Axiomatic foundations of Quantum Mechanics revisited: the case of systems

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Abstract

We present an axiomatization of non-relativistic Quantum Mechanics for a system with an arbitrary number of components. The interpretation of our system of axioms is realistic and objective. The EPR paradox and its relation with realism is discussed in this framework. It is shown that there is no contradiction between realism and recent experimental results.

KEYWORDS: Axiomatics - Philosophy of Quantum Mechanics

1 INTRODUCTION

The interpretation of Quantum Mechanics (QM) has been a controversial subject over the last fifty years. A central point in this controversy is the debate between the realistic position and the orthodox line of the Copenhagen school. In recent years, this discussion has been reheated by some experiments that enabled the testing of the implications of the paradox formulated by Einstein, Podolsky, and Rosen (EPR) (1935). There is a widespread belief that the results of those experiments imply the refutation of realism and favour a subjectivistic vision of QM. However, these conclusions originate in an informal analysis of the structure of the theory. Any conclusion in the aforementioned sense should be a consequence of a careful study of a formalized theory of QM, in such a way that all the presuppositions and interpretation rules be explicit.

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Only in this case one can determine whether the realistic interpretation of the statements is consistent with the experimental results.

In a previous work (Perez-Bergliaffa *et al.* 1993), we presented a realistic and objective axiomatization of QM for a single microsystem from which the main theorems can be deduced. Problems such as those arising from the EPR paradox cannot be discussed in that axiomatic frame, because they involve systems with more than one component. We develop here a generalization of our preceding paper for the case of systems with an arbitrary number of components. Armed with this new axiomatization, we analyze some interpretational issues of QM.

We briefly present in Section 2 the ontological background of our interpretation, because it is of the utmost importance in all our argumentations (for details, see Bunge 1977, 1979). In Section 3 we set forth the axiomatization of the theory, with its presuppositions, its axiomatic basis, the pertinent definitions, and some representative theorems. In Section 4 we discuss the relation between the EPR paradox and realism, and then, we shortly sketch some items of the "consistent interpretation of QM" that can be deduced from our axiomatization.[1]

2 ONTOLOGICAL BACKGROUND

A consistent axiomatic treatment of nonrelativistic QM for systems with an arbitrary number of components presuposses a *theory of systems*. This in turn can only be constructed on the basis of an accurate caracterization of the concept of *individual* and its *properties*. In this section we caracterize a *physical system*. We shall assume the realistic ontology of Bunge (a complete and detailed analysis can be found in Bunge 1977, 1979).

The concept of individual is the basic primitive concept of any ontological theory. Individuals associate themselves with other individuals to yield new individuals. It follows that they satisfy a calculus, and that they are rigorously characterized only through the laws of such calculus. These laws are set with the aim of reproducing the way real things associate. Specifically, it is postulated that every individual is an element of a set S in such a way that the structure $\mathcal{S} =< S, \circ, \Box >$ is a *commutative monoid of idempotents*. In the structure $\mathcal{S}$, S is to be interpreted as the set of all the individuals, the element $\Box \in S$ as the null individual, and the binary operation $\circ$ as the association of individuals. It is easy to see that there are two classes of individuals: *simple* and *composed*.

D₁ $x \in S$ is composed $\Leftrightarrow \exists\, y, z \in S \ni x = y \circ z$.

[1] Some of the formal tools used in this work have been described in Perez-Bergliaffa *et al.* (1993) (mainly mathematical tools, such as $\mathcal{H}$, G, etc.)

D₂ $x \in S$ is simple $\Leftrightarrow \neg \exists\, y, z \in S \ni x = y \circ z$.

D₃ $x \sqsubset y \Leftrightarrow x \circ y = y$ (x is part of $y \Leftrightarrow x \circ y = y$).

D₄ $\mathcal{C}(x) \equiv \{y \in S \ni y \sqsubset x\}$ (composition of x).

Real things differentiate from abstract individuals because they have a number of properties in addition to their capability of association. These properties can be *intrinsic* ($P_{\rm i}$) or *relational* ($P_{\rm r}$). The intrinsic properties are inherent and they are represented by predicates or unary applications, while relational properties are represented by n-ary predicates, with n>1, as long as nonconceptual arguments are considered. For instance, the position and the velocity of a particle are relational properties, but its charge is an intrinsic property.

P is called a *substantial property* if and only if some individual x possesses P:

D₅ $P \in \mathcal{P} \Leftrightarrow (\exists x)(x \in S \wedge Px)$.

Here $\mathcal{P}$ is the set of all the substantial properties. The set of the properties of a given individual x is

D₆ $P(x) \equiv \{P \in \mathcal{P} \ni Px\}$.

If two individuals have exactly the same properties they are the same: $\forall\, x, y \in S$ if $P(x) = P(y) \Rightarrow x \equiv y$. Two individuals are identical if their intrinsic properties are the same: $x \overset{id}{\leftrightarrow} y$ (they can differ only in their relational properties).

A detailed account of the theory of properties is given in Bunge (1977). We only give here two useful definitions:

D₇ P is an inherited property of $x \Leftrightarrow P \in P(x) \wedge (\exists y)(y \in \mathcal{C}(x) \wedge y \neq x \wedge P \in P(y))$.

D₈ P is an emergent property of $x \Leftrightarrow P \in P(x) \wedge (\, (\forall\, y)_{\mathcal{C}(x)}(y \neq x\,) \Rightarrow P \notin P(y)\,)$.

