical setups. It took the work of Leonhard Euler to show that. However, this was far from inexorable or smooth. Euler was not Newton’s self-declared disciple—Euler’s mentor, Johann Bernoulli, was a vehement Leibnizian. More notably, some cultural and historical distance separates him from Newton; hence the Briton’s priorities and methods are not those of the Swiss Euler. Moreover, Euler was uniquely at ease in all three traditions of mechanics that come to flourish in the eighteenth century. So, his relation to Newton and the Newtonian version of dynamics is quite complicated. This chapter lays the groundwork for a more accurate image of that relation, and thus a better appreciation of Newton’s and Euler’s respective merits. I have two aims: to illuminate and contextualize (to some extent) Newton’s legacy for mechanics in the Age of Reason; and to remind readers that many supposedly Newtonian elements in modern mechanics really come from Euler.

I begin with a historian’s outline of Euler’s stance relative to Newton. Then I illustrate how Euler extended Newtonian ideas to a field he largely created: rigid-body dynamics. I end with some open questions about his mechanical foundations.

Euler’s relation to Newton

R.W. Home once questioned the wisdom of using seventeenth-century labels —“Newtonian,” “Cartesian,” and the like—for the views of Enlightenment figures. Desirably, they should be explanatory: given a thinker, the label ought to let us anticipate and account for the contour of a particular domain (e.g., optics) in her natural philosophy, based on the broad shape of another domain (e.g., dynamics). But, in this capacity, our labels fail miserably, he noted (Home 1977). For example, even in post-*Principia* Britain, most Newtonians about gravitation espoused Cartesian doctrines of magnetism. Then what good are such labels, if they cannot predict and explain? Do we understand historical figures better by inflicting these markers on them? “The eighteenth century figure who has perhaps been most ill-used by our habit of classifying people into ‘isms’ is Leonhard Euler,” he concluded with dismay (1979, 242).

While Home did not elaborate on his verdict just quoted, the facts bear it out. In hindsight we can see that, as a legacy for the Enlightenment, around 1730 Newtonian natural philosophy would denote a doctrinal package, or set of theses and views connected inferentially. In regard to dynamics, its key ingredients were the following: 1) Absolute space and time are indispensable mechanical foundations. 2) Gravity is a universal, direct action at a distance. 3) The Second Law is a general principle for mechanics. 4) The unit of matter is the rigid atom endowed with repulsive and attractive forces. 5) The proper mathematics for dynamics is geometric methods and models. 6) The concepts and laws of dynamics are a posteriori. 7) Light is an emission of special particles.

To see how inadequate this label is for Euler, consider: only (1) and (3) are part of his natural philosophy. Even so, it is not clear at all that he took (1) from Newton. Euler very likely rediscovered *independently* that absolute space is needed to anchor a concept of true motion (Stan 2012). Moreover, Euler did not embrace (3) out of a *previous, exclusive* conviction that Newton's Second Law is guaranteed to ground all of mechanics. Rather, it took him decades to learn that it is a general principle. And, during that time he found that there are other, *non-Newtonian* general laws, which makes his exclusive commitment to (3) questionable. Thus, in Euler's case, the label "Newtonian" is inadequate even as a group marker. Not only does it not explain, it does not track theoretical filiation or ideological parentage, either. Since asking about Euler's Newtonianism looks unprofitable, I examine instead his relation to Newton's achievement in mechanics. Three facets stand out in this respect: continuation, integration by reconceptualization, and generalization. Let us consider them in turn.

Euler contributed to solving a number of problems that Newton had first broached in the *Principia*. Among them was the motion of the lunar and planetary lines of apsides; the horizontal parallaxes of the moon and sun; solar orbiting around the planetary center of gravity; gravimetric variation with latitude; lunar motion; issues in perturbation theory (e.g., deviations by Jupiter, Saturn, and their satellites from exact Kepler orbits); the motion of the tides; comet trajectories; and the precession of the equinoxes.[1] Euler made very significant advances on all these fronts (Verdun 2015). But, he was not alone in working on them, though he was the most capable; these Newtonian programs engaged then many talented others, like Clairaut, d'Alembert and Lagrange.

As he reconceptualized mechanical problems and ideas into new research programs, Euler ended up integrating Newtonian lines of research with tracks for theory building started by other figures. For instance, his creation of rigid-body dynamics unifies lines of work in Newtonian gravitation theory—the precession of equinoxes; and Bradley's discovery of nutation, which sparked Euler's interest—and inquiries into constrained motion stemming from Huygens. Likewise, his work on hydrodynamics in the 1750s bring together his effort to extend Newton's Second Law beyond its original sphere of application with the keen interest in fluid motion among his Leibnizian mentors, like Jakob Hermann, Daniel and Johann Bernoulli. The latter in fact supplied Euler with just the right starting insight: Johann applied the Second Law to every mass element in a fluid, and he integrated over the volume. "You have given me now the greatest light in this matter, whereas previously I would approach it in a great fog, and was unable to determine it other than by the indirect method," confessed Euler.[2] In a similar vein, his work in elasticity integrates into a broadly Newtonian account previous research—on the vibrating string, the thin rod, the buckling of columns, and the elastic plate—based in distinct, largely non-Newtonian dynamics: virtual work approaches, energy methods, and least action principles (Truesdell 1960; Caparrini and Fraser 2013; Maltese 1992).

