Passive modifications for partial assignment of natural frequencies of mass-spring systems

Huajiang Ouyang[1], Jiafan Zhang[2]
[1]School of Engineering, University of Liverpool
The Quadrangle, Liverpool L69 3GH, U.K.
[2]School of Mechanical Engineering, Wuhan Polytechnic University
Wuhan 430023, China

Abstract: The inverse structural modification for assigning a subset of natural frequencies of a structure to some targeted values has been found to inevitably lead to undesired changes to the other natural frequencies of the original structure that should not have been modified, which is referred to as the frequency "spill-over" phenomenon. Passive structural modifications of mass-spring systems for partial assignment of natural frequencies without frequency "spill-over" are addressed in this paper. For two kinds of lumped mass-spring systems, i.e. simply connected in-line mass-spring systems and multiple-connected mass-spring systems, two solution methods are proposed to construct the required mass-normalised stiffness matrix, which satisfies the partial assignment requirement of natural frequencies and maintains the configuration of the original structure after modifications. The modifications are also physically realisable. Finally, some examples of lumped mass-spring systems are analysed to demonstrate the effectiveness and accuracy of the proposed methods.

1. Introduction

Structural modifications (SM) are a procedure aimed at determining values of physical parameters of a structure to achieve desirable dynamic characteristics (usually modal properties such as natural frequencies and mode shapes, i.e. eigenpairs). One common task of

SM is to predict modal properties as a result of structural modifications. The inverse SM problem, however, aims to determine the necessary structural modifications such that the modified structure has some prescribed desired dynamic behaviour, which usually involves an optimisation procedure looking for right modifications. As is well known, when the frequency of excitation is very close to a natural frequency excessive vibration occurs that may lead to structural failure. In this situation, it is useful to determine the changes of geometrical parameters (such as thickness, length, diameter, etc.) and/or material parameters (such as density, Young's modulus, etc.), and/or consider the addition of any combination of lumped masses and stiffnesses in order to relocate the natural frequencies concerned to other locations. This inverse structural frequency modification problem is known as frequency (eigenvalue) placement or assignment.

Mathematically, it is closely related to inverse eigenvalue problems (IEP), which involve the specification of one or more eigenvalues of a matrix or a matrix pencil and the evaluation of how the elements of the matrix need to change to result in the prescribed eigenvalues. These problems have attracted much attention of researchers over the past thirty years.

1.1 Literature review

Research into structural modifications has been conducted mainly from two aspects: theoretical modelling (such as physical models, modal models, and frequency response function or FRF models) and experimental testing (e.g. modal testing). The goal of having desirable modal properties can be achieved by either passive or active procedures. Of course, it can be carried out by combining the above two procedures in a hybrid approach, in order to achieve the desired changes. It should be noted that even on frequency placement by structural modifications studied in this paper, there exist a large number of publications in the literature, and therefore only a brief review of some relevant papers is attempted below.

The methods for inverse structural modifications are based on the use of modal properties derived from a finite elements solution or experimental modal analysis. He [1], Sestieri and D'Ambrogio [2], and Nad [3] reviewed various structural modification methods. Tsuei and Yee [4] described a method for shifting natural frequencies by using only measured frequency response data at modification points. This is particularly convenient and effective for modal testing. Mottershead [5], and his collaborators [6, 7] studied the relocation of an antiresonance and cancellation of a resonance with an antiresonance, and the assignment of natural frequencies and nodes of normal modes by the addition of grounded springs and concentrated masses using FRF data. Park and Park [8, 9] used FRF formulations to find analytically the necessary multiple mass, stiffness and damping modifications in order to exactly achieve both required eigenvalues and eigenvectors. For other related works based on FRF data, refer to [10–13].

Bucher and Braun [14] derived structural changes to produce prescribed frequencies and/or mode shapes, using incomplete modal data from experimental results. Sivan and Ram [15] and Ram [16] studied the construction of a mass-spring system with prescribed natural frequencies. They [17] developed a new algorithm based on Joseph's work [18]. Gladwell [19] studied finite-element discretised structures and mass-spring structures with tridiagonal mass and stiffness matrices and derived a closed-form solution of reconstructed mass and stiffness matrices. Braun and Ram [20] analysed structures consisting of discrete masses and springs and put forward an approximate method for calculating the modification matrices of the structure.

