# Continuity of higher order commutators generated by maximal Bochner-Riesz operator on Morrey space

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**Abstact:** In this papers ,we use the control method of the maximal fractional integral and obtain the boundedness of higher order commutator generated by maximal Bochner-Riesz operator on Morrey space. Moreover , we get it's continuty from Morrey space to Lipschtz space and from Morrey space to BMO space.

**Key words:** BMO space, higher commutator, Morrey space,Lipschtz space

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## 1 Introduction And Main Results

Let maximal Bochner-Riesz operator $B_\delta^*$ [1] be

$$B_\delta^* f(x) = \sup_{t>0} |B_\delta^t(f)(x)|,$$

where$B_\delta^t(x) = t^{-n} B_\delta(x/t)$ is the kernel of $B_\delta^t$, which satisfies $|\frac{\partial^\beta}{\partial x^\beta} B_\delta(x)| \le C(1+|x|)^{-(\delta+(n+1)/2)}$ for any $x \in \mathbb{R}^n$ and multi-index $\beta \in \mathbb{Z}_+^n$.

For $0 < \beta < 1$ ,let the homogeneous Besov-Lipschitz space $\dot{\Lambda}_\beta$ [2] be the space of the functions satisfying

$$\|f\|_{\dot{\Lambda_\beta}} = \sup_{x,y\in\mathbb{R}^n, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\beta}.$$

Suppose $b \in \dot{\Lambda_\beta}$. For suitable function $f$, define the high order commutator $B_{\delta,*}^{b,m}$ generated by $B_\delta^*$ and $b$ as follows

$$B_{\delta,*}^{b,m} f(x) = \sup_{t>0} \int_{\mathbb{R}^n} B_\delta^t(x-y)[b(x)-b(y)]^m f(y)dy.$$

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For $1 < p < \infty$, $\lambda \geq 0$, the classic Morrey space $M^{p,\lambda}(\mathbb{R}^n)$[3] is the space of function $f \in L^p_{loc}$ such that

$$\|f\|_{M^{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} (\frac{1}{|Q(x,r)|^{\lambda}} \int_{Q(x,r)} |f(y)|^p dy)^{1/p} < \infty.$$

Let $1 \leq p < \infty$. Denote the generalized Morrey space[4] as

$$M^{p,\varphi} = \{f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{M^{p,\varphi}} < \infty\},$$

where

$$\|f\|_{M^{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} (\frac{1}{\varphi(x,r)} \int_Q |f(y)|^p dy)^{1/p},$$

$\varphi$ is a integer increasing function on $\mathbb{R}^n \times \mathbb{R}^+$ satisfying

$$\varphi(x, 2r) \leq D_{\varphi}\varphi(x,r),$$

and$D_{\varphi}$is a constant ,independent of $r$.

For $1 \leq l < \infty$ and $\beta > 0$, we call that

$$T_l^{l\beta}(f)(x) = \sup_{x\in Q} \Big(\frac{1}{|Q|^{1-l\beta/n}} \int_Q |f(y)|^l \,\mathrm{y}\Big)^{1/l}$$

is the maximal fractional integral[5],where $l < p < n/\beta$ and $1/q = 1/p - \beta/n$.

Let $\lambda < 1/n$ and $1 < q < \infty$. We define $f \in L^q_{loc}(\mathbb{R}^n)$ is belong to $BMO^{\lambda,q}$ space if $f$ satisfies

$$\|f\|_{BMO^{\lambda,q}} = \sup_{x\in\mathbb{R}^n,\, r>0} (\frac{1}{|Q(x,r)|^{1+\lambda q}} \int_{Q(x,r)} |f(y) - f_Q|^q dy)^{1/q} < \infty.$$

Also,we have

$$\|f\|_{BMO^{\lambda,q}} \approx \sup_{x\in\mathbb{R}^n, r>0} (\frac{1}{|Q(x,r)|^{1+\lambda}} \int_{Q(x,r)} |f(y) - f_Q|dy) \approx \|f\|_{\dot{\Lambda}_{\beta}},$$

where $f_Q = \frac{1}{|Q|}\int_Q f(y)dy, \lambda = \beta/n$.

