Representation of strongly independent preorders by sets of scalar-valued functions

David McCarthy and Kalle Mikkola and Teruji Thomas

Dept. of Philosophy, University of Hong Kong, Hong Kong, Dept. of Mathematics and Systems Analysis, Aalto University, Finland, Future of Humanity Institute, University of Oxford, United Kingdom

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David McCarthy[†] Kalle Mikkola[‡] Teruji Thomas[§]

Abstract

We provide conditions under which an incomplete strongly independent preorder on a convex set X can be represented by a set of mixture preserving real-valued functions. We allow X to be infinite dimensional. The main continuity condition we focus on is mixture continuity. This is sufficient for such a representation provided X has countable dimension or satisfies a condition that we call Polarization.

1 Introduction

The completeness axiom of expected utility theory has long been regarded as normatively and descriptively implausible. Von Neumann and Morgenstern (1953) themselves found it "very dubious", but claimed that without it, a vector-valued generalization of expected utility could still be obtained. They did not elaborate, but a variety of generalizations have since been produced.

In this article, we present conditions under which imposing a natural and rather weak continuity condition on a strongly independent, incomplete preorder on a convex set leads to generalizations of expected utility. We allow the convex set to be infinite-dimensional. To situate our results, we recall some standard forms of representation.

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†Corresponding author, Dept. of Philosophy, University of Hong Kong, Hong Kong, davidmccarthy1@gmail.com

‡Dept. of Mathematics and Systems Analysis, Aalto University, Finland, kalle.mikkola@iki.fi

§Future of Humanity Institute, University of Oxford, United Kingdom, teru.thomas@oxon.org

Let X be a convex set and $\succsim_X$ a strongly independent preorder on X. A function $u\colon X \to \mathbb{R}$ is *mixture preserving* (MP) if for all x, $y \in X$, and $\alpha \in (0,1)$,

$$u(\alpha x + (1-\alpha)y) = \alpha u(x) + (1-\alpha)u(y).$$

More generally, for any $x_1, \ldots, x_n \in X$ and positive numbers $\alpha_1, \ldots, \alpha_n$ summing to 1, it follows from the above equation that $u(\sum \alpha_i x_i) = \sum \alpha_i u(x_i)$. The following is the abstract form of an expected utility representation.

(R) There exists an MP function $u\colon X \to \mathbb{R}$ such that

$$x \succsim_X y \iff u(x) \geq u(y).$$

We call such an MP function u an *MP representation.* When $\succsim_X$ is incomplete, it cannot satisfy R, but several related representations have been considered. What we call a *weak MP representation* was introduced by Aumann (1962).

(WR) There exists an MP function $u\colon X \to \mathbb{R}$ such that

$$\begin{aligned} x \sim_X y &\implies u(x) = u(y), \text{ and} \\ x \succ_X y &\implies u(x) > u(y). \end{aligned}$$

Aumann (1962) noted the usefulness of weak MP representations for maximization problems: if u attains a maximum at z on a subset Z of X, then z is an $\succsim_X$-maximal element of Z. However, a weak MP representation does not in general enable one to recover the *set* of maximal elements in a given subset, even when it attains a maximum. But the set of maximal elements is central to normative and descriptive applications involving choice from among incomparable alternatives.[1] Thus starting with Shapley and Baucells (1998), focus turned to MP *multi-representations* that fully characterize the preorder. The most discussed version is the following.

(MR) There exists a set U of MP functions $X \to \mathbb{R}$ such that

$$x \succsim_X y \iff u(x) \geq u(y) \text{ for all } u \in U.$$

Such a set U need not consist of weak MP representations of $\succsim_X$. For example, adjoining a constant function to U still yields a MP multi-representation of $\succsim_X$. The special case in which U does consist of weak MP representations is given by the following.

[1]For entries into a large literature, see e.g. Apesteguia and Ballester (2009); Danan, Guerdjikova and Zimper (2012); Eliaz and Ok (2006); Heller (2012); Mandler (2005); Masatlioglu and Ok (2005); Nehring (1997).

(SMR) There exists a set U of MP functions $X \to \mathbb{R}$ such that

$$x \sim_X y \iff u(x) = u(y) \text{ for all } u \in U, \text{ and}$$
$$x \succ_X y \iff u(x) > u(y) \text{ for all } u \in U.$$

We call such a representation a *strict MP multi-representation* of $\succsim_X$. It combines the advantages of WR and MR.[2]

The main result of this article gives conditions for $\succsim_X$ to satisfy MR that satisfy three desiderata: (i) X is allowed to be infinite dimensional; (ii) X is not required to have any further structure beyond convexity; and (iii) continuity and other imposed conditions are required to be 'internal' to $\succsim_X$, in a sense we will indicate below. In addition, we present conditions for $\succsim_X$ to satisfy SMR. These criteria are related to the literature as follows.

First, conditions for MR that satisfy (ii) and (iii) are presented in Baucells and Shapley (2008), but they assume that X is finite dimensional. But there are many problems in which is useful to allow X to be infinite dimensional. Thus this article can be considered an extension of this approach.

Second, results that satisfy (i) and (iii) are given in Dubra, Maccheroni and Ok (2004) and Evren (2008, 2014). But these assume much structure on X beyond convexity. In particular, say that $u\colon X \to \mathbb{R}$ is an *expected utility function* when X is a set of probability measures on some measurable space Y, and there exists a function $v\colon Y \to \mathbb{R}$ such that $u(d\mu) = \int_Y v\, d\mu$ for all $d\mu \in X$. In this setting, the conditions R, WR, MR and SMR specialize to conditions EUR, EUWR, EUMR and EUSMR respectively in which the representing functions are all required to be expected utility functions. The works just mentioned present conditions under which $\succsim_X$ satisfies EUWR, EUMR and EUSMR. They assume that Y is a compact or sigma-compact metric space, and that X is the set of Borel probability measures on Y. But notwithstanding the obvious value of multi-representations consisting of expected utility functions, there are many cases of interest where X is not of this form, even when it consists of representations of uncertainty. For example, even if X is a set of probability measures, the outcome space may not naturally be a metric space. Alternatively, X may not be a set of probability measures at all; for example, it may consist of Anscombe-Aumann 'acts', or explicitly nonprobabilistic representations such as plausibility measures.[3] Thus in requiring no structure at all on X beyond convexity, the approach of this article may be seen as complementary to the expected utility approaches. Our main result does, however, provide expected utility functions when X

[2] See Evren (2014) for a rich discussion of the interpretation of MR and SMR, and the usefulness of SMR in applications.

[3] See e.g. Halpern (2003) for a survey of nonprobabilistic representations.

consists of probability measures with finite support, and a supplementary result provides conditions under which an EUMR representation converts to an EUSMR representation.

Third, to illustrate what we mean by conditions 'internal' to $\succsim_X$, a standard analytical technique is to embed X in a vector space V and extend $\succsim_X$ to a preorder $\succsim_V$ on V that is necessarily a vector preorder. A typical 'external' criterion might then impose a condition on $\succsim_V$, or its positive cone V^+. For example, the approach of Shapley and Baucells (1998) (compare also Kannai, 1963) satisfies (i) and (ii), but assumes that V^+ has a nonempty relative interior. But as noted by Dubra *et al* (2004), it is not easy to understand the normative or behavioral significance of such a condition, so we seek natural conditions that are imposed directly on $\succsim_X$.