Fox and Kapoor [21] provided expressions of both eigenvalue and eigenvector sensitivities with respect to a design parameter, which can be expressed in terms of only the corresponding unmodified modal parameters and the structure's matrices. Smith and Hutton [22] discussed the use of Newton's method and inverse iteration of mode shape updating on the frequency modification in terms of first-order expansions of eigenvalues with respect to design variables. Farahani and Bahai [23] provided algorithms for relocating eigenvalues of structures based on eigenvalue sensitivities and their second-order expansions. Djoudi et al. [24] gave a formulation free from iterations for the inverse modification of bar and truss structures. Olsson and Lidström [25] considered constraints on structures when obtaining desired frequencies. The undamped natural frequencies of a constrained structure were calculated by solving a generalised eigenvalue problem derived from the equations of motion for the constrained system involving Lagrange multipliers. Smith and Hutton [26] and Kim et al. [27] solved inverse modification problems using perturbation theory.

All these above approaches involve assigning a subset of natural frequencies of a structure to some targeted values, and inevitably lead to undesired changes to the other natural frequencies of the original structure that should not have been modified, which is referred to as the frequency “spill-over” phenomenon. For example, it may happen that an unknown frequency would gain an unwanted value, and the effects brought about by the changes in the modified structure are usually difficult to predict when a global or a large local structural modification to large-scale structures is made, because not all eigenvalues or natural frequencies of large-scale structures could be obtained accurately using the state-of-the-art techniques of matrix computations, or be measured using existing experimental facilities due to hardware limitations.

It should be mentioned that a necessary and sufficient condition was proposed for the incremental mass and stiffness matrices that modify some eigenpairs while keeping other eigenpairs unchanged in [28] but these matrices are not guaranteed to lead to physically realisable structural modifications. Additionally, there are several papers devoted to a related problem that a specific natural frequency of a structure does not change after mass and/or stiffness modifications. Çakar [29] studied a situation in which one of the pre-specified natural frequencies can be preserved by attaching a grounded spring to a structure after adding a number of masses to it. He developed a method based on the Sherman-Morrison formula in order to determine the necessary spring constant. Gürgöze and İnceoğlu [30] was concerned with satisfying a design objective such that the fundamental frequency of a cantilever beam remained the same in spite of the addition of a mass at some point on the beam. Mermertaş and Gürgöze [31] investigated the possibility of using springs to preserve the fundamental frequency of a thin rectangular plate carrying any number of point masses.

In active control of structural vibration via eigenvalue assignment techniques, the frequency “spill-over” phenomenon is overcome by using some partial eigenvalue assignment methods, which reallocate some ‘troublesome’ eigenvalues (or natural frequencies) of the open-loop structure to suitable locations, while leaving the remaining eigenvalues and/or corresponding eigenvectors unchanged in the closed-loop structure. The partial eigenvalue assignment problem of the first-order control system has been widely studied from both theoretical and computational view points, for example, see [32, 33]. To describe the dynamics of a structural system, usually a second-order differential equation is used, with structural matrices that are symmetric and sparse. However, transferring second-order equations to first-order configuration doubles the dimension of the system and the structural matrices lose some nice properties, such as positive semi-definiteness and sparsity, and even symmetry. Therefore, a large effort can be seen from the literature to have been made to tackle this problem directly on second-order dynamic system models over the past ten years, for example, see [34-39].

The capability of active control in making partial eigenvalue or eigenstructure assignment has been known. Obviously, it is desirable to make partial eigenvalue (or natural frequencies) assignment, without frequency “spill-over”, by means of passive control or passive structural modification (abbreviated as PEVAPSM in this paper) due to its advantage of low cost and maintenance of system stability. However, this is a far more difficult task. To the authors’ best knowledge, this has not been achieved before and is the major objective of this investigation.

In this paper, two methods for making partial assignment of natural frequencies for undamped mass-spring systems are proposed. Importantly, the configuration of the structure is also kept, that is, the structure of the mass and stiffness matrices is maintained after modifications. This is a very desirable property, meaning that modifications are made to the existing masses and springs (and unconnected masses in the original structure remain unconnected). Of course, it is easier to make partial eigenvalue assignment without keeping the configuration of the structure concerned than keeping it. It should be pointed out that the latter covers the former.