In this paper, $x_0$ is the center of cube $Q$ with the side length $r$. $Q_k$ is the cube with the same center as $Q$ and side length $2^k$ times the side length of $Q$.

**Theorem 1.1** Suppose $\lambda \geq 0$, $\delta > (n-1)/2$, $0 < m\beta < n$, $0 < \beta < 1$ , $1 < p < n/(m\beta)$, $1/p - m\beta/n = 1/q$ and $b \in \dot{\Lambda}_{\beta}$. We have

$$\|B^{b,m}_{\delta,*}(f)\|_{M^{q,\lambda}} \leq C\|f\|_{M^{p,\lambda(1-\beta mp/n)}}.$$

**Corollary 1.2** Let $\varphi_1^{1/p_1}(r) = \varphi_2^{1/p_2}(r)$. Under conditions of theorem 1.1 , we get

$$\|B^{b,m}_{\delta,*}(f)\|_{M^{p_2,\varphi_2}} \leq C\|f\|_{M^{p,\varphi_1}}.$$

**Theorem 1.3** Let $\delta > (n-1)/2$, $b \in \dot{\Lambda}_\beta$, $0 < \beta < 1$, $0 < m\beta < n$, $1/p - m\beta/n = 1/q$ ,$1 < p < n/(m\beta)$, $\zeta = \min\{1, \delta - (n-1)/2\}$, and $-1/q < \lambda < \zeta$ .We have

$$\|B_{\delta,*}^{b,m}\|_{BMO^{q,\lambda}} \leq C\|f\|_{M^{p,1+p(\lambda - m\beta/n)}}.$$

**Theorem 1.4** Suppose $\delta > (n-1)/2$, $1/p - m\beta/n = 1/q$ and $b \in \dot{\Lambda}_\beta$, $0 < \beta < 1$ ,$0 < m\beta < n$, $1 < p < n/(m\beta)$. Let $\zeta = \min\{1, \delta - (n-1)/2\}$ and $1 - (mp\beta)/n < \lambda < 1 + (\zeta - m\beta)p/n$ . We get

$$\|B_{\delta,*}^{b,m}\|_{\dot{\Lambda}_{m\beta+(\lambda-1)n/p}} \leq C\|f\|_{M^{p,\lambda}}.$$

## 2 $[M^{p,\lambda(1-\beta mp/n)}(\mathbb{R}^n), M^{q,\lambda}(\mathbb{R}^n)]$-Type Continuity.

**Lemma 2.1** Let $\delta > (n-1)/2$, $b \in \dot{\Lambda}_\beta$, $0 < m\beta < n$ , $1 < p < n/(m\beta)$ and $1/p - m\beta/n = 1/q$ ,Then there is a constant $C > 0$ ,independent of $f$ ,such that

$$\|T_{\delta,*}^{b,m} f\|_q \leq C\|f\|_p.$$

**Proof of lemma 2.1** For $x \in \mathbb{R}^n$ and $\varepsilon > 0$ with $0 < m\beta - \varepsilon < m\beta + \varepsilon < n$, we choose a $\xi > 0$ such that

$$\xi^{2\varepsilon} = M_{m\beta-\varepsilon} f(x)/M_{m\beta+\varepsilon} f(x).$$

Write

$$\begin{aligned} T_{\delta,t}^{b,m} f(x) &= \int_{|x-y|<\xi} B_\delta^t(x-y)[b(x)-b(y)]^m f(y)dy \\ &\quad + \int_{|x-y|\geq\xi} B_\delta^t(x-y)[b(x)-b(y)]^m f(y)dy = D_1 + D_2. \end{aligned}$$

By the inequality[4] $|B_\delta^t(x-y)| \leq |x-y|^{-n}$ and $b \in \dot{\Lambda}_\beta$, we have

$$\begin{aligned} |D_1| &\leq C\sum_{j=0}^{\infty} \int_{2^{-j-1}\xi \leq |x-y| < 2^{-j}\xi} \frac{|f(y)|}{|2^{-j}\xi|^{n-m\beta}} dy \\ &\leq C\sum_{j=0}^{\infty} \frac{(2^{-j}\xi)^\varepsilon}{|2^{-j}\xi|^{n-m\beta+\varepsilon}} \int_{|x-y|<2^{-j}\xi} |f(y)|dy \\ &\leq C\sum_{j=0}^{\infty} 2^{-j\varepsilon}\xi^\varepsilon M_{m\beta-\varepsilon} f(x) \leq C\xi^\varepsilon M_{m\beta-\varepsilon} f(x). \end{aligned}$$