The specific continuity condition we investigate was introduced by Aumann (1962).

> **(MC)** For x, y, $z \in X$, if $\alpha x + (1-\alpha)y \succ_X z$ for all $\alpha \in (0,1]$, then $y \succsim_X z$.

For strongly independent preorders, this is equivalent to a number of conditions, including the well-known mixture-continuity condition of Herstein and Milnor (1953), that $\{\alpha \in [0,1]\colon \alpha x + (1-\alpha)y \succsim_X z\}$ is closed (in $[0,1]$). The displayed formulation is natural and normatively plausible, and its use is further motivated by the well-known fact that $\succsim_X$ satisfies R if and only if it is strongly independent, mixture continuous, and complete (we prove this in Theorem 2.1(2b) below); it is natural to ask what happens when completeness is abandoned.

Our results are presented in section 2. Section 2.1 gives our main result, which shows that for convex X, MC is sufficient for $\succsim_X$ to satisfy MR when either X has countable dimension, or value differences are in certain sense bounded according to $\succsim_X$, in which case there is no dimensionality restriction. This notion of boundedness is formalized in a condition we call Polarization. We then round out this result as follows. Section 2.2 shows that our main result does not significantly change when we replace MC by a stronger continuity condition involving the weak topology on X, except that then $\succsim_X$ also satisfies MR in the special case in which X has a non-empty relative algebraic interior. Section 2.3 presents conditions in which $\succsim_X$ satisfies SMR. Section 2.4 gives upper and lower bounds on the size of the set U in MR and SMR. Section 2.5 extends the standard uniqueness result of Dubra *et al* (2004) to the case of MR. Section 3 further discusses the literature, and relates our results to other forms of multi-representation. Proofs are given in section 4 onwards.

Works on other aspects of mixture preserving multi-representation should be noted. These include Vind (2000) (using the notion of the expected value of a preference function); Nau (2006), Ok, Ortoleva and Riella (2012) and Galaabaatar and Karni (2013) (working in the subjective framework of Anscombe and Aumann, 1963); Manzini and Mariotti (2008) (using interval-valued functions); and Galaabaatar and Karni (2012) (providing a multi-representation of a strict partial order that extends to one of an induced preorder).

2 Main Results

2.1 Multi-Representability and Mixture Continuity

Let X be a nonempty convex set. We will study *preorders* (reflexive, transitive binary relations) on X. We are going to assume throughout that such a preorder $\succsim_X$ satisfies

Strong Independence (SI). For all $x, y, z \in X$ and $\alpha \in (0, 1]$,

$$x \succsim_X y \iff \alpha x + (1-\alpha)z \succsim_X \alpha y + (1-\alpha)z.$$

So $\succsim_X$ is an *SI preorder.*

The question we consider in this paper is how to represent SI preorders that satisfy MC but are not necessarily complete. In particular, we are interested in when the property MR is satisfied.

Theorem 2.1.

1. *MR $\implies$ MC, but the converse does not hold.*
2. *However, MR and MC are equivalent for an SI preorder $\succsim_X$ if any one of the following conditions holds:*
   (a) *The dimension of X is at most countable.*
   (b) *$\succsim_X$ is complete.*
   (c) *(Polarization.) There exist $P^+, P^- \in X$ such that $P^+ \succsim_X P^-$ and moreover, for any $x, y \in X$ and for all small enough $\kappa > 0$, $(1-\kappa)P^+ + \kappa x \succsim_X (1-\kappa)P^- + \kappa y$.*
3. *If the dimension of X is uncountable, then there exists an SI preorder on X that satisfies MC but not MR.*

The heuristic for condition (2c) in the theorem is that the value difference between any two elements x and y can be probabilistically discounted to be less than that between the two 'poles' P^+ and P^-.

To prove Theorem 2.1, we use the well-known fact (which we explain more in section 4) that we can embed X in the vector space $V = \mathrm{Span}(X - X)$, and, having done so, there is a bijection between SI preorders $\succsim_X$ on X and convex cones $C_{\succsim_V}$ in V. (For us, a convex cone is a convex set that is closed under multiplication by non-negative scalars; by this definition it includes 0.) We prove the following result.

Theorem 2.2.

1. *$\succsim_X$ satisfies MR iff $C_{\succsim_V}$ is weakly closed in V.*
2. *$\succsim_X$ satisfies MC iff $C_{\succsim_V}$ is algebraically closed in V.*

The point is then that weak closure entails algebraic closure, but not vice versa. Recall here that the weak topology on V is the coarsest one such that every linear functional on V is continuous; while a convex set C is algebraically closed if and only if $(v, w] \subset C \implies v \in C$. (In standard notation, $(v, w] := \{(1 - \alpha)v + \alpha w : \alpha \in (0, 1]\}$.)

We now round out the results.

2.2 Weak Continuity

First, we show that parts 1 and 2 of Theorem 2.1 hold even if we replace MC by a slightly stronger condition, weak continuity. We can define the *weak topology* on X to be the coarsest topology such that all MP functions $X \to \mathbb{R}$ are continuous. (If X is finite-dimensional then this is just the usual Euclidean topology; in general it is the restriction of the weak topology on V, as follows from Lemma 4.4 below.)

> **Weak Continuity (WC).** For each $x \in X$, the set $\{y \in X : y \succsim x\}$ is closed in the weak topology on X.

Theorem 2.3.

1. *MR $\implies$ WC $\implies$ MC, but the converses do not hold.*
2. *However, MR and WC are equivalent for an SI preorder $\succsim_X$ if any one of the conditions (2abc) of Theorem 2.1 holds.*
3. *In addition, MR and WC are equivalent if X has a nonempty algebraic interior (relative to its affine hull). In particular, they are equivalent when X is a vector space.*

Recall here that $x \in X$ is said to be in the algebraic interior or 'intrinsic core' of X if, for all $v \in V$, there is $\epsilon > 0$ with $[x, x + \epsilon v) \subset X$. Having put the weak topology on the affine hull of X (which we can identify with the vector space V), the algebraic interior contains the topological interior of X relative to V; in fact, it coincides with the topological interior of X with respect to the finest locally convex linear topology on V.[4]

Note the contrast between part (3) of this theorem and part (3) of Theorem 2.1. When X is a vector space of uncountable dimension, MR and WC are equivalent with each other but not with MC.

Remark 2.4. Our version of Weak Continuity is a condition on the set of elements that are greater than each basepoint z. But a common version of this principle places an analogous condition also on the set of elements less than z. We do not know whether this stronger version of Weak Continuity suffices for MR.

Some versions of Mixture Continuity similarly place a condition on the set of elements less than z. However, for SI preorders, this further condition actually follows from MC. In terms of Theorem 2.2(2), this is because $C_{\succsim_V}$ is algebraically closed if and only if $-C_{\succsim_V}$ is.

2.3 Strict Multi-Representations

Second, we consider strict MP multi-representations. Recall from the introduction that these are multi-representations containing only weak MP representations.

Theorem 2.5.

1. *If X has uncountable dimension, then there exists an SI preorder with an MP multi-representation but no weak MP representation (hence no strict MP multi-representation).*

2. *Suppose that $\succsim_X$ has an MP multi-representation. Then $\succsim_X$ has a strict MP multi-representation if either of the following is true:*

   (a) $\succsim_X$ has a weak MP representation.

   (b) $\dim X/{\sim_X}$ is countable.

