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Some investigations on mbC and mCi

MARCELO E. CONIGLIO
TARCÍSIO G. RODRIGUES

Introduction

The *Logics of Formal Inconsistency* (**LFI**s, from now on) were introduced by W. Carnielli and J. Marcos in (Carnielli and Marcos 2002) as a class of paraconsistent logics able to internalize in the object language the notions of *consistency* and *inconsistency* by means of specific connectives (which are primitives or not). This approach to paraconsistency generalizes the original ideas of N.C.A. da Costa behind his well-known hierarchy of systems C_n — see (Costa 1963).

In (Carnielli, Coniglio, and Marcos, 2007) the study of **LFI**s started with a propositional logic called **mbC**, defined on a language containing a paraconsistent negation $\neg$, a conjunction $\wedge$, a disjunction $\vee$, an implication $\rightarrow$ and an unary connective $\circ$ for *consistency*. All the other systems studied in (Carnielli and Marcos 2002) and (Carnielli, Coniglio, and Marcos 2007) are extensions of **mbC** obtained by adding appropriate axioms.

We propose here a new axiomatization of the logic **mbC** formulated in the signature $\{\bot, \rightarrow, \neg, \circ\}$, where $\bot$ is a bottom. This simpler formulation allows to see in a clear way that **mbC** is in fact an extension of propositional classical logic obtained by adding a paraconsistent negation $\neg$ and a consistency operator $\circ$.

We also present sequent calculi for **mbC** and its extension **mCi**, both defined in the new signature, which are shown to admit cut elimination. As a consequence of this, two new results are proved for these logics: just like in classical logic, a negated formula $\neg\alpha$ is a theorem of **mbC** (resp., of **mCi**)

Mortari, C. A. (org.) *Tópicos de lógicas não clássicas*.
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iff α has no models. The other result states that the logic **mbC** is not controllably explosive. This gives a negative answer to an open problem known in the literature of **LFI**s.

1. The logics mbC and mCi

In this paper we will deal with the so-called *Tarskian logics* — see, for instance, (Wójcicki 1984):

Definition 1 (Tarskian Logic)**.** A logic $\mathscr{L}$ defined over a language $\mathcal{L}$ and with a consequence relation $\vdash$ is *Tarskian* if it satisfies the following properties:

(i) if $\alpha \in \Gamma$ then $\Gamma \vdash \alpha$;
(ii) if $\Gamma \vdash \alpha$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash \alpha$;
(iii) if $\Delta \vdash \alpha$ and $\Gamma \vdash \beta$ for every $\beta \in \Delta$ then $\Gamma \vdash \alpha$.

A Tarskian logic $\mathscr{L}$ is *finitary* if it also satisfies:

(iv) if $\Gamma \vdash \alpha$ then there exists a finite subset Γ_0 of Γ such that $\Gamma_0 \vdash \alpha$.

Finally, a Tarskian logic $\mathscr{L}$ defined over a propositional language $\mathcal{L}$ generated by a signature from a set of propositional variables is called *structural* if it also satisfies:

(v) if $\Gamma \vdash \alpha$ then, for every substitution ε of formulas for variables, $\varepsilon[\Gamma] \vdash \varepsilon(\alpha)$.

A propositional logic is *standard* if it is a Tarskian, finitary and structural — see (Wójcicki 1984).

As mentioned in the previous section, **LFI**s are paraconsistent logics which can express, at the language level, the property of some formula to be consistent or inconsistent. To give a precise definition, we will slightly adapt Definition 23 in (Carnielli, Coniglio, and Marcos 2007), as it was done in (Coniglio and Silvestrini 2014) and (Coniglio, Esteva, and Godo 2014).

Definition 2. Let $\mathscr{L} = \langle For, \vdash \rangle$ be a standard logic. Assume that $\mathscr{L}$ is defined in a signature containing a negation $\neg$, and let $\bigcirc(p)$ be a nonempty set of formulas depending exactly on the propositional variable p. Then $\mathscr{L}$ is an **LFI** (with respect to $\neg$ and $\bigcirc(p)$) if the following holds (here, $\bigcirc(\varphi) = \{\psi(\varphi) \mid \psi(p) \in \bigcirc(p)\}$):

(i) $\varphi, \neg\varphi \nvdash \psi$ for some φ and ψ, i.e., $\mathscr{L}$ is not explosive w.r.t. $\neg$;
(ii) $\bigcirc(\varphi), \varphi \nvdash \psi$ for some φ and ψ;
(iii) $\bigcirc(\varphi), \neg\varphi \nvdash \psi$ for some φ and ψ; and
(iv) $\bigcirc(\varphi), \varphi, \neg\varphi \vdash \psi$ for every φ and ψ.

Principle (iv) is usually called *gently explosiveness* w.r.t. $\neg$ and $\bigcirc(p)$. When $\bigcirc(p)$ is a singleton, its element will be denoted by $\circ p$, and $\circ$ is called a *consistency* operator. The general definition above encompasses a wide range of paraconsistent logics. Any logic featuring a consistency connective must present, in order to formally express the properties of consistency, a set of logical axiom schemes or semantic rules governing this connective. Along these lines, in (Carnielli, Coniglio, and Marcos 2007) it were introduced **mbC** and **mCi**, two fundamental propositional **LFI**s. Starting from positive classical logic plus *tertium non datur* ($\alpha \vee \neg\alpha$), **mbC** is intended to comply with the above definition in a minimal way: an axiom scheme called (**bc1**) is added just describing the expected behavior of the consistency operator $\circ$ (see Definition 5). By its turn, the logic **mCi** is obtained extending **mbC** in order to express inconsistency as the (paraconsistent) negation of consistency (see Definition 6). In what follows, these logics will be briefly exposed in their original language along with the statement of soundness and completeness theorems with respect to paraconsistent bivaluations.

Definition 3 ($\Sigma^{\wedge,\vee}$ and $\mathcal{L}_{\Sigma^{\wedge,\vee}}$)**.** Let $Var = \{p_1, p_2, \ldots\}$ be a denumerable set of propositional variables (which will kept fixed along the paper). The propositional signature $\Sigma^{\wedge,\vee}$ is the set $\{\wedge, \vee, \to, \neg, \circ\}$ formed by connectives for conjunction, disjunction, implication, negation and consistency. The propositional language generated by $\Sigma^{\wedge,\vee}$ from Var will be denoted by $\mathcal{L}_{\Sigma^{\wedge,\vee}}$.

Definition 4 (Formula Complexity)**.** The complexity of a given formula $\varphi \in \mathcal{L}_{\Sigma^{\wedge,\vee}}$, denoted by $l(\varphi)$, is defined recursively as follows:

1. If $\varphi = p$, where $p \in Var$, then $l(\varphi) = 0$;
2. If $\varphi = \neg\alpha$, then $l(\varphi) = l(\alpha) + 1$;

3. If $\varphi = \circ\alpha$, then $l(\varphi) = l(\alpha) + 2$;
4. If $\varphi = \alpha\#\beta$, where $\# \in \{\wedge, \vee, \rightarrow\}$, then $l(\varphi) = l(\alpha) + l(\beta) + 1$.

Definition 5 (**mbC**$^{\wedge\vee}$)**.** The calculus **mbC**$^{\wedge\vee}$ — or **mbC**, as introduced in (Carnielli, Coniglio, and Marcos 2007) — is defined over the language $\mathcal{L}_{\Sigma^{\wedge,\vee}}$ by the following Hilbert calculus:

Axiom schemes:

$$\alpha \rightarrow (\beta \rightarrow \alpha) \qquad \textbf{(Ax1)}$$

$$\big(\alpha \rightarrow \beta\big) \rightarrow \big((\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\alpha \rightarrow \gamma)\big) \qquad \textbf{(Ax2)}$$

$$\alpha \rightarrow \big(\beta \rightarrow (\alpha \wedge \beta)\big) \qquad \textbf{(Ax3)}$$

$$(\alpha \wedge \beta) \rightarrow \alpha \qquad \textbf{(Ax4)}$$

$$(\alpha \wedge \beta) \rightarrow \beta \qquad \textbf{(Ax5)}$$

$$\alpha \rightarrow (\alpha \vee \beta) \qquad \textbf{(Ax6)}$$

$$\beta \rightarrow (\alpha \vee \beta) \qquad \textbf{(Ax7)}$$

$$\big(\alpha \rightarrow \gamma\big) \rightarrow \big((\beta \rightarrow \gamma) \rightarrow ((\alpha \vee \beta) \rightarrow \gamma)\big) \qquad \textbf{(Ax8)}$$

$$(\alpha \rightarrow \beta) \vee \alpha \qquad \textbf{(Ax9)}$$

$$\alpha \vee \neg\alpha \qquad \textbf{(Ax10)}$$

$$\circ\alpha \rightarrow \big(\alpha \rightarrow (\neg\alpha \rightarrow \beta)\big) \qquad \textbf{(bc1)}$$

Inference rule:

$$\frac{\alpha \quad \alpha \rightarrow \beta}{\beta} \qquad \textbf{(MP)}$$

Definition 6 (**mCi**$^{\wedge\vee}$)**.** The calculus **mCi**$^{\wedge\vee}$ — or **mCi**, as introduced in (Carnielli, Coniglio, and Marcos 2007) — is defined over the language $\mathcal{L}_{\Sigma^{\wedge,\vee}}$ by adding to **mbC**$^{\wedge\vee}$ the following axiom schemes, for $n \geq 0$:[1]

$$\neg\circ\alpha \rightarrow (\alpha \wedge \neg\alpha) \qquad \textbf{(ci)}$$

$$\circ\neg^{n}\circ\alpha \qquad (\textbf{cc}_n)$$

Observe that **Ax1**-**Ax9** plus **MP** constitutes a Hilbert calculus for positive classical logic (**CPL**$^{+}$), which is in fact the basis for **mbC** and its extensions such as **mCi**.

The above logics are sound and complete with relation to a suitable bivaluation semantics, to be defined now.

Definition 7 (Bivaluations for $\mathbf{mbC}^{\wedge\vee}$)**.** A function $v : \mathcal{L}_{\Sigma^{\wedge,\vee}} \to \{0,1\}$ is a bivaluation for $\mathbf{mbC}^{\wedge\vee}$ if it satisfies the following clauses:

$$v(\neg\varphi) = 0 \quad \Longrightarrow \quad v(\varphi) = 1 \tag{1}$$
$$v(\circ\varphi) = 1 \quad \Longrightarrow \quad v(\varphi) = 0 \text{ or } v(\neg\varphi) = 0 \tag{2}$$
$$v(\alpha \to \beta) = 1 \quad \Longleftrightarrow \quad v(\alpha) = 0 \text{ or } v(\beta) = 1 \tag{3}$$
$$v(\alpha \wedge \beta) = 1 \quad \Longleftrightarrow \quad v(\alpha) = 1 \text{ and } v(\beta) = 1 \tag{4}$$
$$v(\alpha \vee \beta) = 1 \quad \Longleftrightarrow \quad v(\alpha) = 1 \text{ or } v(\beta) = 1 \tag{5}$$

The set of all such valuations is designated by $V^{\mathbf{mbC}^{\wedge\vee}}$.

Definition 8 (Bivaluations for $\mathbf{mCi}^{\wedge\vee}$)**.** A function $v : \mathcal{L}_{\Sigma^{\wedge,\vee}} \to \{0,1\}$ is a bivaluation for $\mathbf{mCi}^{\wedge\vee}$ if it is a paraconsistent bivaluation for $\mathbf{mbC}^{\wedge\vee}$ and satisfies also the following:

$$v(\neg\circ\alpha) = 1 \quad \Longrightarrow \quad v(\alpha) = 1 \text{ and } v(\neg\alpha) = 1 \tag{6}$$
$$v(\circ\neg^{n}\circ\alpha) = 1 \quad (\text{for } n \geq 0) \tag{7}$$

The set of all such bivaluations is designated by $V^{\mathbf{mCi}^{\wedge\vee}}$.

If $V \in \{V^{\mathbf{mbC}^{\wedge\vee}}, V^{\mathbf{mCi}^{\wedge\vee}}\}$ we define, for every $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\Sigma^{\wedge,\vee}}$, the following semantic consequence relation w.r.t. the set of bivaluations V: $\Gamma \vDash_V \varphi$ iff, for every $v \in V$, if $v(\gamma) = 1$ for every $\gamma \in \Gamma$ then $v(\varphi) = 1$. The sets collecting the bivaluations just defined, associated respectively to $\mathbf{mbC}^{\wedge\vee}$ and $\mathbf{mCi}^{\wedge\vee}$, form a sound and complete semantics for the respective logic:

Theorem 9. *Let* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\Sigma^{\wedge,\vee}}$*. Then:*

$$\Gamma \vdash_{\mathbf{mbC}^{\wedge\vee}} \varphi \quad \Longleftrightarrow \quad \Gamma \vDash_{V^{\mathbf{mbC}^{\wedge\vee}}} \varphi$$
$$\Gamma \vdash_{\mathbf{mCi}^{\wedge\vee}} \varphi \quad \Longleftrightarrow \quad \Gamma \vDash_{V^{\mathbf{mCi}^{\wedge\vee}}} \varphi$$

For a proof of the above theorem the reader is referred to (Carnielli, Coniglio, and Marcos 2007), Theorems 56, 61, 85 and 88.

2. A New Language for mbC and mCi

The present approach simplifies the propositional axioms from Definitions 5 and 6 by the way of a simplification in the propositional signature: in the place of the above set $\Sigma^{\wedge,\vee}$ of connectives it is made use of a new simpler one, namely $\Sigma^{\perp} = \{\perp, \to, \neg, \circ\}$. The propositional language generated by $\Sigma^{\perp}$ from Var will be denoted by $\mathcal{L}_{\Sigma^{\perp}}$. The notion of complexity of a formula in $\mathcal{L}_{\Sigma^{\perp}}$ is defined analogously to Definition 4:

Definition 10 (Formula Complexity in $\mathcal{L}_{\Sigma^{\perp}}$)**.** The complexity of a given formula $\varphi \in \mathcal{L}_{\Sigma^{\perp}}$, denoted by $l(\varphi)$, is defined recursively as follows:

1. If $\varphi = p$, where $p \in Var \cup \{\perp\}$, then $l(\varphi) = 0$;
2. If $\varphi = \neg\alpha$, then $l(\varphi) = l(\alpha) + 1$;
3. If $\varphi = \circ\alpha$, then $l(\varphi) = l(\alpha) + 2$;
4. If $\varphi = \alpha \to \beta$, then $l(\varphi) = l(\alpha) + l(\beta) + 1$.

As observed above, positive classical logic **CPL**$^{+}$ may be axiomatized by axioms **Ax1**-**Ax9** plus **MP**. As a consequence, the connectives $\wedge$, $\vee$ and $\to$, as defined by these axioms, are the classical ones and they could, in principle, be defined in terms of just $\to$ and a bottom particle $\perp$, as in classical logic. Despite there is no $\perp$ in **CPL**$^{+}$, in **mbC** and all its extensions, any formula α defines a bottom $\perp_\alpha \stackrel{\text{def}}{=} \alpha \wedge (\neg\alpha \wedge \circ\alpha)$, because of the axiom **bc1**.

As there is such a bottom particle in **mbC**, it is possible to consider from the beginning a 0-ary connective $\perp$ and the axiom schemes for **CPL** in the signature $\Sigma^{\perp}$, as well as the corresponding axiom schemes for the paraconsistent negation $\neg$ and the consistency operator $\circ$ without modifying the logics in question, and therefore to use the signature above to axiomatize **mbC** and its extensions.

A justification for the language proposed here, besides the simplification achieved (for instance, in the proofs by induction on the complexity of a formula), is that $\perp$, being so important in the context of **LFI**s, is usually defined with respect to a formula α as $\perp_\alpha$ and so there is an infinitude of such bottom particles. Same observation applies to the classical negation ($\sim$), which is defined as $\sim_\alpha\beta \stackrel{\text{def}}{=} \beta \to \perp_\alpha$ and so there are infinite classical negations inside **mbC** and its extensions.[2] Therefore, the inclusion of bottom $\perp$ in the signature allows to define a distinguished classical negation:

Definition 11 (Classical Negation)**.**

$$\sim\alpha \stackrel{\text{def}}{=} \alpha \to \bot$$

From this, $\bot$ and $\sim$ can be considered as canonical choices for bottom and the classical negation inside **mbC** and its extensions. Moreover, these logics can be considered as extensions of classical propositional logic **CPL** (defined in the signature $\{\to, \bot\}$) by adding a paraconsistent negation and a consistency operator. This allows to see these **LFI**s as a kind of bimodal logics based on **CPL**. Despite this, these logics are not *self-extensional* in Wójcicki's sense — see (Wójcicki, 1979) — that is, (weak)replacement does not hold: from $\alpha \vdash \beta$ and $\beta \vdash \alpha$ does not follow in general that $\#\alpha \vdash \#\beta$ and $\#\beta \vdash \#\alpha$, for $\# \in \{\neg, \circ\}$. Of course this disappointing feature is already present in the original formulation of **mbC** and **mCi** — see (Carnielli, Coniglio, and Marcos 2007).

Definition 12 (**mbC**$^\bot$)**.** The calculus **mbC**$^\bot$ is defined over the language $\mathcal{L}_{\Sigma^\bot}$ by the following Hilbert calculus:

Axiom schemes:

$$\alpha \to (\beta \to \alpha) \qquad (\textbf{Ax1})$$

$$\big(\alpha \to (\beta \to \gamma)\big) \to \big((\alpha \to \beta) \to (\alpha \to \gamma)\big) \qquad (\textbf{Dst})$$

$$\sim\sim\alpha \to \alpha \qquad (\textbf{Dne})$$

$$\sim\alpha \to \neg\alpha \qquad (\sim\neg)$$

$$\circ\alpha \to (\neg\alpha \to \sim\alpha) \qquad (\textbf{bc1}^\bot)$$

Inference rule:

$$\frac{\alpha \quad \alpha \to \beta}{\beta} \qquad (\textbf{MP})$$

Remark 13. *The axiom schemes* ***Ax1****,* ***Dst*** *and* ***Dne*** *plus* ***MP*** *constitute an axiomatization of* **CPL** *in the signature* $\{\to, \bot\}$*, which is usually atributed to Church (taking* $\sim$ *as in Definition 11).*

Definition 14 ($\mathbf{mCi}^\perp$)**.** The calculus $\mathbf{mCi}^\perp$ is defined over the language $\mathcal{L}_{\Sigma^\perp}$ by adding to $\mathbf{mbC}^\perp$ (Definition 12) the following axiom schemes, for $n \geq 0$:

$$\neg\circ\alpha \to \alpha \qquad (\mathbf{ci}^1)$$

$$\neg\circ\alpha \to \neg\alpha \qquad (\mathbf{ci}^2)$$

$$\circ\neg^n\circ\alpha \qquad (\mathbf{cc}_n)$$

The deduction meta-theorem (MTD) holds for these logics. This is a consequence of a well-known result that states that any Hilbert calculus with **MP** as its only inference rule and where **Ax1** and **Dst** are derivable, satisfies MTD:

Theorem 15 (Deduction Meta-Theorem)**.** *Let* $\mathscr{L} \in \{\mathbf{mbC}^\perp, \mathbf{mCi}^\perp\}$*. Then, for every* $\Gamma \cup \{\varphi, \psi\} \subseteq \mathcal{L}_{\Sigma^{\wedge,\vee}}$*:*

$$\Gamma \cup \{\varphi\} \vdash_{\mathscr{L}} \psi \qquad \Longleftrightarrow \qquad \Gamma \vdash_{\mathscr{L}} \varphi \to \psi\,.$$

The next technical lemma is required for establishing the completeness theorem in the next section.

