On a Theory of Truth and on the Regress Problem

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Abstract

A theory of truth is introduced for the first–order language L of set theory. Fully interpreted metalanguages which contain their truth predicates are constructed for L. The presented theory is free from infinite regress, whence it provides a proper framework to study the regress problem. In proofs only ZF set theory, concepts definable in L and classical two-valued logic are used. Similar truth theory can be presented also for each consistent theory of every countable first–order formal language.

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1 Introduction

In this paper we construct for the first–order formal language $L = \{\in\}$ of set theory logically consistent metalanguages which contain their truth predicates. Valuations of metalanguages conform to that of L determined by the model M for Zermelo Fraenkel (ZF) set theory constructed in [2]. Classical two-valued logic is used in reasoning. The regress problem is studied within the framework of such a metalanguage. Examples of infinite regresses are presented.

The paper is organized as follows. In Section 2 we construct metalanguages for L. They are included in the expansion $\mathcal{L} = \{\in, T\}$ of L, where T is a monadic predicate. Considerations are influenced by the derivation of Kripke's theory of truth for arithmetics presented in [5]. Denote by X the set of numerals of Gödel numbers of sentences of $\mathcal{L}$ in its fixed Gödel numbering. X is a subset of M because M contains finite ordinals. Let $\lceil A \rceil$ denote the numeral of the Gödel number of a sentence A of $\mathcal{L}$. Corresponding to each subset U of X we construct a metalanguage $\mathcal{L}_U$ of L. U is called consistent if for no sentence A of $\mathcal{L}$ both $\lceil A \rceil$ and $\lceil \neg A \rceil$ are in U.

In Section 3 we first valuate the sentences of constructed metalanguages. Standard truth tables of two-valued logic hold for logical connectives of sentences of $\mathcal{L}_U$ if U is consistent. We call T a truth predicate for $\mathcal{L}_U$ if T-biconditionality: $A \leftrightarrow T(\lceil A \rceil)$ is true for all sentences A of $\mathcal{L}_U$. To every formula $\theta(x)$ of L there is such a sentence A in $\mathcal{L}$ that the sentence $A \leftrightarrow \neg\theta(\lceil A \rceil)$ is true in M (cf., e.g., [11, Lemma IV.5.30]). Thus the language L does not contain its own truth predicate with respect to the valuation determined by M.

Denote by $G(U)$ the set of numerals of Gödel numbers of true sentences of $\mathcal{L}_U$. We shall show that if U is consistent, and is a fixed point of G, i.e., $U = G(U)$, then T is a truth predicate for $\mathcal{L}_U$. Moreover, the truth in M, as defined, e.g., in [11] and the truth in the metalanguage $\mathcal{L}_U$ are shown to be connected as follows: For each sentence A of L, either

(a) A is true in M, equivalently, A is true in $\mathcal{L}_U$, equivalently, $T(\lceil A \rceil)$ is true in $\mathcal{L}_U$, or

(b) $\neg A$ is true in M, equivalently, $\neg A$ is true in $\mathcal{L}_U$, equivalently, $\neg T(\lceil A \rceil)$ is true in $\mathcal{L}_U$.

In Section 4 we prove that G has consistent fixed points including the smallest one. Denoting it by U, the sentences of $\mathcal{L}_U$ are called *grounded*. In proofs only ZF set theory, concepts definable in L, sets of M and classical logic are used.

Section 5 is devoted to the study of the regress problem within the language of grounded sentences. In the study we question the following conclusion stated in [16]: "it is logically impossible for there to be an infinite parade of justifications". Notwithstanding this conclusion we present examples of infinite parades (regresses) of justifications that satisfy the conditions imposed on them in [15]. Examples are inconsistent with the above quoted conclusion that is used in [15, 16] as a basic argument to refute Principles of Sufficient Reasons. An example of a transfinite regress is also presented. Tarski's theory of truth is not a suitable framework because of infinite regress of metalanguages. Three-valued inner logic in Kripke's theory of truth makes the use of it problematic.

Relations of the constructed theory to these two truth theories and to the norms presented for such theories in [12] are studied in Section 6.

2 Construction of metalanguages

First we shall construct a family of metalanguages for the language $L = \{\in\}$ of set theory. As for the used terminology, cf. e.g., [6, 11]. Let $\mathrm{Th}(M)$ be comprised by all those sentences of L which are true in the model M of ZF set theory constructed in [2] in the sense that $M \models A$ (cf. [11, Subsection II.2.7 and p. 237]). $\mathrm{Th}(M)$ is by [11, Lemma II.2.8.22] complete with respect to L, i.e., for each sentence A of L either A or its negation $\neg A$ is true in M. Given a subset U of the set X of numerals of Gödel numbers of sentences of $\mathcal{L} = \{\in, T\}$, define subsets $G(U)$ and $F(U)$ of X by following rules (iff abbreviates if and only if):

(r1) If A is a sentence of L, then $\lceil A \rceil$ is in $G(U)$ iff $M \models A$.

(r2) If A is a sentence of $\mathcal{L}$, then $\lceil T(\lceil A \rceil) \rceil$ is in $G(U)$ iff $\lceil A \rceil$ is in U.

(r3) If A is a sentence of $\mathcal{L}$, then $\lceil T(\lceil A \rceil) \rceil$ is in $F(U)$ iff $\lceil \neg A \rceil$ is in U.

Sentences determined by rules (r1)–(r3), i.e., sentences A of L for which $M \models A$ and sentences $T(\lceil A \rceil)$ of $\mathcal{L}$ for which $\lceil A \rceil$ or $\lceil \neg A \rceil$ is in U, are called *atomic sentences*.
Next rules deal with sentences containing logical constants. Let A and B be sentences of $\mathcal{L}$.

(r4) Negation rule: $\lceil \neg A \rceil$ is in $G(U)$ iff $\lceil A \rceil$ is in $F(U)$, and in $F(U)$ iff $\lceil A \rceil$ is in $G(U)$.

(r5) Disjunction rule: $\lceil A \vee B \rceil$ is in $G(U)$ iff $\lceil A \rceil$ or $\lceil B \rceil$ is in $G(U)$, and in $F(U)$ iff $\lceil A \rceil$ and $\lceil B \rceil$ are in $F(U)$.