By far the most important aspect of his work in relation to Newton's is generalization. I mean the unification of local accounts into a *systematic* and *general* theory of forces and torques moving and deforming masses at subrelativistic speeds—broadly speaking, Newtonian mechanics as we know it. Here caution is needed, though. There are really two facets to Euler's generalizing program, formal and explanatory. Only the latter has anything to do with Newton. The former amounts in essence to Euler repudiating Newton's tenet (5) stated earlier. This formal facet regards the mathematics of dynamics. Already in the late 1720s, as he was drafting *Mechanica*, the young Euler had voiced frustration with the geometric takes on mechanical problems valued by Newton, the British Newtonians, and even Leibnizians like Jakob Hermann. Their approach, he complained, failed to yield a general method for finding solutions (Euler 1736, preface). As remedy, he translated all the results in one-particle dynamics then extant into the formalism of the *Leibnizian* calculus. His subsequent discovery of the concept of function made this early, fateful choice even more effective: it produced the insight that all mechanical processes are functional dependencies between variables (Maronne & Panza 2014). His systematic use of this formalism originated and typified magisterially the modern notion that the general task of mechanics is to reduce problems to sets of differential equations and integrate them. However, this is ultimately a vindication of Leibniz, not Newton. By the early 1700s, Leibniz had already extolled his calculus as *cogitatio caeca*, or a rule-driven quasi-mechanical (thus "blind") algorithm, which he contrasted favorably with Newton's geometric methods and models.

The second, explanatory facet regards Euler's choice of dynamical laws for a general theory. His practice in the long run was to rest mechanics on robustly Newtonian laws of impressed force and torque. Still, here, too, the matter is not simple or clear cut. Contrary to much current prejudice, Euler's choice was not predetermined. In the early Enlightenment, it was neither inexorable nor evident that Newton's *axiomata, sive leges motus* can ground a general mechanics, by entailing equations of motion and equilibrium for all systems, statical and dynamical.[3] Specifically, two very serious obstacles prevented Newton's laws from becoming the inevitable, universal basis for mechanics.

One was in kinematics. From Huygens and Newton to the late 1740s, the problems studied largely concerned plane motion. Theorists treated it in so-called natural coordinates: at every point on the trajectory, they would set up a two-frame naturally suggested by the tangent and the normal at that location. This approach had two limitations. First, it made the choice of frame context-dependent: skill, not method, was needed to find the frame most "natural" to the particular problem at hand. Second, it was inapplicable to extended bodies moving in three dimensions. This is because the "natural" frame must be reoriented at subsequent points on the trajectory—and so, for the equations of motion thus written to be integrable, relations between frames at different locations must be given. For 3D motion, this task in general exceeded the resources of Enlightenment mathematics. What they needed, we can see in retrospect, was the concepts of arc-length, curvature, torsion, and the Frenet-Serret equations to connect them. It took Euler to overcome these barriers, with two brilliant conceptual innovations. First, he revolutionized early modern kinematics, by making systematic use of the Cartesian three-frame external to the system, fixed in space. This yielded for him a *general*, context-*independent* way to describe a physical setup (cf. also Verdun 2015, § 2.3.1). And, as an added benefit, it gave him a dynamical insight: with the motion referred to an external frame, he could then see that force is that which always correlates with the second derivative of position, or the acceleration. Second, he extended the old field of spherical trigonometry to infinitesimal motions, so as to find out, before the advent of differential geometry, how to connect two frames—related by the "Euler angles"—rotating in respect to each other.[4] This allowed him to deal with the unique challenges of rigid-body motion, which is really change of attitude, not just "change of place," as motion was vaguely defined at the time. (In turn, the absence of a properly general kinematics would hobble Euler, too, just as it had Newton. Euler never developed a concept of strain and deformation. In consequence, his results in 3D elasticity were rather minor. This lacuna would not be filled until the 1820s, by Cauchy, Navier, Poisson, and Saint-Venant.)

The second obstacle was in dynamics. Throughout the 1700s, mechanical principles proliferated, and Newton's laws were just one choice among many. Before 1750, other candidates were Conservation of Vis Viva, Torricelli's Principle, d'Alembert's Principle, the Law of the Lever, the Principle of Virtual Work, and the Principle of Least Action (Lagrange 1811; Nakata 2002).[5] Now, all these laws had a common shortcoming: their obvious application was limited to the motion of a centroid, (i.e., a *single* "representative point") in an extended body or mechanical system.[6] Euler was aware of this flaw:

> These principles are of no use in the study of motion, unless the bodies are infinitesimally small, hence *the size of a point*—or at least we can *regard* them as such without much error: which happens when the direction of the soliciting power *passes through the center of gravity*.… But if the direction does not pass through that center, we *cannot* determine the entire effect of these powers. That is all the more so when the body to be moved is *not free*, or is constrained by some obstacle, depending on its structure.
>
> (Euler 1745, §17; my emphasis)

In other words, those laws fail to predict the motion of mass elements in continua and of individual parts in constrained systems. And so, the dynamical principles at hand in the 1740s, including Newton's own, lacked any clear mark of generality. It was anyone's guess how, or even whether, those laws might apply to every possible motion of every part in any possible body. It took Euler half a century to show that such principles exist and are broadly Newtonian. Let us see next an instance of how he did so.

