A mathematical theory of truth and an application to the regress problem

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April 20, 2015

Shorttitle: A mathematical theory of truth and an application

Abstract

In this paper a class of languages which are formal enough for mathematical reasoning is introduced. Its languages are called mathematically agreeable (shortly MA). Languages containing a given MA language L, and being sublanguages of L augmented by a monadic predicate, are constructed. A mathematical theory of truth (shortly MTT) is formulated for some of these languages. MTT makes them MA languages which posses their own truth predicates. MTT is shown to conform well with the eight norms presented for theories of truth in the paper 'What Theories of Truth Should be Like (but Cannot be)', by Hannes Leitgeb. MTT is also free from infinite regress, providing a proper framework to study the regress problem. Main tools used in proofs are Zermelo-Fraenkel (ZF) set theory and classical logic.

MSC: 00A30, 03A05, 03B10, 03F50, 47H04, 47H10

Keywords: language, fully interpreted, truth, fixed point, regress problem.

1 Introduction

In this paper a theory of truth is formulated for a class of languages. The regress problem is studied within the framework of that theory.

A language L is called mathematically agreeable (shortly MA), if it satisfies the following three conditions.

(i) L contains a countable syntax of the first-order predicate logic with equality (cf., e.g., [17, Definitions II.5.1–5.2.6]), natural numbers in variables and their names, numerals in terms.
(ii) L is fully interpreted, i.e., every sentence of L is interpreted either as true or as false.
(iii) Classical truth tables (cf. e.g., [17], p.3) are valid for the logical connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$ and $\leftrightarrow$ of sentences of L, and classical rules of truth hold for applications of quantifiers $\forall$ and $\exists$ to formulas of L.

These properties ensure that every MA language is formal enough for mathematical reasoning. Any countable first-order formal language, equipped with a consistent theory interpreted by a countable model, and containing natural numbers and numerals, is an MA language. A classical example is the language of arithmetic with its standard model and interpretation.
Basic ingredients of the present approach are:
1. An MA language L (base language).
2. A monadic predicate T having the set X of numerals as its domain of discourse.
3. The language $\mathcal{L}$, which has sentences of L, $T(\mathbf{n})$, $\mathbf{n} \in X$, $\forall x T(x)$ and $\exists x T(x)$ as its basic sentences, and which is closed under logical connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$ and $\leftrightarrow$.
4. The set D of Gödel numbers of sentences of $\mathcal{L}$ in its fixed Gödel numbering.
The paper is organized as follows.
In Section 2 we construct to each subset U of D new subsets $G(U)$ and $F(U)$ of D. Let $\mathcal{L}_U$ be the language of the sentences whose Gödel numbers are in $G(U) \cup F(U)$. It contains L.
In Section 3 results on the existence and construction of consistent fixed points of G, i.e. consistent sets satisfying $U = G(U)$, including the smallest one, are presented. U is called consistent if for no sentence A of $\mathcal{L}$ the Gödel numbers of both A and $\neg A$ are in U.
In Section 4 a mathematical theory of truth (shortly MTT) is defined for languages $\mathcal{L}_U$, where U is a consistent fixed point of G. A sentence A of $\mathcal{L}_U$ is interpreted as true if its Gödel number $\#A$ is in $G(U)$, and as false if $\#A$ is in $F(U)$. This makes $\mathcal{L}_U$ an MA language. T is called a truth predicate for $\mathcal{L}_U$. Biconditionality: $A \leftrightarrow T(\lceil A \rceil)$, where $\lceil A \rceil$ is the numeral of the Gödel number of A, is shown to be true for all sentences A of $\mathcal{L}_U$. Since both L and $\mathcal{L}_U$ are fully interpreted, their sentences are either true or false. Moreover, a sentence A of L is either true or false in the interpretation of L if and only if A is either true or false in $\mathcal{L}_U$.
Section 5 is devoted to the study of the regress problem within the framework of MTT. We present an example of an infinite regress (parade) of justifications that satisfies the conditions imposed on them in [22]. Example is inconsistent with the following conclusion stated in [23]: "it is logically impossible for there to be an infinite parade of justifications". That conclusion is used in [22, 23] as a basic argument to refute Principles of Sufficient Reasons.
In Section 6 we shall first introduce some benefits of MTT compared with some other theories of truth. The lack of Liar-like sentences makes MTT mathematically acceptable. MTT is shown to conform well with the eight norms presented in [18] for theories of truth. Connections of obtained results to mathematical philosophy and epistemology are also presented.