Similarly,we get

$$
\begin{aligned}
|D_2| &\le C\sum_{j=1}^{\infty}\int_{2^{j-1}\xi\le|x-y|<2^j\xi}\frac{|2^j\xi|^{m\beta}}{|2^{j-1}\xi|^n}|f(y)|dy\\
&\le C\sum_{j=1}^{\infty}\frac{(2^j\xi)^{-\varepsilon}}{|2^j\xi|^{n-m\beta-\varepsilon}}\int_{|x-y|<2^j\xi}|f(y)|dy\\
&\le C\sum_{j=0}^{\infty}2^{-j\varepsilon}\xi^{-\varepsilon}M_{m\beta+\varepsilon}f(x)\le C\xi^{-\varepsilon}M_{m\beta+\varepsilon}f(x).
\end{aligned}
$$

Thus,by the above selection of $\xi$ we get

$$|T_{\delta,t}^{b,m}f(x)|\le C(\xi^{\varepsilon}M_{m\beta-\varepsilon}f(x)+\xi^{-\varepsilon}M_{m\beta+\varepsilon}f(x))=C[M_{m\beta+\varepsilon}f(x)]^{1/2}[M_{m\beta+\varepsilon}f(x)]^{1/2}.$$

Noting $1<p<n/(m\beta)$, there is an $\varepsilon>0$, such that $1<p<n/(m\beta+\varepsilon)$. Let $1/q_1=1/p-(m\beta-\varepsilon)/n$, $1/q_2=1/p-(m\beta+\varepsilon)/n$, $l=2q_1/q$, $l^{'}=2q_2/q$, then $q_1,q_2>0$,$l^{'}>1$ and $1/l+1/l^{'}=1$. Thus ,by inequality (1) ,we have

$$
\begin{aligned}
&\|T_{\delta,t}^{b,m}f\|_q^q\le C\int_{\mathbb{R}^n}|M_{m\beta-\varepsilon}f(x)|^{q/2}|M_{m\beta+\varepsilon}f(x)|^{q/2}dx\\
&\le C(\int_{\mathbb{R}^n}|M_{m\beta-\varepsilon}f(x)|^{ql/2}dx)^{1/l}(\int_{\mathbb{R}^n}|M_{m\beta+\varepsilon}f(x)|^{ql^{'}/2}dx)^{1/l^{'}}\\
&\le C(\int_{\mathbb{R}^n}|M_{m\beta-\varepsilon}f(x)|^{q_1}dx)^{q/2q_1}(\int_{\mathbb{R}^n}|M_{m\beta+\varepsilon}f(x)|^{q_2}dx)^{q/2q_2}\\
&\le C(\int_{\mathbb{R}^n}|f(x)|^{p}dx)^{q/2p}(\int_{\mathbb{R}^n}|f(x)|^{p}dx)^{q/2p}=C\|f\|_p^q.
\end{aligned}
$$

**Lemma 2.2**[6] Let $T_l^{l\beta}$ be the maximal fractional operator , $\lambda\ge 0$, $1\le l<p<n/\beta$ and $1/p-\beta/n=1/q$, we have

$$\|T_l^{l\beta}(f)\|_{M^{q,\lambda}}\le C\|f\|_{M^{p,\lambda(1-\beta p/n)}}.$$

**Proof of Theorem 1.1** Let $f=f\chi_{2Q}+f\chi_{(2Q)^c}=f_1+f_2$. Write

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q|B_{\delta,t}^{b,m}f(y)-B_{\delta,t}^{b,m}f_2(x_0)|dy\\
&\le C\frac{1}{|Q|}\int_Q|B_{\delta,t}^{b,m}f_1(y)|dy+C\frac{1}{|Q|}\int_Q|B_{\delta,t}^{b,m}f_2(y)-B_{\delta,t}^{b,m}f_2(x_0)|dy\\
&=E_1+E_2.
\end{aligned}
$$