Extending the *Principia*

In depth and range, Euler's work in mechanics is immense; a book may barely hope to summarize, let alone interpret it. His results in the dynamics of particles and continua have received a fair amount of attention.[7] Less well known is his creative application of the Second Law to rigid-body motion. This story has never been told in full or contextualized properly. I can give it only the barest of outlines below.[8]

Before we glance at his results, we should ask, What drove Euler to study this topic? One cause was a general problem that Newton had left untouched: how to treat the motion of celestial bodies as *extended* objects. In this latter capacity, they exhibit gravitational phenomena—precession, nutation—simply beyond the conceptual reach of Newton's point-particle dynamics.[9] To quantify them, Euler modeled the planet as a rigid body, as did d'Alembert, then Lagrange in regard to yet another phenomenon, lunar libration. Another cause was a research program that goes back to Huygens and Jakob Bernoulli: the behavior of constrained rigid bodies. (To wit, the motion of a compound pendulum, for Huygens in *Horologium Oscillatorium*; and the equilibrium condition for a bent lever, for Bernoulli in *Meditatio de natura centri oscillationis*.) A lack of general kinematics forced these figures to reduce their problem to the motion of a centroid, viz. the "center of oscillation." The third cause was prosaic, rooted in the engineering needs of Continental powers at the time: the steering of a ship, which Euler modeled as a rigid body; or the transmission of angular momentum in assemblies of movable rigid parts. In connection with the latter, the French Academy in 1739 had launched a prize essay competition for the improved design of a winch, used to lift anchors on ships. Euler co-won it, with *Dissertation sur la meilleure construction du cabestan*.

Against this backdrop, Euler's effort to tame the rigid body starts with his discovery, wholly unknown to his predecessors, that a Newtonian principle—force equals mass times acceleration—about the causes of *straight-line* motions has a counterpart for rigid-body *rotation*.[10] In *Scientia Navalis* he proves that, for a rigid body moving around a fixed axis, the "rotary force" generated in a "little time" equals the sum of the "moments of the applied powers” divided by the “sum of the body’s particles” times the “square of their distances” from the axis of rotation (1749, § 160). In modern terms, the angular acceleration equals the torque divided by the moment of inertia, a term that Euler was to coin later. This is strikingly analogous to the Newtonian Second Law, restated as: linear acceleration equals the force divided by the mass. Euler had his new insight between 1736 and 1738, and later broadcast its analogy with Newton’s law:

> The moment of the soliciting powers, divided by the moment of the matter, gives the force of rotation—just as, in rectilinear motion, the power divided by the body’s matter gives the accelerating force. This great analogy deserves well to be noted.
>
> (Euler 1745, § 28)

Torque is like force, in that both induce accelerations. Moment of inertia is analogous to inertial mass: they are kinds of resistance to a changing state of motion. And, just as Newtonian *vis* accelerates a body *in directum*, Eulerian torque accelerates it *in gyrum*. As $\mathbf{F}=M\mathbf{a}$ for Newton, so does $\mathbf{H}=I\boldsymbol{\alpha}$ for Euler. Let us call them the “Force Law” and the “Torque Law” for rigid bodies.[11]

Though revolutionary, Euler’s insight about extending the Second Law is not yet fully clear. He knows how to apply it just to the special case where the “centrifugal forces” of the body’s parts “mutually destroy each other,”[12] hence the axis of rotation is *fixed*: for instance, a ship made to pitch downward by the wind pushing it from behind (1752, §8). But the *general* solution, when a torque moves a rigid body around a *variable* axis, eluded him until 1749 (Verdun 2015, 462ff.). To d’Alembert, he confessed in 1750:

> I have grappled with the [precession of the equinoxes] several times, but I was always defeated, both by the great number of parameters to consider and, more importantly, by this Problem: *If a body turns around an arbitrary free axis, and is acted on by an oblique force, to find the change in the axis of rotation and the speed*. Solving that problem is absolutely required for the topic you have developed so felicitously.[13]

The problem was so hard that Euler had given up on it, until d’Alembert’s work gave him hope that it could be solved and offered “some light” as to how.[14]

In the same letter, Euler announced his discovery of the relevant equations of motion. He presented it in *Découverte d’un nouveau principe de mécanique*, an epochal paper read out on September 3, 1750, at the Royal Academy in Berlin. There, he tackles the key problem—to find equations of motion for a rigid body moving around a variable axis—by building on several unprecedented insights. Some are kinematic: for one, now he knows that there exists an instantaneous axis of spin; more importantly, for each “element” in the body, Euler represents its position as three functions of coordinates relative to orthogonal axes fixed *in space* intersecting at the body’s center of mass. Another insight is dynamical: he applies Newton’s Second Law, which he calls the “general principle of motion,” to the *net* force acting on each element, resolved along each axis. Given a *mass-point*, Euler explains, the “general principle” entails, for each force component P, Q, R, that:[15]