2 Construction of languages

Let basic ingredients L, T, $\mathcal{L}$ and D be as in the Introduction. We shall construct a family of sublanguages for the language $\mathcal{L}$. As for the used terminology, cf. e.g., [17]. Let U be a subset of D. Define subsets $G(U)$ and $F(U)$ of D by following rules, which are similar to those presented in [11] ('iff' abbreviates 'if and only if'):

(r1) If A is a sentence of L, then the Gödel number $\#A$ of A is in $G(U)$ iff A is true in the interpretation of L, and in $F(U)$ iff A is false in the interpretation of L.

(r2) Let $\mathbf{n}$ be a numeral. $T(\mathbf{n})$ is in $G(U)$ iff $\mathbf{n} = \lceil A \rceil$, where A is a sentence of $\mathcal{L}$ and $\#A$ is in U. $T(\mathbf{n})$ is in $F(U)$ iff $\mathbf{n} = \lceil A \rceil$, where A is a sentence of $\mathcal{L}$ and $\#[\neg A]$ is in U.

Sentences determined by rules (r1) and (r2), i.e., all sentences A of L and those sentences $T(\lceil A \rceil)$ of $\mathcal{L}$ for which $\#A$ or $\#[\neg A]$ is in U, are called *basic sentences*.
Next rules deal with logical connectives. Let A and B be sentences of $\mathcal{L}$.

(r3) Negation rule: $\#[\neg A]$ is in $G(U)$ iff $\#A$ is in $F(U)$, and in $F(U)$ iff $\#A$ is in $G(U)$.

(r4) Disjunction rule: $\#[A \vee B]$ is in $G(U)$ iff $\#A$ or $\#B$ is in $G(U)$, and in $F(U)$ iff $\#A$ and $\#B$ are in $F(U)$.

(r5) Conjunction rule: $\#[A \wedge B]$ is in $G(U)$ iff $\#[\neg A \vee \neg B]$ is in $F(U)$ iff (by (r3) and (r4)) both $\#A$ and $\#B$ are in $G(U)$. Similarly, $\#[A \wedge B]$ is in $F(U)$ iff $\#[\neg A \vee \neg B]$ is in $G(U)$ iff $\#A$ or $\#B$ is in $F(U)$.

(r6) Implication rule: $\#[A \rightarrow B]$ is in $G(U)$ iff $\#[\neg A \vee B]$ is in $G(U)$ iff (by (r3) and (r4)) $\#A$ is in $F(U)$ or $\#B$ is in $G(U)$. $\#[A \rightarrow B]$ is in $F(U)$ iff $\#[\neg A \vee B]$ is in $F(U)$ iff $\#A$ is in $G(U)$ and $\#B$ is in $F(U)$.

(r7) Biconditionality rule: $\#[A \leftrightarrow B]$ is in $G(U)$ iff $\#A$ and $\#B$ are both in $G(U)$ or both in $F(U)$, and in $F(U)$ iff $\#A$ is in $G(U)$ and $\#B$ is in $F(U)$ or $\#A$ is in $F(U)$ and $\#B$ is in $G(U)$.

Rule (r1) is applicable for sentences of L formed by applications of universal and existential quantifiers to formulas of L. Thus it suffices to set rules for $\exists x T(x)$ and $\forall x T(x)$.

(r8) $\#[\exists x T(x)]$ is in $G(U)$ iff $\#T(\mathbf{n})$ is in $G(U)$ for some numeral $\mathbf{n}$. $\#[\exists x T(x)]$ is in $F(U)$ iff $\#T(\mathbf{n})$ is in $F(U)$ for every numeral $\mathbf{n}$.

(r9) $\#[\forall x T(x)]$ is in $G(U)$ iff $\#T(\mathbf{n})$ is in $G(U)$ for every numeral $\mathbf{n}$, and $\#[\forall x T(x)]$ is in $F(U)$ iff $\#T(\mathbf{n})$ is in $F(U)$ at least for one numeral $\mathbf{n}$.