We choose $l$ satisfying $1\le l<p<n/(m\beta)$. Take a $s$ such that $1/s=1/l-m\beta/n$. By the lemma 2.1 and Hölder inequality, we have

$$
\begin{aligned}
E_1&\le C(\frac{1}{|Q|}\int_Q|T_{\delta,t}^{b,m}f_1(y)|^sdy)^{1/s}\le C\frac{1}{|Q|^{1/s}}(\int_Q|f(y)|^ldy)^{1/l}\\
&\le C(\frac{1}{|Q|^{1-m\beta l/n}}\int_Q|f(y)|^ldy)^{1/l}\le CT_l^{m\beta l}f(x).
\end{aligned}
$$

We turn to $E_2$. Noticing $z \in (2Q)^c, y \in Q$ and $|y-z| \sim |x_0 - z|$,we have

**case 1** $t < r$

$$
\begin{aligned}
&|B^{b,m}_{\delta,t} f_2(y) - B^{b,m}_{\delta,t} f_2(x_0)| \leq C \sum_{k=1}^{\infty} |\int_{Q_{k+1}\setminus Q_k} B^t_\delta(y-z)[b(y)-b(z)]^m f(z)dz| \\
&\leq C \sum_{k=1}^{\infty} |2^k r|^{m\beta} t^{-n} \int_{Q_{k+1}\setminus Q_k} (1+\frac{|x-z|}{t})^{-(\delta+\frac{n+1}{2})} |f(z)|dz \\
&\leq C \sum_{k=1}^{\infty} 2^{-k[\delta-(n-1)/2]} (\frac{1}{|Q_{k+1}|^{1-m\beta l/n}} \int_{Q_{k+1}} |f(z)|^l dz)^{1/l} \leq C T^{m\beta l}_l f(x).
\end{aligned}
$$

**case 2** $t \geq r$

(1)We conside $\delta < (n+1)/2$. By the mean value theorem, we obtain

$$
\begin{aligned}
&|T^{b,m}_{\delta,t} f_2(y) - T^{b,m}_{\delta,t} f_2(x_0)| \\
&\leq C \sum_{k=1}^{\infty} |2^k r|^{m\beta} |\int_{Q_{k+1}\setminus Q_k} B^t_\delta(y-z) f(z)dz - \int_{Q_{k+1}\setminus Q_k} B^t_\delta(x_0-z) f(z)dz| \\
&\leq C \sum_{k=1}^{\infty} |2^k r|^{m\beta} t^{-n-1} \int_{Q_{k+1}\setminus Q_k} (1+\frac{|x_0-z|}{t})^{-(\delta+\frac{n+1}{2})} |y-x_0||f(z)|dz \\
&\leq C \sum_{k=1}^{\infty} (\frac{r}{t})^{\frac{n+1}{2}-\delta} 2^{-k[\delta-(n-1)/2]} (\frac{1}{|Q_{k+1}|^{1-m\beta l/n}} \int_{Q_{k+1}} |f(z)|^l dz)^{1/l} \leq C T^{m\beta l}_l f(x).
\end{aligned}
$$

(2)When $\delta \geq (n+1)/2$, we also obtain

$$
\begin{aligned}
&|T^{b,m}_{\delta,t} f_2(y) - T^{b,m}_{\delta,t} f_2(x_0)| \\
&\leq C \sum_{k=1}^{\infty} |2^k r|^{m\beta} t^{-n-1} r \int_{Q_{k+1}\setminus Q_k} (1+\frac{|x_0-z|}{t})^{-(\delta+\frac{n+1}{2})} |f(z)|dz \\
&\leq C \sum_{k=1}^{\infty} 2^{-k} T^{m\beta l}_l f(x) \leq C T^{m\beta l}_l f(x).
\end{aligned}
$$

Then,by the lemma 2.2,we gain

$$
\begin{aligned}
\|B^{b,m}_{\delta,t}(f)\|_{M^{q,\lambda}} &\leq C\|M(B^{b,m}_{\delta,t})(f)\|_{M^{q,\lambda}} \leq C\|(B^A_{\delta,t})^\sharp(f)(x)\|_{M^{q,\lambda}} \\
&\leq C\|T^{m\beta l}_l(f)\|_{M^{q,\lambda}} \leq C\|f\|_{M^{p,\lambda(1-m\beta p/n)}}.
\end{aligned}
$$

Taking the supreme of the left side about $E_1$ and $E_2$ for any $t > 0$ , this complets the proof of theorem 1.1.