(r6) Conjunction rule: $\lceil A \wedge B \rceil$ is in $G(U)$ iff $\lceil (\neg A) \vee (\neg B) \rceil$ is in $F(U)$ iff (by (r4) and (r5)) both $\lceil A \rceil$ and $\lceil B \rceil$ are in $G(U)$. Similarly, $\lceil A \wedge B \rceil$ is in $F(U)$ iff $\lceil (\neg A) \vee (\neg B) \rceil$ is in $G(U)$ iff $\lceil A \rceil$ or $\lceil B \rceil$ is in $F(U)$.

(r7) Implication rule: $\lceil A \rightarrow B \rceil$ is in $G(U)$ iff $\lceil \neg A \vee B \rceil$ is in $G(U)$ iff (by (r4) and (r5)) $\lceil A \rceil$ is in $F(U)$ or $\lceil B \rceil$ is in $G(U)$. $\lceil A \rightarrow B \rceil$ is in $F(U)$ iff $\lceil \neg A \vee B \rceil$ is in $F(U)$ iff $\lceil A \rceil$ is in $G(U)$ and $\lceil B \rceil$ is in $F(U)$.

(r8) Biconditionality rule: $\lceil A \leftrightarrow B \rceil$ is in $G(U)$ iff $\lceil A \rceil$ and $\lceil B \rceil$ are both in $G(U)$ or both in $F(U)$, and in $F(U)$ iff $\lceil A \rceil$ is in $G(U)$ and $\lceil B \rceil$ is in $F(U)$ or $\lceil A \rceil$ is in $F(U)$ and $\lceil B \rceil$ is in $G(U)$.

When $A(x)$ is a formula in L, then quantifications $\exists x A(x)$ and $\forall x A(x)$ are sentences of L. Rules (r1)–(r8) are applicable to them and their logical connectives with sentences of $\mathcal{L}$. Remaining formulas $A(x)$ in $\mathcal{L}$ form its subset, denoted by $\mathcal{F}$. Assume that X is their domain of discursion, and that

(r9) $\lceil \exists x A(x) \rceil$ is in $G(U)$ iff $\lceil A(\mathbf{n}) \rceil$ is in $G(U)$ for some $\mathbf{n} \in X$, and in $F(U)$ iff $\lceil A(\mathbf{n}) \rceil$ is in $F(U)$ for every $\mathbf{n} \in X$.

(r10) $\lceil \forall x A(x) \rceil$) is in $G(U)$ iff $\lceil A(\mathbf{n}) \rceil$ is in $G(U)$ for every $\mathbf{n} \in X$, and in $F(U)$ iff $\lceil A(\mathbf{n}) \rceil$ is in $F(U)$ at least for one $\mathbf{n} \in X$.

Rules (r1)–(r10) and induction on the complexity of formulas determine uniquely subsets $G(U)$ and $F(U)$ of X whenever U is a subset of X. Languages $\mathcal{L}_U$ formed by those sentences A of $\mathcal{L}$ for which $\lceil A \rceil$ is in $G(U)$ or in $F(U)$ are metalanguages for L.
We say that a subset U of X is *consistent* iff both $\lceil A \rceil$ and $\lceil \neg A \rceil$ are not in U for any sentence A of $\mathcal{L}$. For instance, the empty set $\emptyset$ is consistent.
The following two lemmas have counterparts in [5], and proofs are similar.

Lemma 2.1. *Let U be a consistent subset of X. Then $G(U) \cap F(U) = \emptyset$.*

Proof. Consider first atomic sentences. Rule (r1) puts no member to $F(U)$.
By rules (r2) and (r3) $\lceil T(\lceil A \rceil) \rceil$ is in $G(U)$ iff $\lceil A \rceil$ is in U, and in $F(U)$ iff $\lceil \neg A \rceil$ is in U. Thus $\lceil T(\lceil A \rceil) \rceil$ cannot be both in $G(U)$ and in $F(U)$ because U is consistent.
Make an induction hypothesis:

(h0) A and B are such sentences of $\mathcal{L}$ that neither $\lceil A \rceil$ nor $\lceil B \rceil$ is in $G(U) \cap F(U)$.

As shown above, (h0) holds if A and B are atomic sentences.
If $\lceil \neg A \rceil$ is in $G(U) \cap F(U)$, then $\lceil A \rceil$ is in $F(U) \cap G(U)$. Hence, if (h0) holds, then $\lceil \neg A \rceil$ is not in $G(U) \cap F(U)$.
If $\lceil A \vee B \rceil$ is in $G(U) \cap F(U)$, then $\lceil A \rceil$ or $\lceil B \rceil$ is in $G(U)$, and both $\lceil A \rceil$ and $\lceil B \rceil$ are in $F(U)$ by (r5), so that $\lceil A \rceil$ or $\lceil B \rceil$ is in $G(U) \cap F(U)$. Hence, if (h0) holds, then $\lceil A \vee B \rceil$ is not in $G(U) \cap F(U)$.
$\lceil A \wedge B \rceil$ cannot be in $G(U) \cap F(U)$, for otherwise both $\lceil A \rceil$ and $\lceil B \rceil$ are in $G(U)$, and at least one of $\lceil A \rceil$ and $\lceil B \rceil$ is in $F(U)$, contradicting with (h0).
If $\lceil \neg A \rceil$ is in $G(U) \cap F(U)$, then $\lceil A \rceil$ is in $F(U) \cap G(U)$, and (h0) is not valid. Thus, under the hypothesis (h0) neither $\lceil \neg A \rceil$ nor $\lceil B \rceil$ is in $G(U) \cap F(U)$. This result and the above result for disjunction imply that $\lceil \neg A \vee B \rceil$, or equivalently, $\lceil A \rightarrow B \rceil$, is not in $G(U) \cap F(U)$.
Similarly, $\lceil A \leftrightarrow B \rceil$ is not in $G(U) \cap F(U)$, for otherwise, both $\lceil A \rceil$ and $\lceil B \rceil$ would be in $G(U) \cap F(U)$ by rule (r8), contradicting with (h0).
It remains to show that if $A(x)$ is a formula in $\mathcal{F}$, then $\lceil \exists x A(x) \rceil$ and $\lceil \forall x A(x) \rceil$ are not in $G(U) \cap F(U)$. If $A(\mathbf{n})$ is an atomic sentence for every $\mathbf{n} \in X$, the above proof implies that the following induction hypothesis holds:

(h1) $\lceil A(\mathbf{n}) \rceil$ is not in $G(U) \cap F(U)$ for any $\mathbf{n} \in X$.