Rules (r0)–(r9) and induction on the complexity of formulas determine uniquely subsets $G(U)$ and $F(U)$ of D whenever U is a subset of D. Denote by $\mathcal{L}_U$ the language formed by all those sentences $\mathcal{L}$ whose Gödel numbers are in $G(U)$ or in $F(U)$. $\mathcal{L}_U$ contains by rule (r1) all sentences of the base language L.

3 Fixed point results

We say that a subset U of D is *consistent* if for no sentence A of $\mathcal{L}$ the Gödel numbers of both A and $\neg A$ are in U. For instance, the empty set $\emptyset$ is consistent. Let $\mathcal{P}$ denote the family of all consistent subsets of the set D of Gödel numbers of sentences of $\mathcal{L}$.
The following three lemmas can be proved as the corresponding results in [11], replacing 'true in M' by 'true in the interpretation of L'.

Lemma 3.1. *([11, Lemma 2.1]) If $U \in \mathcal{P}$, then $G(U) \in \mathcal{P}$, $F(U) \in \mathcal{P}$, and $G(U) \cap F(U) = \emptyset$.*

According to Lemma 3.1 the mapping $G := U \mapsto G(U)$ maps $\mathcal{P}$ into $\mathcal{P}$. Assuming that $\mathcal{P}$ is ordered by inclusion, we have the following result.

Lemma 3.2. *([11, Lemma 4.2]) G is order preserving in $\mathcal{P}$, i.e., $G(U) \subseteq G(V)$ whenever U and V are sets of $\mathcal{P}$ and $U \subseteq V$.*

Lemma 3.3. *([11, Lemma 4.3]) If $\mathcal{W}$ is a chain in $\mathcal{P}$, then $\cup\mathcal{W} = \cup\{U \mid U \in \mathcal{W}\}$ is in $\mathcal{P}$.*

Fixed points of the set mapping $G := U \mapsto G(U)$ from $\mathcal{P}$ to $\mathcal{P}$, i.e., those $U \in \mathcal{P}$ for which $U = G(U)$, have a central role in the formulation of MTT. In the formulation our main fixed point theorem we use transfinite sequences of $\mathcal{P}$ indexed by von Neumann ordinals. Such a sequence $(U_\lambda)_{\lambda\in\alpha}$ of $\mathcal{P}$ is said to be strictly increasing if $U_\mu \subset U_\nu$ whenever $\mu \in \nu \in \alpha$. A set V of $\mathcal{P}$ is called *sound* iff $V \subseteq G(V)$.
The following fixed point theorem is proved in [11].

Theorem 3.1. *([11, Theorem 4.1]) If $V \in \mathcal{P}$ is sound, then there exists the smallest of those consistent fixed points of G which contain V. This fixed point is the last member of the union of those transfinite sequences $(U_\lambda)_{\lambda\in\alpha}$ of $\mathcal{P}$ which satisfy*

(C) $(U_\lambda)_{\lambda\in\alpha}$ is strictly increasing, $U_0 = V$, and if $0 \in \mu \in \alpha$, then $U_\mu = \bigcup_{\lambda\in\mu} G(U_\lambda)$.

As a consequence of Theorem 3.1 we obtain.

Corollary 3.1. *Let W be the set of Gödel numbers of all those sentences of L which are true in its interpretation, and let V be any subset of W.*
(a) V is a sound and consistent subset of D.
(b) The union of the transfinite sequences which satisfy (C) is the smallest consistent fixed point of G.

Proof. (a) Rule (r1) and Lemma 3.2 imply that $V \subseteq G(\emptyset) \subseteq G(V)$, so that V is sound. It is also consistent, as a subset of a consistent set $G(\emptyset)$.
(b) V is by (a) sound and consistent. If U is a consistent fixed point of G, then $V \subseteq G(\emptyset) \subset G(U) = U$. Thus V is contained in every consistent fixed point of G. By Theorem 3.1, the union of those transfinite sequences $(U_\lambda)_{\lambda\in\alpha}$ of $\mathcal{P}$ which satisfy (C) is the smallest consistent fixed point of G that contains V. This proves (b). □

Remarks 3.1. The smallest members of $(U_\lambda)_{\lambda\in\alpha}$ satisfying (C) are n-fold iterations $U_n = G^n(V)$, $n \in \mathbb{N} = \{0, 1, \dots\}$. If they form a strictly increasing sequence, the next member U_ω is their union, $U_{\omega+n} = G^n(U_\omega)$, $n \in \mathbb{N}$, and so on.