According to these definitions, mass is an inherited property and viscosity is an emergent property of a classical fluid.

An individual with its properties make up a *thing* X:

D₉ $X \overset{Df}{=} < x, P(x) >$.

The laws of association of things follow from those of the individuals. The association of all things is the *Universe* (σ_U). It should not be confused with the set of all things; this is only an abstract entity and not a thing. Given a thing $X = < x, P(x) >$, a conceptual object named *model* X_m of the thing X can be constructed by a nonempty set M and a finite sequence $\mathcal{F}$ of mathematical functions over M, each of them formally representing a property of x:

D$_{10}$ $X_m \overset{Df}{=} < M, \mathcal{F} >$, where $\mathcal{F} =< \mathcal{F}_1, \ldots, \mathcal{F}_n > \ni \mathcal{F}_i : M \rightarrow V_i$, $1 \leq i \leq n, V_i$ vector space $, \mathcal{F}_i \hat{=} P_i \in P(x)$.

It is said then that X_m represents X: $X_m \hat{=} X$ (Bunge 1977).

The *state* of the thing X can be characterized as follows:

D$_{11}$ Let X be a thing with model $X_m =< M, \mathcal{F} >$, such that each component of the function

$$\mathcal{F} =< \mathcal{F}_1, \ldots, \mathcal{F}_n >: M \rightarrow V_1 \times \ldots \times V_n.$$

represents some $P \in P(x)$. Then $\mathcal{F}_i$ $(1 \leq i \leq n)$ is named i-th state function of X, $\mathcal{F}$ is the *total state function* of X, and the value of $\mathcal{F}$ for some $m \in M$, $\mathcal{F}(m)$, represents the *state of* X *at* m *in the representation* X_m.

If all the V_i, $1 \leq i \leq n$, are vector spaces, $\mathcal{F}$ is the state vector of X in the representation X_m, and $V = V_1 \times \ldots \times V_n$ is the state space of X in the representation X_m.

The concept of physical law can be introduced as follows:

D$_{12}$ Let $X_m =< X, \mathcal{F} >$ be a model for X. Any restriction on the possible values of the components of $\mathcal{F}$ and any relation between two or more of them is a *physical law* if and only if it belongs to a consistent theory of the X and has been satisfactorily confirmed by the experiment.

We say that a thing X *acts* on a thing Y if X modifies the path of Y in its space state ($X \triangleright Y$: X acts on Y).

We say that two things X and Y are *connected* if at least one of them acts on the other. We come at last to the definition of *system*:

D$_{13}$ A system is a thing composed by at least two connected things.

In particular, a physical system is a system ruled by physical laws. A set of things is not a system, because a system is a physical entity and not a set. A system may posses emergent properties with respect to the component subsystems. The composition of the system σ with respect to a class A of things is (at the instant t):

$$C_A(\sigma, t) = \{X \in A \ni X \sqsubset \sigma\}$$

.

D$_{14}$ $\overline{\sigma}_A(\sigma, t) = \{X \in A \ni X \notin \mathcal{C}_A(\sigma, t) \wedge (\exists Y)_{\mathcal{C}_A(\sigma,t)} \wedge (X \triangleright Y \vee Y \triangleright X)\ )\}$ is the A-environment of σ at t.

If $\overline{\sigma}_A(\sigma, t) = \emptyset \Rightarrow \sigma$ is *closed* at the instant t. In any other case we say that it is open.

A specific physical system will be characterized by expliciting its space of physical states. This is done in the axiomatic basis of the physical theory. In what follows we pay particular attention to a special type of systems: the *q-systems*.

3 AXIOMATICS OF QM

We present in this section the axiomatic structure of the theory following the main lines of our previous paper. The advantadges of an axiomatic formulation are discussed in Bunge (1967a).

3.1 FORMAL BACKGROUND

$\mathbf{P_1}$ Ordinary bivalued logic.

$\mathbf{P_2}$ Formal semantics (Bunge 1974a,b).

$\mathbf{P_3}$ Mathematical analysis with its presuppositions and theory of generalized functions (Gel'fand 1964).

$\mathbf{P_4}$ Probability theory.

$\mathbf{P_5}$ Group theory.

$\mathbf{P_6}$ Association theory (Bunge 1977).

3.2 MATERIAL BACKGROUND

$\mathbf{P_7}$ Cronology.

$\mathbf{P_8}$ Physical theory of probabilities (Popper 1959).

$\mathbf{P_9}$ Dimensional analysis.

$\mathbf{P_{10}}$ Systems theory (Bunge 1977, 1979).

3.3 GENERATING BASIS

The conceptual space of the theory is generated by the basis B of primitive concepts, where

$$\mathrm{B}=\{\overline{\Sigma}, \mathrm{E}_3, \mathrm{T}, \mathcal{H}_E, \mathcal{P}, \Sigma_1, \mathrm{A}, \mathrm{G}, \Pi, \hbar\}$$

The elements of the basis will be semantically interpreted by means of the axiomatic basis of the theory, with the aid of some conventions.

3.4 AXIOMATIC BASIS

QM is a finite-axiomatizable theory, whose axiomatic basis is

$$\mathcal{B}_A(QM) = \bigwedge_{i=1}^{36} \mathbf{A_i}$$

where the index i runs on the axioms.

3.5 DEFINITIONS

$\mathbf{D_{15}}$ $K \stackrel{Df}{=}$ set of physical reference systems.

$\mathbf{D_{16}}$ $|\Psi(\sigma, k)> \in \Psi_\sigma \stackrel{Df}{=}$ is the representative of the ray Ψ_σ that corresponds to the system σ with respect to $k \in K$.