Then neither $\lceil \exists x A(x) \rceil$ nor $\lceil \forall x A(x) \rceil$ is in $G(U) \cap F(U)$, for otherwise, it would follow from (r9) and (r10) that $\lceil A(\mathbf{n}) \rceil$ is in $G(U) \cap F(U)$ for some $\mathbf{n} \in X$, contradicting with (h1).
The above results and induction on the complexity of formulas imply that $\lceil A \rceil$ is not in $G(U) \cap F(U)$ for any sentence A of $\mathcal{L}$. □

Lemma 2.2. *If a subset U of X is consistent, then $G(U)$ is consistent.*

Proof. If $G(U)$ is not consistent, then there is such a sentence A of $\mathcal{L}$, that $\lceil A \rceil$ and $\lceil \neg A \rceil$ are in $G(U)$. Because $\lceil \neg A \rceil$ is in $G(U)$, then $\lceil A \rceil$ is also in $F(U)$ by rule (r4), and hence in $G(U) \cap F(U)$. But then, by Lemma 2.1, U is not consistent. Consequently, if U is consistent, then $G(U)$ is consistent. □

3 A theory of truth

We shall first define valuations and a concept of truth predicate for metalanguages $\mathcal{L}_U$ formed by those sentences A of $\mathcal{L}$ for which $\lceil A \rceil$ is in one of the subsets $G(U)$ and $F(U)$ of X constructed in Section 2.

Definition 3.1. *Let U be a consistent subset of X. We say that a sentence A of $\mathcal{L}_U$ is true iff $\lceil A \rceil$ is in $G(U)$, and false iff $\lceil A \rceil$ is in $F(U)$. T is called a truth predicate for the language $\mathcal{L}_U$ iff T-biconditionality: $A \leftrightarrow T(\lceil A \rceil)$ is true for every sentence A of $\mathcal{L}_U$.*

In view of the given valuation '$\lceil A \rceil$ is in $G(U)$' can be replaced by 'A is true' and '$\lceil A \rceil$ is in $F(U)$' by 'A is false' in (r4)–(r10). Standard truth tables of classical two-valued logic hold for logical connectives of sentences of $\mathcal{L}_U$ when 'true' is replaced by '1' and 'false' by '0'.
In the study whether T is a truth predicate for $\mathcal{L}_U$, the consistent subsets U of X which are fixed points of the set mapping $G := U \mapsto G(U)$, i.e., $U = G(U)$, play the crucial role.

Lemma 3.1. *If $U \subset X$ is a consistent fixed point of G, then T is a truth predicate for $\mathcal{L}_U$.*

Proof. Assume that $U \subset X$ is consistent, and let A be a sentence of $\mathcal{L}_U$. Applying rules (r2), (r3) and (r4), it follows that if $U = G(U)$, then
(i) $\lceil A \rceil$ is in $G(U)$ iff $\lceil A \rceil$ is in U iff $\lceil T(\lceil A \rceil) \rceil$ is in $G(U)$;
(ii) $\lceil A \rceil$ is in $F(U)$ iff $\lceil \neg A \rceil$ is in $G(U)$ iff $\lceil \neg A \rceil$ is in U iff $\lceil T(\lceil A \rceil) \rceil$ is in $F(U)$.
(i) and (ii) imply that $\lceil A \rceil$ and $\lceil T(\lceil A \rceil) \rceil$ are either both in $G(U)$ or both in $F(U)$. Thus $\lceil A \leftrightarrow T(\lceil A \rceil) \rceil$ is by rule (r8) in $G(U)$, so that $A \leftrightarrow T(\lceil A \rceil)$ is true by Definition 3.1. This holds for every sentence A of $\mathcal{L}_U$, whence T is a truth predicate for $\mathcal{L}_U$ by Definition 3.1. □

Our main result on the connection between valuations in L determined by M and those determined by Definition 3.1 reads as follows:

Theorem 3.1. *Let U be a consistent fixed point of G. If A is a sentence of L, then either*
(a) $M \models A$ (A is true in M), equivalently, A is true, equivalently, $T(\lceil A \rceil)$ is true, or
(b) $M \models \neg A$ (A is false in M), equivalently, A is false, equivalently, $T(\lceil A \rceil)$ is false.

Proof. Assume that A is a sentence of L. Because theory $\mathrm{Th}(M)$ is complete with respect to L by [11, Lemma II.8.22], then either $M \models A$ or $M \models \neg A$.
(a) $M \models A$ iff $\lceil A \rceil$ is in $G(U)$, by rule (r1), iff A is true, by Definition 3.1, iff $T(\lceil A \rceil)$ is true, by Lemma 3.1.
(b) $M \models \neg A$ iff $\lceil \neg A \rceil$ is in $G(U)$, by rule (r1), iff $\lceil A \rceil$ is in $F(U)$, by rule (r4), iff A is false, by Definition 3.1, iff $T(\lceil A \rceil)$ is false, by Lemma 3.1. □

It remains to show that G has consistent fixed points. Denote by $\mathcal{P}$ the family of consistent subsets of X. In the formulation and the proof of our main fixed point theorem we use transfinite sequences of $\mathcal{P}$ indexed by von Neumann ordinals. Such a sequence $(U_\lambda)_{\lambda \in \alpha}$ of $\mathcal{P}$ is said to be strictly increasing if $U_\mu \subset U_\nu$ whenever $\mu \in \nu \in \alpha$, and strictly decreasing if $U_\nu \subset U_\mu$ whenever $\mu \in \nu \in \alpha$. A set V of $\mathcal{P}$ is called *sound* iff $V \subseteq G(V)$. The following fixed point theorem is proved in Section 4.

Theorem 3.2. *If V is a sound subset of $\mathcal{P}$, then there exists the smallest of those consistent fixed points of G which contain V. This fixed point is obtained as the last member of the union of those transfinite sequences $(U_\lambda)_{\lambda\in\alpha}$ of $\mathcal{P}$ which satisfy*

(C) $(U_\lambda)_{\lambda\in\alpha}$ is strictly increasing, $U_0 = V$, and if $0 \in \mu \in \alpha$, then $U_\mu = \bigcup_{\lambda\in\mu} G(U_\lambda)$.

The following Theorem shows that every consistent subset C of X has the greatest sound and consistent subset. The proof is presented in Section 4.