[1]

$$2Md^2x = \pm Pdt^2 \quad 2Md^2y = \pm Qdt^2$$
$$2Md^2z = \pm Rdt^2$$

His strategy is a two-step attack on the problem. First, he applies the Second Law (i.e. the formula [1] just quoted) to each “element,” or infinitesimal mass *d*M. Second, he forms the vector (cross) product of **r**, the distance to a point *O* fixed in space, with each member of the equations [1] above. Thereby, Euler obtains the torque and angular acceleration around *O* for a mass element. Then he needs only integrate the new equation over the body, to find the whole motion (Euler 1752a).[16]

He faces two challenges. One is dynamical: the net force on an element is the resultant of *two kinds* of forces: external, applied to the body; and internal, exerted by other elements. The latter are generally unknown. Euler bypasses them entirely, by invoking the rigidity condition to infer that the internal forces “destroy each other mutually,” so their particular laws need not be known beforehand (1750, §42). Another, greater challenge, is kinematic: he must compute the second derivatives, or accelerations, of the three coordinates *x, y, z* for any element. This is daunting, because the elements are in *constrained* motion: rigidity limits their kinematic possibilities. Yet another breakthrough, known as “Euler’s kinematic equations,” allows Euler at last to derive the change in angular velocity for an element about each coordinate axis. By integration, he then obtains the increment of angular momentum for the whole body. For instance, relative to the *x*-axis, the increment is:[17]

[2]

$$2M[h^2\nu' - m^2\lambda' - l^2\mu' + \mu\nu m^2 - \lambda\nu l^2 - (\lambda^2 - \mu^2)n^2 + \lambda\mu(g^2 - f^2)]$$

To keep track of this quantity, label it “$\mathbf{L'_x}$”. Euler sets it equal to **P***a*, the *x*-component of the force times its arm *a* around the axis of rotation. He calls that the “moment of the force”—we call it the “torque.” So, relative to the *x*-axis, $\mathbf{P}a=\mathbf{L'_x}$ expresses Euler’s insight that the impressed torque equals the increment of angular momentum, by analogy with Newton’s law that the impressed force equals the increment of linear momentum. Seven decades after the *Principia*, Euler, armed with the "general principle" that **F**=M**a**, conquered rigid rotation, that fortress impregnable to all but d'Alembert before him.

And yet, completeness of insight still eluded Euler in 1750. "Solving these equations would lead us to formulas that are too long," he says coyly (1752, §56). What he really means is that his newfound equations of motion are not integrable, so they're all but useless. The cause is his fateful choice to refer the kinematics to a *single* frame fixed *in space*. Because of that, the products and moments of inertia for any mass element (his coefficients *f, g, h, l, m, n* in the preceding extracted formula) are *unknown* functions of time, not constants as they should be for the equation to be integrable.[18] It took him years to see that, speaking modernly, he needs to refer the change in angular momentum to a frame *co-moving* with the body; and then to find kinematic transformations relating the co-moving frame to an inertial frame at rest.

For nine years after *Découverte d'un nouveau principe de mécanique*, Euler kept getting closer to a satisfactory account. In 1751, he grasped that the equation of motion would be simpler with respect to a frame fixed in the *body*. Of all these frames, he found in 1758, one makes the equation simpler and *integrable*: the frame with the origin at the center of mass and the axes set by the body's principal axes of inertia. In the same year, Euler coined the phrase "moment of inertia" to express the resistance of mass to rotation, and showed how to compute it for some simple shapes. All these he collected and systematized in his 1765 compendium, *Theoria Motus Corporum Rigidorum*.[19]

Euler kept learning for another decade after that. In *Formulas generales pro translatione quacunque corporum rigidorum*, he saw that the study of rigid bodies really has two sides: "one belongs in Geometry, or rather Stereometry"; the other "properly belongs in Mechanics." Failure to keep them apart in *Theoria Motus* "made the whole account there quite cumbersome and tangled," whereas now he will set out the geometry first, "so that the mechanical part can be dispatched more easily" (1776a, §1).

The "geometric" part contains a kinematic discovery: any finite rigid motion is equivalent to a translation and a rotation.[20] The translation moves an arbitrary reference point in the rigid body by a rectilinear displacement with components (*f, g, h*). The rotation turns the body by three component angles (η, η', η'') around three axes intersecting at the reference point.[21] Thus, Euler's "general formulas for the arbitrary motion of a rigid body" are three equations conveying what we, beneficiaries of linear algebra, express by a translation vector and a rotation matrix. Now called "Euler's geometric equations," the formulas lay out clearly the kinematics of rigid motion. Euler tackled the dynamics in a companion piece, the *Nova methodus motum corporum rigidorum determinandi*. There, his newly gained clarity let him spell out lucidly for the first time how external causes change the motion of a rigid body. He refers the change to the principal frame, fixed in the body. His procedure is, again, to compute the action on an infinitesimal element, and to integrate the action over the body. So, he obtains six integrals, two for each axis. That is, relative to the *x*-axis, the total change is:[22]