Lemma 16. *All the following formulas are theorems of* $\mathbf{mbC}^\perp$ *and* $\mathbf{mCi}^\perp$*:*

1. $\bot \to \alpha$
2. $((\alpha \to \beta) \to \alpha) \to \alpha$
3. $(\alpha \to \gamma) \to \Big((\beta \to \gamma) \to \big(((\alpha \to \beta) \to \beta) \to \gamma\big)\Big)$
4. $(\alpha \to \gamma) \to \Big((\beta \to \gamma) \to \big(((\alpha \to \bot) \to \beta) \to \gamma\big)\Big)$

Proof. All these formulas are classic tautologies and therefore they can be derived in both logics, from Remark 13. □

3. Completeness for Bivaluation Semantics

Definition 17 (Bivaluations for $\mathbf{mbC}^\perp$)**.** A function $v : \mathcal{L}_{\Sigma^\perp} \to \{0, 1\}$ is a bivaluation for $\mathbf{mbC}^\perp$ if it satisfies the following clauses:

$$v(\bot) = 0 \qquad (1)$$

$$v(\neg\varphi) = 0 \quad\Longrightarrow\quad v(\varphi) = 1 \tag{2}$$

$$v(\circ\varphi) = 1 \quad\Longrightarrow\quad v(\varphi) = 0 \text{ or } v(\neg\varphi) = 0 \tag{3}$$

$$v(\alpha \to \beta) = 1 \quad\Longleftrightarrow\quad v(\alpha) = 0 \text{ or } v(\beta) = 1 \tag{4}$$

Definition 18 (Bivaluations for $\mathbf{mCi}^{\perp}$)**.** A function $v : \mathcal{L}_{\Sigma^{\perp}} \to \{0, 1\}$ is a bivaluation for $\mathbf{mCi}^{\perp}$ if it is a bivaluation for $\mathbf{mbC}^{\perp}$ and satisfies also the following:

$$v(\neg\circ\varphi) = 1 \quad\Longrightarrow\quad v(\varphi) = 1 \text{ and } v(\neg\varphi) = 1 \tag{5}$$

$$v(\circ\neg^{n}\circ\varphi) = 1 \quad (\text{for } n \geq 0) \tag{6}$$

Proposition 19. *The bivaluations for* $\mathbf{mCi}^{\perp}$ *are the mappings* $v : \mathcal{L}_{\Sigma^{\perp}} \to \{0, 1\}$ *satisfying clauses (1), (2), (4) and (6) from the two previous definitions, plus the folllowing:*

$$v(\circ\varphi) = 1 \quad\Longleftrightarrow\quad v(\varphi) = 0 \textit{ or } v(\neg\varphi) = 0 \tag{7}$$

Proof. Let $v : \mathcal{L}_{\Sigma^{\perp}} \to \{0, 1\}$ be a mapping satisfying clauses (2) and (6). Then, it is straightforward to prove that v satisfies clauses (3) and (5) iff it satisfies clause (7). □

Now, a technical result is given whose demonstration will be used latter on, in the proof of Theorem 35.

Lemma 20. *Let* $v_0 : Var \to \{0, 1\}$ *be a mapping. Then, there exists bivaluations* $v_b^{\perp} \in V^{\mathbf{mbC}^{\perp}}$, $v_i^{\perp} \in V^{\mathbf{mCi}^{\perp}}$, $v_b^{\wedge\vee} \in V^{\mathbf{mbC}^{\wedge\vee}}$ *and* $v_i^{\wedge\vee} \in V^{\mathbf{mCi}^{\wedge\vee}}$, *all of them extending* v_0.

Proof. The values of $v_b^{\perp}(\psi)$ and $v_i^{\perp}(\psi)$, for $\psi \in \mathcal{L}_{\Sigma^{\perp}}$, and those of $v_b^{\wedge\vee}(\psi)$ and $v_i^{\wedge\vee}(\psi)$, for $\psi \in \mathcal{L}_{\Sigma^{\wedge,\vee}}$, are defined by induction on $l(\psi)$. To begin with, if ψ is such that $l(\psi) = 0$, then $\psi \in Var$ (if $\psi \in \mathcal{L}_{\Sigma^{\perp}}$ or $\psi \in \mathcal{L}_{\Sigma^{\wedge,\vee}}$), or $\psi = \perp$ (if $\psi \in \mathcal{L}_{\Sigma^{\perp}}$). The bivaluations, for this cases, are defined as:

1. $v_b^{\perp}(p) = v_i^{\perp}(p) = v_b^{\wedge\vee}(p) = v_i^{\wedge\vee}(p) = v_0(p)$, for all $p \in Var$;
2. $v_b^{\perp}(\perp) = v_i^{\perp}(\perp) = 0$.

Suppose now that $l(\psi) = n$, $n > 1$, and that the bivaluations are defined for all ψ' such that $l(\psi') < n$. According to the main connective of ψ the definition goes as follows:

1. If $\psi = \alpha \to \beta$, then, for $v \in \{v_b^{\wedge\vee}, v_i^{\wedge\vee}, v_b^{\perp}, v_i^{\perp}\}$:

$$v(\alpha \to \beta) = 1 \quad \Longleftrightarrow \quad v(\alpha) = 0 \text{ or } v(\beta) = 1$$

2. If $\psi = \alpha \wedge \beta$, then, for $v \in \{v_b^{\wedge\vee}, v_i^{\wedge\vee}\}$:

$$v(\alpha \wedge \beta) = 1 \quad \Longleftrightarrow \quad v(\alpha) = 1 \text{ and } v(\beta) = 1$$

3. If $\psi = \alpha \vee \beta$, then, for $v \in \{v_b^{\wedge\vee}, v_i^{\wedge\vee}\}$:

$$v(\alpha \vee \beta) = 1 \quad \Longleftrightarrow \quad v(\alpha) = 1 \text{ or } v(\beta) = 1$$

4. If $\psi = \neg\gamma$, there are two cases:

   (a) If, on the one hand, $v \in \{v_b^{\wedge\vee}, v_b^{\perp}\}$ and γ is arbitrary or, on the other hand, $v \in \{v_i^{\wedge\vee}, v_i^{\perp}\}$ and $\gamma \neq \neg^k \circ \gamma'$, for all $k \geq 0$ and formula γ', then:

$$v(\neg\gamma) = \begin{cases} 1 & \text{if } v(\gamma) = 0, \text{ or} \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

   (b) If $v \in \{v_i^{\wedge\vee}, v_i^{\perp}\}$ and $\gamma = \neg^k \circ \gamma'$, for some $k \geq 0$ and formula γ', then:

$$v(\neg\gamma) = 1 \quad \Longleftrightarrow \quad v(\gamma) = 0$$

5. If $\psi = \circ\gamma$, then, for $v \in \{v_b^{\wedge\vee}, v_b^{\perp}\}$:

$$v(\circ\gamma) = \begin{cases} 0 & \text{if } v(\gamma) = v(\neg\gamma) = 1, \text{ or} \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

   and, for $v \in \{v_i^{\wedge\vee}, v_i^{\perp}\}$:

$$v(\circ\gamma) = 0 \quad \Longleftrightarrow \quad v(\gamma) = v(\neg\gamma) = 1$$

It is left to the reader to check that the above definitions result indeed on bivaluations in $V^{\mathbf{mbC}^{\wedge\vee}}$, $V^{\mathbf{mCi}^{\wedge\vee}}$, $V^{\mathbf{mbC}^{\perp}}$ or $V^{\mathbf{mCi}^{\perp}}$, as required. □

Observe that in the process of the inductive definition above, it is possible to choose arbitrarily some values. Along these lines, in Theorem 35 a number of bivaluations are defined, modifying Lemma 20 only on those cases for which the value is choosen arbitrarily.

Now we will prove that the new logics are sound and complete for the semantics defined at the beginning of the present section.

Theorem 21 (Soundness)**.** *Let $\mathscr{L}$ be* $\mathbf{mbC}^{\perp}$ *or* $\mathbf{mCi}^{\perp}$*. Then, for every* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\Sigma^{\perp}}$*:*

$$\Gamma \vdash_{\mathscr{L}} \varphi \qquad \Longrightarrow \qquad \Gamma \vDash_{\mathscr{L}} \varphi$$

Proof. This proof presents no difficulty and it is left to the reader to check that the value of any bivaluation for $\mathbf{mbC}^{\perp}$ or $\mathbf{mCi}^{\perp}$ is always 1 for any instance of the axioms of Definitions 17 or 18, respectively. Additionally, if the value given by a bivaluation to the two premises of **MP** is 1 then the value given to the conclusion must be 1. ◻

The proof of completeness needs some definitions and results. Recall from Definition 1 the notion of Tarskian Logic.

Definition 22. For a given Tarskian logic $\mathscr{L}$ over the language $\mathcal{L}$, let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$. The set Γ is called *maximal non-trivial with relation to* φ if $\Gamma \nvdash_{\mathscr{L}} \varphi$ but $\Gamma, \psi \vdash_{\mathscr{L}} \varphi$ for any $\psi \notin \Gamma$.

A set of formulas Γ is *closed* in a Tarskian logic $\mathscr{L}$ if it holds, for every formula ψ: $\Gamma \vdash_{\mathscr{L}} \psi$ iff $\psi \in \Gamma$. The proof of the following result is straightforward:

Lemma 23. *Any set of formulas maximal non-trivial with relation to φ in $\mathscr{L}$ is closed, provided that $\mathscr{L}$ is Tarskian.*

In (Wójcicki 1984), Theorem 22.2, there is a proof of the following classical result:

Theorem 24 (Lindenbaum-Łos)**.** *Let $\mathscr{L}$ be a Tarskian and finitary logic over the language $\mathcal{L}$. Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$ such that $\Gamma \nvdash_{\mathscr{L}} \varphi$. Then, there exists a set Δ such that $\Gamma \subseteq \Delta \subseteq \mathcal{L}$ with Δ maximal non-trivial with relation to φ in $\mathscr{L}$.*

Every logic $\mathscr{L}$ defined by a Hilbert calculus where the inference rules are finitary is Tarskian and finitary, and so Theorem 24 holds in $\mathscr{L}$. In particular, Theorem 24 holds for $\mathbf{mbC}^{\perp}$ and $\mathbf{mCi}^{\perp}$.

Theorem 25. *Let* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\Sigma^{\perp}}$*, with* Γ *maximal non-trivial with relation to* φ *in* $\mathbf{mbC}^{\perp}$ *(resp. in* $\mathbf{mCi}^{\perp}$*). The mapping* $v : \mathcal{L}_{\Sigma^{\perp}} \to \{0, 1\}$ *defined by:*

$$v(\psi) = 1 \quad \Longleftrightarrow \quad \psi \in \Gamma$$

for all $\psi \in \mathcal{L}_{\Sigma^{\perp}}$ *is a bivaluation for* $\mathbf{mbC}^{\perp}$ *(resp. for* $\mathbf{mCi}^{\perp}$*).*

Proof. Let $\psi \in \mathcal{L}_{\Sigma^{\perp}}$ be an arbitrary formula. The cases common to both $\mathbf{mbC}^{\perp}$ and $\mathbf{mCi}^{\perp}$ will be firstly analyzed:

1. $\psi = \perp$. Suppose, by contradiction, that $\perp \in \Gamma$. As $\vdash_{\mathscr{L}} \perp \to \varphi$ (Lemma 16, Item 1) then $\perp \to \varphi \in \Gamma$, by Lemma 23. By **MP** and Lemma 23 again it follows that $\varphi \in \Gamma$, a contradiction. Therefore $\perp \notin \Gamma$ and so $v(\perp) = 0$.

2. $\psi = \neg\alpha$. Suppose $\neg\alpha \notin \Gamma$ and, by contradiction, that also $\alpha \notin \Gamma$. As Γ is maximal, it follows that $\Gamma, \neg\alpha \vdash_{\mathscr{L}} \varphi$ and $\Gamma, \alpha \vdash_{\mathscr{L}} \varphi$. By the Deduction Theorem, $\Gamma \vdash_{\mathscr{L}} \alpha \to \varphi$ and $\Gamma \vdash_{\mathscr{L}} \neg\alpha \to \varphi$. Now, by Lemma 16, Item 4, $\Gamma \vdash_{\mathscr{L}} ((\alpha \to \perp) \to \neg\alpha) \to \varphi$. However, $(\alpha \to \perp) \to \neg\alpha$ is an instance of Axiom $\sim\neg$, and then $\Gamma \vdash_{\mathscr{L}} \varphi$, a contradiction. Therefore:

$$v(\neg\alpha) = 0 \quad \Longrightarrow \quad v(\alpha) = 1\ .$$

3. $\psi = \circ\alpha$. Suppose $\circ\alpha \in \Gamma$ and, by contradiction, that both $\alpha \in \Gamma$ and $\neg\alpha \in \Gamma$. Then, by Axiom $\mathbf{bc1}^{\perp}$ and Lemma 23, $\sim\alpha \in \Gamma$. By definition of $\sim$ and by **MP** this implies that $\perp \in \Gamma$. By Lemma 16 Item 1 it follows that $\varphi \in \Gamma$, a contradiction. Therefore:

$$v(\circ\alpha) = 1 \quad \Longrightarrow \quad v(\alpha) = 0 \text{ or } v(\neg\alpha) = 0\ .$$

4. $\psi = \alpha \to \beta$. Suppose $\alpha \to \beta \in \Gamma$. If $\alpha \in \Gamma$ then $\beta \in \Gamma$, by **MP** and Lemma 23. Therefore:

$$v(\alpha \to \beta) = 1 \quad \Longrightarrow \quad v(\alpha) = 0 \text{ or } v(\beta) = 1\ .$$

Now, suppose $\alpha \notin \Gamma$ or $\beta \in \Gamma$. If $\beta \in \Gamma$ then $\alpha \to \beta \in \Gamma$ by Axiom **Ax1**, **MP** and Lemma 23. If $\alpha \notin \Gamma$ then, by the maximality of Γ, it follows that

$\Gamma, \alpha \vdash_{\mathscr{L}} \varphi$. Now, suppose, by contradiction, that $\alpha \to \beta \notin \Gamma$. Then, $\Gamma, \alpha \to \beta \vdash_{\mathscr{L}} \varphi$. By the Deduction Meta-Theorem, both $\Gamma \vdash_{\mathscr{L}} (\alpha \to \beta) \to \varphi$ and $\Gamma \vdash_{\mathscr{L}} \alpha \to \varphi$. By Lemma 16, Item 3, $\Gamma \vdash_{\mathscr{L}} \big(((\alpha \to \beta) \to \alpha) \to \alpha\big) \to \varphi$ and, by Item 2, $\Gamma \vdash_{\mathscr{L}} \varphi$, a contradiction. Therefore:

$$v(\alpha) = 0 \text{ or } v(\beta) = 1 \qquad \Longrightarrow \qquad v(\alpha \to \beta) = 1\ .$$

Now, the cases valid only for $\mathbf{mCi}^{\perp}$:

5. $\psi = \neg\circ\alpha$. Suppose $\neg\circ\alpha \in \Gamma$. This implies, by Axioms $\mathbf{ci}^1$ and $\mathbf{ci}^2$, that $\alpha \in \Gamma$ and $\neg\alpha \in \Gamma$. Therefore:

$$v(\neg\circ\alpha) = 1 \qquad \Longrightarrow \qquad v(\alpha) = 1 \text{ and } v(\neg\alpha) = 1\ .$$

6. $\psi = \circ\neg^n\circ\alpha$. By Axiom $\mathbf{cc}_n$ and Lemma 23, it follows that:

$$v(\circ\neg^n\circ\alpha) = 1\ .$$

□

Corollary 26 (Completeness)**.** *Let* $\mathscr{L}$ *be* $\mathbf{mbC}^{\perp}$ *or* $\mathbf{mCi}^{\perp}$*, then:*

$$\Gamma \vDash_{\mathscr{L}} \varphi \qquad \Longrightarrow \qquad \Gamma \vdash_{\mathscr{L}} \varphi$$

Proof. Suppose $\Gamma \nvdash_{\mathscr{L}} \varphi$ and let Δ be a set maximal non-trivial with relation to φ in $\mathscr{L}$ extending Γ (see Theorem 24). By Theorem 25, there is a bivaluation for $\mathscr{L}$ satisfying Γ (as $\Gamma \subseteq \Delta$) but not φ (as $\varphi \notin \Delta$). Therefore $\Gamma \nvDash_{\mathscr{L}} \varphi$ and so the theorem follows by contraposition. □

4. Equivalence between both formulations

In this section $\mathbf{mbC}^{\perp}$ and $\mathbf{mCi}^{\perp}$ will be shown to be equivalent to their counterparts $\mathbf{mbC}^{\wedge\vee}$ and $\mathbf{mCi}^{\wedge\vee}$. To achieve this, the formalism to compare logics known as *conservative translations between logics*, introduced in (Silva, D'Ottaviano, and Sette 1999), will be used. In what follows, if * is a mapping defined on formulas and Γ is a set of formulas then $\Gamma^* \overset{\text{def}}{=} \{\gamma^* \mid \gamma \in \Gamma\}$.

Definition 27 (Translation between Logics (Silva, D'Ottaviano, and Sette 1999))**.** Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be logics with sets of formulas $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. A mapping $^*\colon \mathcal{L}_1 \to \mathcal{L}_2$ is said to be a *translation from* $\mathscr{L}_1$ *to* $\mathscr{L}_2$ if, for every $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}_1$:

$$\Gamma \vdash_{\mathscr{L}_1} \alpha \qquad \Longrightarrow \qquad \Gamma^* \vdash_{\mathscr{L}_2} \alpha^*\ .$$

And it is called a *conservative translation* if it satisfies the stronger property:

$$\Gamma \vdash_{\mathscr{L}_1} \alpha \qquad \Longleftrightarrow \qquad \Gamma^* \vdash_{\mathscr{L}_2} \alpha^* .$$

Recall the notion of Tarskian logic (Definition 1). A logic satisfying Item (ii) of that definition is called monotonic, and if it satisfies Item (iv) is called finitary. Then:

Theorem 28. *Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be monotonic logics, where $\mathscr{L}_1$ is also finitary, such that both logics have implications $\to$ and $\to'$ respectively, satisfying the Deduction Meta-Theorem MTD (see Theorem 15). Suppose that $^*\colon \mathcal{L}_1 \to \mathcal{L}_2$ is a mapping for which:*

$$\vdash_{\mathscr{L}_1} \alpha \qquad \Longrightarrow \qquad \vdash_{\mathscr{L}_2} \alpha^* ,$$

and this mapping is such that $(\alpha \to \beta)^* = \alpha^* \to' \beta^*$. Then * is a translation from $\mathscr{L}_1$ to $\mathscr{L}_2$. Moreover, if $\mathscr{L}_2$ is also compact and * satisfies the stronger property:

$$\vdash_{\mathscr{L}_1} \alpha \qquad \Longleftrightarrow \qquad \vdash_{\mathscr{L}_2} \alpha^* ,$$

then the mapping * is also a conservative translation.

Proof. Suppose $\Gamma \vdash_{\mathscr{L}_1} \alpha$. By the finitariness of $\mathscr{L}_1$, there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathscr{L}_1} \alpha$. Now, suppose that $\Gamma_0 = \{\gamma_1, \ldots, \gamma_n\}$ is non-empty. Then, from the fact that $\to$ satisfies MTD, $\vdash_{\mathscr{L}_1} \gamma_1 \to \big(\ldots \to (\gamma_n \to \alpha)\ldots\big)$. From the hypothesis on *, it is the case that:

$$\vdash_{\mathscr{L}_2} \Big(\gamma_1 \to \big(\ldots \to (\gamma_n \to \alpha)\ldots\big)\Big)^*$$

and:

$$\vdash_{\mathscr{L}_2} \gamma_1^* \to' \big(\ldots \to' (\gamma_n^* \to' \alpha^*)\ldots\big) .$$

From the fact that $\to'$ satisfies MTD:

$$\gamma_1^*, \ldots, \gamma_n^* \vdash_{\mathscr{L}_2} \alpha^*$$

and, from the monotonicity of $\mathscr{L}_2$, $\Gamma^* \vdash_{\mathscr{L}_2} \alpha^*$. The case when Γ_0 is empty is even simpler. The other statements are proved similarly. □

Two mappings will now be defined by induction on the formula complexity. They will be proved to be conservative translations latter on, on the present section.