**Proof of Corollary 1.2**

$$(\frac{1}{\varphi_2(r)}\int_Q |B^{b,m}_{\delta,t}f(x)|^q dx)^q = \frac{|Q|^{\lambda/q}}{\varphi_2^{1/q}(r)}(\frac{1}{|Q|^{\lambda}}\int_Q |B^{b,m}_{\delta,t}f(x)|^q dx)^{1/q}$$
$$\leq \frac{|Q|^{\lambda/q}\varphi_1(r)^{1/p}}{\varphi_2(r)^{1/q}|Q|^{\lambda/p-\lambda m\beta/n}}\Big(\frac{1}{\varphi_1(r)}\int_Q |f(x)|^p dx\Big)^{1/p} = C\Big(\frac{1}{\varphi_1(r)}\int_Q |f(x)|^p dx\Big)^{1/p}.$$

## 3 $[M^{p,1+p(\lambda-m\beta/n)}(\mathbb{R}^n)$,$BMO^{\lambda,q}(\mathbb{R}^n)]$-Type Continuity .

**Proof of Theorem 1.3** Write $f = f\chi_{2Q} + f\chi_{(2Q)^c} = f_1 + f_2$, then

$$(\frac{1}{|Q|^{1+q\lambda}}\int_Q |B^{b,m}_{\delta,t}f(x) - B^{b,m}_{\delta,t}f_2(x_0)|^q dx)^{1/q} \leq C(\frac{1}{|Q|^{1+q\lambda}}\int_Q |B^{b,m}_{\delta,t}f_1(x)|^q dx)^{1/q}$$
$$+ C(\frac{1}{|Q|^{1+q\lambda}}\int_Q |B^{b,m}_{\delta,t}f_2(x) - B^{b,m}_{\delta,t}f_2(x_0)|^q dx)^{1/q} = F_1 + F_2.$$

By the lemma 2.1,we have

$$F_1 \leq C\frac{1}{|Q|^{1/q+\lambda}}(\int_Q |f_1(x)|^p dx)^{1/p} \leq C\|f\|_{M^{p,1+p(\lambda-m\beta/n)}}.$$

Now,we consider $F_2$. Noticing $x \in Q$, we have $y \in (2Q)^c$.

**case 1** $t < r$ .We then obtain

$$|B^{b,m}_{\delta,t}f_2(x) - B^{b,m}_{\delta,t}f_2(x_0)|$$
$$\leq C\sum_{k=1}^{\infty}|2^{k+1}Q|^{1+\lambda}t^{\delta-(n-1)/2}|2^k r|^{-[\delta+(n+1)/2]}(\frac{1}{|Q_{k+1}|^{1+p(\lambda-m\beta/n)}}\int_{Q_{k+1}}|f(y)|^p dy)^{1/p}$$
$$\leq C(\frac{t}{r})^{\delta-(n-1)/2}\sum_{k=1}^{\infty}2^{-k[\delta-(n-1)/2-n\lambda]}r^{n\lambda}\|f\|_{M^{p,1+p(\lambda-m\beta/n)}} \leq Cr^{n\lambda}\|f\|_{M^{p,1+p(\lambda-m\beta/n)}}.$$

**case 2** $t \geq r$.

(1)When $\delta < (n+1)/2$, by the mean value theorem, we have

$$|T^{b,m}_{\delta,t}f_2(x) - T^{b,m}_{\delta,t}f_2(x_0)|$$
$$\leq C\sum_{j=2}^{\infty}|2^{k_0+j}|^{m\beta}t^{-n-1}\int_{B_{k_0+j+1}\backslash B_{k_0+j}}|x-x_0|(1+\frac{|x-y|}{t})^{-[\delta+(n+1)/2]}|f(y)|dy$$
$$\leq C\sum_{k=1}^{\infty}2^{-k[\delta-(n-1)/2-n\lambda]}r^{n\lambda}\|f\|_{M^{p,1+p(\lambda-m\beta/n)}} \leq Cr^{n\lambda}\|f\|_{M^{p,1+p(\lambda-m\beta/n)}}.$$