[3]

$$\int \cdot\, dM\mathbf{x}'' = i\mathbf{P}$$

[4]

$$\int \cdot\, \mathbf{z}dM\mathbf{y}'' - \int \cdot\, \mathbf{y}dM\mathbf{z}'' = i\mathbf{S}$$

Ignorance of Enlightenment mechanics long misled us into taking these integrals to be obviously Newtonian principles. And yet, they are the late, hard-won fruit of *Euler's* struggle, who first extended Newton's concept of force beyond the reach of the *Principia*. In the two formulas just stated, the left side of [3] is the total increment in linear momentum; of [4], in angular momentum. The right side is the force and the torque, respectively. Adding up the relevant vector components, Euler's discovery is that, when forces act on a rigid body, its whole motion is ruled by two general laws. In differential form, they are:[23]

[3b]

$$\mathbf{F} = d\mathbf{p}/dt$$

[4b]

$$\mathbf{H} = d\mathbf{l}/dt$$

These are the Force Law, or Newton's *Lex Secunda*, made general; and the Torque Law, Euler's far-reaching extension of Newton's principle to rotation. With Euler's solid Newtonian basis in place, rigid-body dynamics could then grow into the theory that now enables us to land spacecraft on Mars. More significantly, it made possible Cauchy's later extension of Newtonian theory. He broadened the reach of Euler's two principles to cover elastic 3D bodies, and thereby turned them into Cauchy's Laws of Motion, the two dynamical laws of continuum mechanics in its Newtonian version. (The other formulation rests on the Principle of Virtual Work, which goes back to Lagrange.) Thus, we must see Euler as a very major yet by no means final figure in the growth of impressed-force mechanics.

Some Open Problems

My views and results outlined in this chapter are all preliminary. Euler's work was immense, so assessing his relation to Newton accurately is still a task for the future. Moreover, his unprecedented expansion of mechanics, his synthesis of formerly local theories, and his openness to alternative foundations prompt us to ask about the unity of his theoretical edifice. Specifically, is his mechanics a monolithic account? What theory of matter is compatible with his dynamical laws? What *were* his dynamical laws, after all? These questions are complex, far from easy, and presuppose answers to other questions. I outline and suggest lines of inquiry into them.

Axiomatic structure.

One area of great interest is still *terra incognita*—namely, the structure of Euler's mechanics. In the high Enlightenment, largely thanks to him, there is a growing collective awareness that mechanical theory comprises four levels of articulation. In order of generality and abstraction, they were i) equations of motion; ii) local principles of restricted validity (e.g., Hooke's Law, the Law of the Lever); iii) general dynamical laws, entailing equations of motion for a broad class of mechanical systems; and also iv) conceptual foundations, that is, metaphysical posits about inertial structure and non-empirical assumptions about material constitution.

However, the vocabulary for these various levels was rather fluid, and possibly equivocal. For instance, items in the classes (iii) and (iv) were called "laws," "principles," and also "axioms" rather indiscriminately, and Euler is no exception. Likewise, for class (i), the terms "formulas," "rules of motion," and "equations" were used. We ought to study Euler's language in this area, so as to clarify his meanings. More importantly, we ought to try and reconstruct his views on the architecture of mechanics—its structure, scope, and intra-theoretic relations. Achieving clarity on these aspects will have historical and philosophical payoffs. It will help us assess precisely the extent and novelty of Euler's contributions and alleged innovations in mechanics.[24] In turn, this assessment will let us understand better his relation to Newton—and also to towering figures after him, like Lagrange, Poisson, and Cauchy. Not least, a global image of Euler's theory would be a better context and basis for evaluating philosophical reflection on mechanics, such as that achieved by Kant, Hume, Christian Wolff, and Emilie du Châtelet.

Matter theory.

Yet another unresolved issue is Euler's theory of matter. Three ontologies for mechanics emerge in the Age of Reason. Their basic entity is, respectively, the mass-point, the rigid body, and the deformable continuum. These ontologies are essentially distinct, so they cannot be equally basic.[25] Euler *modeled* mathematically the physics for all three kinds, but he also had a realist metaphysic. We should learn his views about the unit of matter, to decide how coherent his natural philosophy is. In *Mechanica* and some later works, Euler extensively treated mass-points, which he first defined. And yet, it is unlikely that he regarded them as real, despite the reassurances of some historians.[26] Mass-points exert exclusively action at a distance, whose objective reality Euler denied resolutely. Plus, bodies made of mass-points never make kinematic contact, while Euler was firm that actual bodies do. This leaves two contestants in play, but the question is just as hard to answer. His talk of bodies as made up of "elements," "molecules," or "particles" is elusive, and the evidence pulls the reader in two directions. [i] In *Anleitung*, Euler claims that matter is infinitely divisible. Elsewhere, he calls the elements of rigid bodies *corpuscula infinite parva*. And, he takes rigidity to be relative to applied force: increase the force sufficiently, and the body will fracture or bend. All this suggests he might have taken the parts of gross bodies to be *deformable infinitesimal* volumes dV. (Then his talk of rigid bodies would denote merely a type of stiff deformables whose internal stresses are very high.) [ii] On the other hand, Euler takes flexible bodies to have borrowed their elasticity from a different medium: an ether whose vortex centrifugal force, when trapped in the pores of compressed bodies, makes them rebound. And, he claims that the "proper matter" of gross bodies is incompressible. This coheres with viewing bodies as assemblies of *finite rigid* parts connected through external force-closure, by the pressure of the ambient ether.[27] To confound readers further, Euler assigns to his elements sometimes a finite mass M, sometimes an infinitesimal mass dM.[28] More research is needed to elucidate this issue.