$\mathbf{D_{17}}$ $\Sigma_N = \Sigma_1 \times \Sigma_1 \times \ldots \times \Sigma_1$ (N times) is the set of all the systems composed by elements of Σ_1.[2]

$\mathbf{D_{18}}$ $\Sigma^* = \{\Sigma_2, \Sigma_3, \ldots, \Sigma_N, \ldots\}$

Remark 1 With the aim of avoiding unnecesary complexity in notation we are not going to explicit the dependence of the operators and the eigenvalues on the reference system. **Remark 2** The domain of the cuantified variables is explicited by means of subindexes of the quantification parenthesis. For instance, $(\forall\ x)_A(\exists y)_B(Rxy)$, means that for all x in A, exists y in B such that Rxy. **Remark 3** The symbol $\stackrel{d}{=}$ is used for the relation of denotation (see Bunge 1974a for details).

3.6 AXIOMS

GROUP I: SPACE AND TIME

$\mathbf{A_1}$ $\mathrm{E}_3 \equiv$ tridimensional euclidean space.

$\mathbf{A_2}$ $\mathrm{E}_3 \hat{=}$ physical space.

$\mathbf{A_3}$ $\mathrm{T} \equiv$ interval of the real line $\mathcal{R}$.

$\mathbf{A_4}$ $\mathrm{T} \hat{=}$ time interval.

$\mathbf{A_5}$ The relation $\leq$ that orders T means "before to" $\vee$ "simultaneous with".

[2]The set Σ_1 will be characterized by $\mathbf{A_7}$.

GROUP II: Q-SYSTEMS AND STATES

$\mathbf{A_6}$ $\Sigma_1, \overline{\Sigma}$: nonempty numerable sets.

$\mathbf{A_7}$ $(\forall\sigma)_{\Sigma_1}(\sigma \stackrel{d}{=}$ simple microsystem$)$. [3]

$\mathbf{A_8}$ $(\forall\, \sigma)_{\Sigma=\Sigma_1\cup\Sigma^*}$ $(\sigma \stackrel{d}{=}$ q-system$)$.[4]

$\mathbf{A_9}$ $(\forall\, \overline{\sigma})_{\overline{\Sigma}}$ $(\overline{\sigma} \stackrel{d}{=}$ environment of some q-system$)$. [5]

$\mathbf{A_{10}}$ $(\exists\, K)(K \subset \overline{\Sigma} \wedge$ the configuration of each $k \in K$ is independent of time$)$.

$\mathbf{A_{11}}$ $(\forall\, k)_K\, (\exists\, b)\, (\overline{b} \hat{=}\, k)$.

$\mathbf{A_{12}}$ $(\forall\, \sigma)_\Sigma\, (\forall\, k)_K (k \not\triangleleft\!\triangleright \sigma)$.

$\mathbf{A_{13}}$ $(\forall <\sigma,\overline{\sigma}>)_{\Sigma\times\overline{\Sigma}}\, (\exists\, \mathcal{H}_E)(\mathcal{H}_E =< \mathcal{S},\ \mathcal{H},\ \mathcal{S}' > \equiv$ rigged Hilbert space$)$.

$\mathbf{A_{14}}$ There exists a one-to-one correspondence between physical states of $\sigma \in \Sigma$ and rays $\Psi_\sigma \subset\ \mathcal{H}$.

GROUP III: OPERATORS AND PHYSICAL QUANTITIES

$\mathbf{A_{15}}$ $\mathcal{P} \equiv$ nonempty family of applications over Σ.

$\mathbf{A_{16}}$ A $\equiv$ ring of operators over $\mathcal{H}_E$.

$\mathbf{A_{17}}$ $(\forall\, P)_\mathcal{P}(\exists\sigma)_\Sigma(P \in P(\sigma))$.

$\mathbf{A_{18}}$ $(\forall\, P)_\mathcal{P}(\exists\, \hat{A})_A\, (\hat{A}\, \hat{=} P)$.

$\mathbf{A_{19}}$ (Hermiticity and linearity)

$(\forall\, \sigma)_\Sigma\, (\forall\hat{A})_A\, (\forall\, P)_\mathcal{P}\, (\forall\, k)_K\, (\hat{A}\hat{=}P \wedge |\Psi(\sigma_1,k)>,\ |\Psi(\sigma_2,k)> \in \mathcal{H}_E \Rightarrow$

1. $\ni \hat{A}[\lambda_1|\Psi(\sigma_1,k)> +\lambda_2|\Psi(\sigma_2,k)>] = \lambda_1\hat{A}|\Psi(\sigma_1,k)> +\lambda_2\hat{A}|\Psi(\sigma_2,k)>$ with $\lambda_1,\lambda_2 \in \mathcal{C}$
2. $\hat{A}^\dagger = \hat{A})$.

$\mathbf{A_{20}}$ (Probability densities)

$(\forall\, <\sigma,\overline{\sigma}>)_{\Sigma\times\overline{\Sigma}}\, (\forall\hat{A})_A\, (\forall\, P)_\mathcal{P}\, (\forall\, |a>)_{\mathcal{H}_E}\, (\forall\, |\Psi(\sigma,k)>)_{\mathcal{H}_E}\, (\hat{A}\hat{=}P\ \wedge\ \hat{A}|a>= a|a> \Rightarrow$ the probability

[3] σ satisfies the set of axioms of our previous paper, so the definitions concerning σ given there are still valid. Also note that what is denoted here Σ_1 was denoted Σ in that paper.