4 A mathematical theory of truth

Recall that D denotes the set of Gödel numbers of sentences of the language $\mathcal{L}$. Given a subset U of D, let $G(U)$ and $F(U)$ be the subsets of D constructed in Section 3. In the next definition, which is the same as presented in [11] in a special case, we formulate our mathematical theory of truth (shortly MTT).

Definition 4.1. *Assume that U is a consistent subset of D, and that $U = G(U)$. Denote by $\mathcal{L}_U$ the language containing those sentences A of $\mathcal{L}$ for which $\#A$ is in $G(U)$ or in $F(U)$. A sentence A of $\mathcal{L}_U$ is interpreted as true iff $\#A$ is in $G(U)$, and as false iff $\#A$ is in $F(U)$. T is called a truth predicate for $\mathcal{L}_U$.*

In view of Definition 4.1, '$\#A$ is in $G(U)$' can be replaced by 'A is true' and '$\#A$ is in $F(U)$' by 'A is false' in (r1)–(r9). This replacement, the construction of $G(U)$ and $F(U)$ and Lemma 3.1 imply that $\mathcal{L}_U$ is an MA language, having thus those syntactical and semantical properties which are assumed for the base language L.
The following result justifies to call T as a truth predicate of $\mathcal{L}_U$.

Lemma 4.1. *If U is a consistent subset of D, and if $U = G(U)$, then T-biconditionality: $A \leftrightarrow T(\lceil A \rceil)$ is true, and $A \leftrightarrow \neg T(\lceil A \rceil)$ is false for every sentence A of $\mathcal{L}_U$.*

Proof. Assume that $U \subset D$ is consistent, and that $U = G(U)$. Let A be a sentence of $\mathcal{L}_U$. Applying rules (r2) and (r3), and the assumption $U = G(U)$, we obtain
– $\#A$ is in $G(U)$ iff $\#A$ is in U iff $\#T(\lceil A \rceil)$ is in $G(U)$ iff $\#\neg T(\lceil A \rceil)$ is in $F(U)$;
– $\#A$ is in $F(U)$ iff $\#[\neg A]$ is in $G(U)$ iff $\#[\neg A]$ is in U iff $\#T(\lceil A \rceil)$ is in $F(U)$ iff $\#\neg T(\lceil A \rceil)$ is in $G(U)$.
The above results, rule (r7) and Definition 4.1 imply that $A \leftrightarrow T(\lceil A \rceil)$ is true, and that $A \leftrightarrow \neg T(\lceil A \rceil)$ is false. This holds for every sentence A of $\mathcal{L}_U$. □

Our main result on the connection between the valuations determined by the interpretation of L and that of $\mathcal{L}_U$ defined in Definition 4.1 reads as follows:

Lemma 4.2. *Let U be a consistent fixed point of G. If A is a sentence of L, then either*
(a) A is true in the interpretation of L, iff A is true, iff $T(\lceil A \rceil)$ is true, or
(b) A is false in the interpretation of L, iff A is false, iff $T(\lceil A \rceil)$ is false.