[4]This coincidence uses the fact that X is convex. The point is that, if x is in the algebraic interior, then $A := (X - x) \cap (x - X)$ is an absolutely convex, absorbing set containing 0, and therefore $x + A \subset X$ is a neighbourhood of x in V with respect to some locally convex topology – see e.g. Holmes (1975, Lemma in §10A, and Exercise 2.10(g)).

The following example shows how the EUMR representation presented in Dubra *et al* (2004) converts to an EUSMR representation involving expected utility functions with continuous integrands.

Example 2.6. Let X be the set of Borel probability measures on a compact metric space Y. Let $C(Y)$ be the set of continuous real functions on Y. Define the map $I\colon C(Y) \mapsto X^*$ by $I(v)(d\mu) = \int_Y v\, d\mu$ for $v \in C(Y)$. Dubra *et al* (2004) give independence and continuity conditions on $\succsim_X$ under which there is a convex subset $V \subset C(Y)$ such that $U := I(V)$ is an MP multi-representation of $\succsim_X$. They also note that there is a function $v_0 \in C(Y)$ such that $u_0 := I(v_0)$ is a MP weak representation of $\succsim_X$. By setting $U' := \{u_0 + nu\colon n \in \mathbb{N}, u \in U\}$ (cf. the proof of Theorem 2.5(2a)), we obtain a strict MP multi-representation of X.

2.4 Cardinality

Next, we give upper and lower bounds for the necessary size (i.e. number of functions in) an MP multi-representation or strict MP multi-representation of a given SI preorder.

Theorem 2.7. *Suppose that $\succsim_X$ has an MP multi-representation U.*

1. *$\#U$ is large enough that $\dim \mathbb{R}^U \geq \dim X/{\sim_X}$. If U is finite, then this means $\#U \geq \dim X/{\sim_X}$; otherwise, $2^{\#U} \geq \dim X/{\sim_X}$.*

2. *There exists an MP multi-representation $U' \subset U$ such that $\#U' \leq \max(\#\mathbb{N}, \dim X/{\sim_X})$.*

3. *If $\succsim_X$ has a weak MP representation, it has a strict MP multi-representation U' with $\#U' \leq \max(\#U, \#\mathbb{N})$.*

4. *If $\dim X/{\sim_X}$ is countable, then $\succsim_X$ has a strict MP multi-representation U' with $\#U' \leq \#\mathbb{N}$.*

2.5 Uniqueness

Let us say that an MP function $f\colon X \to \mathbb{R}$ is a *one-way* representation of $\succsim_X$ if $x \succsim_X y \implies f(x) \geq f(y)$. Thus an MP multi-representation consists of one-way representations. Moreover, if $\succsim_X$ has an MP multi-representation, it has a unique *maximal* one, consisting of *all* one-way representations of $\succsim_X$. But in general there need be little relation between different MP multi-representations of the same preorder, even if they are minimal in cardinality. For one thing, it is always possible to replace a one-way representation u by a positive affine transformation $au+b$ ($a, b \in \mathbb{R}, a > 0$). But there is significant leeway even ignoring such transformations.

Example 2.8. Let $X = V = \mathbb{R}^3$. Let $\succsim_X$ be the SI preorder such that the set of non-negative elements is $C_{\succsim_V} = \{(x, y, z) : z^2 \geq x^2 + y^2\}$. Consider the set of all linear functionals u that are non-negative on $C_{\succsim_V}$ and vanish on a hyperplane tangent to it. Each such functional u vanishes on a line in the xy-plane; let $s(u)$ be the slope of that line (possibly ∞). One MP multi-representation consists of all u such that $s(u)$ is a rational number; another consists of all u such that $s(u) - \pi$ is a rational number. These multi-representations are as small as possible in cardinality, but do not overlap. Note also that no minimal MP multi-representations exist: one can always remove elements as long as $s(U)$ is dense in $\mathbb{R} \cup \{\pm\infty\}$.

Example 2.9. If $\succsim_X$ happens to have a *finite* MP multi-representation, then it has a minimal one. If it further happens that $C_{\succsim_V}$ spans V (as one generically expects) then the elements of this minimal representation U are essentially unique: it contains one one-way representation for each facet of $C_{\succsim_V}$. The one-way representation should be constant on the corresponding facet; this determines it up to positive affine transformations. But if $C_{\succsim_V}$ does not span V, then about the best one can say is that the restriction of U to $\operatorname{Span} C_{\succsim_V}$ is essentially unique, in the same sense.

However, we can still prove a kind of uniqueness result, analogous to that in Dubra *et al* (2004). Two MP multi-representations represent the same preorder if and only if they, along with the positive constants, generate the same closed convex cone.

To be more precise, MP functions $X \to \mathbb{R}$ form a vector space $\hat{X}$, which we can endow with the weak-$*$ topology (the coarsest one such that, for each $x \in X$, $u \mapsto u(x)$ is continuous on $\hat{X}$). For each set U of MP functions on X, let $C(U)$ be the smallest convex cone in $\hat{X}$ containing both U and all positive constant functions.

Theorem 2.10. *Suppose given sets U and U' of MP functions on X, representing SI preorders $\succsim_X$ and $\succsim'_X$. Then $\succsim_X$ and $\succsim'_X$ coincide if and only if $C(U)$ and $C(U')$ have the same closure in $\hat{X}$.*

In applications, it can be useful to note that, if Z is a subset of $\hat{X}$ containing both $C(U)$ and $C(U')$, then $C(U)$ and $C(U')$ have the same closure in $\hat{X}$ if and only if they have the same closure in Z.

Example 2.11. Suppose X is a convex set of probability measures on a measurable space Y. Let $\mathcal{F}$ be a vector space of functions on Y that are integrable against every $d\mu \in X$, and which includes the constant functions; for example, $\mathcal{F}$ could be the space of all bounded measurable functions on Y. Integration gives a linear map $I\colon \mathcal{F} \to \hat{X}$, i.e.

$$I(f)(d\mu) = \int_Y f\, d\mu.$$

Given subsets W, W' of $\mathcal{F}$, $U := I(W)$ and $U' := I(W')$ are MP representations of some SI preorders $\succsim_X$ and $\succsim'_X$ on X. We can apply Theorem 2.10, and the observation following it, using the set $Z := I(\mathcal{F})$. We can state the result in terms of $\mathcal{F}$ rather than $\hat{X}$, as follows. Endow $\mathcal{F}$ with the "weak" topology induced by X, i.e. the coarsest one such that $f \mapsto I(f)(d\mu)$ is continuous for every $d\mu \in X$. Let $C(W)$ be the smallest convex cone in $\mathcal{F}$ containing W and all constant functions. Then we find that $\succsim_X$ and $\succsim'_X$ coincide if and only if $C(W)$ and $C(W')$ have the same closures in $\mathcal{F}$.

3 Discussion

3.1 Prior results.

Aumann (1962) raised the question of what happens when a SI preorder $\succsim_X$ satisfies the very weak continuity condition

> **(AC).** For x, y, $z \in X$, if $\alpha x + (1-\alpha) y \succ_X z$ for all $\alpha \in (0, 1]$, then $z \not\succ_X y$.