(2)When $\delta \geq (n+1)/2$, we similarly gain

$$|T^{b,m}_{\delta,t}f_2(x) - T^{b,m}_{\delta,t}f_2(x_0)| \leq C\sum_{k=1}^{\infty}2^{-k(1-n\lambda)}r^{n\lambda}\|f\|_{M^{p,1+p(\lambda-m\beta/n)}}$$
$$\leq Cr^{n\lambda}\|f\|_{M^{p,1+p(\lambda-m\beta/n)}}.$$

So,we get estimation for $F_2$ which is

$$F_2 \le C\|f\|_{M^{p,1+p(\lambda-m\beta/n)}}.$$

These yields the desired result.

## 4 $[M^{p,\lambda}(\mathbb{R}^n),\dot{\Lambda}_{[m\beta+n/p(\lambda-1)]/n}(\mathbb{R}^n)]$-Type Continuity .

**Proof of Theorem 1.4** We write

$$\int_Q |B^{b,m}_{\delta,t} f(x) - T^{b,m}_{\delta,t} f_2(x_0)|dx \le C\int_Q |T^{b,m}_{\delta,t} f_1(x)|dx$$
$$+\int_Q |T^{b,m}_{\delta,t} f_2(x) - T^{b,m}_{\delta,t} f_2(x_0)|dx = G_1 + G_2.$$

By the *Hölder* inequality and the theorem 1.1 ,we get

$$G_1 \le C(\int_Q |T^{b,m}_{\delta,t} f_1(x)|^q dx)^{\frac{1}{q}} |Q|^{1-1/q} \le C|Q|^{1+[m\beta+(\lambda-1)n/p]/n}\|f\|_{M^{p,\lambda}}.$$

Now,we estimate $G_2$. Noting $x \in Q$, $y \in (2Q)^c$, we know $|x-y| \sim |x_0-y|$.

**case 2** $t < r$. We get

$$|T^{b,m}_{\delta,t} f_2(x) - T^{b,m}_{\delta,t} f_2(x_0)|$$
$$\le C\sum_{k=1}^{\infty} 2^{-k[\delta-(n-1)/2-m\beta+(1-\lambda)n/p]}|Q|^{m\beta/n-1/p+\lambda/p}\|f\|_{M^{p,\lambda}}$$
$$\le C|Q|^{m\beta/n-1/p+\lambda/p}\|f\|_{M^{p,\lambda}}.$$

**case 2** $t \ge r$.

(1)When $\delta < (n+1)/2$,we have

$$|T^{b,m}_{\delta,t} f_2(x) - T^{b,m}_{\delta,t} f_2(x_0)|$$
$$\le C(\frac{r}{t})^{(n+1)/2-\delta}\sum_{k=1}^{\infty} 2^{-k[\delta-(n-1)/2-m\beta+(1-\lambda)n/p]}|Q|^{m\beta/n-1/p+\lambda/p}\|f\|_{M^{p,\lambda}}$$
$$\le C|Q|^{m\beta/n-1/p+\lambda/p}\|f\|_{M^{p,\lambda}}.$$

(2)When $\delta \ge (n+1)/2$,we have

$$|T^{b,m}_{\delta,t} f_2(x) - T^{b,m}_{\delta,t} f_2(x_0)|$$
$$\le C\sum_{k=1}^{\infty} 2^{-k[1-m\beta+(1-\lambda)n/p]}|Q|^{m\beta/n-1/p+\lambda/p}\|f\|_{M^{p,\lambda}}$$
$$\le C|Q|^{m\beta/n-1/p+\lambda/p}\|f\|_{M^{p,\lambda}}.$$

We gain

$$G_2 \le C|Q|^{1+[m\beta+(\lambda-1)n/p]/n}\|f\|_{M^{p,\lambda}}.$$

Together with $G_1$ and $G_2$, we have

$$\frac{1}{|Q|^{1+[m\beta+(\lambda-1)n/p]/n}}\int_Q |B_{\delta,t}^{b,m}f(x)-T_{\delta,t}^{b,m}f_2(x_0)|dx \le C\|f\|_{M^{p,\lambda}}.$$

Taking the supreme of the left side for any $t>0$ ,we get the ideal result.This finishes the proof of theorem 1.4.

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