[4] Not every $\sigma \in \Sigma$ is necessarily a system as defined by $\mathbf{D_{13}}$. However, with the aim of avoiding complex notation, we commit an abuse of language in this respect.

[5] In particular, $\overline{\sigma}_0 \stackrel{d}{=}$ the empty environment, $<\sigma,\overline{\sigma}_0>\stackrel{d}{=}$ a free q-system, and $<\sigma_0,\overline{\sigma}_0>\stackrel{d}{=}$ the vacuum.

density $<\psi|a><a|\psi>$ corresponds to the property P when σ is associated to $\overline{\sigma}$), that is, $\int_{a_1}^{a_2} <\psi|a> <a|\psi>\ da$ is the probability for σ to have a value of P in the interval $[a_1, a_2]$.

$\mathbf{A_{21}}$ $(\forall\ \sigma)_\Sigma\ (\forall \hat{A})_A\ (\forall\ a)_\mathcal{R}\ (eiv\ \hat{A} = a \wedge \hat{A} \hat{=} P \Rightarrow a$ is the sole value that P takes on σ).

$\mathbf{A_{22}}$ $\hbar\ \in \mathcal{R}^+$.

$\mathbf{A_{23}}$ $[\hbar] = LMT^{-1}$.

GROUP IV: SYMMETRIES AND GROUP STRUCTURE

$\mathbf{A_{24}}$ (Unitary operators)

$(\forall <\sigma, \overline{\sigma}>)_{\Sigma \times \overline{\Sigma}}\ (\forall \hat{A})_A\ (\forall\ P)_\mathcal{P}\ (\forall\ \hat{U})(\hat{A} \hat{=} P \wedge \hat{U}$ is an operator on $\mathcal{H}_E \wedge \hat{U}^\dagger = \hat{U}^{-1} \Rightarrow\ \hat{U}^\dagger \hat{A} \hat{U} \hat{=} P)$.

$\mathbf{A_{25}}$ $(\forall <\sigma, \overline{\sigma}_0>)_{\Sigma \times \overline{\Sigma}}\ \exists\ \hat{D}(\tilde{G})(\hat{D}(\tilde{G})$ is a unitary representation of rays of some central nontrivial extension of the universal covering group $\bar{G}$ of a Lie group G by an abelian unidimensional group on $\mathcal{H}_E$).

$\mathbf{A_{26}}$ The Lie algebra $\mathcal{G}$ of the group G is generated by $\{\hat{H},\ \hat{P}_i,\ \hat{K}_i,\ \hat{J}_i\} \subset A$.

$\mathbf{A_{27}}$ (Algebra structure)

The structure of $\tilde{\mathcal{G}}$, Lie algebra of $\tilde{G}$ is:

$$[\hat{J}_i, \hat{J}_j] = i\hbar\epsilon_{ijk}\hat{J}_k \qquad [\hat{J}_i, \hat{K}_j] = i\hbar\epsilon_{ijk}\hat{K}_k \qquad [\hat{J}_i, \hat{P}_j] = i\hbar\epsilon_{ijk}\hat{P}_k$$

$$[\hat{K}_i, \hat{H}] = i\hbar\hat{P}_i \qquad [\hat{K}_i, \hat{P}_j] = i\hbar\delta_{ij}\hat{M}$$

$$[\hat{J}_i, \hat{H}] = 0 \quad [\hat{K}_i, \hat{K}_j] = 0 \quad [\hat{P}_i, \hat{P}_j] = 0 \quad [\hat{P}_j, \hat{H}] = 0$$

$$[\hat{J}_i, \hat{M}] = 0 \quad [\hat{K}_i, \hat{M}] = 0 \quad [\hat{P}_i, \hat{M}] = 0 \quad [\hat{H}, \hat{M}] = 0$$

where $\hat{M}$ is an element of the Lie algebra of some one-parameter subgroup (which is used to extend $\overline{G}$).

$\mathbf{A_{28}}$ $\hat{M}$ has a discrete spectrum of real and positive eigenvalues.

GROUP V: GAUGE TRANSFORMATIONS AND ELECTRIC CHARGE

$\mathbf{A_{29}}$ $(\forall <\sigma, \overline{\sigma}>)_{\Sigma \times \overline{\Sigma}}\ (\forall\ \hat{A})_A (\exists\ \hat{Q})_A\ (\hat{Q} \neq \hat{I} \wedge\ ([\hat{Q}, \hat{A}] = 0)$.

$\mathbf{A_{30}}$ $\hat{Q}$ has a discrete spectrum of real eigenvalues.

A$_{31}$ $\hat{Q}$ is the generator of gauge transformations of the first kind.

A$_{32}$ There exists a sole normalizable state with *eiv* $\hat{Q} = 0$, called neutral state.

A$_{33}$ There exists a sole normalizable state, called vacuum, that is invariant under $\hat{D}(\tilde{G})$ and under gauge transformations of the first kind.

GROUP VI: COMPOSITION AXIOMS

A$_{34}$ (Product Hilbert space)

$(\forall <\sigma,\overline{\sigma}>)_{\Sigma\times\overline{\Sigma}}\,(\mathcal{C}(\sigma) = \{\sigma_1,\ldots,\sigma_n\} \Rightarrow \mathcal{H}_E = \bigotimes_{i=1}^{n} \mathcal{H}_{Ei})$.