1.2 Problem definition

Problem PEVAPSM: For an n-degree-of-freedom (DOF) undamped vibrating system with a given theoretical model $\{\mathbf{M}_{0,}\ \mathbf{K}_0\}$, a set of its associated eigenpairs $(\lambda_{i,}\ \mathbf{x}_i)$ $(i = 1,2,\ldots,p)$ with $p < n$, and another set of targeted (or modified) eigenvalues (μ_i) $(i = 1,2,\ldots,p)$, find a *physically realisable* n-DOF undamped vibrating system with the theoretical model $\{\mathbf{M}, \mathbf{K}\}$, which has *the same structured form as* $\{\mathbf{M}_{0,}\ \mathbf{K}_0\}$, such that:

(1) The modified model $\{\mathbf{M}, \mathbf{K}\}$ now has eigenvalues (μ_i) $(i = 1, 2, \ldots,\ p)$;

(2) The remaining (unknown) $n - p$ eigenvalues of the modified model $\{\mathbf{M}, \mathbf{K}\}$ are the same as those of the original model $\{\mathbf{M}_{0,}\ \mathbf{K}_0\}$.

where $\mathbf{M}_0, \mathbf{M}$, and $\mathbf{K}_0, \mathbf{K}$ are, respectively, the mass and stiffness matrices of the models of the original and modified structures, and $\mathbf{M}_0 = \mathbf{M}_0^{\mathrm{T}} > \mathbf{0}$, $\mathbf{K}_0 = \mathbf{K}_0^{\mathrm{T}} \geq \mathbf{0}$, $\mathbf{M} = \mathbf{M}^{\mathrm{T}} > \mathbf{0}$, $\mathbf{K} = \mathbf{K}^{\mathrm{T}} \geq \mathbf{0}$.

PEVAPSM discussed here is seemingly similar to a model updating problem (MUP) [40, 41]. The essential differences between them are that (1) PEVAPSM involve modifications to structures and these must be *physically realisable* and frequency changes and the needed modifications can be very big, while model updating involves small modifications of system parameters; (2) PEVAPSM has the freedom in choosing which masses or springs to modify and the solution is not unique, which allows other design constraints to be considered to achieve other desirable functions of the structure concerned, but model updating is restricted to a prescribed set of sensitive system parameters; (3) the modified structure allows *addition of new members* (*springs*) to the original structure with its number of degrees-of-freedom unchanged in PEVPPSM, which can be considered an extension of (2); (4) Parametric model updating always leads to spill-over, while direct model updating usually does not result in physically realisable modifications.

Two kinds of lumped mass-spring systems are considered in this paper: (1) simply connected in-line mass-spring systems; (2) multiple-connected mass-spring systems. Their solutions are obtained from different numerical construction procedures. The former applies Lanczos method of tridiagonalisation reported in [42–44] to the real symmetric matrix constructed from the mass and stiffness matrices of the original structure and the eigenvalues to be assigned; while the latter exploits the gradient flow method for inverse eigenvalue problems with prescribed entries [45, 46]. In what follows, a real symmetric matrix satisfying the eigenvalue demands of PEVAPSM is constructed, and the solution of simply connected mass-spring systems is presented with a numerical example in Section 2. In Section 3, multiple-connected mass-spring vibrating systems are tackled, and the solution method and conditions for realising PEVAPSM are introduced with two numerical examples. Finally, some conclusions are drawn in Section 4.

2. PEVAPSM solution for simply connected mass-spring systems

2.1 Construction of a real symmetric matrix $\mathbf{J}_s$

For an n-DOF undamped vibrating system with a given theoretical model $\{\mathbf{M}_0, \mathbf{K}_0\}$, its dynamics is characterised by the following eigenvalue equation:

$$(\mathbf{K}_0 - \lambda \mathbf{M}_0)\mathbf{x} = \mathbf{0}. \tag{1}$$

Introducing $\mathbf{D}_0$ so that $\mathbf{M}_0 = \mathbf{D}_0^2$, and $\mathbf{u} = \mathbf{D}_0\mathbf{x}$, Eq.(1) is rewritten as follows:

$$\mathbf{D}_0^{-1}(\mathbf{K}_0 - \lambda \mathbf{D}_0^2)\mathbf{D}_0^{-1}\mathbf{u} = \mathbf{0},$$

that is

$$(\mathbf{J}_0 - \lambda\mathbf{I})\mathbf{u} = \mathbf{0}, \tag{2}$$

where

$$\mathbf{J}_0 = \mathbf{D}_0^{-1}\mathbf{K}_0\mathbf{D}_0^{-1}. \tag{3}$$

$\mathbf{J}_0$ is known as the mass-normalised stiffness matrix and it has the same eigenvalues as $\{\mathbf{M}_{0,}\mathbf{K}_0\}$. For simply connected mass-spring systems (which means a mass is connected via a spring to only an adjacent mass and the system is in the form of a chain of consecutive masses and springs; a mass is connected to at most two other masses), $\mathbf{J}_0$ is a Jacobi matrix in a tridiagonal form as follows:

$$\begin{bmatrix} a_1 & -b_1 & 0 & \cdots & 0 \\ -b_1 & a_2 & -b_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -b_{n-1} \\ 0 & \cdots & 0 & -b_{n-1} & a_n \end{bmatrix}, \tag{4}$$

where $a_i > 0$, $b_i > 0$.

In what follows, a real symmetric matrix $\mathbf{J}_s$ is constructed first such that it has (μ_i) $(i = 1, 2, \ldots, p)$, and (λ_i) $(i = p+1, p+2, \ldots, n)$ (unmodified eigenvalues of $\{\mathbf{M}_{0,}\mathbf{K}_0\}$) as its eigenvalues. Let

$$\mathbf{\Lambda}_1 = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_p),\ \mathbf{\Lambda}_2 = \mathrm{diag}(\lambda_{p+1}, \lambda_{p+2}, \cdots, \lambda_n),\ \mathbf{\Sigma}_1 = \mathrm{diag}(\mu_1, \mu_2, \cdots, \mu_p),$$

$\mathbf{X}_1 = \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_p\}$, $\mathbf{X}_2 = \{\mathbf{x}_{p+1}, \mathbf{x}_{p+2}, \cdots, \mathbf{x}_n\}$. $\mathbf{X} = (\mathbf{X}_1\ \mathbf{X}_2)$ is the mass-normalised eigenvector matrix of Eq.(1). Correspondingly, $\mathbf{U} = (\mathbf{U}_1, \mathbf{U}_2)$ is the normalised eigenvector matrix of Eq.(2), partitioned corresponding to $\mathbf{X}_1$ and $\mathbf{X}_2$. From the spectral decomposition theorem of symmetric matrices [47], $\mathbf{J}_s$ can be constructed as follows:

$$\mathbf{J}_s = \mathbf{U}_1\mathbf{\Sigma}_1\mathbf{U}_1^{\mathrm{T}} + \mathbf{U}_2\mathbf{\Lambda}_2\mathbf{U}_2^{\mathrm{T}}. \tag{5}$$

Using the condition $\mathbf{U}_1 = \mathbf{D}_0\mathbf{X}_1$ and $\mathbf{U}_2\mathbf{\Lambda}_2\mathbf{U}_2^{\mathrm{T}} = \mathbf{J}_0 - \mathbf{U}_1\mathbf{\Lambda}_1\mathbf{U}_1^{\mathrm{T}}$, one has

$$\mathbf{J}_s = \mathbf{J}_0 + \mathbf{D}_0\mathbf{X}_1\,(\mathbf{\Sigma}_1 - \mathbf{\Lambda}_1)\mathbf{X}_1^{\mathrm{T}}\mathbf{D}_0. \tag{6}$$

Note that (1) the constructed matrix $\mathbf{J}_s$ has the same eigenvectors as $\mathbf{J}_0$; (2) $\mathbf{J}_s$ usually is not in a Jacobi matrix form as in (4), and thus a *physically realisable* simply connected mass-spring system cannot be reconstructed from $\mathbf{J}_s$. This calls for an alternative mass-normalised reconstructed stiffness matrix $\mathbf{J} = \mathbf{D}^{-1}\mathbf{K}\mathbf{D}^{-1}$ that possesses the same eigenvalues as $\mathbf{J}_s$ and at the same time is in a Jacobi form (4) (so that the modifications will be physically realisable), where $\mathbf{D}^2 = \mathbf{M}$, and $\mathbf{M}$ and $\mathbf{K}$ are mass and stiffness matrices of the modified system, respectively.