Theorem 3.3. *The equation $V = C \cap G(V)$ has for each consistent subset C of X the greatest solution V in $\mathcal{P}$. It is the greatest sound subset of $\mathcal{P}$ that is contained in C. V is the last member of the union of those transfinite sequences $(V_\lambda)_{\lambda\in\alpha}$ of $\mathcal{P}$ which satisfy*

(D) $(V_\lambda)_{\lambda\in\alpha}$ is strictly decreasing, $V_0 = C$, and if $0 \in \mu \in \alpha$, then $V_\mu = C \cap (\bigcap_{\lambda\in\mu} G(V_\lambda))$.

The next result is a special case of Theorem 3.2.

Theorem 3.4. *G has the smallest consistent fixed point.*

Proof. The empty set $\emptyset$ is both sound and consistent. Thus, by Theorem 3.2, there is the smallest consistent fixed point of G that contains $\emptyset$. It is the smallest consistent fixed point of G, since every fixed point of G contains $\emptyset$. □

Sentences of the language $\mathcal{L}_U$, where U is the smallest consistent fixed point of G, are in every other fixed point language. These sentences are called *grounded*, as in [5, 10].

A result for formalists: Replace $M \models A$ in rule (r1) by $ZF \vdash A$ (A is provable in ZF set theory). *Interpreting in the so obtained language of grounded sentences a sentence A as true if $\lceil A \rceil$ is in $G(U)$, then T recognizes a sentence of L as true if it is provable in ZF.*

Remarks 3.1. The smallest members of $(U_\lambda)_{\lambda\in\alpha}$ satisfying (C) are n-fold iterations $U_n = G^n(V)$, $n \in \mathbb{N} = \{0, 1, \dots\}$. If they form a strictly increasing sequence, the next member U_ω is their union, $U_{\omega+n} = G^n(U_\omega)$, $n \in \mathbb{N}$, and so on.

Let V be a set of numerals of Gödel numbers of those sentences of L which are true in M. Then $V \subset G(\emptyset) \subset G(V)$, whence V is sound. Moreover, V is also consistent. If U is a fixed point of G, then $V \subset G(\emptyset) \subset G(U) = U$. In particular, the smallest consistent fixed point of G contains V.

If the set C is finite, then the longest sequence $(V_\lambda)_{\lambda\in\alpha}$ satisfying (D) is finite. Its members are determined by the finite algorithm:

(A) $V_0 = C$. *For n from 0 while $V_n \neq C \cap G(V_n)$ do: $V_{n+1} = C \cap G(V_n)$.*

The above results hold also when M is replaced by any other countable model of ZF set theory, including "the basic Cohen model" and "the second Cohen model" described, e.g., in [8, p. 10]. Similar truth theory can be presented also for each consistent theory of every countable first–order formal language, M being a countable model of the theory in question. Such a model exist by Downward Löwenheim-Skolem Theorem. For instance, the language of set theory can be replaced by the language of arithmetic, and M by a standard model of arithmetic. Models for other set theories, for arithmetic and for other important theories of contemporary mathematics are presented in [11].

As for axiomatic theories of truth, see, e.g., [4] and the references therein.

4 The proofs of Theorems 3.2 and 3.3

Let $\mathcal{P}$ denote the family of all consistent subsets of the set X. Before the proofs of Theorems 3.2 and 3.3 we prove some auxiliary results.

Lemma 4.1. *Let U and V be sets of $\mathcal{P}$, and assume that $U \subseteq V$. If A is a sentence of $\mathcal{L}$, then $\lceil A \rceil$ is in $G(V)$ whenever it is in $G(U)$, and $\lceil A \rceil$ is in $F(V)$ whenever it is in $F(U)$.*

Proof. Assume that $U \subseteq V$. Consider first atomic sentences. Let A be a sentence of L. By rule (r1) $\lceil A \rceil$ is in $G(U)$ and also in $G(V)$ iff $M \models A$. Rule (r1) leaves both $F(U)$ and $F(V)$ empty.
Let A be a sentence of $\mathcal{L}$. If $\lceil T(\lceil A \rceil) \rceil$ is in $G(U)$, then $\lceil A \rceil$ is in U by rule (r2). Because $U \subseteq V$, then $\lceil A \rceil$ belongs to V, whence $\lceil T(\lceil A \rceil) \rceil$ is in $G(V)$ by rule (r2).
If $\lceil T(\lceil A \rceil) \rceil$ is in $F(U)$, then $\lceil \neg A \rceil$ is in U by rule (r3). Since $U \subseteq V$, then $\lceil \neg A \rceil$ belongs to V, so that $\lceil T(\lceil A \rceil) \rceil$ is in $F(V)$ by rule (r3).
Thus all atomic sentences satisfy the lemma.
Assume that A is a sentence of $\mathcal{L}$. If $\lceil \neg A \rceil$ is in $G(U)$ but not in $G(V)$, then $\lceil A \rceil$ is in $F(U)$ but not in $F(V)$ by rule (r4). If $\lceil \neg A \rceil$ is in $F(U)$ but not in $F(V)$, then $\lceil A \rceil$ is in $G(U)$ but not in $G(V)$ by rule (r4). Hence, if A satisfies the lemma, then also $\neg A$ satisfies it.
Make an induction hypothesis:

(h2) A and B are such sentences of $\mathcal{L}$ that $\lceil A \rceil$ is in $G(V)$ if it is in $G(U)$, $\lceil A \rceil$ is in $F(V)$ if it is in $F(U)$, $\lceil B \rceil$ is in $G(V)$ if it is in $G(U)$, and $\lceil B \rceil$ is in $F(V)$ if it is in $F(U)$.