A$_{35}$ $(\forall\ <\sigma,\overline{\sigma}>)_{\Sigma\times\overline{\Sigma}}\,(\forall\,|\Psi>)_{\mathcal{H}_E}(\exists U_\Pi)(U_\Pi$ is a representation of a symmetric group Π by unitary operators $\hat{U}_\Pi \wedge$

$$\begin{aligned}\hat{U}_\Pi|\Psi> &= \hat{U}_\Pi\{|\psi_1^a> \otimes|\psi_2^b> \otimes \ldots \otimes |\psi_n^l>\} \\ &= |\psi_{\alpha_1}^a> \otimes \ldots \otimes |\psi_{\alpha_n}^l>\end{aligned}$$

where $\{\alpha_1,\ldots \alpha_n\}$ is a permutation P of $\{1,\ldots n\}$).

A$_{36}$ $(\forall\ \ <\sigma,\overline{\sigma}\ >)_{\Sigma\times\overline{\Sigma}}\ (\forall\ \ \hat{A})_A\ (\forall\ \ |\Psi\ >)_{\mathcal{H}_E}\ (\mathcal{C}(\sigma)\ =\ \{\sigma_1,\ldots\sigma_n\} \wedge \sigma_i \overset{id}{\leftrightarrow} \sigma_j \wedge\ \ |\Psi\ >=\ \hat{U}_\Pi|\Psi'\ > \Rightarrow$ $<\Psi|\hat{A}|\Psi>=<\Psi'|\hat{A}|\Psi'>)$.

Remark: note that $\tilde{\mathcal{G}}$ is a representation by operators of the extended Lie algebra of the Galilei group that acts on a Hilbert space. For other representations, see Levy-Leblond (1963).

3.7 DEFINITIONS

D$_{19}$ $(\forall <\sigma,\overline{\sigma}>)_{\Sigma\times\overline{\Sigma}}\,(\forall\,\epsilon)_{\mathcal{R}}\,(\hat{H}|\Psi(\sigma,k)>= \epsilon\,|\Psi(\sigma,k)>)\,,\epsilon \overset{Df}{=}$ energy of σ in the state $|\Psi(\sigma,k)>$ with respect to $k \in K$ when it is influenced by $\bar{\sigma}$.

D$_{20}$ $(\forall <\sigma,\overline{\sigma}>)_{\Sigma\times\overline{\Sigma}}\,(\forall\,p_i)_{\mathcal{R}}\,(\hat{P}_i|\Psi(\sigma,k)>= p_i|\Psi(\sigma,k>)\,,p_i \overset{Df}{=}$ i-th component of the lineal momentum of σ in the state $|\Psi(\sigma,k)>$ with respect to $k \in K$ when it is influenced by $\bar{\sigma}$.

D$_{21}$ $(\forall <\sigma,\overline{\sigma}>)_{\Sigma\times\overline{\Sigma}}\,(\forall\,j_i)_{\mathcal{R}}\,(\hat{J}_i|\Psi(\sigma,k)>= j_i|\Psi(\sigma,k)>),\ j_i \overset{Df}{=}$ i-th component of the angular momentum of σ in the state $|\Psi(\sigma,k)>$ with respect to $k \in K$ when it is influenced by $\bar{\sigma}$.

D$_{22}$ $(\forall <\sigma,\overline{\sigma}>)_{\Sigma\times\overline{\Sigma}}\,(\forall\,m)_{\mathcal{R}}\,(\hat{M}|\Psi(\sigma,k)>= m|\Psi(\sigma,k)>)\,,m \overset{Df}{=}$ mass of σ.

D$_{23}$ $\hat{X}_i \overset{Df}{=} \frac{1}{m}\hat{K}_i$.

$\mathbf{D_{24}}$ $(\forall <\sigma,\overline{\sigma}>)_{\Sigma\times\overline{\Sigma}}(\forall\, x_i)_{\mathcal{R}}\,(\hat{X}_i|\Psi(\sigma,k)>= x_i|\Psi(\sigma,k)>)\,, x_i \stackrel{Df}{=}$ i-th component of the position of the *center of mass* of σ in the state $|\Psi(\sigma,k)>$ with respect to $k\in K$ when it is influenced by $\bar{\sigma}$.

$\mathbf{D_{25}}$ $(\forall\, q)_{\mathcal{R}}\,(\hat{Q}|\Psi(\sigma,k)>= q|\Psi(\sigma,k)>)\;, q \stackrel{Df}{=}$ electric charge of σ when it is influenced by $\bar{\sigma}$.

$\mathbf{D_{26}}$ $\mathcal{H}_S \equiv \{|\Psi>\ni |\Psi>\in \mathcal{H}_E \wedge \hat{U}_T|\Psi>=|\Psi>,\, T$ transposition$\}$.

$\mathbf{D_{27}}$ $\mathcal{H}_A \equiv \{|\Psi>\ni |\Psi>\in \mathcal{H}_E \wedge \hat{U}_T|\Psi>=-|\Psi>,\, T$ transposition$\}$.

$\mathbf{D_{28}}$ $\mathcal{H}_{\mathcal{PS}} \stackrel{Df}{=}$ space of accesible states to a given physical system $\sigma\in\Sigma$.

Remark 1 The names given to the eigenvalues in the above definitions are merely conventional and they do not imply that our axiomatics presupposes any concept of classical physics. Any identification between a property of the q-systems and a macroscopical property of classical physics must be justified *a posteriori*.
Remark 2 The meaning of the expression *center of mass* can be established by means of $\mathbf{T_1}$.