He showed that while $\succsim_X$ always satisfies WR when X is finite dimensional, it may not when X has uncountable dimension, and Kannai (1963) showed that $\succsim_X$ may fail to satisfy WR even when X is countably infinite dimensional. Baucells and Shapley (2008) showed that when AC is strengthened to MC and X is finite dimensional and closed (in the Euclidean topology), $\succsim_X$ satisfies MR. Our results improve by dropping the requirement that X is closed, and showing that under MC, $\succsim_X$ satisfies MR when X has countable dimension, or when $\succsim_X$ satisfies Polarization, in which case there is no restriction on the dimension of X. In addition, $\succsim_X$ satisfies WR under MC provided $X/\sim_X$ has countable dimension, and when it satisfies both WR and MR, it must also satisfy SMR.

3.2 Archimedean conditions

The condition MC is obtained from AC by replacing '$z \not\succ_X y$' with '$y \succsim_X z$'. This appears to be a fairly modest and normatively natural strengthening. Another natural strengthening of AC is

($\mathbf{Ar^+}$). For x, y, $z \in X$, if $x \succ_X z$, then $(1-\alpha)x + \alpha y \succ_X z$ for all sufficiently small $\alpha \in (0, 1]$.

Aumann regarded MC and Ar^+ as comparably plausible, because for SI preorders, they are equivalent, respectively to '$\{\alpha \in [0,1] \colon \alpha x + (1-\alpha)y \succsim_X z\}$ is closed in $[0,1]$' and '$\{\alpha \in [0,1] \colon \alpha x + (1-\alpha)y \succ_X z\}$ is open in $[0,1]$'. It is therefore natural to investigate the possibilities for MP multi-representation under each condition separately, and under them jointly.

Since SMR entails MR, and by Theorem 2.1, MR entails MC, one would need a different form of multi-representation for SI preorders that satisfy Ar^+ but not MC. Consider

(PMR) There exists a set U of MP functions $X \to \mathbb{R}$ such that

$$x \succ_X y \iff u(x) > u(y) \text{ for all } u \in U.$$

We might think of this as a *full* multi-representation of a strict *partial* order $\succ_X$, or a *partial* multi-representation of a *preorder* $\succsim_X$. Consider the following Archimedean axiom, Ar. It is weaker than, but for SI strict partial orders, equivalent to the standard Archimedean condition. Still assuming SI, it is strictly weaker than Ar^+, but strictly stronger than AC.

(Ar). For x, y, $z \in X$, if $x \succ_X y \succ_X z$, there exists $\alpha \in (0, 1)$ such that $(1-\alpha)x + \alpha z \succ_X y$.

Results establishing the sufficiency of Ar for PMR are given in Galaabaatar and Karni (2012) and McCarthy, Mikkola, and Thomas (2017b).

One might hope that by combining MC with Ar or Ar^+, one could strengthen results concerning multi-representation. But in these cases the scope for incompleteness is rather limited. Say that $\succsim_X$ is 'nontrivial' if $\succ_X \neq \varnothing$. If an SI preorder $\succsim_X$ satisfies MC and Ar, comparability must be an equivalence relation; and if it is nontrivial and satisfies MC and Ar^+, it must be complete.[5][6]

3.3 Vector-valued multi-representations.

The literature has generally approached MP multi-representations by asking what happens when completeness is dropped from the triad of strong

[5]These claims are proved in McCarthy, Mikkola, and Thomas (2017c). Both claims strengthen an observation of Aumann (1962); the second is proved by Dubra (2011) in the special case where X is finite dimensional.

[6]For a possible escape route from these limitations, see Karni (2011) and Galaabaatar and Karni (2012).

independence, completeness and some form of continuity. But if one regards strong independence alone as the canonical expected utility axiom, it is natural to ask what difference adding continuity to strong independence makes. In McCarthy, Mikkola, and Thomas (2017a) we show that a SI preorder $\succsim_X$ has a multi-representation consisting of a family of vector-valued mixture preserving functions into a lexicographically ordered subspace of $\mathbb{R}^C$, where C is an ordered set of possibly infinite cardinality.[7] It is reasonable to say that $\succsim_X$ is lexicographic when $\#C > 1$ has to hold, so it is natural to ask when continuity ensures we can take $\#C = 1$, that is, when all the mixture preserving functions can be taken to be scalar-valued;[8] but that is the question this article has been considering.

4 Preliminaries: SI Preorders and Convex Cones

We first recall some basic facts about preorders and convex cones.

Let W be a vector space, and $X \subset W$ a convex nonempty set. Choosing any $x_0 \in X$, the function $x \mapsto x - x_0$ embeds X into the vector space $\operatorname{Span}(X - X) \subset W$. So from now on we assume that X is given to us as a convex subset of a vector space V such that $V = \operatorname{Span}(X - X) = \operatorname{Span} X$. Here is a useful way of representing elements of V.

Lemma 4.1. $V = \{\lambda(x - y) : \lambda \in (0, \infty), x, y \in X\}$.

Proof. The right-hand side is clearly contained in the left. Conversely, suppose given $v \in V$. Write v in the form $v = \sum_{i=1}^n r_i(x_i - x_i')$, with each $x_i, x_i' \in X$ and $r_i \in \mathbb{R}$. Exchange x_i with x_i' if necessary to have $r_i \geq 0$ for all i. Set $\lambda := \sum_i r_i$. If $\lambda = 0$, then $v = 0$, which can be written as $1(x - x)$ for any $x \in X$. Otherwise, set $x := \sum \frac{r_i}{\lambda} x_i$, $y := \sum \frac{r_i}{\lambda} x_i'$. These are elements of X, since it is convex. Then v is of the desired form $v = \lambda(x - y)$. □

Now we explain the correspondence between SI preorders on X, SI preorders on V, and convex cones in V. Start with an SI preorder $\succsim_V$ on V. It defines an SI preorder $\succsim_X$ on X by restriction. It also determines a set

$$C_{\succsim_V} := \{v \in V : v \succsim_V 0\}$$

in V.

[7] This generalizes Hausner and Wendel (1952), which provides one basis for proving the result.

[8] For precise statements, and use in the characterization of a generalized form of Harsanyi-style utilitarianism, see McCarthy, Mikkola, and Thomas (2016, §3.4)

Lemma 4.2. *$C_{\succsim_V}$ so defined is a convex cone. Moreover, for any $v, w \in V$, $v \succsim_V w \iff v - w \in C_{\succsim_V}$.*

(This shows that an SI preorder on V is the same thing as a linear preorder, i.e. it makes V into a 'preordered vector space'.)

Proof. Convexity is a simple consequence of SI. Also $0 \in C_{\succsim_V}$ since $\succsim_V$ is reflexive. Next we check that $C_{\succsim_V}$ is invariant under positive rescaling. Since $C_{\succsim_V}$ is convex and contains 0, it will suffice to show that if $v \in C_{\succsim_V}$ and $\alpha > 1$ then $\alpha v \in C_{\succsim_V}$. But SI yields $\alpha v \succsim_V 0 \Leftrightarrow \frac{1}{\alpha}(\alpha v)+(1-\frac{1}{\alpha})0 \succsim_V 0 \Leftrightarrow v \succsim_V 0$.

As for the last statement of the lemma, an application of SI gives

$$v \succsim_V w \iff \tfrac{1}{2}v + \tfrac{1}{2}(-w) \succsim_V \tfrac{1}{2}w + \tfrac{1}{2}(-w).$$

The right-hand side simplifies to $\frac{1}{2}(v-w) \succsim_V 0$. Since $C_{\succsim_V}$ is invariant under positive scalars, this is equivalent to $v - w \in C_{\succsim_V}$. □

Lemma 4.3.