One may worry that my question is anachronistic in regard to Euler. Still, my point is not out of place; Euler and his contemporaries had a fair grasp of these differences in material structure. Boscovich and Kant could very well tell mass-points from deformable continua. Euler had a wholly modern grasp of how mass distribution in discrete and continuous bodies differs. He recognized that flexible continua support internal shear forces (which he quantified; Euler 1771), unlike mass-points or rigid bodies. Kant in 1786 knows that continuous bodies exert two kinds of actions, viz. body force and contact force (2002, 226). More generally, the Age of Reason had distinct names for these three kinds: "physical monad," "hard body," and "infinitely divisible matter." So, my question stands: What was Euler's considered theory of matter?

Dynamical laws.

We must also ask about the basic laws of Euler's mechanics. In the late *Nova Methodus*, he wrote integral formulas for the Force Law and the Torque Law. In the 1960s, Truesdell submitted that Euler offered them as "fundamental, general and independent laws of mechanics, for all kinds of motions of all kinds of bodies" (1968, 260). Many have hailed and repeated his words, but seldom have they assessed or explained them with much care. Truesdell's threefold claim is overdue for careful scrutiny.

Take independence: the Torque Law ($\mathbf{H}=d\mathbf{l}/dt$) is not necessarily independent of Newton's Second Law. It *can* be proven from the latter, by two strategies: (i) start with mass-points and the "strong" Third Law; (ii) start with continuous matter and posit that the stress tensor is symmetric. But route (i) would be unacceptable to Euler because it requires action-at-a-distance forces; and proof-strategy (ii) needs a premise—the symmetry of the stress tensor—not spelled out explicitly until the 1900s, by Boltzmann and Hamel.[29] So, did Euler take the Torque Law to be independent for good reasons or just *faute de mieux*?[30]

Take generality: Euler's two laws are enough to predict the behavior of systems of *free* mass-points, deformable bodies, and *single* rigid bodies. But can they handle *systems* of rigid bodies and *constrained* systems in general? Some in the Enlightenment, such as Lagrange and his followers at the Ecole Polytechnique, thought not: so, they anchored mechanics in the Principle of Virtual Work. Objection: Euler did not live to see the Lagrangian synthesis. Answer: First, he did not see the final product, but he did read the statement of mission in 1762, then saw a first implementation in 1763. And, he grasped Lagrange's approach impeccably and quite early; see the next section.[31] Second, *Méchanique analitique* was Lagrange's *second* unification of mechanics. By 1757, he had drafted a unified theory from a *different* yet equally general law—the Principle of Least Action—and published it in a terse, stripped-down version in 1762. Euler knew very well of this general theory—based in a *non-Newtonian* law that he too had defended, in the 1750s—which Lagrange had offered as an explicit extension of Euler's variational approach to mechanics of 1744. In relation to the latter, we must ask: Why did Euler—who had endorsed the principle earlier, and praised Lagrange's calculus of variations—react so coolly to his generalized dynamics based in the Principle of Least Action?[32] After all, it happened as he was instructing his German princess that "in all changes which happen in nature, the cause which produces them is the least that can be"; ergo the Principle of Least Action is "perfectly founded in the nature of body, and those who deny it are very much in the wrong" (Euler 1802, 303). Here is another, serious objection to Truesdell. Euler also extolled as his *favorite* general method the essentially *non-*

Newtonian virtual-work approach of analytic mechanics. He grasped and described it with amazing clarity, before Lagrange did so in the 1788 *Méchanique analitique*.[33] And so, to sum up, it turns out that in the same period Euler championed *three* distinct basic laws for dynamics: the Newtonian laws of force and torque; the Principle of Least Action; and the Principle of Virtual Work. Then, is Truesdell still right about him?

Finally, take fundamentality: how did Truesdell mean it? That for Euler the two laws are axioms? Then caution is in order: the term "axiom" is ambiguous. For us moderns, mechanical axioms are indemonstrables posited for the sake of deriving equations of motion, and *these* we confront then with experience. Euler through the 1760s took his axioms—the Force Law and the Least Action Principle—to be *demonstrable*. The foundations of his mechanics were not free-floating, as they are for us, but fastened to a metaphysical doctrine, the *Naturlehre*, and derivable a priori from it. Knowing all this, we must follow up on Truesdell, and ask: In what sense are the Force Law and the Torque Law fundamental for Euler in 1776? Did he still believe them to be provable from deeper premises? If so, how? Or had he given up on that Enlightenment-rationalist idea, thus opening the way for the modern view of axioms in mechanics?

Some of these questions may have no conclusive answers; still, we must try to answer them. My list is by no means exhaustive, but it is hopefully a good start. I offer it as a call to arms, an invitation to explore, and a promise of scholarly delights. Among them is learning how we became Eulerians, all the while thinking we were heirs to Newton.