3.8 THEOREMS

In this section we give some illustrative theorems that can be deduced from the axiomatic basis. We are not going to repeat here the theorems valid in the case of simple microsystems (for instance, probability amplitudes, Schrödinger equation, Heisenberg inequalities, Heisenberg equation, superselection rules, spin). Such theorems can be found in our previous paper.

$\mathbf{T_1}$ (Additivity theorem)

$(\forall <\sigma,\overline{\sigma}>)_{\Sigma\times\overline{\Sigma}}\,(\forall\, k)_K\,(\mathcal{C}(\sigma)=\{\sigma_1,\ldots\sigma_n\}\wedge$

$[\hat{P}_i,\hat{X}_{jr}]=i\hbar\delta_{ij} \qquad [\hat{J}_i,\hat{X}_{jr}]=i\hbar\epsilon_{ijk}\hat{X}_{kr} \qquad [\hat{P}_i,\hat{P}_{jr}]=0 \qquad [\hat{J}_i,\hat{P}_{jr}]=i\hbar\epsilon_{ijk}\hat{P}_{kr}$

$[\hat{K}_i,\hat{X}_{jr}]=0 \qquad [\hat{K}_i,\hat{P}_{jr}]=i\hbar\delta_{ij}m_r \qquad (i,j=1,2,3; r=1,2\ldots n)\Rightarrow$

$\hat{P}_i=\sum_{s=1}^n \hat{P}_{is}\wedge \hat{J}_i=\sum_{s=1}^n \hat{J}_{is}\wedge \hat{K}_i=\sum_{s=1}^n \hat{K}_{is}\wedge \hat{M}=\sum_{s=1}^n \hat{M}_s)$

Proof: from $\mathbf{P_5}$ and $\mathbf{A_{35}}$.

$\mathbf{T_2}$ $\mathcal{H}_S\oplus\mathcal{H}_A\subset\mathcal{H}_E$ is a vector subspace of $\mathcal{H}_E$.

Proof: from the definitions given above.

$\mathbf{T_3}$ (Symmetrization theorem)

$(\forall <\sigma,\overline{\sigma}>)_{\Sigma\times\overline{\Sigma}}\;(\mathcal{H}_{PS}=\mathcal{H}_S\oplus\mathcal{H}_A)\;.$

Proof: Let $|\Psi(\sigma)>$ be such that

$$\hat{U}_T|\Psi(\sigma)>=\lambda_T|\Psi(\sigma)>\;. \tag{1}$$

Applying two transpositions T_1 and T_2

$$\hat{U}_{T_1}\hat{U}_{T_2}|\Psi(\sigma)\rangle = \lambda_{T_1}\lambda_{T_2}|\Psi(\sigma)\rangle\ .$$

Besides, applying the transposition T_1T_2,

$$\hat{U}_{T_1T_2}|\Psi(\sigma)\rangle = \lambda_{T_1T_2}|\Psi(\sigma)\rangle\ .$$

From $\mathbf{A_{36}}$ and $\mathbf{P_5}$, $\lambda_{T_1T_2} = \lambda_{T_1}\lambda_{T_2}$, and then λ_T is a scalar representation of the group Π.

There exist only two scalar representations of Π (Cornwell 1984):

$$(\forall\ T)(\lambda_T = 1) \vee (\lambda_T = +1,\ T \text{ even} \wedge \lambda_T = -1,\ T \text{ odd}).$$

Then, from (1),

$$(|\Psi(\sigma)\rangle \in \mathcal{H}_{\mathcal{S}}) \vee (|\Psi(\sigma)\rangle \in \mathcal{H}_{\mathcal{A}})\,. \tag{2}$$

Let now be $|\Psi(\sigma)\rangle_S \in \mathcal{H}_{\mathcal{S}}$ and $|\Psi(\sigma)\rangle_A \in \mathcal{H}_{\mathcal{A}}$, then,

$$_S\langle\Psi(\sigma)|\Psi(\sigma)\rangle_A =\ _S\langle\Psi(\sigma)|U_T^{-1}U_T|\Psi(\sigma)\rangle_A =\ _S\langle\Psi(\sigma)U_T|U_T\Psi(\sigma)\rangle_A = -\ _S\langle\Psi(\sigma)|\Psi(\sigma)\rangle_A\ .$$

That is to say,

$$\mathcal{H}_{\mathcal{A}} \perp \mathcal{H}_{\mathcal{S}} \tag{3}$$

Finally, from (2) and (3), $\mathcal{H}_{\mathcal{PS}} = \mathcal{H}_{\mathcal{A}} \oplus \mathcal{H}_{\mathcal{S}}$

COROLLARY: (Pauli's Exclusion Theorem)

$(\forall\ \langle\sigma,\overline{\sigma}\rangle)_{\Sigma\times\overline{\Sigma}}\ (\mathcal{C}(\sigma) = \{\sigma_1,\dots\sigma_n\} \wedge \sigma_i \overset{id}{\leftrightarrow} \sigma_j \Rightarrow |\Psi(\sigma)\rangle \in \mathcal{H}_{\mathcal{PS}})$