If $\lceil A \vee B \rceil$ is in $G(U)$, then $\lceil A \rceil$ or $\lceil B \rceil$ is in $G(U)$ by rule (r5). By (h2) $\lceil A \rceil$ or $\lceil B \rceil$ is in $G(V)$, so that $\lceil A \vee B \rceil$ is in $G(V)$. If $\lceil A \vee B \rceil$ is in $F(U)$, then $\lceil A \rceil$ and $\lceil B \rceil$ are in $F(U)$ by rule (r5), and hence also in $F(V)$, by (h2), so that $\lceil A \vee B \rceil$ is in $F(V)$. Thus $A \vee B$ satisfies the lemma if (h2) holds.
If $\lceil A \wedge B \rceil$ is in $G(U)$, then both $\lceil A \rceil$ and $\lceil B \rceil$ are in $G(U)$ by rule (r6), and hence also in $G(V)$, by (h2). Thus $\lceil A \wedge B \rceil$ is in $G(V)$. If $\lceil A \wedge B \rceil$ is in $F(U)$, then $\lceil A \rceil$ or $\lceil B \rceil$ is in $F(U)$ by rule (r6), and hence also in $F(V)$, by (h2), whence $\lceil A \wedge B \rceil$ is in $F(V)$. Thus $A \wedge B$ satisfies the lemma if (h2) holds.
If $\lceil A \rightarrow B \rceil$ is in $G(U)$, then $\lceil \neg A \rceil$ or $\lceil B \rceil$ is in $G(U)$, i.e., $\lceil A \rceil$ is in $F(U)$ or $\lceil B \rceil$ is in $G(U)$. Then, by (h2), $\lceil A \rceil$ is in $F(V)$ or $\lceil B \rceil$ is in $G(V)$, i.e., $\lceil \neg A \rceil$ or $\lceil B \rceil$ is in $G(V)$. Thus $\lceil A \rightarrow B \rceil$ is in $G(V)$. If $\lceil A \rightarrow B \rceil$ is in $F(U)$, then $\lceil A \rceil$ is in $G(U)$ and $\lceil B \rceil$ is in $F(U)$. This implies by (h2) that $\lceil A \rceil$ is in $G(V)$ and $\lceil B \rceil$ is in $F(V)$, so that $\lceil A \rightarrow B \rceil$ is in $F(V)$. Thus $A \rightarrow B$ satisfies the lemma if (h2) holds. Similarly, it can be shown that $B \rightarrow A$ satisfies the lemma, so that also $A \leftrightarrow B$ satisfies the lemma if (h2) holds.
When $A(x)$ is a formula in $\mathcal{F}$, make an induction hypothesis:

(h3) For every $\mathbf{n} \in X$, $\lceil A(\mathbf{n}) \rceil$ is in $G(V)$ whenever it is in $G(U)$, and in $F(V)$ whenever it is in $F(U)$.

If $\lceil \exists x A(x) \rceil$ is in $G(U)$, then (r9) implies that at least one $\lceil A(\mathbf{n}) \rceil$ belongs to $G(U)$. This $\lceil A(\mathbf{n}) \rceil$ is by (h3) also in $G(V)$, whence $\lceil \exists x A(x) \rceil$ is in $G(V)$, by (r9). If $\lceil \exists x A(x) \rceil$ is in $F(U)$, it follows from (r9) that $\lceil A(\mathbf{n}) \rceil$ is in $F(U)$, and hence, by (h3), also in $F(V)$, for every $\mathbf{n} \in X$, whence $\lceil \exists x A(x) \rceil$ is in $F(V)$, by (r9).

If $\lceil \forall x A(x) \rceil$ is in $G(U)$, then every $\lceil A(\mathbf{n}) \rceil$ is in $G(U)$, by (r10). Thus, by (h3), every $\lceil A(\mathbf{n}) \rceil$ is in $G(V)$, so that $\lceil \forall x A(x) \rceil$ is in $G(V)$, by (r10). If $\lceil \forall x A(x) \rceil$ is in $F(U)$, then, by (r10), some $\lceil A(\mathbf{n}) \rceil$ is in $F(U)$, and hence also in $F(V)$, by (h3). This implies by (r10) that $\lceil \forall x A(x) \rceil$ is in $F(V)$.
Consequently, if (h3) holds, then both $\lceil \exists x A(x) \rceil$ and $\lceil \forall x A(x) \rceil$ are in $G(V)$ whenever they are in $G(U)$, and in $F(V)$ whenever they are in $F(U)$.
Because (h2) and (h3) hold for atomic sentences, the above results and induction on the complexity of expressions imply the conclusion of the lemma. □

According to Lemma 2.2 the mapping $G := U \mapsto G(U)$ maps $\mathcal{P}$ into $\mathcal{P}$. Assuming that $\mathcal{P}$ is ordered by inclusion, the above lemma implies the following result.

Lemma 4.2. *G is order preserving in $\mathcal{P}$, i.e., $G(U) \subseteq G(V)$ whenever U and V are sets of $\mathcal{P}$ and $U \subseteq V$.*

Lemma 4.3. *(a) If $\mathcal{W}$ is a chain in $\mathcal{P}$, then the union $\cup\mathcal{W} = \cup\{U \mid U \in \mathcal{W}\}$ is consistent.*
(b) The intersection $\cap\mathcal{W} = \cap\{U \mid U \in \mathcal{W}\}$ of every nonempty subfamily $\mathcal{W}$ of $\mathcal{P}$ is a consistent subset of X.

Proof. (a) Assume on the contrary that $\cup\mathcal{W}$ is not consistent. Then there is a such a sentence A of $\mathcal{L}$ that both $\lceil A \rceil$ and $\lceil \neg A \rceil$ are in $\cup\mathcal{W}$. Thus $\mathcal{W}$ has a member, say U, which contains $\lceil A \rceil$, and a member, say V, which contains $\lceil \neg A \rceil$. If $\mathcal{W}$ is a chain, then $U \subseteq V$ or $V \subseteq U$. In former case V and in latter case U contains both $\lceil A \rceil$ and $\lceil \neg A \rceil$. But this is impossible because $\mathcal{W}$ is a subfamily of $\mathcal{P}$. This proves (a).
(b) $\cap\mathcal{W}$ is a subset of X, and is contained in every member of $\mathcal{W}$. Hence $\cap\mathcal{W}$ is consistent, for otherwise there is such a sentence A in $\mathcal{L}$ that both $\lceil A \rceil$ and $\lceil \neg A \rceil$ are in $\cap\mathcal{W}$. Then every member of $\mathcal{W}$ would also contain both $\lceil A \rceil$ and $\lceil \neg A \rceil$. But this is impossible because every member of $\mathcal{W}$ is consistent. This proves (b). □

As an application of Lemmas 2.2, 4.2 and 4.3 we shall prove our main fixed point theorem.