Definition 29. Fix an arbitrary propositional variable in Var, for instance p_1. The mapping $^{\circledast} : \mathcal{L}_{\Sigma^{\perp}} \to \mathcal{L}_{\Sigma^{\wedge,\vee}}$ is defined inductively for all $\varphi \in \mathcal{L}_{\Sigma^{\perp}}$ as follows:

$$\begin{aligned} q^{\circledast} &= q, \text{ if } q \in Var; \\ \perp^{\circledast} &= p_1 \wedge (\neg p_1 \wedge \circ p_1); \\ (\#\alpha)^{\circledast} &= \#(\alpha^{\circledast}) \text{ for } \# \in \{\neg, \circ\}; \\ (\alpha \to \beta)^{\circledast} &= \alpha^{\circledast} \to \beta^{\circledast}. \end{aligned}$$

Definition 30. The mapping $^* : \mathcal{L}_{\Sigma^{\wedge,\vee}} \to \mathcal{L}_{\Sigma^{\perp}}$ is defined by induction on $l(\varphi)$, for all $\varphi \in \mathcal{L}_{\Sigma^{\wedge,\vee}}$ as follows:

$$\begin{aligned} q^* &= q, \text{ if } q \in Var; \\ (\#\alpha)^* &= \#(\alpha^*) \text{ for } \# \in \{\neg, \circ\}; \\ (\alpha \to \beta)^* &= \alpha^* \to \beta^*; \\ (\alpha \vee \beta)^* &= (\alpha^* \to \perp) \to \beta^*; \\ (\alpha \wedge \beta)^* &= (\alpha^* \to (\beta^* \to \perp)) \to \perp. \end{aligned}$$

The injectivity of these mappings needs to be established, in order to be possible to properly define the bivaluations of Theorem 35. But, first, an intermediary result is given:

Lemma 31. *There is no formulas $\varphi, \psi \in \mathcal{L}_{\Sigma^{\wedge,\vee}}$ satisfying the following equation:*

$$\varphi^* \to \perp = \psi^*$$

Proof. Suppose, by contradiction, there is a solution in $\mathcal{L}_{\Sigma^{\wedge,\vee}}$ for the above identity and let φ and ψ be such a solution with minimum value of $l(\varphi)+l(\psi)$. Observe now that $\perp$ is not on the image of * and so $\psi \neq \alpha \to \beta$ and $\psi \neq \alpha \vee \beta$, for any of these would imply $\beta^* = \perp$. Therefore, the only way to get the image of ψ to be $\varphi^* \to \perp$ is with $\psi = \psi_1 \wedge \psi_2$. Therefore $\varphi^* \to \perp = (\psi_1^* \to (\psi_2^* \to \perp)) \to \perp$, and so $\varphi^* = \psi_1^* \to (\psi_2^* \to \perp)$.

Now, there are two cases:

1. $\varphi = \varphi_1 \to \varphi_2$. Therefore $\varphi^* = \varphi_1^* \to \varphi_2^*$, $\varphi_1^* = \psi_1^*$ and $\varphi_2^* = \psi_2^* \to \bot$.
2. $\varphi = \varphi_1 \vee \varphi_2$. Therefore $\varphi^* = (\varphi_1^* \to \bot) \to \varphi_2^*$, $\varphi_1^* \to \bot = \psi_1^*$ and $\varphi_2^* = \psi_2^* \to \bot$.

In both cases (ψ_2, φ_2) is a solution to the equation in question with $l(\psi_2) + l(\varphi_2) < l(\varphi) + l(\psi)$, a contradiction. □

Theorem 32. *The mappings* $^{\circledast} : \mathcal{L}_{\Sigma^{\bot}} \to \mathcal{L}_{\Sigma^{\wedge,\vee}}$ *and* $^{*} : \mathcal{L}_{\Sigma^{\wedge,\vee}} \to \mathcal{L}_{\Sigma^{\bot}}$ *from Definitions 29 and 30 are injective.*

Proof. Let $\varphi, \psi \in \mathcal{L}_{\Sigma^{\bot}}$ be such that $\varphi^{\circledast} = \psi^{\circledast}$. By induction on $l(\varphi)+l(\psi)$ it is easy to prove that $\varphi = \psi$. It is a consequence of the fact that, by Definition 29, there are no two different equations producing, to the right, formulas with the same main connective.

Now, let $\varphi, \psi \in \mathcal{L}_{\Sigma^{\wedge,\vee}}$ be such that $\varphi^* = \psi^*$. The proof is by induction, analogous to that for $^{\circledast}$. However, the induction step for which the main connective of both sides of the above equation is $\to$ is a bit more complicated. In fact, there are three equations on Definition 30 producing, to the right, a formula with $\to$ as the main connective. So, let $\varphi^* = \alpha' \to \beta' = \psi^*$. Then, there are the following possibilities:

a) $\varphi = \alpha \to \beta$ and $\psi = \gamma \to \delta$. Therefore, $\varphi^* = \alpha^* \to \beta^* = \gamma^* \to \delta^* = \psi^*$. By unique readability, it follows that $\alpha^* = \gamma^*$ and $\beta^* = \delta^*$. The result is then obtained by the induction hypothesis: $\alpha = \gamma$ and $\beta = \delta$, which implies that $\varphi = \psi$.

b) $\varphi = \alpha \to \beta$ and $\psi = \gamma \vee \delta$. Therefore, $\varphi^* = \alpha^* \to \beta^* = (\gamma^* \to \bot) \to \delta^* = \psi^*$. By unique readability, $\alpha^* = \gamma^* \to \bot$, which is impossible by Lemma 31.

c) $\varphi = \alpha \to \beta$ and $\psi = \gamma \wedge \delta$. This is impossible, for it would imply $\beta^* = \bot$.

d) $\varphi = \alpha \vee \beta$ and $\psi = \gamma \vee \delta$. Then, like in item a), $\alpha = \gamma$ and $\beta = \delta$, from the fact that $\alpha^* \to \bot = \gamma^* \to \bot$ and $\beta^* = \delta^*$. Then $\varphi = \psi$.

e) $\varphi = \alpha \vee \beta$ and $\psi = \gamma \wedge \delta$. This is impossible, for it would imply $\beta^* = \bot$.

f) $\varphi = \alpha \wedge \beta$ and $\psi = \gamma \wedge \delta$. Then, $\alpha^* \to (\beta^* \to \bot) = \gamma^* \to (\delta^* \to \bot)$, which implies that $\alpha = \gamma$ and $\beta = \delta$. Therefore $\varphi = \psi$. □

Corollary 33.
1. Let $\varphi = \#\gamma \in \mathcal{L}_{\Sigma^{\perp}}$*, with* $\# \in \{\neg, \circ\}$*. If* $\varphi \in Im(^*) = \{\psi^* \mid \psi \in \mathcal{L}_{\Sigma^{\wedge,\vee}}\}$*, there exists a unique formula* $\delta \in \mathcal{L}_{\Sigma^{\wedge,\vee}}$ *such that* $\varphi = (\#\delta)^*$*.*

2. Let $\varphi = \#\gamma \in \mathcal{L}_{\Sigma^{\wedge,\vee}}$*, with* $\# \in \{\neg, \circ\}$*. If* $\varphi \in Im(^{\circledast}) = \{\psi^{\circledast} \mid \psi \in \mathcal{L}_{\Sigma^{\perp}}\}$*, there exists a unique formula* $\delta \in \mathcal{L}_{\Sigma^{\perp}}$ *such that* $\varphi = (\#\delta)^{\circledast}$*.*

Proof. It is a direct consequence of the injectivity and the very definition of the mappings * and $^{\circledast}$. □

Lemma 34.
1. Let $v \in V^{\mathbf{mbC}^{\wedge\vee}}$ *(resp.* $v \in V^{\mathbf{mCi}^{\wedge\vee}}$*). Then the mapping* $v' : \mathcal{L}_{\Sigma^{\perp}} \to \{0,1\}$ *defined by* $v'(\varphi) \stackrel{def}{=} v(\varphi^{\circledast})$ *is such that* $v' \in V^{\mathbf{mbC}^{\perp}}$ *(resp.* $v' \in V^{\mathbf{mCi}^{\perp}}$*).*

2. Let $v \in V^{\mathbf{mbC}^{\perp}}$ *(resp.* $v \in V^{\mathbf{mCi}^{\perp}}$*). Then the mapping* $v' : \mathcal{L}_{\Sigma^{\wedge,\vee}} \to \{0,1\}$ *defined by* $v'(\varphi) \stackrel{def}{=} v(\varphi^*)$ *is such that* $v' \in V^{\mathbf{mbC}^{\wedge\vee}}$ *(resp.* $v' \in V^{\mathbf{mCi}^{\wedge\vee}}$*).*

Proof. 1. Let $\varphi \in \mathcal{L}_{\Sigma^{\perp}}$ be an arbitrary formula. We will prove that v' satisfies the clauses from Definition 17 (also from Definition 18, if $v \in V^{\mathbf{mCi}^{\wedge\vee}}$). Firstly, the cases common to both $V^{\mathbf{mbC}^{\perp}}$ and $V^{\mathbf{mCi}^{\perp}}$ will be analyzed:

a) $\varphi = \bot$. Then $\varphi^{\circledast} = p_1 \wedge (\neg p_1 \wedge \circ p_1)$ and so $v(\varphi^{\circledast}) = 0$ for any bivaluation for $\mathbf{mbC}^{\wedge\vee}$ or $\mathbf{mCi}^{\wedge\vee}$. Therefore $v'(\bot) = v(\varphi^{\circledast}) = 0$.

b) $\varphi = \neg\alpha$. Then $\varphi^{\circledast} = \neg(\alpha^{\circledast})$ and therefore, if $v'(\neg\alpha) = 0$, then $v(\neg(\alpha^{\circledast})) = 0$ (by definition of v'). Now, as v is a bivaluation for $\mathbf{mbC}^{\wedge\vee}$ or $\mathbf{mCi}^{\wedge\vee}$, it follows that $v(\alpha^{\circledast}) = 1$, and so $v'(\alpha) = 1$.

c) $\varphi = \circ\alpha$. Then $\varphi^{\circledast} = \circ(\alpha^{\circledast})$ and therefore, if $v'(\circ\alpha) = 1$, then $v(\circ(\alpha^{\circledast})) = 1$. Now, as v is a bivaluation for $\mathbf{mbC}^{\wedge\vee}$ or $\mathbf{mCi}^{\wedge\vee}$, $v(\alpha^{\circledast}) = v'(\alpha) = 0$ or $v(\neg(\alpha^{\circledast})) = v'(\neg\alpha) = 0$.

d) $\varphi = \alpha \to \beta$. Then $\varphi^{\circledast} = \alpha^{\circledast} \to \beta^{\circledast}$ and, therefore, $v'(\alpha \to \beta) = 1$ if, and only if, $v(\alpha^{\circledast} \to \beta^{\circledast}) = 1$. But the last occurs exactly when $v(\alpha^{\circledast}) = 0$ or $v(\beta^{\circledast}) = 1$, that is, exactly when $v'(\alpha) = 0$ or $v'(\beta) = 1$.

Now, the cases valid only for $\mathbf{mCi}^{\perp}$:

e) $\varphi = \neg\circ\alpha$. Then, $\varphi^{\circledast} = \neg\circ(\alpha^{\circledast})$ and, therefore, if $v'(\neg\circ\alpha) = 1$, also $v(\neg\circ(\alpha^{\circledast})) = 1$. Now, as v is a bivaluation for $\mathbf{mCi}^{\wedge\vee}$, $v'(\alpha) = v(\alpha^{\circledast}) = 1$ and $v'(\neg\alpha) = v(\neg(\alpha^{\circledast})) = 1$.

f) $\varphi = \circ\neg^n\circ\alpha$. Then, $\varphi^{\circledast} = \circ\neg^n\circ(\alpha^{\circledast})$ and, as v is a bivaluation for $\mathbf{mCi}^{\wedge\vee}$, $v(\varphi^{\circledast}) = 1$. This implies that $v'(\circ\neg^n\circ\alpha) = 1$.

2. Let $\varphi \in \mathcal{L}_{\Sigma^{\wedge,\vee}}$ be an arbitrary formula. We will prove that v' is an $\mathbf{mbC}^{\wedge\vee}$-valuation. If φ is of the form $\neg\alpha$, $\circ\alpha$, $\alpha \to \beta$, $\neg\circ\alpha$ or $\circ\neg^n\circ\alpha$, the proof is similar to that of Item 1 above. Now, for the remaining cases, common to both $\mathbf{mbC}^{\wedge\vee}$ and $\mathbf{mCi}^{\wedge\vee}$:

a) $\varphi = \alpha \vee \beta$. Then $\varphi^* = (\alpha^* \to \bot) \to \beta^*$ and therefore, $v'(\alpha \vee \beta) = v((\alpha^* \to \bot) \to \beta^*)$. Since $\to$ and $\bot$ are interpreted as in propositional classical logic, it follows that $v'(\alpha\vee\beta) = v(\varphi^*) = 1$ iff $v'(\alpha) = v(\alpha^*) = 1$ or $v'(\beta) = v(\beta^*) = 1$.

b) $\varphi = \alpha \wedge \beta$. Then $\varphi^* = \big(\alpha^* \to (\beta^* \to \bot)\big) \to \bot$ and therefore, $v'(\alpha \wedge \beta) = v\Big(\big(\alpha^* \to (\beta^* \to \bot)\big) \to \bot\Big)$. By an argument as in the previous item, it follows that $v'(\alpha \wedge \beta) = v(\varphi^*) = 1$ iff $v'(\alpha) = v(\alpha^*) = 1$ and $v'(\beta) = v(\beta^*) = 1$. □

The next lemma establishes that, given a model (or counter-model) for a formula φ in the logics defined in the $\Sigma^\bot$ signature, there also exists a model (or counter-model) for $\varphi^{\circledast}$ in the logics defined in the $\Sigma^{\wedge,\vee}$ signature. Similarly, given a model (or counter-model) for a formula φ in the logics defined in the $\Sigma^{\wedge,\vee}$ signature, there also exists a model (or counter-model) for φ^* in the logics defined in the $\Sigma^\bot$ signature. As it will become clear later on, this result suffices to prove that the translations in question are conservative ones.

Lemma 35.
1. Let $v \in V^{\mathbf{mbC}^{\wedge\vee}}$ (resp. $v \in V^{\mathbf{mCi}^{\wedge\vee}}$). Therefore exists $v' \in V^{\mathbf{mbC}^{\bot}}$ (resp. $v' \in V^{\mathbf{mCi}^{\bot}}$) such that $v'(\varphi^*) = v(\varphi)$, for every $\varphi \in \mathcal{L}_{\Sigma^{\wedge,\vee}}$.

2. Let $v \in V^{\mathbf{mbC}^{\bot}}$ (resp. $v \in V^{\mathbf{mCi}^{\bot}}$). Therefore exists $v' \in V^{\mathbf{mbC}^{\wedge\vee}}$ (resp. $v' \in V^{\mathbf{mCi}^{\wedge\vee}}$) such that $v'(\varphi^{\circledast}) = v(\varphi)$, for every $\varphi \in \mathcal{L}_{\Sigma^\bot}$.

Proof. 1. Let v be a bivaluation for $\mathbf{mbC}^{\wedge\vee}$. Define a mapping $v' : \mathcal{L}_{\Sigma^\bot} \to \{0,1\}$ by induction on the complexity of the formulas $\psi \in \mathcal{L}_{\Sigma^\bot}$ as follows:

- If $\psi = q \in Var$ then $v'(q) = v(q)$.
- If $\psi = \bot$ then $v'(\bot) = 0$.
- If $\psi = \delta \to \gamma$ then $v'(\delta \to \gamma) = 1$ iff $v'(\delta) = 0$ or $v'(\gamma) = 1$.
- If $\psi = \neg\gamma$, then

$$v'(\neg\gamma) = \begin{cases} 1 & \text{if } v'(\gamma) = 0 \\ v(\neg\delta) & \text{if } \neg\gamma = (\neg\delta)^* \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$

- If $\psi = \circ\gamma$, then

$$v'(\circ\gamma) = \begin{cases} 0 & \text{if } v'(\gamma) = v'(\neg\gamma) = 1 \\ v(\circ\delta) & \text{if } \circ\gamma = (\circ\delta)^* \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$

Using Corollary 33 it is easy to prove, by induction on the complexity of formulas, that v' is well-defined and $v'(\varphi^*) = v(\varphi)$ for every $\varphi \in \mathcal{L}_{\Sigma^{\wedge,\vee}}$. Additionally, $v' \in V^{\mathbf{mbC}^\perp}$, by the proof of Lemma 20.

Now, if $v \in V^{\mathbf{mCi}^{\wedge\vee}}$, the definition of v' is as above, but with the following modifications:

- If $\psi = \neg\gamma$ but $\gamma \neq \neg^k\circ\delta$ for every $k \geq 0$ and every δ, then $v'(\neg\gamma)$ is defined as above. Otherwise, if $\psi = \neg\gamma$ for $\gamma = \neg^k\circ\delta$ then $v'(\neg\gamma) = 1$ iff $v'(\gamma) = 0$.
- If $\psi = \circ\gamma$, then $v'(\circ\gamma) = 0$ iff $v'(\gamma) = v'(\neg\gamma) = 1$.

By induction again, it is easy to prove that v' is a well-defined bivaluation for $\mathbf{mCi}^{\wedge\vee}$ such that $v'(\varphi^*) = v(\varphi)$ for every formula $\varphi \in \mathcal{L}_{\Sigma^{\wedge,\vee}}$.

2. Let v be a bivaluation for $\mathbf{mbC}^\perp$. Consider a mapping $v' : \mathcal{L}_{\Sigma^{\wedge,\vee}} \to \{0, 1\}$ defined by induction as follows:

- If $\psi = q \in Var$ then $v'(q) = v(q)$.
- If $\psi = \delta \wedge \gamma$ then $v'(\delta \wedge \gamma) = 1$ iff $v'(\delta) = v'(\gamma) = 1$.
- If $\psi = \delta \vee \gamma$ then $v'(\delta \vee \gamma) = 0$ iff $v'(\delta) = v'(\gamma) = 0$.
- If $\psi = \delta \to \gamma$ then $v'(\delta \to \gamma) = 1$ iff $v'(\delta) = 0$ or $v'(\gamma) = 1$.
- If $\psi = \neg\gamma$, then

$$v'(\neg\gamma) = \begin{cases} 1 & \text{if } v'(\gamma) = 0 \\ v(\neg\delta) & \text{if } \neg\gamma = (\neg\delta)^{\circledast} \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$

- If $\psi = \circ\gamma$, then

$$v'(\circ\gamma) = \begin{cases} 0 & \text{if } v'(\gamma) = v'(\neg\gamma) = 1 \\ v(\circ\delta) & \text{if } \circ\gamma = (\circ\delta)^{\circledast} \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$

By Corollary 33 it is straightforward to prove, by induction on the complexity of formulas, that v' is well-defined and $v'(\varphi^{\circledast}) = v(\varphi)$ for every $\varphi \in \mathcal{L}_{\Sigma^{\perp}}$. Moreover, $v' \in V^{\mathbf{mbC}^{\wedge\vee}}$, by the proof of Lemma 20.

Now, if $v \in V^{\mathbf{mCi}^{\perp}}$, the definition of v' is modified as in the proof item 1. □

The equivalence between these logics in the different languages can then be established in Theorem 37 as a consequence of the following:

Lemma 36. *The functions* $^{*} : \mathcal{L}_{\Sigma^{\wedge,\vee}} \to \mathcal{L}_{\Sigma^{\perp}}$ *and* $^{\circledast} : \mathcal{L}_{\Sigma^{\perp}} \to \mathcal{L}_{\Sigma^{\wedge,\vee}}$ *satisfy the following:*

$$\begin{array}{rcl} \vdash_{\mathbf{mbC}^{\perp}} \varphi & \iff & \vdash_{\mathbf{mbC}^{\wedge\vee}} \varphi^{\circledast} \\ \vdash_{\mathbf{mCi}^{\perp}} \varphi & \iff & \vdash_{\mathbf{mCi}^{\wedge\vee}} \varphi^{\circledast} \\ \vdash_{\mathbf{mbC}^{\wedge\vee}} \varphi & \iff & \vdash_{\mathbf{mbC}^{\perp}} \varphi^{*} \\ \vdash_{\mathbf{mCi}^{\wedge\vee}} \varphi & \iff & \vdash_{\mathbf{mCi}^{\perp}} \varphi^{*} \end{array}$$

Proof. Only the first and last statements will be proved, as the others have a similar demonstration. As a consequence of the completeness for the logics in both languages, the present lemma can be proved by using bivaluation semantics, namely:

$$\begin{array}{rcl} \models_{V^{\mathbf{mbC}^{\perp}}} \varphi & \iff & \models_{V^{\mathbf{mbC}^{\wedge\vee}}} \varphi^{\circledast} \\ \models_{V^{\mathbf{mCi}^{\wedge\vee}}} \varphi & \iff & \models_{V^{\mathbf{mCi}^{\perp}}} \varphi^{*} \end{array}$$

or equivalently, by contraposition:

$$\begin{array}{rcl} \exists v \in V^{\mathbf{mbC}^{\perp}} : v(\varphi) = 0 & \iff & \exists v \in V^{\mathbf{mbC}^{\wedge\vee}} : v(\varphi^{\circledast}) = 0 \\ \exists v \in V^{\mathbf{mCi}^{\wedge\vee}} : v(\varphi) = 0 & \iff & \exists v \in V^{\mathbf{mCi}^{\perp}} : v(\varphi^{*}) = 0 \,. \end{array}$$

For the first equivalence, suppose that there exists $v \in V^{\mathbf{mbC}^{\perp}}$ such that $v(\varphi) = 0$. By Lemma 35, Item 2, there exists $v' \in V^{\mathbf{mbC}^{\wedge\vee}}$ such that $v'(\varphi^{\circledast}) = v(\varphi) = 0$. On the other hand, if $v(\varphi^{\circledast}) = 0$ for some $v \in V^{\mathbf{mbC}^{\wedge\vee}}$ then, by Lemma 34, Item 1, there exists $v' \in V^{\mathbf{mbC}^{\perp}}$ such that $v'(\varphi) = v(\varphi^{\circledast}) = 0$.