$\mathbf{T_4}$ $(\forall\ \langle\sigma,\overline{\sigma}_0\rangle)_{\Sigma\times\overline{\Sigma}}\ (\mathcal{C}(\sigma) = \{\sigma_1\dots\sigma_n\} \Rightarrow$

$$\hat{H} = \frac{1}{2}\sum_{i=1}^{n}\frac{\hat{p}_i^2}{m_i} + \sum_{i<j}[V(r_{ij}) + V(\hat{\vec{s}}_i,\hat{\vec{s}}_j)]$$

with

$$V(\hat{\vec{s}}_i,\hat{\vec{s}}_j) = V_1(r_{ij}) + V_2(r_{ij})(\hat{\vec{s}}_i.\hat{\vec{s}}_j) + V_3(r_{ij})[3(\hat{\vec{s}}_i.\vec{n}_{ij})(\hat{\vec{s}}_j.\vec{n}_{ij}) - \hat{\vec{s}}_i.\hat{\vec{s}}_j]$$

where

$r_{ij} \overset{Df}{=} |\vec{x}_i - \vec{x}_j|$ $\qquad$ $\vec{s}_i = \frac{\hbar}{2}\vec{\tau}_i$ $\qquad$ $\vec{n}_{ij} \overset{Df}{=} \frac{\vec{r}_{ij}}{r_{ij}}$

and $\vec{\tau}_i$ are the Pauli matrices)

Proof: from $\mathbf{A_{28}}$, $\mathbf{P_5}$, and $\mathbf{T_1}$.

Remark 1: The first (second) group of commutation relations in $\mathbf{T_1}$ means that the behaviour of each simple mycrosystem under a Euclidean motion (instantaneous Galilean transformations) is unaffected by the presence of interactions. **Remark 2:** If $\sigma \in \Sigma$ such that $C(\sigma) = \{\sigma_1, ...\sigma_n\}$, and σ_i interacts weakly with σ_j $\Rightarrow \hat{H}_\sigma = \sum_i \hat{H}_{\sigma_i} + O(\lambda)$, where λ is some coupling constant. **Remark 3:** $\mathbf{T_3}$ is the so-called symmetrization postulate. Here it is a theorem implied by the axiomatic core. **Remark 4:** There exist some systems whose representative kets have no definite symmetry when a physical space with non-trivial topology is considered (Girardeau 1965). Such systems are excluded in the present work because of $\mathbf{A_1}$ and $\mathbf{A_2}$: it is possible to build a coordinate representation of the operator U_T in E_3 without any additional restriction.

4 DISCUSSION

The axiomatization of QM in the case of a q-system with an arbitrary number of components developed in this work is realistic and objective. It is realistic because it assumes that the objects contained in the ontology (that is, the set $\Sigma \cup \bar{\Sigma}$) exist independently of sensorial experience (contrary to the fundamental thesis of idealism). It is objective because knowing subjects or observers do not belong to the domain of quantification of the bound variables of the theory .

It is worth noticing that the realistic thesis does not imply that all the functions that represent properties of real objects must have definite values simultaneously, as classicism requires (Bunge 1989). This is clearly seen in Heisenberg's inequalities (they follow from $\mathbf{A_{28}}$, see Perez-Bergliaffa *et al.* 1993): they have nothing to do with measuring devices. They reflect an inherent property of every microsystem.

At this point, an important difference should be remarked between realism and classicism. The former is a philosophical conception regarding the nature of the objects studied by the theory, while the latter is only a specific feature of certain theories (see Bunge 1989).

In recent years, it has been argued that the fall of Bell's inequalities leads to the conclusion that realism is inconsistent with experiment. However, as we show in the next section, such a refutation does not threaten in any way the realistic thesis adopted here.

4.1 EPR AND REALISM

Let $\sigma \in \Sigma \ni \mathcal{C}(\sigma) = \{\sigma_1, \sigma_2\} \Rightarrow \hat{P} = \hat{P}_1 + \hat{P}_2$ by $\mathbf{T_1}$. It follows from $\mathbf{A_{28}}$ that $[\hat{X}_1 - \hat{X}_2, \hat{P}] = 0$, and then, from $\mathbf{T_8}$ of Perez-Bergliaffa *et al.* (1993), the quantities associated to the operators $\hat{X}_1 - \hat{X}_2$ and $\hat{P}$ are simultaneously well-defined and can be measured with as much precision as the state-of-the-art allows. Let's suppose now that the components σ_1 and σ_2 are far away from each other in such a way that, for the purpose of experiment, they can be considered as isolated. Solving Schrödinger's equation ($\mathbf{T_4}$ of Perez-Bergliaffa *et al.* 1993) in the center of mass system of σ for a null potential (see for instance De la Peña

1979), we find (in the coordinate representation)

$$\Psi(x_1, x_2) = \delta(x - a)e^{ip(x_1+x_2)/2\hbar} \tag{4}$$

where a is the relative separation between σ_1 and σ_2. If we now measure the position of σ_1 we can infer (from the relation $x_1 - x_2 = a$) which value would be found if we measure the position of σ_2 immediately after the first measure has been carried out. Assuming that there is no action-at-distance in a quantum sense (i.e. that two subsystems apart enough from each other can be considered as isolated, an assumption known as locality or separability), the inference of x_2 is made without perturbing σ_2 in any way. It follows then that the position of σ_2 has a definite predetermined value not included in (4). This implies that the description given by QM is incomplete. By the same reasoning, it can be inferred that the lineal momentum of σ_2 has also a definite value, at variance with Heisenberg's inequalities. Then both the position and the lineal momentum of σ_2 have a definite predetermined value: we do not have to work out any additional measure to know them. This clearly contradicts the subjetivistic interpretation of Copenhagen.