Conclusions

In dynamics, Newton left a rich and robustly successful legacy for the Age of Reason. However, it would be a mistake to think of it as a Kuhnian paradigm-breakthrough followed by normal science. If Newtonian mechanics seems paradigmatic, it is an illusion of hindsight. His legacy was not exclusive; Leibniz and Huygens left theirs, too. And, it was not obvious. Various obstacles—mathematical, kinematic, and dynamical—stood in the way of turning Newton's principles into a basis for mechanics in toto. Euler carried out most of the work needed for that accomplishment. In doing so, he had to create new theoretical fields, invent new formalisms wholesale, and reconceptualize the main task of mechanical theory. The edifice he co-created rightfully deserves the name "Newton-Euler dynamics."

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Notes:

([1]) See, respectively, Newton (1999), Book III, Propositions 3 and 14; 4, 8 and 37; 12 and 15; 19 and 20; 22 and 26–35; 13 and 23; 24 and 37; 40–42; and 39. I am indebted to George E. Smith for an illuminating account of these matters.

([2]) Euler to Johann Bernoulli, May 5, 1739; Eneström (1905), 25. The “indirect method” was the attempt to derive equations of motion from an integral principle, here the Conservation of Vis Viva. In contrast, the Second Law is a differential law.

([3]) In fact, it was Euler who made the formulation and integration of equations of motion into *the* method of mechanics; see Verdun (2015), § 2.3.2.

([4]) Euler’s results in spherical kinematics are catalogued in Koetsier (2007).

([5]) In fact, proliferation of principles obtains *after* Euler’s “Newtonian” unification, too. It is remarkable, though seldom discussed, that Lagrange anchored his general theory of 1788 in the Principle of Virtual Work, not in any laws in the *Principia*. His choice became the norm in French mechanics for a century. Then in the 1830s, yet another non-Newtonian version arose, Hamilton-Jacobi theory. These alternatives to Newton became the “first-” and “second formalism” of modern analytic mechanics.

([6]) In planetary dynamics, the preferred centroid was the mass center. In the compound pendulum, it was the “center of oscillation.” For rigid collision, it was the “center of percussion,” that is, the instantaneous center of zero velocity.

([7]) Hepburn (2007) is a philosophical study of Euler’s *Mechanica*; Verdun (2015) is the definitive account of Euler’s celestial mechanics; Truesdell (1954) and (1960) are classic studies of Euler’s continuum mechanics; Romero Chacon (2007) explores Euler’s researches on fluids in the 1750s. Cannon and Dostrovsky (1981) analyze Euler’s contributions to vibration theory before 1742.

([8]) Episodes in Euler’s creation of rigid-body dynamics are recounted in Maltese (1996); Wilson (1987); Langton (2007); Verdun (2015, § 4.1.1) is the most comprehensive account to date. There is as yet no study on this topic to rival Truesdell on Euler’s mechanics of continua.

([9]) A centroid has no internal spin. But, precession, nutation, and libration are all types of change in a body’s spin angular velocity.

([10]) To be sure, *Newton* did not put his *Lex Secunda* thus. “A change in motion is proportional to the motive force impressed,” he says (Newton [1999], 416). Leibniz’s followers on the Continent, however, who influenced the young Euler, wrote the Second Law in terms of infinitesimal momentum increments, more in line with my statement above.

([11]) **F** is force, **H** is torque; M is mass, I is moment of inertia; **a** is linear-, **α** is angular acceleration. Boldface indicates vectors, and the prime symbol denotes time differentiation.

([12]) By that, Euler really means that the products of inertia around the axis of rotation vanish, so the axis remains fixed. He thinks of products of inertia as “forces” pulling the individual parts radially away from the axis.

([13]) Euler to d’Alembert, March 7, 1750; cf. Euler ([1980], 306). D’Alembert had solved it in his 1749 *Recherches sur la precession des equinoxes*, by applying “d’Alembert’s Principle” to a rigid Earth acted on by a torque from the sun and the moon. But, in his March 7th letter, Euler confesses himself unable to follow d’Alembert’s solution, so he sought his own.

([14]) That “light” appears to have been d’Alembert’s proof that, in rigid spin, there exists always an instantaneous axis of rotation—for the entire body—relative to some inertial frame. I thank Andreas Verdun for pressing me on this issue.

([15]) The plus or minus sign marks whether the acting force tends to move the particle closer to or away from the origin of the frame. Verdun ([2015], 198ff.) explains the need for the factor two in [1].

([16]) For limpid analysis, including previously unavailable evidence, see Verdun (2015), § 4.1.1.

([17]) In the expression below λ, μ, ν are components of the angular velocity vector; *f, g, h* are the radii of gyration relative to the three axes; *l, m, n* are such that Ml^2, Mm^2, Mn^2 are the body’s products of inertia relative to the planes *xy, xz*, and *yz*, respectively.

([18]) This is because they are referred to as axes fixed in space, relative to which each part of the body has different moments and products of inertia over time. Hence, to integrate the equation of motion, these factors would have to be recalculated at every instant.

([19]) See, in order, Euler (1767) and Euler (1765a, b, and c). So far, the best account of these developments is Verdun (2015), § 4.1.1.2; a briefer account is in Maltese (1996), Chapter 11.5–6.