Now, suppose that there exists $v \in V^{\mathbf{mCi}^{\wedge\vee}}$ such that $v(\varphi) = 0$. By Lemma 35, Item 1, there exists $v' \in V^{\mathbf{mCi}^{\perp}}$ such that $v'(\varphi^{*}) = v(\varphi) = 0$. Conversely, if $v(\varphi^{*}) = 0$ for some $v \in V^{\mathbf{mCi}^{\perp}}$ then, by Lemma 34, Item 2, there exists $v' \in V^{\mathbf{mCi}^{\wedge\vee}}$ such that $v'(\varphi) = v(\varphi^{*}) = 0$. □

Theorem 37. *The mapping* $^{\circledast} : \mathcal{L}_{\Sigma^{\perp}} \to \mathcal{L}_{\Sigma^{\wedge,\vee}}$ *is a conservative translation from* $\mathbf{mbC}^{\perp}$ *to* $\mathbf{mbC}^{\wedge\vee}$ *and from* $\mathbf{mCi}^{\perp}$ *to* $\mathbf{mCi}^{\wedge\vee}$*. The mapping* $^{*} : \mathcal{L}_{\Sigma^{\wedge,\vee}} \to \mathcal{L}_{\Sigma^{\perp}}$ *is a conservative translation from* $\mathbf{mbC}^{\wedge\vee}$ *to* $\mathbf{mbC}^{\perp}$ *and from* $\mathbf{mCi}^{\wedge\vee}$ *to* $\mathbf{mCi}^{\perp}$.

Proof. It is a direct consequence of Theorem 28 and Lemma 36. □

Remark 38. *The last result deserves some comments. Observe that E. Jeřábek proved recently in (Jeřábek 2012) that almost any two reasonable deductive systems (namely, extensions of a certain fragment of full Lambek calculus FL) can be conservatively translated into each other. Thus, the existence of conservative translations as the ones we found above should not be surprising.*

As a consequence of Jeřábek's result, several positions could be adopted. Under a pessimistic vision, conservative translations would be useless as it is always possible to find such a mapping between two given logics. But there is another, more interesting perspective: one of the main questions on the subject of translation between logics, namely "there exists a conservative translation between logics $\mathscr{L}_1$ *and* $\mathscr{L}_2$*", has changed to "this specific function is a conservative translation between logics* $\mathscr{L}_1$ *and* $\mathscr{L}_2$*". That is, the existence of a conservative translation between two given logics is no longer interesting (as it is always true), but the important point now is to establish a conservative translation with informational content, as the ones we obtained in Theorem 37. In the present case, they state that, in fact,* $\mathbf{mbC}^{\perp}$ *is a reformulation of* **mbC** *in the signature* $\Sigma^{\perp}$*. The same holds for* $\mathbf{mCi}^{\perp}$ *and* **mCi**.

It is worth noting that the definition of $\vee$ inside $\mathbf{mbC}^{\perp}$ must be exactly as we propose: if disjunction is interpreted as usual just in terms of the implication, the resulting mapping is no longer a conservative translation:

Proposition 39. *Let* $^{*} : \mathcal{L}_{\Sigma^{\wedge,\vee}} \to \mathcal{L}_{\Sigma^{\perp}}$ *be the translation mapping of Definition 30 except for the clause for* $\vee$*, which is replaced by the following:* $(\alpha \vee \beta)^{*} = (\alpha^{*} \to \beta^{*}) \to \beta^{*}$*. Then the mapping* * *thus defined, even though it is a translation from* $\mathbf{mbC}^{\wedge\vee}$ *to* $\mathbf{mbC}^{\perp}$*, it is not a conservative one. The same result holds for* $\mathbf{mCi}^{\wedge\vee}$ *and* $\mathbf{mCi}^{\perp}$.

Proof. First observe that both formulas $\alpha \vee \beta$ and $(\alpha \to \beta) \to \beta$ are translated into the same formula:

$$(\alpha \vee \beta)^* = (\alpha^* \to \beta^*) \to \beta^* = ((\alpha \to \beta) \to \beta)^*,$$

thus the translation is not injective. Moreover, there is a way to choose a formula such that its translation under * is a theorem, but there is some other formula translated to the same theorem which is not a theorem of the source logic. Consider, for instance, the formula:

$$\varphi = \neg\Big(\circ(\alpha \vee \beta) \wedge \neg(\alpha \vee \beta) \wedge (\alpha \vee \beta)\Big)\,.$$

It is easy to see that φ^* is a theorem of $\mathbf{mbC}^{\perp}$, and φ is also a theorem of $\mathbf{mbC}^{\wedge\vee}$ (the same holds for **mCi**). But now consider the following formula:

$$\psi = \neg\Big(\circ((\alpha \to \beta) \to \beta) \wedge \neg(\alpha \vee \beta) \wedge (\alpha \vee \beta)\Big)\,.$$

It is straightforward to prove that $\psi^* = \varphi^*$, but ψ is not a theorem in the source logic. This shows that, if $\mathscr{L} \in \{\mathbf{mbC}^{\wedge\vee}, \mathbf{mCi}^{\wedge\vee}\}$ represents some of the two logics in the old signature and $\mathscr{L}'$ is the same logic in the new signature, then

$$\vdash_{\mathscr{L}} \psi \qquad \not\Longleftarrow \qquad \vdash_{\mathscr{L}'} \psi^*. \qquad \square$$

This illustrates the impact of a logic not being self-extensional (see Section 2), and draws our attention to the care required when dealing with this kind of logics. As the proposition above shows, the right translation of disjunctions inside $\mathbf{mbC}^{\perp}$ is through the schema formula that uses $\to$ and $\perp$.

5. Sequents for mbC and mCi

Some work grounded on the sequent formalism has already been made for the **LFI**s. For instance, the logics **bC** and **Ci**, which respectively extend $\mathbf{mbC}^{\wedge\vee}$ and $\mathbf{mCi}^{\wedge\vee}$ by the addition of the axiom scheme:

$$\neg\neg\alpha \to \alpha \qquad \textbf{(cf)}$$

were formulated as the sequents systems **BC** and **CI** in (Gentilini 2011) and proved to admit cut elimination (as well as many other **LFI**s extending them). In (Rodrigues 2010), it can be found a proof of the cut elimination for **QMBC**, the first order extension of the fragment of **BC** suited to characterize $\mathbf{mbC}^{\wedge\vee}$. A method for obtaining cut-free sequent calculi for C-systems characterizable by finite Nmatrices is presented in (Avron, Konikowska, and Zamansky 2013) — although the method developed there is of a general character, the basic logic considered is slightly stronger than **mbC**.[3] In the present section sequents systems for $\mathbf{mbC}^{\perp}$ and $\mathbf{mCi}^{\perp}$ are presented and proved to admit cut-elimination. This is an intentionally self-contained section, and so it is long. The reader already acquainted with the notions and techniques of sequent calculi and proofs of cut-elimination can ignore most of the concepts and basic facts described herein.

5.1. Sequents Systems MBC and MCI

Along the present section, sets of formulas will be understood as *multi-sets*, that is, sets in which the elements can appear with multiplicity. In a given multi-set, some formula φ may have more than one *occurrence*, and each occurrence is distinct from the other, even if they correspond to the same formula. As with the usual sets, the order in which the elements occur in a multi-set does not matter.

Definition 40 (Sequent)**.** Sequents are pairs of multi-sets of formulas and are designated by the following notation:

$$\Gamma \longmapsto \Delta\,,$$

in which Γ and Δ are multi-sets of formulas.

Sometimes it is necessary to draw attention to some formula occurrences in a given sequent. Then such occurrences are indicated by meta-variables for occurrences (lower case greek letters), contrasting with those for multi-sets (upper case greek letters). Such occurrences are called *designated occurrences*. For instance, in the following sequent:

$$\Delta, \delta \longmapsto \Gamma, \gamma$$

the occurrence of δ to the left and γ to the right are designated occurrences. That being said, sequent calculi can be defined by enumerating their *sequent rules*:

Definition 41 (Sequent Rule, Antecedent, Succedent)**.** *Sequent rules* are pairs whose first element is a sequence of sequents, called the *antecedent*, and the second is a single sequent, called the *succedent*, and these pairs are restricted to the conditions of Definition 43.

It is allowed for the antecedent of a sequent rule to be the empty sequence, and this is the simplest case of a sequent rule. In such cases the sequent rule is called an *axiom*:

Definition 42 (Axiom)**.** An *axiom* is any sequent rule whose antecedent is the empty sequence.

Although, there are some conditions for a given pair of antecedent and succedent to be considered a sequent rule:

Definition 43 (Occurrences Consumed and Produced)**.** In any sequent rule all occurrences of formulas present in a given side of some sequent in the antecedent, if there are any, must also be present on the same side of the succedent, except maybe for some occurrences which are said to have been *consumed*. Also all occurrences of formulas present in a given side of the succedent are taken from this same side on some of the antecedent's sequents, except maybe for one or two which are said to have been *produced*. Axiom rules are the only rules allowed to produce more than one formula occurrence.

This induces a relation on the formula occurrences of a sequent rule:

Definition 44 (Successor, Predecessor, Principal Formula Occurrence)**.** Each formula occurrence in the succedent of a sequent rule not produced by it is the *successor* of the corresponding occurrence in the antecedent, which is called its *predecessor*. The occurrences produced by the rule are the successors of those consumed and these, their predecessors. The *principal formula occurrences* of a sequent rule are those produced by this rule.

From this definition it follows that in a rule with empty antecedent, all formula occurrences in the succedent are principal and have no predecessors. Observe that it can be rules producing formula occurrences without consuming any and also rules consuming and not producing. The rules of sequent calculi are designed to be chained together in order to constitute proofs:

Definition 45 (Sequent Derivation)**.** For a given set of sequent rules, a *sequent derivation* is an upside down finite tree whose nodes are sequent rules and an edge between them can be established whenever a sequent in the antecedent of some node is the same as the succedent of some other node; in the case in which an edge is actually established the sequents are said to *participate* in this given edge. It is not allowed to be any node with a non participating sequent, except for the succedent of the root node, as well as any sequent must not participate in more than one edge. It is said that a given sequent derivation *derives*, is a *derivation for*, or *concludes* its root's succedent. It is said that the root's succedent is the *conclusion* of the sequent derivation.

From this definition it is clear that in the leaves of a sequent derivation tree are present only axioms from the set of the sequent rules. A sequent calculus $\mathscr{S}$ is identified with the enumeration of its sequent rules. For a given sequent calculus $\mathscr{S}$, it is represented that some sequent derivation ϖ derives the sequent $\Delta \longmapsto \Gamma$ by the following notation:

$$\begin{array}{c} \vdots\, \varpi \\ \Gamma \longmapsto \Delta \end{array}$$

or that ϖ derives $\Delta \longmapsto \Gamma$, its last rule is $\mathbf{R}$ and ϖ' is one of its subderivations:

$$\left.\begin{array}{c} \vdots\, \varpi' \\ \hline \Gamma \longmapsto \Delta \end{array} : \mathbf{R} \right\} \varpi$$

Concrete examples of sequent rules are the rules for contractions, whose definition is neecessary to be given earlier than those for the other rules, in order to properly define what means for formula to be introduced by some rule in a derivation:

Definition 46 (Contraction Rules)**.** For any multi-sets Γ and Δ of formulas of a given language $\mathcal{L}$ and formula $\alpha \in \mathcal{L}$, the following are contraction rules:

$$\frac{\Gamma, \alpha, \alpha \ \longmapsto\ \Delta}{\Gamma, \alpha \ \longmapsto\ \Delta} \ (\textbf{Ct-L}) \qquad \frac{\Gamma \ \longmapsto\ \Delta, \alpha, \alpha}{\Gamma \ \longmapsto\ \Delta, \alpha} \ (\textbf{Ct-R})$$

The relations of predecessor and successor on the formula occurrences of the sequent rules induce other relations on the occurrences of the entire derivation:

Definition 47 (Ancestor, Descendant, Formula Introduced by Rule)**.** An occurrence of a formula α in a sequent derivation is called the *ancestor* of a occurrence of a formula ω if these occurrences are the same occurrence of the same sequent or if the occurrence of α is the predecessor of an ancestor of the occurrence of ω. The occurrence of ω is then called a *descendant* of the occurrence of α and it is called a *integral descendant* if $\alpha = \omega$, in which case the occurrence of α is called a *direct ancestor* of the one of ω. If the occurrence of some formula ω has a direct ancestor which is principal for some application of a rule **R** different from a contraction, then this occurrence is said to be *introduced by **R***.

Observe that, due to contractions, a formula occurrence may be introduced by several rules.

Definition 48 (Derivation Height)**.** The *height of a derivation* ϖ is the height of the tree which constitutes it and it is denoted by $h(\varpi)$.

Now, the sequent calculi subject of the present section are introduced by the enumeration of their rules. Let Γ and Δ, followed or not by primes ($'$), be any multi-sets of formulas of $\mathcal{L}_{\Sigma^\perp}$ and φ, α or β be any formulas in this same language. The sequent calculi here presented are formed by the following sequent rules:

Definition 49 (**MBC**)**.**

- **Axioms**

$$\frac{}{\varphi \ \longmapsto\ \varphi} \ (\textbf{Ax}) \qquad \frac{}{\perp \ \longmapsto} \ (\perp\textbf{L})$$

- **Structural Rules**

$$\frac{\Gamma \ \longmapsto \ \Delta}{\Gamma, \alpha \ \longmapsto \ \Delta} \quad (\textbf{Wk-L}) \qquad \frac{\Gamma \ \longmapsto \ \Delta}{\Gamma \ \longmapsto \ \Delta, \alpha} \quad (\textbf{Wk-R})$$

$$\frac{\Gamma, \alpha, \alpha \ \longmapsto \ \Delta}{\Gamma, \alpha \ \longmapsto \ \Delta} \quad (\textbf{Ct-L}) \qquad \frac{\Gamma \ \longmapsto \ \Delta, \alpha, \alpha}{\Gamma \ \longmapsto \ \Delta, \alpha} \quad (\textbf{Ct-R})$$

$$\frac{\Gamma \ \longmapsto \ \Delta, \alpha \quad \alpha, \Gamma' \ \longmapsto \ \Delta'}{\Gamma, \Gamma' \ \longmapsto \ \Delta, \Delta'} \quad (\textbf{Cut})$$

- **Classic Logical Rules**

$$\frac{\Gamma \ \longmapsto \ \Delta, \alpha \quad \beta, \Gamma' \ \longmapsto \ \Delta'}{\alpha \to \beta, \Gamma, \Gamma' \ \longmapsto \ \Delta, \Delta'} \quad (\to\textbf{E}) \qquad \frac{\Gamma, \alpha \ \longmapsto \ \beta, \Delta}{\Gamma \ \longmapsto \ \alpha \to \beta, \ \Delta} \quad (\to\textbf{D})$$

$$\frac{\alpha, \Gamma \ \longmapsto \ \Delta}{\Gamma \ \longmapsto \ \Delta, \neg\alpha} \quad (\neg\textbf{R})$$

- **Paraconsistent Logical Rule**

$$\frac{\circ\alpha, \Gamma \ \longmapsto \ \Delta, \alpha}{\circ\alpha, \neg\alpha, \Gamma \ \longmapsto \ \Delta} \quad (\neg\textbf{L})$$

Definition 50 (**MCI**). All the rules from **MBC** plus the following:

- **Paraconsistent Logical Rules**

$$\frac{\Gamma \;\longmapsto\; \Delta,\ \neg^{n}\circ\alpha}{\neg^{n+1}\circ\alpha,\ \Gamma \;\longmapsto\; \Delta}\ (\neg\circ\mathbf{L}) \qquad \frac{\alpha,\ \neg\alpha,\ \Gamma \;\longmapsto\; \Delta}{\Gamma \;\longmapsto\; \Delta,\ \circ\alpha}\ (\circ\mathbf{R})$$

In the above definitions, all designated formula occurrences stand for those produced or consumed by the rule in which they appear, except for $\circ\alpha$ in $\neg$**L**. In this rule, the occurrence of α present on the right of the unique sequent of its antecedent is consumed and the one of $\neg\alpha$, present on the left of its succedent, is produced. The designated occurrence of $\circ\alpha$ to the left of the sequents is not modified by the rule, but is present to restrict the applicability of the rule only to sequents in which the assumption of the consistency of α is granted. This occurrence is called, following (Gentilini 2011), a *constraint formula occurrency*.

Definition 51. For a given bivaluation v a sequent $\Gamma \;\longmapsto\; \Delta$ is said to *hold* for v if there is an occurrence of $\gamma \in \Gamma$ such that $v(\gamma) = 0$ or there is an occurrence of $\delta \in \Delta$ such that $v(\delta) = 1$.

Theorem 52 (Soundness of the Sequent Calculi)**.** *If* $\Gamma \;\longmapsto\; \Delta$ *is derivable in* **MBC** *(resp.* **MCI***), then* $\Gamma \;\longmapsto\; \Delta$ *holds for all bivaluations* $v \in V^{\mathbf{mbC}^{\perp}}$ *(resp.* $v \in V^{\mathbf{mCi}^{\perp}}$*).*

Proof. It is left for the reader to check that for all bivaluations $v \in V^{\mathbf{mbC}^{\perp}}$ and instances r of rules of **MBC**, if $\Gamma_i \;\longmapsto\; \Delta_i$ holds for v for the sequents $\Gamma_i \;\longmapsto\; \Delta_i$ in the antecedent of r, then the sequent $\Gamma \;\longmapsto\; \Delta$ in the succedent of r also holds for it. The same is required for **MCI** and $V^{\mathbf{mCi}^{\perp}}$. □

The remaining of this subsection is devoted to show that the sequent calculi defined are indeed equivalent to the corresponding hilbertian logics.

Lemma 53. *Let* $\mathscr{S}$ *be* **MBC** *or* **MCI**. *If* $\Gamma \;\longmapsto\; \alpha \to \beta$ *is derivable in* $\mathscr{S}$, *then so it is* $\Gamma, \alpha \;\longmapsto\; \beta$.

Proof. It is left for the reader (only **Ax**, $\to$**E** and **Cut** are required). □

Corollary 54. *Let $\mathscr{S}$ be* **MBC** *of* **MCI***, the sets $\Gamma, \Delta \subseteq \mathcal{L}_{\Sigma^\perp}$, Γ being a finite multi-set such that $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ and $\psi \in \mathcal{L}_{\Sigma^\perp}$. Thus, there exists a derivation ϖ in $\mathscr{S}$ for the sequent $\Delta, \Gamma \longmapsto \psi$ if, and only if, there exists a derivation ϖ' in $\mathscr{S}$ for the sequent $\Delta \longmapsto \gamma_1 \to (\gamma_2 \to \dots (\gamma_n \to \psi) \dots)$.*

Proof. It also presents no difficulties. Proceed by induction on n, aplying Lemma 53 and $\to$**D**. □

Lemma 55. *Let $\mathscr{L}$ be* **mbC**$^\perp$ *or* **mCi**$^\perp$ *and $\mathscr{S}$ be, respectively,* **MBC** *or* **MCI***. Then φ is derivable in $\mathscr{L}$ if, and only if, there exists a sequent derivation ϖ in $\mathscr{S}$ for the sequent $\longmapsto \varphi$.*

Proof.