The argument given above is a brief account of the so-called "EPR paradox". In short, it states that if locality is accepted in QM then the theory must be incomplete. In other words, the theory must have hidden variables (Bohm 1953). Besides, a theorem due to Bell (1966) shows that the predictions of deterministic, local theories that have hidden variables can be compared, by means of a given class of experiments, with the predictions of QM. Experiments of such a class have been carried out by Aspect *et al.* (1991, 1992), and their results are in complete agreement with QM.

The reader should note that these results do not affect the realistic philosophy that underlies our axiomatization. In fact, as it was shown by Clauser and Shimony (1978),

$$\text{(Hidden Variables} \wedge \text{Separability)} \Rightarrow \text{(Bell's inequalities)}$$

It follows that if Bell's inequalities are refuted by recourse to the experiment, then (1) theories with hidden variables are false (i.e. QM is complete) or (2) the theory is non-local or (3) both (1) and (2) are true. The axiomatization we present here assumes non-locality and completeness, so it *predicts* that Bell's inequalities are false. The non-locality originates in the systemic point of view adopted in the background material (more precisely, in $\mathbf{P_{10}}$; see Section 2 for details), while completeness is introduced through $\mathbf{A_{19}}$, according which every property of the physical system under study has its mathematical counterpart uniquely defined in the theory.

In brief, the axiomatization we present here is realistic, objective, non-local, and complete. These features are essential for the study of quantum cosmology, a subject in which the orthodox (subjetivistic) interpretation cannot be applied succesfully.

The system formed by the association of all the things is the Universe (σ_U, see Section 2). By definition, the environment of σ_U is the empty environment: $\bar{\sigma}_U = \overline{\sigma}_0$. It follows that any interpretation of QM that requires external observers to produce the collapse of the wave function cannot be applied to the study of σ_U. In this case it is mandatory to have at our disposal an objective interpretation. The usual approach (based on the wave function) presuposses the interpretation of Everett (1957) or variations of it (see for instance Halliwell 1992). Our axiomatization shares with Everett's interpretation the realism and the needless of Von Neumann's projection postulate. However, the theory of measurement that follows from our axiomatization does not entail the introduction of the "Many Worlds", as will be discussed elsewhere.

4.2 SOME REMARKS ON THE "CONSISTENT INTERPRETATION"

Recently, Griffiths (1984), Omnès (1992), and Gell-Mann and Hartle (1990) have developed a new formulation of QM: the so-called "consistent interpretation". They claim it is both realistic and objective. In the following, we shall argue that their main physical results can be obtained as theorems in our formalism, although detailed proofs, which are lengthy, will be presented elsewhere.

In the consistent interpretation, the density matrix plays a central role. This concept is secondary in our axiomatization because the notion of partition of a system σ in two subsystems (i.e. $\sigma = \sigma_1 \dot{+} \sigma_2$, where the symbol $\dot{+}$ means physical sum, see Bunge 1967b) has been incorporated to the ontological background. Starting from this partition, it is possible to show that the state of each subsystem is represented by a density operator ρ_i (see Balian 1982 for a nonrigorous proof).

The existence of the classical limit can be proved in our formulation essentially in the same way as in Omnés (1992). Specifically, there exists a many-to-one partial function $\mathcal{C}$ that associates a function $\mathcal{A}(p, q)$ (that depends on classical phase space variables) to operators $\hat{A}(\hat{p}, \hat{q})$. The function $\mathcal{A}(p, q)$ is the classical counterpart of $\hat{A}(\hat{p}, \hat{q})$. The function $\mathcal{C}$ is many to one because, due to the lack of commutativity of the operator ring, several operators have the same classical counterpart, and it is partial because dynamical variables such as spin have no classical counterpart.

With these elements and the aid of our axiomatics, we could construct a "theory of measurement". If the system σ is decomposed as follows:

$$\sigma = \sigma_S \dot{+} \sigma_A \dot{+} \bar{\sigma}_S \tag{5}$$

where σ_S is the subsytem on which the measure is performed, σ_A is the "apparatus" and $\bar{\sigma}_S$ is the "environment", then, with suitable restrictions on the three subsystems, the main results of measurement theory could be deduced as in Omnès (1992).[6]

[6]We should remark that the resulting measurement theory does not apply to real situations but to the analysis of highly idealized typical experiments: it can predict accurately no outcome of a single real experiment (Bunge 1967b)

"Wave packet reduction" can be expressed as a trace on the density matrix of the "apparatus" subsystem (Lüders 1951, Omnès 1992). This is probably the closest one can get to a proof of "von Neumann's projection postulate" in our formulation. However, no physical process is involved in the reduction: it is a mathematical device to describe a subset of initial conditions (Omnès 1992). [7]

5 CONCLUDING REMARKS

We finally would like to point out here that certain realistic interpretations of QM cannot face succesfully the refutation of Bell's inequalities. This is true for deterministic interpretations, i.e. interpretations that imply the existence of hidden variables that complete the classical characterization of the state of the particles that compose the statistical *ensembles*. This failure is avoided by a literal (i.e. strictly quantum) interpretation. We have shown here that such an interpretation is possible. Moreover, our axiomatics offers a well-suited frame for the analysis of recent attempts focused on obtaining the classical limit as an emergent property in a macroscopical system from the constituent microsystems, by means of a decoherence process. This line of research will be developed elsewhere.

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[7]The main role of the environment is to produce decoherence on the density matrix of the other two subsystems, forcing them into a diagonal form (Omnés 1992, Paz 1994). There should exist a many-one function mapping (sets of) states of Σ_S into well defined states of Σ_A.

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