Proof of Theorem 3.2. Let $V \in \mathcal{P}$ be sound, i.e., $V \subseteq G(V)$. Transfinite sequences of $\mathcal{P}$ having properties (C) of Theorem 3.2 are called G-sequences. We shall first show that G-sequences are nested:

(1) *Assume that $(U_\lambda)_{\lambda\in\alpha}$ and $(V_\lambda)_{\lambda\in\beta}$ are G-sequences, and that $\{U_\lambda\}_{\lambda\in\alpha} \not\subseteq \{V_\lambda\}_{\lambda\in\beta}$. Then $(V_\lambda)_{\lambda\in\beta} = (U_\lambda)_{\lambda\in\beta}$.*

By the assumption of (1) $\mu = \min\{\lambda \in \alpha \mid U_\lambda \notin \{V_\lambda\}_{\lambda\in\beta}\}$ exists, and $\{U_\lambda\}_{\lambda\in\mu} \subseteq \{V_\lambda\}_{\lambda\in\beta}$. Properties (C) imply by transfinite induction that $U_\lambda = V_\lambda$ for each $\lambda \in \mu$. To prove that $\mu = \beta$, make a counter-hypothesis: $\mu \in \beta$. Since $\mu \in \alpha$ and $U_\lambda = V_\lambda$ for each $\lambda \in \mu$, it follows from properties (C) that $U_\mu = \bigcup\limits_{\lambda\in\mu} G(U_\lambda) = \bigcup\limits_{\lambda\in\mu} G(V_\lambda) = V_\mu$, which is impossible, since $V_\mu \in \{V_\lambda\}_{\lambda\in\beta}$, but $U_\mu \notin \{V_\lambda\}_{\lambda\in\beta}$. Consequently, $\mu = \beta$ and $U_\lambda = V_\lambda$ for each $\lambda \in \beta$, whence $(V_\lambda)_{\lambda\in\beta} = (U_\lambda)_{\lambda\in\beta}$.
By definition, every G-sequence $(U_\lambda)_{\lambda\in\alpha}$ is a function $\lambda \mapsto U_\lambda$ from α into $\mathcal{P}$. Property (1) implies that these functions are compatible. Thus their union is by [7, Theorem 2.3.12]

a function with values in $\mathcal{P}$, the domain being the union of all index sets of G-sequences. Because these index sets are ordinals, then their union is also an ordinal. Denote it by γ. The union function can be represented as a sequence $(U_\lambda)_{\lambda\in\gamma}$ of $\mathcal{P}$. It is strictly increasing as an union of strictly increasing nested sequences.

To show that γ is a successor, assume on the contrary that γ is a limit ordinal. Given $\nu \in \gamma$, then $\mu = \nu \cup \{\nu\}$ and $\alpha = \mu \cup \{\mu\}$ are in γ, and $(U_\lambda)_{\lambda\in\alpha}$ is a G-sequence. Denote $U_\gamma = \bigcup\limits_{\lambda\in\gamma} G(U_\lambda)$. G is order preserving by Lemma 4.2, and $(U_\lambda)_{\lambda\in\gamma}$ is a strictly increasing sequence of $\mathcal{P}$. Thus $\{G(U_\lambda)\}_{\lambda\in\gamma}$ is a chain in $\mathcal{P}$, whence U_γ is consistent by Lemma 4.3(a). Moreover, $U_\nu \subset U_\mu = \bigcup\limits_{\lambda\in\mu} G(U_\lambda) \subseteq U_\gamma$. This holds for each $\nu \in \gamma$, whence $(U_\lambda)_{\lambda\in\gamma\cup\{\gamma\}}$ is a G-sequence. This is impossible, since $(U_\lambda)_{\lambda\in\gamma}$ is the union of all G-sequences. Consequently, γ is a successor, say $\gamma = \alpha\cup\{\alpha\}$. Thus U_α is the last member of $(U_\lambda)_{\lambda\in\gamma}$, $U_\alpha = \max\{U_\lambda\}_{\lambda\in\gamma}$, and $G(U_\alpha) = \max\{G(U_\lambda)\}_{\lambda\in\gamma}$. Moreover, $(U_\lambda)_{\lambda\in\gamma}$ is a G-sequence, for otherwise $(U_\lambda)_{\lambda\in\alpha}$ would be the union of all G-sequences. In particular, $U_\alpha = \bigcup\limits_{\lambda\in\alpha} G(U_\lambda) \subseteq \bigcup\limits_{\lambda\in\gamma} G(U_\lambda) = G(U_\alpha)$, so that $U_\alpha \subseteq G(U_\alpha)$. Equality holds, since otherwise the longest G-sequence $(U_\lambda)_{\lambda\in\gamma}$ could be extended by $U_\gamma = \bigcup\limits_{\lambda\in\gamma} G(U_\lambda)$. Thus U_α is a fixed point of G in $\mathcal{P}$.

Assume that $W \in \mathcal{P}$ is a fixed point of G, and that $V \subseteq W$. Then $U_0 = V \subseteq W$. If $0 \in \mu \in \gamma$, and $U_\lambda \subseteq W$ for each $\lambda \in \mu$, then $G(U_\lambda) \subseteq G(W)$ for each $\lambda \in \mu$, whence $U_\mu = \bigcup\limits_{\lambda\in\mu} G(U_\lambda) \subseteq G(W) = W$. Thus, by transfinite induction, $U_\mu \subseteq W$ for each $\mu \in \gamma$. In particular, $U_\alpha = \max\{U_\lambda\}_{\lambda\in\gamma} \subseteq W$. This proves that U_α is the smallest consistent fixed point of G that contains V. $\square$

Proof of Theorem 3.3. Assume that C is a consistent subset of X. Like the proof that the union of all G-sequences is a G-sequence one can prove that the union of all transfinite sequences which have properties (D) given in Theorem 3.3 has property (D). Let $(V_\lambda)_{\lambda\in\gamma}$ be that sequence, i.e.,

(D_γ) $(V_\lambda)_{\lambda\in\gamma}$ is strictly decreasing, $V_0 = C$, and if $0 \in \mu \in \gamma$, then $V_\mu = C \cap (\bigcap\limits_{\lambda\in\mu} G(V_\lambda))$.

Denote $V = C \cap (\bigcap\limits_{\lambda\in\gamma} G(V_\lambda))$. Because C and the sets V_λ, $\lambda \in \gamma$, are consistent, it follows from Lemma 2.2 and Lemma 4.3(b) that V is consistent. Moreover, $V \subseteq V_\lambda$ for each $\lambda \in \gamma$. If $V \subset V_\lambda$ for each $\lambda \in \gamma$, then the choice $V_\lambda = V$ implies that $(V_\lambda)_{\lambda\in\gamma\cup\{\gamma\}}$ satisfies (D) when $\alpha = \gamma\cup\{\gamma\}$. But this is impossible because of the choice of $(V_\lambda)_{\lambda\in\gamma}$. Thus $V = \min\{V_\lambda\}_{\lambda\in\gamma}$, and V is the last member of $(V_\lambda)_{\lambda\in\gamma}$ because this sequence is strictly decreasing. Since G is order preserving, then $G(V) = \min\{G(V_\lambda)\}_{\lambda\in\gamma} = \bigcap\limits_{\lambda\in\gamma} G(V_\lambda)$. Thus $V = C \cap G(V)$, so that $V \subseteq G(V)$, i.e., V is sound and is contained in C.