1. Suppose $\vdash_{\mathscr{L}} \varphi$ and proceed by induction on the length of the demonstration to obtain $\longmapsto \varphi$. Here will only be given the cases when the demonstration ends in the Axiom **bc1**$^\perp$ from **mbC**$^\perp$ and in **MP**:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\alpha \longmapsto \alpha}\ \textbf{Ax}}{\circ\alpha, \alpha \longmapsto \alpha, \bot}\ \textbf{Wk-L}, \textbf{Wk-R}}{\circ\alpha, \neg\alpha, \alpha \longmapsto \bot}\ \neg\textbf{L}}{\longmapsto \circ\alpha \to (\neg\alpha \to (\alpha \to \bot))}\ \to\textbf{D}(3x)$$

   Suppose now that the derivation $\vdash_{\mathscr{L}} \varphi$ ends in an application of **MP** on the formulas γ and $\gamma \to \varphi$. By induction hypothesis, there are derivations ϖ and ϖ' in $\mathscr{S}$ for the sequents $\longmapsto \gamma$ and $\longmapsto \gamma \to \varphi$, respectively. Now it is going to be proved that $\mathscr{S}$ can simulate this **MP**. The following derivation can then be constructed in $\mathscr{S}$:

$$\dfrac{\begin{array}{c}\vdots\ \varpi \\ \longmapsto \gamma\end{array} \qquad \dfrac{\begin{array}{c}\vdots\ \varpi' \\ \longmapsto \gamma \to \varphi\end{array}}{\gamma \longmapsto \varphi}\ \text{Theorem 53}}{\longmapsto \varphi}\ \textbf{Cut}$$

2. If there is a derivation ϖ in $\mathscr{S}$ for $\longmapsto \varphi$, then, by Theorem 52, $v(\varphi) = 1$ for all bivaluations for $\mathscr{L}$. By completeness, φ is a theorem of $\mathscr{L}$.

□

Theorem 56. *Let* $\mathscr{L}$ *be* $\mathbf{mbC}^{\perp}$ *or* $\mathbf{mCi}^{\perp}$ *and* $\mathscr{S}$ *be, respectively,* **MBC** *or* **MCI**. *Then* $\Gamma \vdash_{\mathscr{L}} \psi$ *if, and only if, there exists a finite subset* $\Gamma^{\circ} \subseteq \Gamma$ *such that* $\Gamma^{\circ} \longmapsto \psi$ *is derivable in* $\mathscr{S}$.

Proof. By the finitariness and monotonicity of $\mathscr{L}$, $\Gamma \vdash_{\mathscr{L}} \psi$ if, and only if, there is a finite subset $\Gamma^{\circ} \subseteq \Gamma$ such that $\Gamma^{\circ} \vdash_{\mathscr{L}} \psi$. Now, let $\Gamma^{\circ} = \{\gamma_1, \ldots, \gamma_n\}$. As $\to$ in these logics observes the MTD, $\Gamma^{\circ} \vdash_{\mathscr{L}} \psi$ if, and only if, $\vdash_{\mathscr{L}} \gamma_1 \to (\gamma_2 \to \ldots(\gamma_n \to \psi)\ldots)$. By Lemma 55, this is the case if, and only if, there is a derivation in $\mathscr{S}$ for $\longmapsto \gamma_1 \to (\gamma_2 \to \ldots(\gamma_n \to \psi)\ldots)$. By Corollary 54, this is the case if, and only if, there is a derivation in $\mathscr{S}$ for $\Gamma^{\circ} \longmapsto \psi$. □

5.2. Cut Elimination for MBC and MCI

In this subsection, the cut-elimination theorem is established for both systems **MBC** and **MCI**. The method used here is drawn from the one found in (Girard, Taylor, and Lafont, 1989) for classical logic, although it does not fit so elegantly as in the original. The reader not interested in syntactical details can safely skip this section.

Definition 57 (Cut Complexity, Cutting Formula)**.** The *complexity of an application of the **Cut** rule* is the complexity of the formula whose occurrences are consumed by this **Cut**, which is called its *cutting formula*.

Definition 58 (Derivation Complexity)**.** The *cut complexity of a sequent derivation* is the maximum value of the cut complexity of all cut applications occurring on it. If there is no such an aplication, the cut complexity of the derivation is 0.

Lemma 59. *Let* π *be a derivation in* $\mathscr{S} \in \{\mathbf{MBC}, \mathbf{MCI}\}$ *of the sequent* $\Gamma \longmapsto \Delta$ *and* φ *be a formula occurring in* Δ *introduced only by* ***Wk-R***. *Thus it can be constructed a proof in* $\mathscr{S}$ *for the sequent* $\Gamma \longmapsto \Delta - \varphi$ *with the same cut complexity of* π *and in which all formula occurrences in the concluding sequent* $\Gamma \longmapsto \Delta - \varphi$ *are introduced by the same rules introducing the corresponding occurrences in the concluding sequent* $\Gamma \longmapsto \Delta$ *of* π.

Proof. It suffices to remove from π all ancestors of the occurrences of φ in Δ, and maybe some applications of **Ct-R** on them, up to the introductory application(s) of **Wk-R** and then removing this(these) very application(s). It can be seem that this process will not break any rules or result in a ill-formed derivation. □

Corollary 60. *Let π be a derivation in $\mathscr{S} \in \{$***MBC***,* **MCI***$\}$ of the sequent $\Gamma \longmapsto \Delta$. Then there exists a derivation π' in $\mathscr{S}$ of the sequent $\Gamma \longmapsto \Delta - \bot$ with the same cut complexity of π and in which all formula occurrences of the concluding sequent $\Gamma \longmapsto \Delta - \bot$ are introduced by the same rules introducing the corresponding occurrences in the concluding sequent $\Gamma \longmapsto \Delta$ of π.*

Proof. Observe that there is no rule in **MBC** or **MCI** introducing $\bot$ to the right. Thus all occurrences of it in Δ must be introduced only by **Wk-R** and the result follows from Lemma 59. □

Lemma 61. *Let $\mathscr{S}$ be* **MBC** *or* **MCI***, π be a sequent derivation in $\mathscr{S}$ for $\Gamma \longmapsto \Delta, \varphi$ and ϖ be a sequent derivation in $\mathscr{S}$ for $\Lambda, \varphi \longmapsto \Xi$. If the designated occurrence of φ is introduced in π by a selected application of* ***Ax*** *and some other rules (including maybe another different applications of* ***Ax***), *then there is a sequent derivation π' in $\mathscr{S}$ for $\Gamma, \Lambda \longmapsto \Delta, \Xi, \varphi$:*

$$\begin{array}{c} \vdots\, \pi' \\ \Gamma, \Lambda \longmapsto \Delta, \Xi, \varphi \end{array}$$

whose cut complexity is the maximum of those of π and ϖ and in which the occurrence of φ designated in the concluding sequent is introduced by the same rules as those introducing the designated one in the conclusion of π, except for the selected application of ***Ax***. *Also all other formula occurrences in the concluding sequent of π' are introduced exactly by the same rules which introduce the corresponding occurrences in the conclusions of π or ϖ. Moreover, if the occurrence of φ in the conclusion of π is introduced only by the selected application of* ***Ax***, *then the proof π' can be constructed for $\Gamma, \Lambda \longmapsto \Delta, \Xi$:*

$$\begin{array}{c} \vdots\, \pi' \\ \Gamma, \Lambda \longmapsto \Delta, \Xi \end{array}$$

and, as well, the cut-complexity of π' is the maximum of those of π and ϖ and in this proof all formula occurrences in the conclusion are introduced *exactly by the same rules which introduce the corresponding occurrences in the conclusions of π or ϖ.*

Proof. Let r be the last rule of π. The proof goes by induction on $h(\pi)$. If $h(\pi) = 1$, the derivation π is restricted to a single application of **Ax** or $\bot$**L**. If $r = \bot\mathbf{L}$, then $\varphi = \bot$ and it is not introduced by **Ax** and then the result trivially holds. If $r = \mathbf{Ax}$, by the hypothesis on the concluding sequent of π, the rule r must be an application of **Ax** introducing φ, and also $\Gamma = \{\varphi\}$ and $\Delta = \{\}$. Observe that this is the only rule introducing φ and thus it suffices to take π' as ϖ. Suppose now that $h(\pi) > 1$ and that the result holds good for all sequent derivations ρ for which $h(\rho) < h(\pi)$.

Suppose now that the designated occurrence of φ is not produced by r and that r is any rule in which there is only one sequent in its antecedent. Let π_1 be the sub-derivation for the unique sequent in the antecedent of r, the sets Θ_1 and Θ_2 be the multi-sets formed by the formula occurrences consumed by r, and Π_1 and Π_2 the multi-sets formed by the occurrence produced by r. Thus, the proof π can be depicted as follows:

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \Gamma',\Theta_1 \;\longmapsto\; \Delta',\Theta_2,\varphi\end{array}}{\Gamma',\Pi_1 \;\longmapsto\; \Delta',\Pi_2,\varphi}\,r$$

in which the multi-sets of the succedent of r are such that $\Gamma = \Gamma' \cup \Pi_1$ and $\Delta = \Delta' \cup \Pi_2$. Observe that the designated occurrence of φ in the conclusion of π is introduced exactly by the same rules which introduce the designated occurrence of φ in the conlcusion of π_1.

Observe also that, as the rules considered produce exactly one formula occurrence, the multi-sets Π_i cannot be both different from $\emptyset$ at the same time for the same rule r. For instance, if $r = \to\mathbf{D}$, then $\Pi_1 = \emptyset$, $\Pi_2 = \alpha \to \beta$ and the multi-sets of consumed occurrences are then $\Theta_1 = \{\alpha\}$ and $\Theta_2 = \{\beta\}$. If $r = \mathbf{Wk\text{-}L}$, then $\Pi_2 = \Theta_1 = \Theta_2 = \emptyset$ and Π_1 is the singleton formed by the formula produced by r. Now, let $\Phi = \{\}$ if the designated occurrence of φ is introduced only by the selected application of **Ax** or $\Phi = \{\varphi\}$ otherwise. By induction hypothesis on π_1 and ϖ, there is a sequent derivation π_1' for $\Gamma', \Theta_1, \Lambda \;\longmapsto\; \Delta', \Theta_2, \Xi, \Phi$. The cut-complexity of π_1' is the maximum of those of π_1 and ϖ. If $\Phi \neq \{\}$, the occurrence of $\varphi \in \Phi$ in the conclusion of π_1'

is introduced by the same rules as those introducing the one in the conclusion of π_1, which coincide with those introducing φ in the conclusion of π, except for the selected application of **Ax**. Also all other formula ocurrences are introduced exactly by the same rules that introduce the corresponding occurrences in π_1 or ϖ. Now, let π' be the derivation constructed from π'_1 by the application of r:

$$\left.\dfrac{\begin{array}{c}\vdots\, \pi'_1 \\ \Gamma', \Theta_1, \Lambda \;\longmapsto\; \Delta', \Theta_2, \Xi, \Phi\end{array}}{\Gamma', \Pi_1, \Lambda \;\longmapsto\; \Delta', \Pi_2, \Xi, \Phi}\, r \right\} \pi'$$

This is possible because the only requirements for r to be applied, except in the case in which $r = \neg\mathbf{L}$, are satisfied by the presence of the multi-sets Θ_i of formula occurrences to be consumed by it. If $r = \neg\mathbf{L}$, introducing, say, an occurrence of a formula $\neg\alpha$, there also must be an occurrence of $\circ\alpha \in \Gamma'$ in order for r to be applied. However, this is always the case, as r is already employed in π, and this guarantees the occurrence of $\circ\alpha$ in Γ'. Observe that all formula ocurrences in the conclusion of π', other than the one in Φ, are introduced exactly by the same rules that introduce the corresponding occurrences in the conclusions of π or ϖ.

Now, suppose r has two sequents in its antecedent and does not produce the designated occurrence of φ. Therefore $r \in \{\rightarrow\mathbf{E}, \mathbf{Cut}\}$. Let π_1 and π_2 be the sequent derivations of the sequents in the antecedent of r and suppose the designated occurrence of φ in the conclusion of π has as its predecessor an occurrence in the conclusion of π_2. Let Π be the set formed by the occurrence produced by r (or the empty set, if there is none):

$$\left.\dfrac{\begin{array}{c}\vdots\, \pi_1 \\ \Gamma_1 \;\longmapsto\; \Delta_1, \alpha\end{array} \qquad \begin{array}{c}\vdots\, \pi_2 \\ \Gamma_2, \beta \;\longmapsto\; \Delta_2, \varphi\end{array}}{\Gamma_1, \Gamma_2, \Pi \;\longmapsto\; \Delta_1, \Delta_2, \varphi}\, r \right\} \pi$$

Observe that the designated occurrence of φ in the conclusion of π is introduced exactly by the same rules which introduce the designated occurrence of φ in the conlcusion of π_2. Now, let $\Phi = \{\}$ if the designated occurrence of φ is introduced only by the selected application of **Ax** or $\Phi = \{\varphi\}$ otherwise.

Let π_2' be obtained by the induction hypothesis on π_2 and ϖ. Thus, π' is the derivation constructed from π_1 and π_2' by the application of r on α and β:

$$\left.\dfrac{\dfrac{\vdots\, \pi_1}{\Gamma_1 \longmapsto \Delta_1, \alpha} \qquad \dfrac{\vdots\, \pi_2'}{\Gamma_2, \beta, \Lambda \longmapsto \Delta_2, \Xi, \Phi}}{\Gamma_1, \Gamma_2, \Pi, \Lambda \longmapsto \Delta_1, \Delta_2, \Xi, \Phi}\, r \right\} \pi'$$

The case in which the designated occurrence of φ has as its predecessor an occurrence in the conclusion of π_1 is similar. Also observe that the above derivation fits the other requirements of the present lemma.

It only remains to address the cases in which the designated occurrence of φ is produced by r. The rule r cannot be a logical rule or a **Wk-R**, as φ would then not be introduced by **Ax**. Thus, the only possible case is $r =$ **Ct-R** and then the derivation π is as follows:

$$\dfrac{\dfrac{\vdots\, \pi_1}{\Gamma \longmapsto \Delta, \varphi, \varphi}}{\Gamma \longmapsto \Delta, \varphi}\, r$$

The selected application of **Ax** introduce only one of the designated occurrences of φ in the conclusion of π_1. Let then $\Phi = \{\}$ if this occurrence is introduced only by the selected application of **Ax** or $\Phi = \{\varphi\}$ otherwise. Let π_1' be the sequent derivation obtained by induction hypothesis on π_1 and ϖ for the occurrence of φ introduced by the selected application of **Ax**:

$$\dfrac{\vdots\, \pi_1'}{\Gamma, \Lambda \longmapsto \Delta, \Xi, \Phi, \varphi}$$

Depending on whether $\Phi = \{\varphi\}$ or $\Phi = \{\}$, **Ct-R** is applied to the end of π_1' or not (respectively):

$$\dfrac{\dfrac{\vdots\, \pi_1'}{\Gamma, \Lambda \longmapsto \Delta, \Xi, \Phi, \varphi}}{\Gamma, \Lambda \longmapsto \Delta, \Xi, \varphi}\, (\textbf{Ct-R})^{?}$$

in which (**Ct-R**)$^?$ represents one or zero applications of **Ct-R**. It is not hard to see that the designated occurrence of φ in the conclusion of the above derivation is introduced by the same rules as those introducing φ in π, except for the selected application of **Ax** and that all other occurrences are introduced by the same rules introducing the corresponding occurrences in π or ϖ. The above derivation has the same cut-complexity as π_1' and so, by induction hypothesis, this is the maximum of π_1 and ϖ, what coincides with the maximum of π and ϖ. □

Corollary 62. *Let $\mathscr{S}$ be* **MBC** *or* **MCI** *and π be a sequent proof in $\mathscr{S}$ for $\Gamma \longmapsto \Delta$. Let ϖ be a sequent proof in $\mathscr{S}$ for $\Lambda, \varphi \longmapsto \Xi$ and $\widehat{\Delta}$ be the multiset obtained from Δ removing all occurrences of φ in Δ introduced only by* ***Ax*** *in the conclusion of π. Then there is a sequent derivation π' in $\mathscr{S}$ for $\Gamma, \Lambda \longmapsto \widehat{\Delta}, \Xi$ whose cut-complexity is the maximum of those of π and ϖ:*

$$\begin{array}{c} \vdots\, \pi' \\ \Gamma, \Lambda \longmapsto \widehat{\Delta}, \Xi \end{array}$$

in which all occurrences of φ in $\widehat{\Delta}$ (if there are any) are not introduced by ***Ax***.

Proof. The proof π' can be obtained by the repeatedly application of Lemma 61 on π and ϖ. □

Lemma 63. *Let ϖ be sequent proof in* **MBC** *or* **MCI** *of $\Gamma \longmapsto \Delta, \neg\varphi$:*

$$\begin{array}{c} \vdots\, \varpi \\ \Gamma \longmapsto \Delta, \neg\varphi \end{array}$$

in which the designated ocurrence of the formula $\neg\varphi$ is not introduced by ***Ax***. *Therefore there is a sequent proof ϖ' of $\Gamma, \varphi \longmapsto \Delta$ with the same cut-complexity of ϖ:*

$$\begin{array}{c} \vdots\, \varpi' \\ \Gamma, \varphi \longmapsto \Delta \end{array}$$

in which all formula ocurrences in the concluding sequent, except the one of φ designated on the left, are introduced exactly by the same rules introducing them in ϖ.

Proof. The proof goes by induction on $h(\varpi)$. If $h(\varpi) = 1$ the proof is trivial for ϖ is restricted to a single application of **Ax** or $\bot$**L** and in these cases the derivation ϖ does not fit the assumptions stated in the hypothesis. Suppose $h(\varpi) > 1$ and that the result holds good for all sequent derivations π for which $h(\pi) < h(\varpi)$. Let r be the last rule of ϖ and suppose that the designated occurrence of $\neg\varphi$ is not produced by r.

If r has only one sequent in its antecedent the proof is as follows. As in the proof of Lemma 61, let ϖ_1 be the sub-proof of the unique sequent in the antecedent of r, Θ_1 and Θ_2 be the multi-sets formed by the formulas occurrences consumed by r, and Π_1 and Π_2 multi-sets formed by the occurrence produced by r. Thus, the proof ϖ is as follows:

$$\dfrac{\begin{array}{c}\vdots\,\varpi_1\\ \Gamma, \Theta_1 \longmapsto \Delta, \Theta_2, \neg\varphi\end{array}}{\Gamma, \Pi_1 \longmapsto \Delta, \Pi_2, \neg\varphi}\, r$$

Now, by induction hypothesis, there is a sequent derivation ϖ_1' of $\Gamma, \Theta_1, \varphi \longmapsto \Delta, \Theta_2$ in which all formula ocurrences are introduced exactly by the same rules that introduce them in ϖ_1. Let ϖ' be the derivation constructed from ϖ_1' by the application of r:

$$\left.\dfrac{\begin{array}{c}\vdots\,\varpi_1'\\ \Gamma, \Theta_1, \varphi \longmapsto \Delta, \Theta_2\end{array}}{\Gamma, \Pi_1, \varphi \longmapsto \Delta, \Pi_2}\, r\right\}\varpi'$$

This is possible because the only requirements for r to be applied, except in the case in which $r = \neg$**L**, are satisfied by the presence of the multi-sets Θ_i of formulas to be consumed by it. If $r = \neg$**L** and its principal occurrence is of a formula $\neg\alpha$, it also must be a formula $\circ\alpha \in \Gamma$. However, this is always the case, as r is already employed in ϖ, what guarantees the presence of $\circ\alpha$ in Γ. It is easy to see that all formula ocurrences in the conclusion of ϖ' are introduced exactly by the same rules that introduce them in ϖ.