Proof. Assume that A is a sentence of L. Because L is completely interpreted, then A is either true or false in the interpretation of L.
– A is true in the interpretation of L iff $\#A$ is in $G(U)$, by rule (r1), iff $\#A$ is in U, because $U = G(U)$, iff $\#T(\lceil A \rceil)$ is in $G(U)$ by rule (r2), iff $T(\lceil A \rceil)$ is true, by Definition 4.1.
– A is false in the interpretation of L iff $\neg A$ is true in the interpretation of L iff $\#[\neg A]$ is in $G(U)$, by rule (r1), iff $\#[\neg A]$ is in U, because $U = G(U)$, iff $\#T(\lceil A \rceil)$ is in $F(U)$, by rule (r2), iff $T(\lceil A \rceil)$ is false, by Definition 4.1.
Consequently, a sentence A of L is true in the interpretation of L iff $T(\lceil A \rceil)$ is true, and false in the interpretation of L iff $T(\lceil A \rceil)$ is false. These results and the result of Lemma 4.1 imply the conclusions (a) and (b). □

5 On the Regress Problem

First of ten theses presented in [1, p. 6] is: "The Regress Problem is a real problem for epistemology." We are going to study the regress problem in the framework of MTT. Given an MA language L, let an MA language $\mathcal{L}_U$ that contains L be determined by Definition 4.1, U being the smallest fixed point of G. We adjust first our terminology to that used in [22] in the study of the regress problem. By statements we mean the sentences of $\mathcal{L}_U$, which are valued by Definition 4.1. A statement A is said to entail B, if it is not possible that A is true and B is false simultaneously. For instance, if $A \rightarrow B$ is true, then A entails B. We say that a statement A justifies a statement B if A confirms the truth of B. For instance, if $A \leftrightarrow \neg B$ is true, then A justifies B iff A is false. If $A \rightarrow B$ is true, then A justifies B iff A is true (Modus Ponens). A is called contingent if the truth value of A is unknown.

Consider an infinite regress

$$\dots F_i, \dots, F_1, F_0 \tag{5.1}$$

of statements F_i, $i \geq 0$, where the statement F_0 is contingent. We shall impose the following conditions on statements F_i, $i > 0$ (cf. [22]):

(i) F_i entails F_{i-1};

(ii) $F_0 \vee \cdots \vee F_{i-1}$ does not entail F_i;

(iii) $F_0 \vee \cdots \vee F_{i-1}$ does not justify F_i.

Regress (5.1) is called justification-saturated if the following condition holds:

(iv) ... what justifies F_{i-1} is $F_i, \dots,$ what justifies F_1 is F_2, what justifies F_0 is F_1.

Lemma 5.1. *Assume that in regress (5.1) the statement F_0 is contingent, and that the statements F_i, $i > 0$, satisfy conditions (i)–(iii).*
(a) If F_1 is false, then F_i is false for each $i > 0$. F_0 is justified iff $F_0 \leftrightarrow \neg F_1$ is true.
(b) If F_n is true for some $n > 0$, then F_i is true when $0 \leq i \leq n$.
(c) The regress (5.1) is justification saturated iff F_i is true for all $i > 0$, in which case F_0 is justified.

Proof. (a) Assume that F_1 is false. If F_i would be true for some $i > 1$, there would be the smallest such an i. Then F_{i-1} would be true by (i). Replacing i by $i-1$, and so on, this reasoning would imply after $i-1$ steps that F_1 is true; a contradiction. Thus all statements F_i, $i > 0$, are false. Because F_1 is false, it confirms the truth of F_0 iff F_0 and $\neg F_1$ have same truth values iff $F_0 \leftrightarrow \neg F_1$ is true.
(b) Assume that F_n is true for some $n > 0$. Since F_i entails F_{i-1}, $i = n, n-1, \dots, 1$, then F_i is true for every $i = n-1, \dots, 0$.
(c) If F_n is false for some $n > 0$, then F_{n+1} is false by property (i), and it does not justify F_n, so that condition (iv) is not valid. On the other hand, condition (i) ensures that condition (iv) is valid if F_i is true for all $i > 0$. In this case F_1 justifies F_0, i.e., F_0 is true. □