1. *The above construction defines bijections between (a) SI preorders $\succsim_X$ on X; (b) SI preorders $\succsim_V$ on V; and (c) convex cones $C_{\succsim_V}$ in V.*
2. *For any SI preorder $\succsim_X$ on X, the corresponding convex cone is*

$$\{\lambda(x-y) : \lambda \in [0, \infty), x, y \in X, x \succsim_X y\}.$$

Proof. Given any convex cone $C \subset V$, we can define $\succsim_V$ on V by $x \succsim_V y \iff x - y \in C$. It is easy to check that this is an SI preorder on V, and that $C_{\succsim_V} = C$. In particular, this establishes the bijection between SI preorders on V and convex cones in V.

Now suppose $\succsim_X$ is an SI preorder on X. Define $C = \{\lambda(x-y) : \lambda \in [0, \infty), x, y \in X, x \succsim_X y\}$ as in part 2 of the lemma. C is clearly a cone; let us check that it is convex. Suppose given $c_1, c_2 \in C$ and $\alpha \in [0, 1]$. Writing $c_1 = \lambda_1(x_1 - y_1)$ and $c_2 = \lambda_2(x_2 - y_2)$, we have

$$\begin{aligned}\alpha c_1 + (1-\alpha)c_2 &= \alpha\lambda_1 x_1 + (1-\alpha)\lambda_2 x_2 - \alpha\lambda_1 y_1 - (1-\alpha)\lambda_2 y_2 \\ &= \lambda([\frac{\alpha\lambda_1}{\lambda}x_1 + \frac{(1-\alpha)\lambda_2}{\lambda}x_2] - [\frac{\alpha\lambda_1}{\lambda}y_1 + \frac{(1-\alpha)\lambda_2}{\lambda}y_2])\end{aligned} \tag{1}$$

where $\lambda := \alpha\lambda_1 + (1-\alpha)\lambda_2$. This shows that any convex combination of c_1 and c_2 is also in C.

Let $\succsim_V$ be the SI preorder on V corresponding to C, as defined in the first paragraph of this proof. It is easy to check that the restriction of $\succsim_V$ to X is $\succsim_X$. This establishes the bijection between SI preorders on X and SI preorders on V, as well as part 2 of the lemma. □

Now we turn to the issue of representing $\succsim_X$. Recall from 2.5 the notion of a one-way representation, and from the introduction that a *weak* MP representation of $\succsim_X$ is a one-way representation with the additional property that $x \succ_X y \implies f(x) > f(y)$.

Lemma 4.4. *Let $\succsim_V$ be an SI preorder on V, and $\succsim_X$ its restriction to X.*

1. *The restriction to X of any affine function $V \to \mathbb{R}$ is an MP function, and every MP function $X \to \mathbb{R}$ arises in this way. (In particular, for $X = V$, MP functions coincide with affine functions.)*[9]

2. *Under this correspondence, one-way representations of $\succsim_X$ correspond to one-way representations of $\succsim_V$; similarly for weak MP representations, and similarly for MP multi-representations.*

Proof. For part 1, the first statement is obvious. For the second, suppose f is an MP function on X. Let $V_1 = \mathbb{R} \oplus V$ and let $X_1 = \{(1, v) : v \in X\} \subset V_1$. Define a linear functional f_1 on $\operatorname{Span} X_1$ by the rule

$$f_1(\sum a_i(1, x_i)) = \sum a_i f(x_i).$$

One can check that this is well defined, using the MP property of f.[10] Moreover, since $V = \operatorname{Span} X$, $V_1 = \operatorname{Span} X_1$. We can now define the extension of f from X to V by $f(v) = f_1(1, v)$ for all $v \in V$. This is an affine function on V, since f_1 is linear on V_1.

For part 2, it should be clear that a one-way representation of $\succsim_V$ restricts to a one-way representation of $\succsim_X$. (Similarly for weak representations and multi-representations.) Conversely, suppose that some affine u restricts to a one-way representation of $\succsim_X$. Let $u_0(v) = u(v) - u(0)$, so that u_0 is linear on V. If $v \succsim_V w$, then, as in Lemma 4.3(2), we can write $v - w$ in the form $v - w = \lambda(x - y)$. Thus $u(v) - u(w) = u_0(v) - u_0(w) = u_0(v - w) = \lambda(u_0(x) - u_0(y)) = \lambda(u(x) - u(y)) \geq 0$. This shows that u is a one-way representation of $\succsim_V$.

[9] An *affine function* f is a real-valued function such that $f(x) - f(0)$ is linear in x.

[10] [Suppose $\sum a_i(1, x_i) - \sum a'_i(1, x_i) = \sum b_i(1, x_i) - \sum b'_i(1, x_i)$, where we have separated out negative coefficients: $a_i, a'_i, b_i, b'_i \geq 0$. Then $\sum a_i(1, x_i) + \sum b'_i(1, x_i) = \sum b_i(1, x_i) + \sum a'_i(1, x_i)$ (*). It follows from this that $\lambda := \sum a_i + \sum b'_i = \sum b_i + \sum a'_i$. Dividing (*) by λ and applying f to both sides, the MP property of f yields $\sum \frac{a_i}{\lambda} f(x_i) + \sum \frac{b'_i}{\lambda} f(x_i) = \sum \frac{a'_i}{\lambda} f(x_i) + \sum \frac{b_i}{\lambda} f(x_i)$. Rearranging, we find

$$\sum a_i f(x_i) - \sum a'_i f(x_i) = \sum b_i f(x_i) - \sum b'_i f(x_i)$$

as desired.]

The case of weak MP representations is similar, using the further observation that if $v \succ_V w$, then $x \succ_X y$, so $u(v) > u(w)$.

Finally, suppose that U is a set of affine functions on V restricting to an MP multi-representation of $\succsim_X$. Suppose given $v, w \in V$ such that $u(v) \geq u(w)$ for all $u \in U$. We can write $v - w = \lambda(x - y)$ in line with Lemma 4.1. Then for each u we must also have $u(x) \geq u(y)$. Since U restricts to an MP multi-representation, we must have $x \succsim_X y$, and therefore $v - w \in C_{\succsim_V}$, using Lemma 4.3(2). Therefore $v \succsim_V w$, as desired. $\square$

5 Proof of Theorem 2.2

Recall from section 4 that corresponding to $\succsim_X$ is an SI preorder $\succsim_V$ on V, and the cone corresponding to $\succsim_X$ is $C_{\succsim_V} = \{v \in V : v \succsim_V 0\}$.

Proof of Part 1

We first show that if $\succsim_X$ satisfies MR, then $C_{\succsim_V}$ is weakly closed. By Lemma 4.4(2), any MP multi-representation is obtained by restriction from an MP multi-representation U of $\succsim_V$. If so, then

$$C_{\succsim_V} = \bigcap_{u \in U} u^{-1}([u(0), \infty))$$

is weakly closed.