Assume that W is consistent, that $W \subseteq G(W)$, and that $W \subseteq C$. Since $V_0 = C$ by (D_γ), then $W \subseteq V_0$. If $0 \in \mu \in \gamma$ and $W \subseteq V_\lambda$ for each $\lambda \in \mu$, then $G(W) \subseteq G(V_\lambda)$ for each $\lambda \in \mu$, whence $W \subseteq C \cap G(W) \subseteq C \cap (\bigcap\limits_{\lambda\in\mu} G(V_\lambda)) = V_\mu$. Thus, by transfinite induction, $W \subseteq V_\lambda$ for each $\lambda \in \gamma$, so that $W \subseteq \min\{V_\lambda\}_{\lambda\in\gamma} = V$. Consequently, V is the greatest sound and consistent subset of X that is contained in C. $\square$

5 On the Regress Problem

First of ten theses presented in [1, p. 6] is: "The Regress Problem is a real problem for epistemology." We shall first adjust our terminology to that used in [15] in the study of the Regress Problem. By statements we mean grounded sentences, i.e., the sentences of $\mathcal{L}_U$, where U is the smallest fixed point of G. A statement A is said to entail B, if it is not possible that A is true and B is false simultaneously. For instance, if $A \to B$ is true, then A entails B. We say that a statement A justifies a statement B if A confirms the truth of B. For instance, if $A \leftrightarrow \neg B$ is true, then A justifies B iff A is false. If $A \to B$ is true, then A justifies B iff A is true (Modus Ponens). A statement A is called contingent if the truth value of A is unknown.

Consider an infinite regress

$$\ldots F_i, \ldots, F_1, F_0 \tag{5.1}$$

of statements F_i, $i \geq 0$, where the statement F_0 is contingent. We shall impose the following conditions on statements F_i, $i > 0$ (cf. [15]):

(i) F_i entails F_{i-1};

(ii) $F_0 \vee \cdots \vee F_{i-1}$ does not entail F_i;

(iii) $F_0 \vee \cdots \vee F_{i-1}$ does not justify F_i.

Regress (5.1) is called justification-saturated if the following condition holds:

(iv) $\ldots$ what justifies F_{i-1} is $F_i, \ldots$, what justifies F_1 is F_2, what justifies F_0 is F_1.

Lemma 5.1. *Assume that in regress (5.1) the statement F_0 is contingent, and that the statements F_i, $i > 0$, satisfy conditions (i)–(iii).*
(a) If F_1 is false, then F_i is false for each $i > 0$. F_0 is justified iff $F_0 \leftrightarrow \neg F_1$ is true.
(b) If F_n is true for some $n > 0$, then F_i is true when $0 \leq i \leq n$.
(c) The regress (5.1) is justification saturated iff F_i is true for all $i > 0$, in which case F_0 is justified.

Proof. (a) Assume that F_1 is false. If F_i would be true for some $i > 1$, there would be the smallest such an i. Then F_{i-1} would be true by (i). Replacing i by $i-1$, and so on, this reasoning would imply after $i-1$ steps that F_1 is true; a contradiction. Thus all statements F_i, $i > 0$, are false. Because F_1 is false, it confirms the truth of F_0 iff F_0 and $\neg F_1$ have same truth values iff $F_0 \leftrightarrow \neg F_1$ is true.
(b) Assume that F_n is true for some $n > 0$. Since F_i entails F_{i-1}, $i = n, n-1, \ldots, 1$, then F_i is true for every $i = n-1, \ldots, 0$.
(c) If F_n is false for some $n > 0$, then F_{n+1} is false by property (i), and it does not justify F_n, so that condition (iv) is not valid. On the other hand, condition (i) ensures that condition (iv) is valid if F_i is true for all $i > 0$. In this case F_1 justifies F_0, i.e., F_0 is true. □

The following examples show that there exist infinite regresses of statements F_i, $i = 0, 1, \dots,$ that satisfy assumptions of Lemma 5.1. Exactly one instance in both examples yields an infinite justification-saturated regress.

Example 5.1. Let $\mathbb{Q}_+$ denote the set nonnegative rational numbers, let Y denote a nonempty subset of $\mathbb{Q}_+$, and let F_i, $i = 0, 1, 2, \dots,$ be the following statements:

$$F_i: \text{ Every number of } Y \text{ is less than } \frac{1}{2^i}, \ i = 0, 1, 2, \dots . \tag{5.2}$$

Show that the following conclusions are valid.
(a) F_0 is contingent.
(b) Conditions (i)–(iii) are valid for the infinite regress (5.1).
(c) If $1 \le p \in Y$, then statement F_i is false for every $i > 0$.
(d) If $0 < p < 1$ for some $p \in Y$, there is an $n \ge 0$ such that F_n is true and F_{n+1} is false.
(e) Regress (5.1),(5.2) is justification-saturated if and only if $Y = \{0\}$.

Solution. (a) Statement F_0 is neither logically necessary (Y can contain numbers ≥ 1) nor logically impossible (all numbers of Y can be < 1). Thus F_0 is contingent, i.e., (a) is valid.
(b) If $p < \frac{1}{2^i}$, then $p < \frac{1}{2^j}$ when $i > j \ge 0$, but not conversely. Thus, for every $i > 0$, F_i entails F_{i-1}, i.e., (i) holds, and F_i is not entailed by $F_0 \vee \cdots \vee F_{i-1}$, so that (ii) holds. If $1 \le i$ and $p = \frac{1}{2^i} \in Y$, then the statement F_j is true when $0 \le j < i$, but the statement F_i is false. Thus F_i is not justified by $F_0 \vee \cdots \vee F_{i-1}$, whence (iii) is valid.
The above results imply that conditions (i)–(iii) are valid, which proves (b).
(c) If $1 \le p \in Y$, then $p \not< \frac{1}{2^i}$ for every $i > 0$, which proves (c).
(d) If $0 < p < 1$ for some $p \in Y$, there exists an $n \ge 0$ such that $\frac{1}{2^{n+1}} \le p < \frac{1}{2^n}$. Then F_n is true and F_{n+1} is false. This proves (d).
(e) It follows from (c) and (d) that condition: $Y = \{0\}$ is necessary for F_i to be true for all $i > 0$. If $Y = \{0\}$, then F_i is true for all $i > 0$. Thus (e) holds by Lemma 5.1. □

Example 5.2. Let β be an element of the 'universe' $S(\omega)$ formed by all natural numbers $0, 1, \dots$ (finite ordinals), plus the smallest infinite ordinal ω. Equip $S(\omega)$ with the natural ordering $<$ of natural numbers plus $n < \omega$ for every natural number n. If Z denotes a nonempty subset of $S(\omega)$, it is easy to verify that the infinite regress (5.1) of statements

$$F_i: \quad i < \beta, \text{ for every } \ \beta \in Z, \quad i = 0, 1, \dots, \tag{5.3}$$

satisfy conditions (i)–(iii), and that F_0 is contingent. Moreover, condition (iv) is valid by Lemma 5.1 if and only if F_i is true for all $i > 0$. This holds if and only if $Z = \{\omega\}$.