Example 5.1. Let L be the first-order language $L = \{\in\}$ of set theory, and M the minimal model of ZF set theory constructed in [3]. M is countable and contains the set ω of natural numbers and their set $S(\omega) = \omega \cup \{\omega\}$ (cf. [3, 14]). We assume that numerals are defined in L, e.g., as in [6]. Interpret a sentence A of L as true in L if $M \models A$, and false in L if $M \models \neg A$, in the sense defined in [17, II.2.7 and p. 237]. By [17, Lemma II.2.8.22] this interpretation makes L fully interpreted. In particular, L is an MA language. Choose L as the base language of theory MTT. Equip $S(\omega)$ with the natural ordering $<$ of natural numbers plus $n < \omega$ for every natural number n. If Z denotes a nonempty subset of $S(\omega)$, it is easy to verify that the infinite regress (5.1) of statements

$$F_i : \quad i < \beta, \text{ for every } \beta \in Z, \quad i = 0, 1, \ldots, \tag{5.2}$$

satisfy conditions (i)–(iii), and that F_0 is contingent. Moreover, condition (iv) is valid by Lemma 5.1 if and only if F_i is true for all $i > 0$. This holds if and only if $Z = \{\omega\}$.

6 Remarks

The main purpose of this paper is to present a mathematical theory of truth (MTT) for a class languages which are formal enough for mathematical reasoning.

To describe properties of MTT and to compare it to some other theories of truth, let $S = (L, \Sigma)$ be a mathematical theory, where L is a first-order formal language, and Σ is a set of axioms. Assume that Σ is consistent, and is either an extension of Robinson arithmetic Q (e.g., Q itself or Peano arithmetic, L being the language of arithmetic), or Q can be interpreted in Σ (e.g., Σ axiomatizes ZF set theory, and L is the language of set theory). By Löwenheim-Skolem theorem that theory has a countable model M. Interpret a sentence A of L as true in L if $M \models A$, and false in L if $M \models \neg A$, in the sense defined in [17, II.2.7]. By [17, Lemma II.2.8.22] this interpretation makes L fully interpreted, and L is an MA language. Tarski's Undefinability Theorem (cf. [25]) implies that L cannot contain its truth predicate, yielding 'Tarski's Commandment' (cf. [19]). Let $\mathcal{L}$ be a formal language obtained by augmenting L with a monadic predicate T. T cannot be a truth predicate of $\mathcal{L}$, for otherwise one could construct a Liar sentence, which implies the 'Liar paradox' (cf. [12]). Thus $\mathcal{L}$ does not contain its truth predicate, either. Many axiomatic theories of truth (cf., e.g, [5]) are constructed for languages which contain a Liar sentence or are subject to the 'Revenge of Liar'. Such languages are not MA languages.

Theory MTT provides an alternative. Given an MA language L, let $\mathcal{L}_U$, where U is a consistent fixed point of G, be an extension of L constructed in Section 3. That construction and the interpretation given for $\mathcal{L}_U$ in Definition 4.1 makes it an MA language. Moreover, $\mathcal{L}_U$ contains by Definition 4.1 a truth predicate T. It follows from Lemma 4.1 that there is no Liar sentence in $\mathcal{L}_U$. As an MA language $\mathcal{L}_U$ *is formal enough for mathematical reasoning.* In particular, the language L of the above theory S is extended in $\mathrm{S}_U =(\mathcal{L}_U,\mathrm{MTT})$ to an MA language $\mathcal{L}_U$ that contains its truth predicate and is free from paradoxes.

If $S = (L, \Sigma)$ is as above, L contains by Gödel's First Incompleteness Theorem a true arithmetical sentence, say B, that is not provable from the axioms of Σ (cf. [24]). By Lemma 4.2 both B and $T(B)$ are true in the interpretation of $\mathcal{L}_U$. Based on the existence of B

the following opinions on mathematical truth presented in [21, Chapter 4]: "The notion of mathematical truth goes beyond the whole concept of formalism. There is something absolute and 'God-given' about mathematical truth. Real mathematical truth goes beyond mere man-made constructions." These opinions are questioned because of the consistency assumption of Σ. (cf. [24]). Despite inability of human mind to see that consistency it is indispensable for reliability mathematical results. Mathematics rests on the belief that its theories are consistent.