Conversely, suppose that $C_{\succsim_V}$ is closed in V (in the weak topology). Here we appeal to the 'Strong Separating Hyperplane Theorem' (Aliprantis and Border, 2006, Theorem 5.79), which specializes thus:

Theorem 5.1. *If C and D are disjoint, non-empty convex subsets of V, and C is closed and D is compact in the weak topology, then there is an affine function $u \colon V \to \mathbb{R}$ such that $u(C) \subset [0, \infty)$ and $u(D) \subset (-\infty, 0)$.*

In particular we can take $C = C_{\succsim_V}$ and $D = \{v\}$ for any $v \notin C_{\succsim_V}$ to find an affine function u such that $u(C_{\succsim_V}) \subset [0, \infty)$ and $u(v) < 0$. Collecting together these functions for different v, we obtain a set U of affine functions on V such that $v \in C_{\succsim_V} \iff u(v) \geq 0$ for all $u \in U$. It follows that U is an MP multi-representation of $\succsim_V$, and restricts to one of $\succsim_X$ by Lemma 4.4.

Proof of Part 2

First suppose $\succsim_X$ satisfies MC. We have to show that $C_{\succsim_V}$ is algebraically closed. Suppose given $v, w \in V$, such that $w_\alpha := (1-\alpha)v + \alpha w$ is in $C_{\succsim_V}$ for all

$\alpha \in (0,1]$. We have to show that $v \in C_{\succsim_V}$, i.e. that $v \succsim_V 0$. As a preliminary, suppose that, for some $\alpha \in (0,1]$, we have $w_\alpha \sim_V 0$, so that $w_\alpha \in -C_{\succsim_V}$. For any $\beta \in (0,\alpha)$ we can find $\kappa > 1$ such that $v = w_\alpha + \kappa(w_\beta - w_\alpha) = (1-\kappa)w_\alpha + \kappa w_\beta$. Since both terms are in $C_{\succsim_V}$, so is v. We are therefore reduced to the case in which $w_\alpha \succ_V 0$ for all $\alpha \in (0,1]$. Now, using Lemma (4.1), write $v = \lambda_1(x_1 - y_1)$ and $w = \lambda_2(x_2 - y_2)$. Since $C_{\succsim_V}$ is a cone, we can simultaneously rescale v and w as necessary to ensure $\lambda_1, \lambda_2 \leq 1/2$. Let $y = (y_1 + y_2)/2$, $x'_1 = (x_1 + y_2)/2$, $x'_2 = (y_1 + x_2)/2$. Then $v = 2\lambda_1(x'_1 - y)$ and $w = 2\lambda_2(x'_2 - y)$, so that $v + y = 2\lambda_1 x'_1 + (1 - 2\lambda_1)y \in X$; similarly, $w + y \in X$. Since $w_\alpha \succ_V 0$, we have $(1-\alpha)(v+y) + \alpha(w+y) = w_\alpha + y \succ_X y$, for all $\alpha \in (0,1]$. By MC, then, $v + y \succsim_X y$, so $v = (v+y) - y$ is in $C_{\succsim_V}$.

Conversely, suppose that $C_{\succsim_V}$ is algebraically closed. We show that $\succsim_X$ satisfies MC. Suppose given $x, y, z \in X$ such that $\alpha x + (1-\alpha)y \succ_X z$ for all $\alpha \in (0,1]$. We have to show $y \succsim_X z$. Define $f(\alpha) = \alpha(x-z) + (1-\alpha)(y-z) = \alpha x + (1-\alpha)y - z$. Then $f(\alpha) \in C_{\succsim_V}$ for all $\alpha \in (0,1]$. By algebraic closedness, $y - z \in C_{\succsim_V}$, so $y \succsim_X z$.

6 Proof of Theorem 2.1

Proof of Parts 1 and 3

First we show that MR entails MC. By Theorem 2.2, it suffices to show that a weakly closed convex cone $C \subset V$ is algebraically closed. Given $v, w \in V$, define $f\colon [0,1] \to V$ by $f(x) = (1-x)v + xw$. This is continuous, so $f^{-1}(C)$ is closed in $[0,1]$. Hence if $f((0,1]) \subset C$ then $f(0) \in C$ as well.

Now we show that MC does not entail MR, giving a construction that works for any X with uncountable dimension, thus establishing parts 1 and 3 of the theorem. The following lemma suffices, in light of Theorem 2.2.

Lemma 6.1. *In any vector space V of uncountable dimension, there is a convex cone K that is algebraically closed but not weakly closed. (For later: for any basis for V, we can choose K to contain only non-negative linear combinations of basis elements.)*

Proof. Suppose B is a basis for V. Choose $b_0 \in B$, and let $V_1 = \mathrm{Span}(B \setminus \{b_0\})$. First we find a convex set $Z \subset V_1$ that is algebraically closed, but not weakly closed. Let Y be the set of all elements of V_0 of the form $n^{-2}\sum_{v \in B_n} v$, for every natural number n and every n-element subset $B_n \subset B \setminus \{b_0\}$. Let Z be the convex hull of Y. By Köthe (1983, pp. 194–195), the "Klee set" Z is algebraically closed in V_1, hence in V, but $0 \in \bar{Z} \setminus Z$ (where $\bar{Z}$ is the closure of Z in the weak topology).

Now let K be the cone generated by $b_0 + Z$: $K = \{\alpha b_0 + \alpha z : \alpha \in [0, \infty), z \in Z\}$. Then K is a convex cone, and K is not weakly closed, since $b_0 \in \bar{K} \setminus K$. It remains to prove that K is algebraically closed. Suppose that $(v_0, v_1]$ is a half-open line segment in K. We need to show that $v_0 \in K$.

Let $V_2 = \mathrm{Span}\{v_0, v_1\}$ and $C = V_2 \cap K$. Since C contains $(v_0, v_1]$, the closure of C in V_2 contains v_0, and it remains to show that C is closed in V_2. Now, C is the convex cone generated by the convex set $L = V_2 \cap (b_0 + Z)$. If L is empty or a singleton, then C is certainly closed; otherwise L is a line segment. Since Z, hence $b_0 + Z$, is algebraically closed, L must contain its endpoints. Being contained in $b_0 + V_1$, these distinct endpoints are linearly independent, so they form a basis for V_2. C is then just the closed positive quadrant of V_2 with respect to this basis. $\square$

6.1 Proof of Part 2

(a) For this we cite (Köthe, 1983, (3) on p. 194): in countably many dimensions, an algebraically closed convex set is the intersection of closed half-spaces, hence it is weakly closed.

(b) Suppose $\succsim_X$ is complete as well as satisfying SI and MC. Then we claim that $\succsim_X$ admits an MP representation u (and therefore an MP multi-representation $\{u\}$). Here we present a proof of this familiar fact using the following standard separation theorem (Holmes, 1975, §1.4.A): if C and D are complementary, non-empty convex sets in V, then the intersection of their algebraic closures is either V or a hyperplane.

Here we take $C = C_{\succsim_V}$. It follows from Lemmas 4.1 and 4.3(2) and the completeness of $\succsim_X$ that $D := V \setminus C$ is contained in $-C_{\succsim_V}$; in fact we must have $D = (-C) \setminus (C \cap (-C)) = \{v \in V : 0 \succ_V v\}$, which is convex by SI. Now, by Theorem 2.2, C and hence $-C$ are algebraically closed. The algebraic closure of D is therefore contained in $-C$. The quoted separation theorem then shows that $Y := C \cap (-C)$ contains a hyperplane. Note that Y is itself a linear subspace of V, so either $Y = V$ or Y is a hyperplane. In the first case, $v \sim_V 0$ for every $v \in V$; we can take $u = 0$ as an MP representation. In the second case, choose any $v_0 \in C$ to have $V = Y + \mathbb{R}v_0$. Let u be the linear functional $\alpha v_0 + y \mapsto \alpha$. Clearly $v \succsim_V w \iff u(v) \geq u(w)$, hence $x \succsim_V y \iff u(x) \geq u(y)$, for any $v, w \in V$, $x, y \in X$.