Now, suppose r has two sequents in its antecedent and, thus, $r \in \{\to$**E**, **Cut**$\}$. Let ϖ_1 and ϖ_2 be the sequent derivations of the hypotheses of r and suppose the designated occurrence of $\neg\varphi$ has as its predecessor an occurrence in the conclusion of ϖ_2. Let Π be the set formed by the occurrence produced by r (or the empty set, if there is none):

$$\left.\dfrac{\dfrac{\vdots\,\varpi_1}{\Gamma_1 \longmapsto \Delta_1, \alpha} \qquad \dfrac{\vdots\,\varpi_2}{\Gamma_2, \beta \longmapsto \Delta_2, \neg\varphi}}{\Gamma_1, \Gamma_2, \Pi \longmapsto \Delta_1, \Delta_2, \neg\varphi}\, r\right\}\varpi$$

Now, let ϖ'_2 be obtained by the induction hypothesis on ϖ_2. Thus, ϖ' is the derivation constructed from ϖ_1 and ϖ'_2 by the application of r on α and β:

$$\left.\dfrac{\dfrac{\vdots\,\varpi_1}{\Gamma_1 \longmapsto \Delta_1, \alpha} \qquad \dfrac{\vdots\,\varpi'_2}{\Gamma_2, \varphi, \beta \longmapsto \Delta_2}}{\Gamma_1, \Gamma_2, \Pi, \varphi \longmapsto \Delta_1, \Delta_2}\, r\right\}\varpi'$$

By induction hypothesis, the formula occurrences in the conclusion of the above derivation, except the one of φ designated on the left, are introduced by the same rules introducing the corresponding ones in the conclusion of ϖ. The case in which the occurrence of $\neg\varphi$ has as its predecessor an occurrence in the conclusion of ϖ_1 is similar. Now, it only remains to address the cases in which the designated occurrence of $\neg\varphi$ is the principal formula occurrence of r:

1. If $r = \neg$**R**, its principal formula occurrence is exactly the occurrence of $\neg\varphi$ in question. It suffices to take ϖ' as the sequent derivation for the only sequent in the antecedent of this application of r.

2. If $r =$ **Ct-R**, ϖ is as follows:

$$\dfrac{\dfrac{\vdots\,\varpi_1}{\Gamma \longmapsto \Delta, \neg\varphi, \neg\varphi}}{\Gamma \longmapsto \Delta, \neg\varphi}\, r$$

   By the induction hypothesis on ϖ_1, there is a sequent derivation ϖ'_1 such that:

$$\dfrac{\vdots\,\varpi'_1}{\Gamma, \varphi \longmapsto \Delta, \neg\varphi}$$

As the designated occurrence of $\neg\varphi$ on the right of the conclusion of ϖ is not introduced by **Ax**, so the occurrences of this same formula designated on the right of the conclusion of ϖ_1 are not. Thus, by the induction hypothesis on ϖ_1, the designated occurrence of $\neg\varphi$ on the right of the conclusion of ϖ_1' is not introduced by **Ax** and, by the induction hypothesis on ϖ_1', there is a sequent derivation ϖ_1'' such that:

$$\begin{array}{c} \vdots\, \varpi_1'' \\ \Gamma, \varphi, \varphi \;\longmapsto\; \Delta \end{array}$$

and the required derivation ϖ' can be obtained from ϖ_1'' by an application of **Ct-L**.

3. If r = **Wk-R**, with principal formula occurrence the designated $\neg\varphi$, ϖ' can be obtained from the proof of the antecedent of ϖ by an application of **Wk-L**.

□

Lemma 64. *Let ϖ be sequent proof in* **MCI** *of $\Gamma \;\longmapsto\; \Delta, \circ\varphi$:*

$$\begin{array}{c} \vdots\, \varpi \\ \Gamma \;\longmapsto\; \Delta, \circ\varphi \end{array}$$

in which the designated ocurrence of the formula $\circ\varphi$ in the concluding sequent is not introduced by ***Ax***. *Therefore there is a sequent proof ϖ' for $\Gamma, \varphi, \neg\varphi \;\longmapsto\; \Delta$ with the same cut-complexity of ϖ:*

$$\begin{array}{c} \vdots\, \varpi' \\ \Gamma, \varphi, \neg\varphi \;\longmapsto\; \Delta \end{array}$$

in which all formula ocurrences in the concluding sequent, except for those of φ and $\neg\varphi$ designated on the left, are introduced exactly by the same rules introducing them in ϖ.

Proof. The proof is very similar to the previous one and is left to the reader.

□

Lemma 65. *Let $\mathscr{S}$ be* **MBC** *or* **MCI** *and ϖ be a sequent derivation in $\mathscr{S}$ whose last rule is the application of a* ***Cut*** *on occurrences of a formula φ. Suppose also that φ is introduced to the right by the last rule r_1 of the derivation ϖ_1 for the first sequent in the antecedent of this* ***Cut****, and to the left by the last rule r_2 of the derivation ϖ_2 for the second sequent in the antecedent of this* ***Cut****:*

$$\dfrac{\dfrac{\vdots\,\varpi_1}{\Gamma_1 \longmapsto \Delta_1, \varphi}\,r_1 \qquad \dfrac{\vdots\,\varpi_2}{\varphi, \Gamma_2 \longmapsto \Delta_2}\,r_2}{\Gamma_1, \Gamma_2 \longmapsto \Delta_1, \Delta_2}\,\textbf{\textit{Cut}}$$

Suppose also that r_1 and r_2 are logical rules, and that the cut complexities of ϖ_1 and ϖ_2 are lower than that of ϖ. Then, there exists a sequent derivation ϖ' in $\mathscr{S}$ for the same sequent derived by ϖ and whose cut complexity is lower than that of ϖ.

Proof. Depending on the cut formula occurrence φ the proof is divided in some cases. The cases in which $\varphi = \bot$, $\varphi \in Var$ or $\varphi = \circ\alpha$ are impossible as there is no logical rule introducing $\bot$ to the right, no logical rule introducing $\circ\alpha$ to the left or introducing propositional variables to any side at all. The case in which $\varphi = \alpha \to \beta$ is the same as for classical logic and is left for the reader to check. For the remaing cases, $\varphi = \neg\alpha$, $r_1 = \neg\mathbf{R}$ and $r_2 \in \{\neg\mathbf{L}, \neg\circ\mathbf{L}\}$:

1. $\varphi = \neg\alpha$, $r_1 = \neg\mathbf{R}$ and $r_2 = \neg\mathbf{L}$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\varpi_1'\\ \Gamma_1, \alpha \longmapsto \Delta_1\end{array}}{\Gamma_1 \longmapsto \Delta_1, \neg\alpha}\,\neg\mathbf{R} \qquad \dfrac{\begin{array}{c}\vdots\,\varpi_2'\\ \Gamma_2', \circ\alpha \longmapsto \Delta_2, \alpha\end{array}}{\Gamma_2', \circ\alpha, \neg\alpha \longmapsto \Delta_2}\,\neg\mathbf{L}}{\Gamma_1, \Gamma_2', \circ\alpha \longmapsto \Delta_1, \Delta_2}\,\mathbf{Cut}$$

Then, ϖ is the following:

$$\dfrac{\dfrac{\vdots\, \varpi_2'}{\Gamma_2', \circ\alpha \longmapsto \Delta_2, \alpha} \qquad \dfrac{\vdots\, \varpi_1'}{\Gamma_1, \alpha \longmapsto \Delta_1}}{\Gamma_1, \Gamma_2', \circ\alpha \longmapsto \Delta_1, \Delta_2}\ \textbf{Cut}$$

2. $\varphi = \neg\alpha$, $r_1 = \neg\mathbf{R}$ and $r_2 = \neg\circ\mathbf{L}$. Therefore $\alpha = \neg^{n+1}\circ\alpha'$:

$$\dfrac{\dfrac{\dfrac{\vdots\, \varpi_1'}{\Gamma_1, \neg^n\circ\alpha' \longmapsto \Delta_1}}{\Gamma_1 \longmapsto \Delta_1, \neg^{n+1}\circ\alpha'}\ \neg\mathbf{R} \qquad \dfrac{\dfrac{\vdots\, \varpi_2'}{\Gamma_2 \longmapsto \Delta_2, \neg^n\circ\alpha'}}{\Gamma_2, \neg^{n+1}\circ\alpha' \longmapsto \Delta_2}\ \neg\circ\mathbf{L}}{\Gamma_1, \Gamma_2 \longmapsto \Delta_1, \Delta_2}\ \textbf{Cut}$$

Then, ϖ is the following:

$$\dfrac{\dfrac{\vdots\, \varpi_2'}{\Gamma_2 \longmapsto \Delta_2, \neg^n\circ\alpha'} \qquad \dfrac{\vdots\, \varpi_1'}{\Gamma_1, \neg^n\circ\alpha' \longmapsto \Delta_1}}{\Gamma_1, \Gamma_2 \longmapsto \Delta_1, \Delta_2}\ \textbf{Cut}$$

□

Lemma 66. *Let $\mathscr{S} \in \{\mathbf{MBC}, \mathbf{MCI}\}$, the derivations ϖ, ϖ', ϖ_1' and ϖ_{Ax}' be sequent derivations in $\mathscr{S}$ with cut-complexities lower than $l(\circ\gamma)$ for the sequent $\Gamma \longmapsto \Delta$, the sequent $\Gamma', \circ\gamma, \neg\gamma \longmapsto \Delta'$, the sequent $\Gamma, \Gamma' - \circ\gamma \longmapsto \Delta - \circ\gamma, \Delta', \gamma$ and the sequent $\circ\gamma, \Gamma' - \circ\gamma, \neg\gamma \longmapsto \Delta'$ respectively. Moreover suppose the last rule of ϖ' is an application of $\neg\mathbf{L}$ introducing the designated occurrence of $\neg\gamma$ on the left. These derivations can be depicted as follows:*

$$\dfrac{\vdots\, \pi}{\Gamma \longmapsto \Delta} \qquad \left.\dfrac{\dfrac{\vdots\, \pi_1'}{\Gamma', \circ\gamma \longmapsto \Delta', \gamma}}{\Gamma', \circ\gamma, \neg\gamma \longmapsto \Delta'}\ \neg\boldsymbol{L}\right\}\pi'$$

$$\dfrac{\vdots\, \varpi_1'}{\Gamma, \Gamma' - \circ\gamma \longmapsto \Delta - \circ\gamma, \Delta', \gamma} \qquad \dfrac{\vdots\, \varpi_{Ax}'}{\circ\gamma, \Gamma' - \circ\gamma, \neg\gamma \longmapsto \Delta'}$$

Thus a derivation ϖ in $\mathscr{S}$ can be obtained for $\Gamma, \Gamma' - \circ\gamma, \neg\gamma \longmapsto \Delta - \circ\gamma, \Delta'$ *with cut-complexity lower than* $l(\circ\gamma)$*.*

Proof. If $\circ\gamma \in \Gamma$, the derivation ϖ can be obtained applying $\neg$**L** to ϖ'_1, resulting in a derivation with the same cut-complexity of ϖ'_1. If $\circ\gamma \notin \Delta$, then $\Delta - \circ\gamma = \Delta$ and the derivation ϖ can be obtained by weakenings on π. In these two cases the derivation constructed has cut-complexity lower than $l(\circ\gamma)$, by the hypothesis on ϖ'_1 or π. For the remaining of the proof it can be assumed that $\circ\gamma \notin \Gamma$ and $\circ\gamma \in \Delta$.

Suppose now that there is no occurrence of $\circ\gamma$ in Δ introduced by **Ax**. In the case in which $\mathscr{S}$ = **MBC**, all occurrences of $\circ\gamma$ in Δ are introduced only by **Wk-R**, as there is no rule introducing the consistency connective $\circ$ to the right in **MBC**. By Lemma 59, it can be found a derivation for the sequent $\Gamma \longmapsto \Delta - \circ\gamma$ with the same cut-complexity of π and the required sequent can be obtained from it only by weakenings. In the case in which $\mathscr{S}$ = **MCI**, by repeatedly using Lemma 64 and some **Ct-L**, a derivation $\hat{\pi}$ with the same cut-complexity of π can be constructed such that:

$$\begin{array}{c} \vdots\, \hat{\pi} \\ \Gamma, \gamma, \neg\gamma \longmapsto \Delta - \circ\gamma \end{array}$$

It follows that a derivation for the required sequent can be constructed applying a **Cut** on the conclusions of ϖ'_1 and $\hat{\pi}$, consuming the designated occurrence of γ on the left of $\hat{\pi}$ and on the right of ϖ'_1, followed by some contractions:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, \varpi'_1 \\ \Gamma, \Gamma' - \circ\gamma \longmapsto \Delta - \circ\gamma, \Delta', \gamma\end{array} \qquad \begin{array}{c}\vdots\, \hat{\pi} \\ \Gamma, \gamma, \neg\gamma \longmapsto \Delta - \circ\gamma\end{array}}{\Gamma, \Gamma' - \circ\gamma, \Gamma, \neg\gamma \longmapsto \Delta - \circ\gamma, \Delta', \Delta - \circ\gamma}\,\textbf{Cut}}{\Gamma, \Gamma' - \circ\gamma, \neg\gamma \longmapsto \Delta - \circ\gamma, \Delta'}\,(\textbf{Ct-L}, \textbf{Ct-R})^*$$

Observe that the cut-complexity of the above derivation is the maximum of $l(\gamma)$, the cut-complexity of $\hat{\pi}$ and the cut-complexity of ϖ'_1, and so lower than $l(\circ\gamma)$. For the remaining of the proof it can then be assumed that there is at least an application of **Ax** in π introducing some occurrence of $\circ\gamma$ in Δ.

By Corollary 62 on π and ϖ'_{Ax}, it can be found a derivation $\widehat{\varpi'_{Ax}}$ for the sequent $\Gamma, \Gamma' - \circ\gamma, \neg\gamma \;\;\longmapsto\;\; \widehat{\Delta}, \Delta'$ in which $\widehat{\Delta} \subseteq \Delta$ stands for the multiset obtained by the removal of all those occurrences of $\circ\gamma$ in Δ introduced exclusively by **Ax** in ϖ'_{Ax}. All the remaining occurrences of $\circ\gamma$ in $\widehat{\Delta}$ (if any) are not introduced by **Ax** in $\widehat{\varpi'_{Ax}}$ and the cut-complexity of the derivation obtained in this way is the maximum of the cut-complexities of π and ϖ'_{Ax}. If there are no remaining occurrences of $\circ\gamma$ in $\widehat{\Delta}$, then $\widehat{\Delta} = \Delta - \circ\gamma$ and to complete the proof it suffices to take $\varpi = \widehat{\varpi'_{Ax}}$. Therefore it can be assumed that there is at least one occurrence of $\circ\gamma$ in $\widehat{\Delta}$. In the case in which $\mathscr{S} =$ **MBC**, as in **MBC** there is no rule introducing the consistency connective $\circ$ to the right, all remaining occurrences of $\circ\gamma$ in $\widehat{\Delta}$ are introduced only by **Wk-R** and, by Lemma 59, it is possible to construct a derivation for the required sequent. It can then be assumed that $\mathscr{S} =$ **MCI** and that there are some occurrences of $\circ\gamma$ in $\widehat{\Delta}$ but none of them introduced by **Ax** in $\widehat{\varpi'_{Ax}}$.

By repeatedly applying Lemma 64 and some **Ct-L**, there is a derivation with the same cut-complexity of $\widehat{\varpi'_{Ax}}$ for the sequent: $\Gamma, \Gamma' - \circ\gamma, \gamma, \neg\gamma \;\longmapsto\; \Delta - \circ\gamma, \Delta'$. By the application of a **Cut**, with cutting formula γ, on this derivation and on ϖ'_1 followed by some contractions, one can construct the following derivation for the required sequent:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \varpi'_1\\ \Gamma, \Gamma' - \circ\gamma \;\longmapsto\; \Delta - \circ\gamma, \Delta', \gamma\end{array} \qquad \begin{array}{c}\vdots\\ \Gamma, \Gamma' - \circ\gamma, \gamma, \neg\gamma \;\longmapsto\; \Delta - \circ\gamma, \Delta'\end{array}}{\Gamma, \Gamma' - \circ\gamma, \Gamma, \Gamma' - \circ\gamma, \neg\gamma \;\longmapsto\; \Delta - \circ\gamma, \Delta', \Delta - \circ\gamma, \Delta'}\ \mathbf{Cut}}{\Gamma, \Gamma' - \circ\gamma, \neg\gamma \;\longmapsto\; \Delta - \circ\gamma, \Delta'}\ (\mathbf{Ct\text{-}L}, \mathbf{Ct\text{-}R})^*$$

in which $(\mathbf{Ct\text{-}L}, \mathbf{Ct\text{-}R})^*$ stands for a number of applications of **Ct-L** and **Ct-R**. Observe that the above derivation has as cut-complexity the maximum of the cut-complexities of ϖ'_1, $\widehat{\varpi'_{Ax}}$ and of $l(\gamma)$, which is still lower than $l(\varphi)$. □

Lemma 67. *Let* $\mathscr{S} \in \{$**MBC**,**MCI**$\}$ *and* $\varphi \in \mathcal{L}_{\Sigma^\perp}$. *Let* π *and* π' *be sequent derivations in* $\mathscr{S}$ *for the sequents* $\Gamma \;\longmapsto\; \Delta$ *and* $\Gamma' \;\longmapsto\; \Delta'$*, respectively, whose cut complexities are lower than* $l(\varphi)$*. Therefore, there exists a sequent derivation* ϖ *in* $\mathscr{S}$ *of the sequent* $\Gamma, \Gamma' - \varphi \;\longmapsto\; \Delta - \varphi, \Delta'$*, whose cut complexity is lower then* $l(\varphi)$*.*

Proof. The proof goes by induction on $h(\pi) + h(\pi')$. The induction basis, for which $h(\pi) + h(\pi') = 2$, is established by itens 1a, 1b and 1c below. Suppose now that $h(\pi) + h(\pi') > 2$ and that, for all derivations ρ and ρ' of $\mathscr{S}$ for which $h(\rho) + h(\rho') < h(\pi) + h(\pi')$, deriving the sequents $\Gamma_\rho \longmapsto \Delta_\rho$ and $\Gamma_{\rho'} \longmapsto \Delta_{\rho'}$ respectively, there exists for all $\psi \in \mathcal{L}_{\Sigma^\perp}$ a sequent derivation in $\mathscr{S}$ for the sequent $\Gamma_\rho, \Gamma_{\rho'} - \psi \longmapsto \Delta_\rho - \psi, \Delta_{\rho'}$ whose cut-complexity is lower than $l(\psi)$. Let r denote the last rule of π and r' the last rule of π'.

1. Suppose $r \in \{$**Ax**$, \perp$**L**$\}$ or $r' \in \{$**Ax**$, \perp$**L**$\}$.

   (a) If $r = \perp$**L**, then $\Gamma = \{\perp\}$ and $\Delta = \{\}$. In such a case, the sequent to prove: $\perp, \Gamma' - \varphi \longmapsto \Delta'$ can be obtained from π by repeatedly weakening.

   (b) If $r' = \perp$**L**, then $\Gamma' = \{\perp\}$ and $\Delta' = \{\}$. In the case that $\varphi \neq \perp$, the sequent to prove: $\Gamma, \perp \longmapsto \Delta - \varphi$ can be obtained from π' by weakenings. If $\varphi = \perp$, the sequent to prove: $\Gamma \longmapsto \Delta - \perp$ can be obtained from π by Lemma 60.

   (c) If $r =$ **Ax** or $r' =$ **Ax**, the proof presents no difficulty and is left to the reader.

   For the remaining of the proof, it can then be assumed that r and r' are both different from **Ax** and $\perp$**L**.