Comments. Examples 5.1 and 5.2 are inconsistent with the conclusion of [16] cited in the Introduction. In these examples the property that regress (5.1) is justification-saturated both implies and is implied by truth of a 'foundational' statements $F_b : Y = \{0\}$ and $Z = \{\omega\}$, respectively. Thus they don't support the form of infinitism presented in [9]: "infinitism holds that there are no ultimate, foundational reasons". On the other hand, they support "impure" infinitism and the form of foundationalism presented in [1, 18].
Examples 5.1 and 5.2 imply that the proofs in [15, 16] to the assertion that "any version of Principle of Sufficient Reason is false" are based on the questionable premise that infinite

regresses of justifications don't exist. In fact, these examples give some support to Principles of Sufficient Reason, as well as to many other arguments whose validity is questioned in [15, 16]. For instance, in the 'universe' $S(\omega)$ of Example 5.2,

- $\{\omega\}$ provides a *sufficient reason* for F_0;
- $\{\omega\}$ affords an *ultimate and foundational reason* that justifies F_0;
- $\{\omega\}$ is the *final explainer* of F_0;
- $\{\omega\}$ gives the *first cause* that makes regress (5.1),(5.3) justification-saturated;
- ω *explains the existence of the 'universe'* $\mathbb{N}$ of natural numbers ($\mathbb{N} = \omega$ by [7]);
- ω and $\{\omega\}$ *explain the existence of the 'universe'* $S(\omega)$ ($S(\omega) = \omega \cup \{\omega\}$ by [7]);
- ω is *something beyond natural numbers*;
- ω is *infinite and greatest* in the 'universe' $S(\omega)$;
- ω is *'self-justified'* (The Axiom of Infinity).

Belief that ω exists is a matter of faith. The basis in the construction of the model M used here is the set $\omega \cup \{\omega\}$, i.e., the set formed by natural numbers and their set (cf. [2, 8]). Notice that this set does not belong to the standard model of arithmetic. Thus a theory of truth for the language of arithmetic is not sufficient framework to Example 5.2 and the next example.

Example 5.3. Let $\{Y_i\}_{i\in S(\omega)}$ be a family of nonempty subsets of M such that $Y_i \subset Y_{i+1}$ for every $i \in \omega$. Let $\mathcal{Z}$ denote a nonempty subfamily of $\{Y_i\}_{i\in S(\omega)}$. Consider an infinite regress (5.1) of statements F_i, $i \in \omega$, given by

$$F_i : \quad Y_i \subset Y \quad \text{for every} \quad Y \in \mathcal{Z}. \tag{5.4}$$

It is easy to verify that F_0 is contingent, and that the conditions (i)–(iii) are valid Moreover, the following condition: (iv) is valid iff $\mathcal{Z} = \{Y_{S(\omega)}\}$, in which case F_0 is true.

6 Final Remarks

A purpose of the truth theory presented in this paper is to establish a proper framework to study the regress problem. Tarski's theory of truth (cf. [17]) does not offer it because that theory itself is not free from infinite regress. According to [14, p.189]: "the most important problem with a Tarskian truth predicate is its demand for a hierarchy of languages. ... within that hierarchy of languages, we cannot seem to have any valid method of ending the regression to introduce the "basic" metalanguage." Pure infinite regress is even refused in ([3, p.13]): "Thus, since there can be no infinite regress, from the point of view of logic mathematics must rest ultimately on some sort of axiomatic foundations."

Kripke's theory of truth is also a problematic framework because of three-valued inner logic. As stated in [12, p.283]: "Classical first-order logic is certainly the default choice for any selection among logical systems. It is presupposed by standard mathematics, by (at least) huge parts of science, and by much of philosophical reasoning." Moreover, T-biconditionality rule does not hold in Kripke's theory of truth because of paradoxical sentences.

Paradoxes led Zermelo to axiomatize set theory. To avoid paradoxes Tarski "excluded all Liar-like sentences from being well-formed", as noticed in [12]. Construction of fixed point languages $\mathcal{L}_U$ serves the same purpose. They are logically consistent and semantically closed, i.e., they contain their truth predicates and codes of all their sequences, violating 'Tarski's Commandment' (cf. [13], p. 531). The presented theory of truth has the following properties corresponding to the eight norms (a)–(h) formulated in [12] for truth theories:

(a) Truth is expressed by a predicate T. Syntax is comprised by logical symbols of first-order predicate logic, nonlogical symbols $\in$ and T, and variables ranging in M.

(b) A theory of truth is added to the theory Th(M), and it proves the latter true, by Theorem 3.1.

(c) Truth predicate is not subject to any restrictions within fixed point languages $\mathcal{L}_U$.

(d) T-biconditionals are derivable unrestrictedly within fixed point languages $\mathcal{L}_U$, by the proof of Lemma 3.1.

(e) Truth is compositional, by Definition 3.1 and rules (r4)–(r10).

(f) The theory allows for standard interpretations. M is a standard model of set theory.

(g) The outer logic and the inner logic coincide. Both are classical. In particular:

(h) The outer logic is classical.

According to [12], "In the best of all (epistemically) possible worlds, some theory of truth would satisfy all of these norms (i.e., norms (a)–(h) formulated in [12]) at the same time. Unfortunately, we do not inhabit such a world." As for mathematical theories of truth the above properties (a)–(h) of the theory presented in this paper indicate that we don't live far away from such a world. Only the following mathematically acceptable additional norm is needed:

(i) Metalanguages for which theories of truth are formulated should be free from contradictions.

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