(c) First we show that $P := P^+ - P^-$ lies in the algebraic interior of $C_{\succsim_V}$. What this means is that given any $v \in V$ and any $\alpha > 0$ small enough, $P + \alpha v \in C_{\succsim_V}$. Lemma 4.1 yields $v = \lambda(x - y)$ with $x, y \in X$, $\lambda > 0$. Set

$\kappa := \alpha\lambda/(1+\alpha\lambda)$ to have $\frac{\kappa}{1-\kappa} = \alpha\lambda$, hence $P+\alpha v = P^+ - P^- + \alpha\lambda(x-y) = \frac{1}{1-\kappa}((1-\kappa)P^+ + \kappa x) - \frac{1}{1-\kappa}((1-\kappa)P^- + \kappa y)$. Note $\kappa \in (0,1)$. Thus

$$P + \alpha v \succsim_V 0 \iff (1-\kappa)P^+ + \kappa x \succsim_X (1-\kappa)P^- + \kappa y.$$

By Polarization, this holds for all small $\kappa > 0$, i.e., for all small $\alpha > 0$.

Now, given that $C_{\succsim_V}$ has an algebraic interior and that $C_{\succsim_V}$ is algebraically closed, every $v \in V \setminus C_{\succsim_V}$ can be strongly separated from $C_{\succsim_V}$ in V, by (Ok, 2007, Corollary G.2.3.4, p. 466). That is to say, there is an affine function $u\colon V \to \mathbb{R}$ with $u(C_{\succsim_V}) \subset [0,\infty)$ and $u(v) < 0$. Thus $C_{\succsim_V}$ is the intersection of closed half-spaces, and is therefore weakly closed.

7 Proof of Theorem 2.3

Proof of Part 1

(MR⇒WC) If $\succsim_X$ satisfies MR, then $C_{\succsim_V}$ is weakly closed, by Theorem 2.2. By Lemma 4.3, $\{y \in X : y \succsim_X x\} = (x + C_{\succsim_V}) \cap X$. This is weakly closed in X, so $\succsim_X$ satisfies WC.

(WC⇏MR) Let V be a vector space of uncountable dimension, with basis B. Let K be the convex cone given by Lemma 6.1; it contains only non-negative linear combinations of basis elements. Let X consist of all non-positive linear combinations of basis elements, and let $\succsim_X$ be the SI preorder corresponding to the cone K. For any $x \in X$, let V_x be the span of those basis elements that have non-zero coefficients in x. Then $K_x := (x+K) \cap X$ is contained in V_x. Since $x \in V_x$ we can write $K_x = (x + K \cap V_x) \cap X$. Since $K \cap V_x$ is algebraically closed and V_x is finite dimensional, $K_x \cap V_x$ is weakly closed in V_x (Ok, 2007, Observation G.1.5.3, p. 450) and therefore in V. Therefore K_x is weakly closed in X, and $\succsim_X$ satisfies WC. But it does not satisfy MR, by Lemma 4.4, since K is not itself weakly closed.

(WC⇒MC) Suppose that WC holds, and we are given x, y, z as in the statement of MC. Define $f(\alpha) = \alpha x + (1-\alpha)y$ for all $\alpha \in [0,1]$. Then f is continuous for the weak topology on X, so, by WC, $\{\alpha : f(\alpha) \succsim_X z\}$ is closed. It therefore contains $\alpha = 0$, i.e. $y \succsim_X z$, as required.

(MC⇏WC) For V, K as in Lemma 6.1, let $X = V$ and define $\succsim_X$ by $C_{\succsim_V} := K$ (Lemma 4.3(1)). Then $\{y \in X : y \succsim_X 0\} = K$ is algebraically but not weakly closed, so WC is false but MC holds, by Theorem 2.2(2).

Proof of Part 2

As WC entails MC (see above), this follows from part 2 of Theorem 2.1.

Proof of Part 3

Suppose X has an algebraic interior X_0. Let τ be the finest locally convex linear topology on V. As explained after the statement of the theorem, X_0 is the interior of X with respect to τ. Translating X if necessary, we can reduce to the case when $0 \in V$ lies in X_0. This implies that, for any $v \in V$, there is some $\lambda > 0$ such that λv is in the X_0 (Holmes, 1975, Lemma in §11A).

Assume that WC holds; by Theorem 2.2 it suffices to show that $C_{\succsim_V}$ is weakly closed. Moreover, by (Aliprantis and Border, 2006, Theorem 5.98), the weak closure of a convex set like $C_{\succsim_V}$ coincides with its τ-closure.

Now, suppose given $v \in V$ in the τ-closure of $C_{\succsim_V}$. We want to show that $v \in C_{\succsim_V}$. For any sufficiently small $\lambda > 0$, we have $\lambda v \in X_0$. Let A be any τ-neighbourhood of λv in X; $\frac{1}{\lambda}A$ is a τ-neighbourhood of v in V, so contains some $v_1 \in C_{\succsim_V}$. This shows that A contains $\lambda v_1 \in C_{\succsim_V} \cap X =: C_0$. Thus λv is in the τ-closure of C_0 in X. However, by Lemma 4.3, $C_0 = \{x \in X : x \succsim_X 0\}$, which, by WC, is weakly closed in X, hence τ-closed in X. So $\lambda v \in C_0 \subset C_{\succsim_V}$, and therefore $v \in C_{\succsim_V}$.

8 Proof of Theorem 2.5

Proof of Part 1

Let B be a basis of V. Write v_b for the coefficient of $b \in B$ in $v \in V$. Let $\geq$ be a well-ordering of B. For each $b \in B$, define a linear functional Λ_b on V by $\Lambda_b(v) := \sum_{i \leq b} v_i$. Let $\succsim_V$ be the SI preorder with

$$C_{\succsim_V} = \{v \in V : \Lambda_b(v) \geq 0 \text{ for all } b \in B\}.$$

To obtain a contradiction, suppose that u is a weak MP representation of $\succsim_V$. Given $b, c \in B \subset X$ with $b < c$, we have $b - c \succ_V 0$, hence $u(b) > u(c)$. Thus, u is strictly decreasing as a function $B \to \mathbb{R}$, so the uncountably many intervals $(u(b+1), u(b)) \subset \mathbb{R}$ are non-empty, open, and disjoint, which is impossible: each open interval must contain a rational number, of which there are countably many.

Proof of Part 2

(a) Let Λ' be the weak MP representation, and U the MP multi-representation. We claim that $\{\Lambda' + nu : n \in \mathbb{N}, u \in U\}$ is then a strict MP multi-representation. First note that, for any $n \in \mathbb{N}$ and $u \in U$, $\Lambda' + nu$ is a weak MP representation. Now, suppose that $\Lambda'(x) + nu(x) \geq \Lambda'(y) + nu(y)$ for all $n \in \mathbb{N}, u \in U$; it remains to show that $x \succsim_X y$. Since, for each u, n can be arbitrarily large, we must have $u(x) \geq u(y)$. Since U is a multi-representation, we find $x \succsim_X y$, as required.