2. Suppose r is a rule that does not produce an occurrence of φ to the right.

   (a) In the case in which r has just one sequent in its antecedent, π and π' are as follows:

$$\left.\begin{array}{c} \vdots\, \pi_1 \\ \Gamma, \Theta_1 \longmapsto \Delta, \Theta_2 \\ \hline \Gamma, \Pi_1 \longmapsto \Delta, \Pi_2 \end{array} r \right\} \pi \qquad\qquad \begin{array}{c} \vdots\, \pi' \\ \Gamma' \longmapsto \Delta' \end{array}$$

   in which the formulas consumed by r are present in the multisets Θ_1 and Θ_2 and the formula produced, in Π_1 or Π_2. By induction hypothesis, as $h(\pi_1) + h(\pi') < h(\pi) + h(\pi')$, the application of the lemma to π_1 and π' results in a sequent derivation

ϖ_1 for the sequent $\Gamma, \Theta_1, \Gamma' - \varphi \longmapsto \Delta - \varphi, \Theta_2 - \varphi, \Delta'$ with cut-complexity lower than $l(\varphi)$. The following derivation can now be constructed:

$$\cfrac{\cfrac{\begin{array}{c}\vdots\ \varpi_1\\ \Gamma, \Theta_1, \Gamma' - \varphi \longmapsto \Delta - \varphi, \Theta_2 - \varphi, \Delta'\end{array}}{\Gamma, \Theta_1, \Gamma' - \varphi \longmapsto \Delta - \varphi, \Theta_2, \Delta'}\ (\mathbf{Wk\text{-}R})^*}{\Gamma, \Pi_1, \Gamma' - \varphi \longmapsto \Delta - \varphi, \Pi_2, \Delta'}\ r$$

in which $(\mathbf{Wk\text{-}R})^*$ stands for zero or more applications of **Wk-R**, whether $\varphi \in \Theta_2$ or not. The above application of r is always possible for, if $r \neq \neg\mathbf{L}$, it needs only to consume the formulas in the sets Θ_1 and Θ_2 to produce the one in Π_1 or Π_2, in the same way it is done in π. In the case in which $r = \neg\mathbf{L}$, it can be applied for its constraint formula occurs in Γ, as it can be observed from the fact that r is also applied as the last rule of π, and in such a derivation the constraint formula can only occur in Γ. Observe now that the concluding sequent of the above derivation is exactly the one required, for $(\Delta, \Pi_2) - \varphi = \Delta - \varphi, \Pi_2$, as $\varphi \notin \Pi_2$, for it is presupposed r not to introduce φ to the right. Also observe that the above derivation has the same cut-complexity of ϖ_1, and so this is lower than $l(\varphi)$.

(b) If r has two sequents in its antecedent then $r \in \{\to\mathbf{E}, \mathbf{Cut}\}$ and does not produce φ to the right. The derivations π and π' are as follows:

$$\left.\cfrac{\begin{array}{c}\vdots\ \pi_1\\ \Gamma \longmapsto \Delta, \alpha\end{array} \qquad \begin{array}{c}\vdots\ \pi_2\\ \Gamma, \beta \longmapsto \Delta\end{array}}{\Gamma, \Pi \longmapsto \Delta}\ r\right\}\pi \qquad\qquad \begin{array}{c}\vdots\ \pi'\\ \Gamma' \longmapsto \Delta'\end{array}$$

in which, in the case that $r = \mathbf{Cut}$, the multi-set Π is empty and the occurrences α and β are from the same formula: the cutting formula. In the case that $r = \to\mathbf{E}$, the multi-set Π collects the occurrence produced by r, that is, $\Pi = \{\alpha \to \beta\}$. By induction hypothesis, there is a derivation ϖ_1 for the sequent

$\Gamma, \Gamma' - \varphi \longmapsto (\Delta, \alpha) - \varphi, \Delta'$ and a derivation ϖ_2 for $\Gamma, \beta, \Gamma' - \varphi \longmapsto \Delta - \varphi, \Delta'$, both of them with cut-complexity lower than $l(\varphi)$. The following derivation can then be constructed:

$$
\dfrac{\dfrac{\dfrac{\vdots\, \varpi_1}{\Gamma, \Gamma' - \varphi \longmapsto (\Delta, \alpha) - \varphi, \Delta'}}{\Gamma, \Gamma' - \varphi \longmapsto \Delta - \varphi, \alpha, \Delta'}\,(\mathbf{Wk\text{-}R})^{?} \qquad \begin{matrix}\vdots\, \varpi_2 \\ \Gamma, \beta, \Gamma' - \varphi \longmapsto \Delta - \varphi, \Delta'\end{matrix}}{\dfrac{\Gamma, \Gamma' - \varphi, \Gamma, \Gamma' - \varphi, \Pi \longmapsto \Delta - \varphi, \Delta', \Delta - \varphi, \Delta'}{\Gamma, \Pi, \Gamma' - \varphi \longmapsto \Delta - \varphi, \Delta'}\,(\mathbf{Ct\text{-}L}, \mathbf{Ct\text{-}R})*}\, r
$$

in which $(\mathbf{Wk\text{-}R})^{?}$ stands for zero or one application of **Wk-R** introducing α, depending whether $\alpha = \varphi$ or not. Observe that, if $r = \rightarrow\mathbf{E}$, the cut-complexity of the above definition is the maximum of the cut-complexities of ϖ_1 and ϖ_2, which are both lower than $l(\varphi)$. If $r = \mathbf{Cut}$, the cut-complexity of the above definition is the maximum of the cut-complexity of ϖ_1, the cut-complexity of ϖ_2 and the complexity of the application of this **Cut**. The cut-complexities of ϖ_1 and ϖ_2 are lower than $l(\varphi)$, by the inductive hypothesis, and the cut complexity of this **Cut** is also lower than $l(\varphi)$, for, by the hypothesis, the cut-complexity of π is lower than $l(\varphi)$.

3. Suppose r' is a rule that does not produce φ to the left.

   (a) Suppose r' is a logical rule different from $\rightarrow$**E** and $\neg$**L** that does not introduce φ to the left, or r' is a structural rule different from **Cut** that does not produce φ to the left. Thus π and π' are as follows:

$$
\begin{matrix}\vdots\, \pi \\ \Gamma \longmapsto \Delta\end{matrix} \qquad\qquad \left.\dfrac{\begin{matrix}\vdots\, \pi'_1 \\ \Gamma', \Theta_1 \longmapsto \Delta', \Theta_2\end{matrix}}{\Gamma', \Pi_1 \longmapsto \Delta', \Pi_2}\, r' \right\} \pi'
$$

   in which the formulas consumed by r' are present in the multisets Θ_1 and Θ_2, and the formula produced by it in Π_1 or Π_2. By induction hypothesis, as $h(\pi) + h(\pi'_1) < h(\pi) + h(\pi')$, the application of the lemma to π and π'_1 gives a sequent derivation ϖ'_1 for $\Gamma, \Gamma' - \varphi, \Theta_1 - \varphi \;\longmapsto\; \Delta - \varphi, \Delta', \Theta_2$ with cut-complexity lower than $l(\varphi)$. The following derivation can now be constructed:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, \varpi'_1 \\ \Gamma, \Gamma' - \varphi, \Theta_1 - \varphi \;\longmapsto\; \Delta - \varphi, \Delta', \Theta_2\end{array}}{\Gamma, \Gamma' - \varphi, \Theta_1 \;\longmapsto\; \Delta - \varphi, \Delta', \Theta_2}\,(\textbf{Wk-L})^*}{\Gamma, \Gamma' - \varphi, \Pi_1 \;\longmapsto\; \Delta - \varphi, \Delta', \Pi_2}\,r'$$

in which $(\textbf{Wk-L})^*$ stands for zero or more applications of **Wk-L** depending whether $\varphi \in \Theta_1$ or not. The above application of r' is always possible for, as $r' \neq \neg\textbf{L}$, it needs only to consume the formulas in the sets Θ_1 and Θ_2 to produce the one in Π_1 or Π_2. Observe now that the concluding sequent of the above derivation is exactly the one required, for $(\Gamma', \Pi_1) - \varphi = \Gamma' - \varphi, \Pi_1$, as $\varphi \notin \Pi_1$, for it is presupposed r' not to produce it to the left. Also observe that it has the same cut-complexity of ϖ'_1, which is lower than $l(\varphi)$.

(b) Suppose r' is $\neg\textbf{L}$ applied to an occurrence of a formula γ that does not introduce φ to the left. If $\varphi \neq \circ\gamma$, the proof goes as in the previous item: just observe, following the above construction, that in such a case $\circ\gamma \in \Gamma' - \varphi$. Thus it remains to address the case in which $\varphi = \circ\gamma$ and the derivations π and π' are as follows:

$$\begin{array}{c}\vdots\, \pi \\ \Gamma \;\longmapsto\; \Delta\end{array} \qquad\qquad \left.\dfrac{\begin{array}{c}\vdots\, \pi'_1 \\ \Gamma', \circ\gamma \;\longmapsto\; \Delta', \gamma\end{array}}{\Gamma', \circ\gamma, \neg\gamma \;\longmapsto\; \Delta'}\,\neg\textbf{L}\right\}\pi'$$

The induction hypothesis on π and π'_1 gives a derivation ϖ'_1 with cut-complexity lower than $l(\circ\gamma)$ such that:

$$\begin{array}{c}\vdots\, \varpi'_1 \\ \Gamma, \Gamma' - \circ\gamma \;\longmapsto\; \Delta - \circ\gamma, \Delta', \gamma\end{array}$$

By induction hypothesis on the one step derivation:

$$\dfrac{}{\circ\gamma \ \longmapsto\ \circ\gamma}\ \textbf{Ax}$$

and π' there is a sequent derivation ϖ'_{Ax} with cut-complexity lower than $l(\varphi)$ such that:

$$\begin{array}{c} \vdots\ \varpi'_{Ax} \\ \circ\gamma, \Gamma' - \circ\gamma, \neg\gamma \ \longmapsto\ \Delta' \end{array}$$

The result then follows from Lemma 66.

(c) Suppose that $r' = \to$**E** and does not introduce φ to the left, or that $r' =$ **Cut**. The proof is similar to one of item 2b and is left for the reader to check.

It can now be assumed that r produces an occurrence of φ to the right and r' produces an occurrence of φ to the left.

4. If $r \in \{$**Wk-R**, **Ct-R**$\}$ and produces an occurrence of φ, the result is obtained by the induction hypothesis on the sub-derivation for the antecedent of r and π'.

5. If $r' \in \{$**Wk-L**, **Ct-L**$\}$ and produces an occurrence of φ, the result is obtained by the induction hypothesis on π and the sub-derivation for the antecedent of r'.

   It can now be assumed that r is a logical rule introducing φ to the right and that r' is a logical rule introducing φ to the left.

6. Suppose r is a logical rule that introduces φ to the right and r' is a logical rule that introduces φ to the left. Let I be the number of sequents in the antecedent of π and J, the number of sequentes in the antecedent of r'. For $i \in \{1, \ldots, I\}$ and $j \in \{1, \ldots, J\}$, π_i and π'_j are, respectively, the sequent derivations of the sequents $\Gamma_i \ \longmapsto\ \Delta_i$ and $\Gamma'_j \ \longmapsto\ \Delta'_j$ in the antecedents of r and r'. For $i \in \{1, \ldots, I\}$, by the induction hypothesis on π_i and π', one obtains the derivation(s) ϖ_i for the sequent(s) $\Gamma_i, \Gamma' - \varphi \ \longmapsto\ \Delta_i - \varphi, \Delta'$. For $j \in \{1, \ldots, J\}$, by the induction hypothesis on π and π'_j, one obtains the derivation(s)

ϖ'_j for the sequent(s) $\Gamma, \Gamma'_j - \varphi \;\longmapsto\; \Delta - \varphi, \Delta'_j$. Now, the rule r can be applied at the end of the derivation(s) ϖ_i yelding the sequent $\Gamma, \Gamma' - \varphi \;\longmapsto\; \Delta - \varphi, \Delta', \varphi$ and r' at the end of the derivation(s) ϖ'_j yelding the sequent $\Gamma, \Gamma' - \varphi, \varphi \;\longmapsto\; \Delta - \varphi, \Delta'$. This can be done because the occurrences consumed by r are present in the multi-sets Γ_i and $\Delta_i - \varphi$, and the occurrences consumed by r', in the multi-sets $\Gamma'_j - \varphi$ and Δ'_j: just observe that the formulas consumed by these rules are different from φ, and so must also occur in $\Delta_i - \varphi$, for r, and in $\Gamma'_j - \varphi$, for r'. Also in the case in which $r' = \neg\mathbf{L}$ producing a formula $\varphi = \neg\gamma$, then $\varphi \neq \circ\gamma$, and so $\circ\gamma \in \Gamma'_1 - \varphi$. It is possible then to construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\vdots\,\varpi_1}{\Gamma_1, \Gamma' - \varphi \longmapsto \Delta_1 - \varphi, \Delta'} \quad \dfrac{\vdots\,\varpi_I}{\Gamma_I, \Gamma' - \varphi \longmapsto \Delta_I - \varphi, \Delta'}}{\Gamma, \Gamma' - \varphi \longmapsto \Delta - \varphi, \Delta', \varphi}\, r \quad \dfrac{\dfrac{\vdots\,\varpi'_1}{\Gamma, \Gamma'_1 - \varphi \longmapsto \Delta - \varphi, \Delta'_1} \quad \dfrac{\vdots\,\varpi'_J}{\Gamma, \Gamma'_J - \varphi \longmapsto \Delta - \varphi, \Delta'_J}}{\Gamma, \Gamma' - \varphi, \varphi \longmapsto \Delta - \varphi, \Delta'}\, r'}{\dfrac{\Gamma, \Gamma' - \varphi, \Gamma, \Gamma' - \varphi \longmapsto \Delta - \varphi, \Delta', \Delta - \varphi, \Delta'}{\Gamma, \Gamma' - \varphi \longmapsto \Delta - \varphi, \Delta'}\,(\mathbf{Ct\text{-}L}, \mathbf{Ct\text{-}R})^*}\,\mathbf{Cut}$$

In the above derivation there is a **Cut** on φ, and the sub-derivations of this cut have cut-complexities lower than $l(\varphi)$. Observe that the sub-derivation ending in this **Cut** fits the hypothesis of Lemma 65, and thus there exists one derivation for the same sequent with lower cut-complexity. To obtain the required derivation, it suffices to replace this derivation obtained from Lemma 65 for the one ending in the **Cut** on φ above.

□

Lemma 68. *If there is a derivation π for $\Gamma \;\longmapsto\; \Delta$ in $\mathscr{S} \in \{\mathbf{MBC}, \mathbf{MCI}\}$ with cut-complexity greater than* 0*, then one can found a proof π' in $\mathscr{S}$ for the same sequent $\Gamma \;\longmapsto\; \Delta$ with a lower cut-complexity.*

Proof. The proof goes by induction on $h(\pi)$. If $h(\pi) = 1$ then it has cut-complexity 0 and the results holds trivially. Suppose now $h(\pi) > 1$ and that the result holds good for all derivations π' for which $l(\pi') < l(\pi)$. Let the following be a sub-derivation of π ending in a cut with maximum complexity:

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \Gamma \longmapsto \Delta,\varphi\end{array}\qquad\begin{array}{c}\vdots\,\pi_2\\ \Gamma',\varphi \longmapsto \Delta'\end{array}}{\Gamma,\Gamma' \longmapsto \Delta,\Delta'}\ \textbf{Cut}$$

If some of the π_i has the same cut complexity of the above derivation, let ϖ_i be the derivation obtained from it with lower cut complexity by the induction hypothesis (or let simply $\varpi_i = \pi_i$ otherwise). Then, by the application of Lemma 67 on ϖ_1 and ϖ_2, it is possible to construct the following sequent derivation:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma,(\Gamma',\varphi)-\varphi \longmapsto (\Delta,\varphi)-\varphi,\Delta'\end{array}}{\Gamma,\Gamma' \longmapsto \Delta,\Delta'}\ (\textbf{Wk-L},\textbf{Wk-R})*$$

with cut-complexity lower than $l(\varphi)$. Now, replace in π the sub-derivation with greater cut-complexity for the above, with lower cut-complexity. To finish, repeat the process for all sub-derivations ending in a cut of complexity $l(\varphi)$. □

Theorem 69 (Cut Elimination)**.** *All sequents derived in* **MBC** *and* **MCI** *(Definitions 49 and 50) can be derived without the use of the* ***Cut*** *rule.*

Proof. For a given sequent derived in one of these calculi, repeatedly apply Lemma 68 on a derivation for it until obtain a proof with cut-complexity 0. □

6. Applications to mbC and mCi

In this section some new results will be established for the logics **mbC** and **mCi**. Taking profit of the results obtained in the previous sections, the results will be firstly proved for the version of these logics over signature $\Sigma^{\perp}$, namely $\mathbf{mbC}^{\perp}$ and $\mathbf{mCi}^{\perp}$. However, due to Theorem 37, these results are also valid for the original formulation of these logics, namely $\mathbf{mbC}^{\wedge\vee}$ and $\mathbf{mCi}^{\wedge\vee}$, respectively.

6.1. mbC is not controllably explosive

Contradiction, with respect to a given negation connective $\neg$, is simply the presence of both a formula φ and its negation $\neg\varphi$ at the same circumstances. From a classical standpoint, the presence of contradictions is inseparable of *triviality* (the fact that all formulas are entailed). Paraconsistency is the study of contradictory yet non-trivial theories. There are levels in which some logic can cope with the compromise between contradictoriness and triviality. The explosive approach, that of classical logic for instance, says that from any contradiction all formulas can be entailed. On the other pole, there are logics not finitely trivializable, that is, for which there is no finite set of formulas entailing all possible formulas. Somewhere between these oposite poles, there are logics for which contradictions involving certain kinds of formula schemes indeed trivialize. To make this point clear, from (Carnielli, Coniglio, and Marcos 2007) it can be taken the following definition:

Definition 70 (Controllable Explosiveness)**.** A standard logic $\mathscr{L}$ over the language $\mathcal{L}$ is said to be *controllably explosive in contact with* some formula $\varphi(p_1, \ldots, p_n) \in \mathcal{L}$ if the following hold:

(i) $\varphi(\alpha_1, \ldots, \alpha_n) \nvdash_{\mathscr{L}} \psi$ for some choice of $\{\alpha_1, \ldots, \alpha_n\}$ and some ψ, i.e., φ is not a bottom formula schema;

(ii) $\neg\varphi(\alpha_1, \ldots, \alpha_n) \nvdash_{\mathscr{L}} \psi$ for some choice of $\{\alpha_1, \ldots, \alpha_n\}$ and some ψ, i.e., $\neg\varphi$ is not a bottom formula schema;

(iii) $\varphi(\alpha_1, \ldots, \alpha_n), \neg\varphi(\alpha_1, \ldots, \alpha_n) \vdash_{\mathscr{L}} \psi$ for every $\{\alpha_1, \ldots, \alpha_n\}$ and ψ.

Definition 71 (Controllably Explosive Logic)**.** A logic $\mathscr{L}$ is said to be *controllably explosive* if there is some formula φ that $\mathscr{L}$ is controllably explosive in contact with φ.

Examples 72.
1. Clearly, a logic is explosive iff it is controllably explosive in contact with $\varphi(p_1) \stackrel{def}{=} p_1$.

2. Da Costa's logic C_1 *— see (Costa 1963) — defined over the signature* $\Sigma^{\wedge,\vee}$ *without the consistency operator* $\circ$*, is an* **LFI** *in which consistency can be defined in terms of the other connectives as follows:* $\alpha^\circ \stackrel{def}{=} \neg(\alpha \wedge \neg\alpha)$.

Being so, it is paraconsistent, but it is controllably explosive in contact with $\varphi(p_1) \stackrel{def}{=} (p_1 \wedge \neg p_1)$ *as, for every* α *and* ψ*,*

$$(\alpha \wedge \neg\alpha), \neg(\alpha \wedge \neg\alpha) \vdash_{C_1} \psi \,.$$

The version of C_1 *in the full signature* $\Sigma^{\wedge,\vee}$ *is called* **Cila***, and it is an extension of* **mCi** *— see (Carnielli, Coniglio, and Marcos 2007).*

3. By its turn, **mCi** *(and every extension of it, including* **Cila***) is controllably explosive in contact with* $\varphi(p_1) \stackrel{def}{=} \circ p_1$*.*

In fact, the following theorem can be established for all non-trivial extensions of **mCi**, relating derivability of consistent formulas with controllable explosiveness.