MTT has properties that conform well with the eight norms formulated in [18] for theories of truth. Truth is expressed by a predicate T. An MA language $\mathcal{L}_U$ contains a syntax of first-order logic with equality, natural numbers in constants and numerals in terms. It is closed under logical connectives and quantifiers. A theory of truth is added to the base language L. If the interpretation of L is determined by a consistent mathematical theory (Peano arithmetic, ZF set theory, e.t.c.), then MTT proves the theory in question true, by Lemma 4.2. Truth predicate T is not subject to any restrictions within a fixed point language $\mathcal{L}_U$. T-biconditional is derivable unrestrictedly within a fixed point language $\mathcal{L}_U$, by the proof of Lemma 4.1. Truth is compositional, by Definition 4.1 and rules (r3)–(r9). The theory allows for standard interpretations if the interpretation of L is standard. In particular, *the outer logic and the inner logic coincide, and they are classical.*
Paradoxes led Zermelo to axiomatize set theory. To avoid paradoxes Tarski "excluded all Liar-like sentences from being well-formed", as noticed in [18]. A fixed point language $\mathcal{L}_U$ does not contain such sentences in the theory MTT. In particular, MTT is immune to 'Tarski's Commandment' (cf. [19]), to Tarski's Undefinability Theorem (cf. [25]), to 'Tarskian hierarchies' (cf. [8]), and to 'Liar paradox' (cf. [12]). The smallest of those languages for which MTT is formulated is $\mathcal{L}_U$, where U is the smallest consistent fixed point of G. It relates to that of the grounded sentences defined in [10, 16] when L is the language of arithmetic. See also [7], where considerations are restricted to signed statements.
A base language L can contain more sentences than first-order formal languages, thus extending the class of languages for which theories of truth are usually formulated.

Another purpose of the presented theory of truth is to establish a proper framework to study the regress problem. Tarski's theory of truth (cf. [25]) does not offer it because that theory itself is not free from infinite regress. According to [20, p.189]: "the most important problem with a Tarskian truth predicate is its demand for a hierarchy of languages. ... within that hierarchy of languages, we cannot seem to have any valid method of ending the regression to introduce the "basic" metalanguage."

Kripke's theory of truth is also a problematic framework because of three-valued inner logic. As stated in [18, p.283]: "Classical first-order logic is certainly the default choice for any selection among logical systems. It is presupposed by standard mathematics, by (at least) huge parts of science, and by much of philosophical reasoning." Moreover, T-biconditionality rule does not hold in Kripke's theory of truth because of paradoxical sentences.
Example 5.1 is inconsistent with the conclusion of [23] cited in the Introduction. In this example the property that regress (5.1) is justification-saturated both implies and is implied by truth of a 'foundational' statement $F_b : Z = \{\omega\}$. Thus it does not support the form of infinitism presented in [15]: "infinitism holds that there are no ultimate, foundational reasons". Pure infinite regress is even refused in [4, p.13]. On the other hand, it supports

"impure" infinitism and the form of foundationalism presented in [1, 26].
Example 5.1 implies that the proofs in [22, 23] to the assertion that "any version of Principle of Sufficient Reason is false" are based on the questionable premise that infinite regresses of justifications don't exist. In fact, this example gives some support to Principles of Sufficient Reason, as well as to many other arguments whose validity is questioned in [22, 23]. For instance, in the 'universe' $S(\omega)$ of Example 5.1,

- $\{\omega\}$ provides a *sufficient reason* for F_0;
- $\{\omega\}$ affords an *ultimate and foundational reason* that justifies F_0;
- $\{\omega\}$ is the *final explainer* of F_0;
- $\{\omega\}$ gives the *first cause* that makes regress (5.1),(5.2) justification-saturated;
- ω *explains the existence of the 'universe'* $\mathbb{N}$ of natural numbers ($\mathbb{N} = \omega$ by [13]);
- ω and $\{\omega\}$ *explain the existence of the 'universe'* $S(\omega)$ ($S(\omega) = \omega \cup \{\omega\}$ by [13]);
- ω is *something beyond natural numbers*;
- ω is *infinite and greatest* in the 'universe' $S(\omega)$;
- ω is *'self-justified'* (The Axiom of Infinity).

Belief that ω exists is a matter of faith. In Example 5.1 we have assumed it because the model M of ZF set theory contains the set $\omega \cup \{\omega\}$. Notice that this set does not belong to the standard model of arithmetic. Thus MTT, where the base language L is the language of arithmetic, is not a sufficient framework for Example 5.1.

Acknowledgments: The author is indebted to Ph.d. Markus Pantsar for valuable discussions on the subject.

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