(b) This follows from part (2a) and the following lemma.

Lemma 8.1. *Assume that* $\dim X/\sim_X$ *is at most countable. If MC holds, then* $\succsim_X$ *has a weak MP representation.*

Proof. Replace V by $V/\sim_V$, and X by its image; we can assume in that way that $C_{\succsim_V} \cap (-C_{\succsim_V}) = \{0\}$, and that $\dim V$ is countable. Then by Theorems 2.1(2a) and 2.2, $C_{\succsim_V}$ is weakly closed.

Step 1. Consider the case when V is finite-dimensional; we can suppose $V = \mathbb{R}^n$. Let S_n be the unit sphere in V. Let H_n be the convex hull of $S_n \cap C_{\succsim_V}$. Since $C_{\succsim_V}$ is closed, $S_n \cap C_{\succsim_V}$ is compact. In finite dimensions, the convex hull of a compact set is compact (Aliprantis and Border, 2006, Cor 5.33, p. 185). So H_n, hence $-H_n$, is compact. Moreover, $-H_n$ is disjoint from $C_{\succsim_V}$, as $v \succ_V 0$ for each $v \in H_n$. By the Separating Hyperplane Theorem (Theorem 5.1), we can therefore find an affine function Λ_n such that $\Lambda_n(v) \geq 0$ for $v \in C_{\succsim_V}$ and $\Lambda_n(v) < 0$ for $v \in -H_n$. An affine function that is non-negative on a cone takes its minimum value on the cone at 0, so the linear functional $\Lambda_n - \Lambda_n(0)$ is also non-negative on $C_{\succsim_V}$ and negative on $-H_n$. Thus we can assume Λ_n is linear.

For $v \in C_{\succsim_V} \setminus \{0\}$, there is some $\alpha < 0$ such that $\alpha v \in -H_n$, so $\Lambda_n(\alpha v) < 0$; it follows that $\Lambda_n(v) > 0$. This shows that Λ_n, restricted to X, is a weak MP representation of $\succsim_X$.

Step 2. Now suppose V has a countably infinite basis $B = \{e_1, e_2, \ldots\}$. Let $V_n \cong \mathbb{R}^n$ be the span of the first n basis vectors, and S_n the unit sphere in V_n. We can, as above, find a linear functional $\Lambda_n \colon V \to \mathbb{R}$ non-negative on $C_{\succsim_V}$ and negative on $-H_n$, where H_n is the convex hull of $S_n \cap C_{\succsim_V}$. Rescaling as necessary, we can also assume that $\Lambda_n(S_n) \subset [-1, 1]$. Now define $\Lambda = \sum_{n \in \mathbb{N}} 2^{-n} \Lambda_n$. This is a well-defined linear functional on V. Moreover, for $v \in C_{\succsim_V}$, every term in the sum $\Lambda(v)$ is non-negative, so $\Lambda(v) \geq 0$; if, moreover, $v \neq 0$, then v is contained in some V_n, and for that n, $\Lambda_n(v) > 0$, as in Step 1 above. Thus Λ is a weak MP representation. $\square$

9 Proof of Theorem 2.7

Proof of Part 1

Define $L\colon V \to \mathbb{R}^U$ by $L(v)_u := u(v)$ for all $v \in V, u \in U$. Then L defines a linear embedding of $V/{\sim_V}$ into $\mathbb{R}^U$, hence $\dim X/{\sim_X} = \dim V/{\sim_V} \leq \dim \mathbb{R}^U$.

When U is finite, we further have $\dim \mathbb{R}^U = \#U$. If U is infinite, we claim $\dim \mathbb{R}^U = 2^{\#U}$. Indeed, by Jacobson (2013, Ch. 9.5, Theorem 2) we have $\dim \mathbb{R}^U = (\#\mathbb{R})^{\#U}$. Since U is infinite, $2 \leq \#\mathbb{R} \leq 2^{\#U}$. Therefore $2^{\#U} \leq (\#\mathbb{R})^{\#U} \leq 2^{\#U \cdot \#U}$. Since $\#U \cdot \#U = \#U$, we find $\dim \mathbb{R}^U = (\#\mathbb{R})^{\#U} = 2^{\#U}$.

Proof of Part 2

Replacing V by $V/{\sim_V}$ if necessary, we can assume that $v \sim_V 0 \iff v = 0$. That is, we can assume $\dim X/{\sim_X} = \dim V$.

First consider the case when V is finite-dimensional, so that the weak topology on V is the usual Euclidean one. Since V is second-countable, its subset $V \setminus C_{\succsim_V}$ is a Lindelöf space. Thus the open cover $A := \{u^{-1}((-\infty, 0)) : u \in U\}$ has a countable subcover, corresponding to some countable $U' \subset U$. This is the required countable MP multi-representation.

Now, if V is infinite-dimensional, let B be a basis, and $\mathcal{P}$ be the set of finite subsets of B. For each $P \in \mathcal{P}$, we can find (by the previous paragraph) a countable subset $U_P \subset U$ such that, for $v \in \operatorname{Span} P$, $v \succsim_V 0 \iff u(v) \geq 0$ for all $u \in U_P$. Since every $v \in V$ is in the span of some P, the union $U' := \bigcup_{P \in \mathcal{P}} U_P$ is an MP multi-representation. We have $\#U' \leq \#\mathcal{P} \times \#\mathbb{N} = \#\mathcal{P}$. It remains to prove that $\#\mathcal{P} = \#B$. There is one 0-element subset of B; for each natural number $n > 0$, the number of n-element subsets of B has cardinality $(\#B)^n = \#B$. Therefore $\#\mathcal{P} = 1 + \#\mathbb{N} \times \#B = \#B$.

Proof of Parts 3 and 4

Parts 3 and 4 of theorem follow from inspection of the construction of strict MP multi-representations in proving part 2 of Theorem 2.5 (use also part 2 for part 4).

10 Proof of Theorem 2.10

The proof relies on the following standard fact. Suppose that (A, B) is a dual pair of vector spaces, each equipped with the corresponding weak topology. For any convex cone $C \subset A$ let C° be the *dual cone* $\{b \in B : \langle a, b\rangle \geq 0\ \forall a \in$

$A\}$. Similarly a convex cone D in B has a dual cone D° in A. The standard fact is that $(C^\circ)^\circ$ is the closure of C. (This is a simple application of the Separating Hyperplane Theorem, or more directly of the Bipolar Theorem, (Aliprantis and Border, 2006, 5.103(2)).)

Remember that by Lemma 4.4 we can identify $\hat{X}$ with the space of affine functions on V. Now, consider the vector space $V_1 = \mathbb{R} \oplus V$. For $(\alpha, v) \in V_1$ and $f \in \hat{X}$ define $\langle f, (\alpha, v) \rangle := \alpha f(0) + f(v) - f(0)$. It is easy to check that this makes $(V_1, \hat{X})$ into a dual pair (i.e., bilinear).

Now, that U is an MP multi-representation of $\succsim_X$ implies that $C_{\succsim_V} \subset V \subset V_1$ is the dual cone of $C(U)$. Therefore the closure of $C(U)$ is the dual cone of $C_{\succsim_V}$. Therefore that closure depends only on $C_{\succsim_V}$, not on U. Moreover, since $C_{\succsim_V}$ is closed, it equals its own double dual. This shows conversely that the closure of $C(U)$ determines $C_{\succsim_V}$.

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