Theorem 73. *Let* $\mathscr{L}$ *be a non-trivial extension of* **mCi** *such that the implication (occurring in the axioms of* **mCi***) satisfies MTD. Then* $\mathscr{L}$ *is controllably explosive in contact with* φ *if, and only if, the formula* $\circ\varphi$ *is a theorem of* $\mathscr{L}$*.*

Proof. See (Carnielli, Coniglio, and Marcos 2007, Theorem 79). □

Regarding now **mbC**, as observed in (Carnielli, Coniglio, and Marcos 2007, p.84), in this logic there are no theorems of the form $\circ\varphi$. So the following question was posed in (Carnielli, Coniglio, and Marcos 2007): is **mbC** a controllably explosive logic? As it will be seen in Theorem 79, the answer is *no*: in **mbC**, any formula $\varphi(p_1, \ldots, p_n)$ satisfying Item (iii) of Definition 70 must be a bottom formula schema, that is, it violates Item (i) of Definition 70.

The proof of Theorem 79 requires some technical results concerning the sequent calculus **MBC**.

Lemma 74. *In a derivation of the sequent calculi defined in the previous section, the only way for a formula* φ *occurring in a sequent of this derivation not to occur at the concluding sequent, or as a sub-formula of an occurrence in the conclusion, is if there is some applications of* ***Cut*** *consuming occurrences of* φ*.*

Proof. Just observe that, in such sequent calculi, **Cut** is the only rule which does not produce a formula occurrence in which the consumed occurrence stands as a sub-formula. □

Lemma 75. *For the sequent calculi defined in the previous section, if some formula occurs on one side of a sequent of a derivation, then it must occur as a proper sub-formula in all of its descendants occurring on a different side.*

Proof. Just observe that in order for any formula occurrence to have descendants on a different side than its own, some descendant of it must have to be consumed by an application of a logical rule. All logical rules only produce an occurrence in which the consumed one occurs as a proper sub-formula. □

Lemma 76. *Let $\mathscr{L}$ be* **MBC** *or* **MCI** *and π be a derivation in $\mathscr{L}$ for the sequent $\Gamma \longmapsto \Delta$, and let φ be a formula occurring in Γ, introduced only by* ***Wk-L*** *and such that $\varphi \neq \circ\gamma$ for all $\gamma \in \mathcal{L}_{\Sigma^\perp}$. Then, it can be constructed a proof in $\mathscr{L}$ for the sequent $\Gamma - \varphi \longmapsto \Delta$ with the same cut-complexity of π and in which all formula occurrences in the concluding sequent are introduced by the same rules introducing the corresponding occurrences in the concluding sequent of π.*

Proof. The proof is similar to the one of Lemma 59. Just observe that, in order to remove the necessary occurrences of φ in π, such a removal must not break any rule, and so φ must be different from $\circ\gamma$ for any γ; removing a formula of type $\circ\gamma$ could break an application of $\neg$**L**. □

Theorem 77. *If there is a sequent derivation in* **MBC** *for the sequent $\sigma, \neg\sigma \longmapsto \bot$, then there is also a derivation for the sequent $\sigma \longmapsto \bot$.*

Proof. From Theorem 69, there is a a cut-free derivation in **MBC** for $\sigma, \neg\sigma \longmapsto \bot$. Now, observe that the only logical rule of **MBC** which introduces a negation to the left is $\neg$**L** and, by applying this rule on σ, the formula $\circ\sigma$ would have to occur in some sequent of the cut-free derivation. But such a thing is impossible by Lemma 74, for $\circ\sigma$ would also have to occur as a sub-formula of a formula occurring in $\sigma, \neg\sigma \longmapsto \bot$. The occurrence of $\neg\sigma$ also could not have been introduced by **Ax**, as this would imply, by Lemma 74, in an occurrence of $\neg\sigma$ as a sub-formula occurring on the right of the concluding sequent, or in an occurrence as a proper sub-formula of some formula occurring on the left. Thus $\neg\sigma$ is introduced in the cut-free derivation only by **Wk-L** and, by Lemma 76, there is a proof in **MBC** for $\sigma \;\longmapsto\; \bot$. □

Corollary 78. *If there is a formula σ such that, for all formulas φ:*

$$\sigma, \neg\sigma \;\vdash_{\mathbf{mbC}} \varphi$$

then, also for all formulas φ it must be that:

$$\sigma \;\vdash_{\mathbf{mbC}} \varphi$$

Proof. For $\mathbf{mbC}^{\bot}$, the proof is a direct consequence of Theorem 77 and Theorem 56. Only in this case the proof will be set for $\mathbf{mbC}^{\wedge\vee}$. Suppose that, for some $\sigma \in \mathcal{L}_{\Sigma^{\wedge,\vee}}$ and for all $\varphi \in \mathcal{L}_{\Sigma^{\wedge,\vee}}$:

$$\sigma, \neg\sigma \;\vdash_{\mathbf{mbC}^{\wedge\vee}} \varphi$$

Taking $\varphi = \circ\sigma$, by Theorem 37 and Definition 30:

$$\sigma^*, \neg(\sigma^*) \;\vdash_{\mathbf{mbC}^{\bot}} \circ(\sigma^*)$$

Now, by Axiom $\mathbf{bc1}^{\bot}$ and some applications of **MP**:

$$\sigma^*, \neg(\sigma^*) \;\vdash_{\mathbf{mbC}^{\bot}} \bot$$

By Theorem 77 and Theorem 56:

$$\sigma^* \;\vdash_{\mathbf{mbC}^{\bot}} \bot$$

By Lemma 16, item 1, taking $\alpha = (\neg\sigma \wedge \circ\sigma)^*$, and some applications of **MP**:

$$\sigma^* \;\vdash_{\mathbf{mbC}^{\bot}} (\neg\sigma \wedge \circ\sigma)^*$$

By Theorem 37:

$$\sigma \;\vdash_{\mathbf{mbC}^{\wedge\vee}} (\neg\sigma \wedge \circ\sigma)$$

Finally, by Axiom **bc1** and some applications of **MP**:

$$\sigma \;\vdash_{\mathbf{mbC}^{\wedge\vee}} \varphi$$

□

Theorem 79. *The logic* **mbC** *is not controllably explosive.*

Proof. Suppose there is a formula $\varphi(p_1, \ldots, p_n)$ satisfying, in **mbC**, the property (iii) from Definition 70. Then it follows from Corollary 78 that $\varphi(\alpha_1, \ldots, \alpha_n)$ must be a bottom formula in **mbC** for all choices of $\alpha_1, \ldots, \alpha_n$, violating clause (i) from Definition 70. Therefore, for every formula $\varphi(p_1, \ldots, p_n)$, **mbC** is not controllably explosive in contact with $\varphi(p_1, \ldots, p_n)$. Thus, **mbC** is not a controllably explosive logic, according to Definition 71. □

6.2. On negated formulas as theorems of mbC and mCi

Now, an application of the results on the cut-elimination theorems established in Section 5 will be given regarding the derivation of negated formulas in **mbC** and **mCi**.

As it is well-known, the problem of provability can be reduced to that of unsatisfiability in classical logic:

Theorem 80. *Let* $\Gamma \cup \{\varphi\}$ *be any set of formulas of the language of* **CPL**. *Then:*

$$\Gamma \vdash_{\mathbf{CPL}} \varphi \qquad \Longleftrightarrow \qquad \Gamma, \neg\varphi \textit{ is unsatisfiable} \qquad (\mathbf{Deriv} \hookrightarrow \mathbf{Unsat})$$

The above theorem states that an arbitrary instance of the problem of derivability for **CPL** can be settled by the solution of the unsatisfiability problem of a related instance concerning the classical negation. However, such an equivalence does not hold, in general, for paraconsistent logics, if we consider the paraconsistent negation instead of the classical one. Paraconsistent logics are most valuable for allowing one to understand how a given formula and its (paraconsistent) negation can be both satisfied at the same circumstances. Therefore, in the paraconsistent setting, there can be a formula φ which is a logical consequence of a set Γ of formulas and, nevertheless, the set $\Gamma \cup \{\neg\varphi\}$ may have a paraconsistent model. In general, for logics satisfying *tertium non datur*, only one side of the above equivalence can be shown to hold. More precisely:

Theorem 81. *Let* $\mathscr{L}$ *be a logic over a language* $\mathcal{L}$ *with a negation* $\neg$. *If* $\mathscr{L}$ *is sound and complete for a semantics* $\mathbb{S} = \langle \mathfrak{M}, \vDash \rangle$ *such that, for all* $\varphi \in \mathcal{L}$, *and for all model $M \in \mathfrak{M}$, either M satisfies φ or M satisfies $\neg\varphi$. Then:*

$$\Gamma \vdash_{\mathscr{L}} \varphi \qquad \Longleftarrow \qquad \Gamma, \neg\varphi \textit{ is unsatisfiable in } \mathbb{S}$$

Proof. It is a direct consequence of the hypothesis and the definitions of completeness, semantical relation and unsatisfiability. □

Concerning the **LFI**s under investigation, it is a well known fact that the negation of any explosive formula in **mbC** or **mCi** is a theorem of this logic. More precisely:

Theorem 82. *Let $\mathscr{L} \in \{\mathbf{mbC}, \mathbf{mCi}\}$. Suppose that φ is a bottom formula, that is: for all formulas ψ, it is the case that $\varphi \vdash_{\mathscr{L}} \psi$. Then: $\vdash_{\mathscr{L}} \neg\varphi$.*

Proof. Observe that:

$$(\varphi \to \neg\varphi) \to \Big((\neg\varphi \to \neg\varphi) \to \big(((\varphi \to \bot) \to \neg\varphi) \to \neg\varphi\big)\Big)$$

is a theorem of $\mathscr{L}$, by Lemma 16, item 4. From the hypothesis, taking $\psi = \neg\varphi$, and by the deduction theorem, it follows that $\varphi \to \neg\varphi$ is also a theorem. Finally, observe that $(\varphi \to \bot) \to \neg\varphi$ is an instance of Axiom $\sim\neg$. The result follows then from **MP**. □

In semantical terms:

Theorem 83. *Let $\mathscr{L}$ be* **mbC** *or* **mCi** *and V be respectively $V^{\mathbf{mbC}^{\bot}}$ or $V^{\mathbf{mCi}^{\bot}}$, if the logics are over the signature $\Sigma^{\bot}$, or let V be respectively $V^{\mathbf{mbC}^{\wedge\vee}}$ or $V^{\mathbf{mCi}^{\wedge\vee}}$, if the logics are over the signature $\Sigma^{\wedge,\vee}$. Then, for every φ:*

$$\forall v \in V : v(\varphi) = 0 \qquad \Longrightarrow \qquad \forall v \in V : v(\neg\varphi) = 1$$

Proof. It is an easy consequence of Theorem 82 and the soundness and completeness theorem of $\mathscr{L}$ with respect to bivaluations. A direct proof is also possible, from the basic clause for $\neg$ required for bivaluations. □

The converses of theorems 82 and 83, our second main result, are far from obvious. However, they are a consequence of Theorem 69:

Theorem 84. *Let $\mathscr{L} \in \{\mathbf{mbC}, \mathbf{mCi}\}$. If some negated formula $\neg\varphi$ is a theorem of $\mathscr{L}$, that is, $\vdash_{\mathscr{L}} \neg\varphi$, then φ is a bottom formula: $\varphi \vdash_{\mathscr{L}} \psi$, for all formulas ψ.*

Proof. Let $\mathscr{S}$ be **MBC** if $\mathscr{L}$ is **mbC**, and let $\mathscr{S}$ be **MCI** if $\mathscr{L}$ is **mCi**. For the logics in the signature $\Sigma^{\perp}$, suppose $\vdash_{\mathscr{L}} \neg\varphi$. From Theorem 56 and Theorem 69, it follows that there is a cut-free derivation for $\longmapsto \neg\varphi$ in $\mathscr{S}$. In such a derivation it is impossible that an application of **Ax** have been used, for it would imply, by Lemma 74, another occurrence of $\neg\varphi$ on the left of the concluding sequent or some occurrence of it as a proper subformula of $\neg\varphi$ to the right, by Lemma 75. From Lemma 63, it follows that the sequent $\varphi \longmapsto$ can be derived in $\mathscr{S}$ and the result follows now from Theorem 56. For the logics in the signature $\Sigma^{\wedge,\vee}$, the proof is similar to the one of Corollary 78. □

In semantical terms, we obtain the converse of Theorem 83:

Theorem 85. *Let $\mathscr{L}$ be **mbC** or **mCi** and V be respectively $V^{\mathbf{mbC}^{\perp}}$ or $V^{\mathbf{mCi}^{\perp}}$, if the logics are over the signature $\Sigma^{\perp}$, or let V be respectively $V^{\mathbf{mbC}^{\wedge\vee}}$ or $V^{\mathbf{mCi}^{\wedge\vee}}$, if the logics are over the signature $\Sigma^{\wedge,\vee}$. Then, for every φ:*

$$\forall v \in V : v(\neg\varphi) = 1 \qquad \Longrightarrow \qquad \forall v \in V : v(\varphi) = 0$$

Proof. It is a direct consequence of the previous theorem and the soundness and completeness theorem of $\mathscr{L}$ with respect to bivaluations. □

Theorems 83 and 85 can be generalized to *any* semantics characterizing **mbC** or **mCi**, assuming that it does not admit trivial models:

Theorem 86. *Let $\mathscr{L}$ be **mbC** or **mCi** and let $\mathbb{S} = \langle \mathfrak{M}, \vDash \rangle$ be a semantics for $\mathscr{L}$,[4] which does not admit trivial models[5]. If $\mathscr{L}$ is sound and complete for $\mathbb{S}$ then, for every φ:*

$$M \text{ satisfies } \neg\varphi, \text{ for every } M \in \mathfrak{M} \iff M \text{ does not satisfy } \varphi, \text{ for every } M \in \mathfrak{M}.$$

Proof. Suppose that M satisfies $\neg\varphi$ for every $M \in \mathfrak{M}$. Then $\vDash \neg\varphi$ and so $\vdash_{\mathscr{L}} \neg\varphi$, by completeness of $\mathscr{L}$ w.r.t. $\mathbb{S}$. By Theorem 84, $\varphi \vdash_{\mathscr{L}} \psi$ for all formulas ψ and, as $\mathbb{S}$ admits no trivial models and $\mathscr{L}$ is sound for it, φ is unsatisfiable in $\mathbb{S}$. That is, M does not satisfy φ for every $M \in \mathfrak{M}$.

Conversely, assume that, for every model $M \in \mathfrak{M}$, M does not satisfy φ. Then $\varphi \vDash \psi$, for every formula ψ. In particular, $\varphi \vDash \neg\varphi$ and so, by completeness of $\mathscr{L}$ w.r.t. $\mathbb{S}$, $\varphi \vdash_{\mathscr{L}} \neg\varphi$. From this, it is easy to prove — see, for instance, (Carnielli, Coniglio, and Marcos 2007) — that $\vdash_{\mathscr{L}} \neg\varphi$. Then, by soundness of $\mathscr{L}$ w.r.t. $\mathbb{S}$ it follows that M satisfies $\neg\varphi$ for every $M \in \mathfrak{M}$. ◻

Theorem 87. *Let* $\mathscr{L}$ *be* **mbC** *or* **mCi** *and let* $\mathbb{S} = \langle \mathfrak{M}, \vDash \rangle$ *be a semantics for* $\mathscr{L}$ *which does not admit trivial models, such that* $\mathscr{L}$ *is sound and complete for* $\mathbb{S}$ *(for instance, let* $\mathbb{S}$ *be the bivaluation semantics for* $\mathscr{L}$*). Then:*

$$\vdash_{\mathscr{L}} \neg\varphi \qquad \Longleftrightarrow \qquad \varphi \text{ is unsatisfiable in } \mathbb{S}$$

Proof. The proof is a direct consequence of Theorem 86, and of the soundness and completeness theorem for $\mathscr{L}$ with respect to $\mathbb{S}$. ◻

It is worth noting that the last theorem cannot be extended to non-empty set of premises or to negated formulas. This is a consequence of the fact that φ and $\neg\neg\varphi$ are inequivalent in $\mathscr{L}$, for $\mathscr{L} \in \{\textbf{mbC}, \textbf{mCi}\}$:

Theorem 88. *There are examples of sets of formulas* Γ *and formulas* φ *such that* $\Gamma \vdash_{\mathscr{L}} \neg\varphi$ *and, nevertheless,* Γ, φ *has models. That is:*

$$\Gamma \vdash_{\mathscr{L}} \neg\varphi \qquad \not\Longrightarrow \qquad \Gamma, \varphi \text{ is unsatisfiable in } \mathbb{S}.$$

Proof. Let $\mathbb{S}$ be the bivaluation semantics for $\mathscr{L}$; let φ be any propositional variable and $\Gamma = \{\neg\varphi\}$. ◻

On the other hand:

Theorem 89. *There are examples of formulas* φ *such that* $\vdash_{\mathscr{L}} \varphi$ *and, nevertheless,* $\neg\varphi$ *has models. That is:*

$$\vdash_{\mathscr{L}} \varphi \qquad \not\Longrightarrow \qquad \neg\varphi \text{ is unsatisfiable in } \mathbb{S}.$$

Proof. Let $\mathbb{S}$ be the bivaluation semantics for $\mathscr{L}$, and let φ be $(\psi \vee \neg\psi)$ with ψ a propositional variable. ◻

7. Concluding Remarks

In this paper we present a new formulation for the logics **mbC** and **mCi**, in a simpler signature which includes the bottom formula $\bot$ as a constant. The immediate effect of this move is that it allows to consider a single classical negation $\sim$, which simplifies the construction and analysis of the logics. Additionally, this allows to see in a clear way that these logics (as well as their extensions) are, in fact, extensions of classical logic by adding two non-truth-functional connectives: the paraconsistent negation $\neg$ and the consistency operator $\circ$. These systems (as well as da Costa's C-systems C_n) are not weaker than classical logic, as one could be naively tempted to believe: they are stronger than classical logic since they *extend* it by adding new conectives of an intensional character. This basic feature frequently remains hidden or ignored, given that the construction of the classical negation within these systems seems to be a secondary, rather unimportant fact. Starting from classical logic from the very beginning intends to clarify this relevant feature of these logics. The rigorous proof of the equivalence between both presentations of **mbC** (and also of **mCi**) is not as easy as one could imagine, because of the non-self-extensionality of the involved logics.

Another contribution of the paper is the presentation of adequate sequent calculi for **mbC** and **mCi** in the new proposed signature. The desired cut-elimination property, as well as other meta-properties of the calculi, are also established with full technical details. From this analysis of the meta-properties of these calculi it is possible to obtain two new and interesting results concerning both logics.

The first new result concerns exclusively **mbC**, and answers (negatively, as expected) an open question about it: is **mbC** a controllably explosive logic? The negative answer to this question means that for no schema φ it is the case that the contradiction $\{\varphi, \neg\varphi\}$ is explosive, and so **mbC** is paraconsistent in a very strong sense (or, from another perspective, the paraconsistent negation in **mbC** is considerably weak).

The second new result, which holds for both **mbC** and **mCi**, states that negated theorems must be unsatisfiable, in any semantics which does not admit trivial models. This is a somewhat surprising feature of these systems, specially in the case of **mbC**, in which the paraconsistent negation is extremely weak, as it was pointed out.

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Notes

[1] Here $\neg^0\alpha \stackrel{\text{def}}{=} \alpha$, and $\neg^{n+1}\alpha \stackrel{\text{def}}{=} \neg\neg^n\alpha$.

[2] Recall from (Carnielli, Coniglio, and Marcos 2007) that $\sim_\alpha\beta \stackrel{\text{def}}{=} \beta \to \bot_\alpha$ satisfies the following: $\beta, \sim_\alpha\beta \vdash \gamma$ for every β and γ; $\vdash \beta \vee \sim_\alpha\beta$, $\vdash \beta \to \sim_\alpha\sim_\alpha\beta$, and $\vdash \sim_\alpha\sim_\alpha\beta \to \beta$, for every β.

[3] The above enumeration is by no means complete: there are other proposals in the literature dealing with sequents for **LFI**s.

[4] As usual, we define $\Gamma \vDash \varphi$ iff, for every $M \in \mathfrak{M}$, if M satisfies ψ for every $\psi \in \Gamma$ then M satisfies φ.

[5] That is: for every $M \in \mathfrak{M}$ there is a formula φ such that M does not satisfy φ.