([20]) By "equivalent," I mean that it takes the body to the same final position, relative to a fixed frame, as the actual displacement, though not necessarily by the same route. At this point Euler does not seem to recognize that this combination itself is equivalent to a screw motion, as Giulio Mozzi had proved already in *Discorso matematico sopra il rotamento momentaneo dei corpi* (1763).

([21]) Euler (1776a), §15. He also proves there that (in our terms) the rotation matrix is orthogonal.

([22]) Euler (1776b), §§27-9. P is force, S is its "moment," or torque; '*i*' is a proportionality factor equal to half the height crossed in a second by a unit body in free fall; recall that he uses units of weight to measure force and mass (Maltese [1996], 197).

([23]) **p** is linear momentum and **l** is angular momentum.

([24]) A case in point is his famous *Découverte d'un nouveau principe de mécanique*. It is still debated what Euler's "new principle" was in that paper, and exactly how novel it was. For conflicting interpretations, see Truesdell (1968) and Langton (2007); Wilson (1987); Maltese ([1992], Ch. 9); and Verdun ([2015], 477–482).

([25]) Mass-points are zero-sized, volume elements are infinitesimal, rigid bodies finite. These three objects have different kinematics: mass-points have three degrees of freedom, rigid bodies have six, deformable continua an infinity. The latter deform; to describe that, we need a concept—strain—undefined for mass-points and rigid bodies. Their dynamics is different as well: mass-points interact only at a distance; rigid bodies can exert contact forces; deformable continua undergo internal stresses, meaningless for the first two stuffs.

([26]) van der Waerden (1983) insists that this was always Euler's theory of matter, but offers no evidence for it. Gaukroger (1982) too credits Euler with mass-points as late as 1765. Others, though not quite explicitly, read Euler's "molecules" as tiny rigid bodies (Arana [1994], 171f).

([27]) Strictly speaking, there are two kinds of stuff, for Euler. Gross bodies are made of "proper matter," deformable at mesoscopic scale and above (it is unclear how far down deformability goes, in his doctrine). Outside these bodies, and inside their "pores," there is a highly elastic physical continuum, or ether. The essential difference between the two is that only the ether is compressible. Cf. Euler (1862). Andreas Verdun informs me (in personal communication) that, in an unpublished piece, Ms. 202, Euler construed rigid bodies as point masses connected by massless, undeformable strings or rods.

([28]) See, respectively, Euler (1765), §§156–63; and Euler (1750), passim.

([29]) The “strong” Third Law is that all forces between particles in a body or system are central and pairwise equilibrated. Binet first adopted this proof strategy, around 1811, then Poisson in 1833, who also made clear that the forces must be pairwise central interactions.

([30]) Maltese ([1996], 197) claims that in *Nova Methodus* Euler just made public what he had thought since the 1740s, viz. that the Torque Law is independent of the Force Law. This seems wrong. In *Découverte*, Euler takes himself to *prove* the Torque Law (restricted to rigids) *from* the Force Law. Had he really believed the law to be independent, he would have announced it as a *second* “general principle of motion,” in 1750. But, Euler there speaks of there being *just one* such principle, namely **F**=M**a**.

([31]) The call to ground *all* of mechanics in the Principle of Virtual Work was first made in the paper *Sur les principes fondamentaux de la méchanique*, ostensibly by Lagrange’s student F. Daviet de Foncenex, but inspired, if not directly drafted, by Lagrange himself. Lagrange first took the principle as his basic law in *Recherches sur la libration de la lune* (1763). His *Mécanique analytique* is wholly grounded in it.

([32]) In 1756, Lagrange showed Euler a sample of problems solved by applying the Principle of Least Action in his variational form. Maupertuis, the original author of the principle, was elated to hear about it. In 1759, Lagrange told Euler he had devised a unified mechanics based on this principle; Euler showed no interest. Then, on seeing Lagrange’s 1762 unification of mechanics, he reacted noncommittally: “Were Mr. Maupertuis still alive, how glad would he be to see his Principle of Least Action brought [by you] to the highest degree of dignity of which it is capable.” (Euler to Lagrange, November 9, 1762.)

([33]) For example, in *De motu corporum circa punctum fixum mobilium* (Euler [1862]: 45f.). The general method he advocates there is this. (1) Suppose the parts of the (constrained) system acquire some actual accelerations, from the external forces acting on them. (2) Multiply these accelerations by their respective masses. (3) Imagine some *fictive* forces, equal to these products but *in the opposite direction*, to act on the system. These imaginary forces will balance exactly the real ones. (4) With the moving system reduced (in thought) to rest, now treat it with the laws of statics. Recall that in Lagrangian mechanics, step (3) is called “D’Alembert’s Principle.” Step (4) is ruled by the Principle of Virtual Work, the basic law in that theory.

Marius Stan

Between 2009 and 2012, Marius was a Mellon postdoctoral fellow at the California Institute of Technology (Caltech). In 2011 he was a visiting fellow at the Max Planck Institute for History of Science (Berlin), as part of Vincenzo de Risi's research group "Modern Geometry and the Concept of Space."