2.2 Tridiagonalisation of $\mathbf{J}_s$ using Lanczos algorithm

The Lanczos algorithm has often been used to reduce symmetric matrices to tridiagonal form in order to solve for their eigenvalues. A variation of it is also used to solve inverse eigenvalue problems of vibrating systems [42, 47], which is employed here. This variation is based on the idea of producing the orthogonal similarity transformation formula as follows:

$$\mathbf{J} = \mathbf{V}^{\mathrm{T}}\mathbf{J}_s\mathbf{V} \quad \text{or} \quad \mathbf{V}\mathbf{J} = \mathbf{J}_s\mathbf{V}, \tag{7}$$

Here $\mathbf{V}$ is an orthogonal matrix and it is built up column by column from $\mathbf{J}_s$. It is known that, if $\mathbf{J}_s$ is positive semi-definite, its eigenvalues are all distinct, and the initial vector $\mathbf{v}_1$ (i.e. the first column of $\mathbf{V}$) is not orthogonal to any eigenvector of $\mathbf{J}_s$, then the algorithm will result in a unique Jacobi matrix (4) for a given $\mathbf{v}_1$ [42]. Additionally, this algorithm has the advantage that numerically it is well conditioned. Now, the algorithm is outlined as follows:

Lanczos algorithm: Given a symmetric matrix $\mathbf{J}_s$, randomly choose a unit vector with its value of each component lying in (-1,1) as an initial Lanczos vector $\mathbf{v}_1$.

Output: a Jacobi matrix $\mathbf{J}$ in the form of (4) and an orthogonal matrix $\mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ such that $\mathbf{J} = \mathbf{V}^{\mathrm{T}}\mathbf{J}_s\mathbf{V}$.

(1) *set* $a_1 := \mathbf{v}_1^{\mathrm{T}}\mathbf{J}_s\mathbf{v}_1$

(2) *for* $i = 1, 2, \dots, n-1$

if $i = 1$ *then* $\mathbf{z}_1 = a_1\mathbf{v}_1 - \mathbf{J}_s\mathbf{v}_1,\ b_1 = \sqrt{\mathbf{z}_1^{\mathrm{T}}\mathbf{z}_1}\ ,\ \mathbf{v}_2 = \mathbf{z}_1/b_1.$

else $a_i = \mathbf{v}_i^{\mathrm{T}}\mathbf{J}_s\mathbf{v}_i,\ \mathbf{z}_i = a_i\mathbf{v}_i - b_{i-1}\mathbf{v}_{i-1} - \mathbf{J}_s\mathbf{v}_i,\ b_i = \sqrt{\mathbf{z}_i^{\mathrm{T}}\mathbf{z}_i}\ ,\ \mathbf{v}_{i+1} = \mathbf{z}_i/b_i.$

end if

end for

(3) *set* $a_n := \mathbf{v}_n^{\mathrm{T}}\mathbf{J}_s\mathbf{v}_n$.

Note that, for a given matrix $\mathbf{J}_s$, the resultant Jacobi matrix $\mathbf{J}$ from the above algorithm is not unique due to randomly chosen initial Lanczos vector $\mathbf{v}_1$.

2.3 Reconstruction of mass-spring systems solving PEVAPSM

Reconstruction of a simply connected lumped mass-spring system from a given Jacobi matrix $\mathbf{J}$ (i.e. the mass-normalised stiffness matrix of the modified system) has been extensively studied during the last thirty years [43, 44]. For three types of end constraint conditions, i.e. "fixed-free", "fixed-fixed", and "free-free", system model $\{\mathbf{M}, \mathbf{K}\}$ can be uniquely determined under certain given conditions on the entries of $\mathbf{M}$ and/or $\mathbf{K}$. For example, for the "fixed-free" type system, if the total mass of the modified system is prescribed, then its system model $\{\mathbf{M}, \mathbf{K}\}$ can be uniquely determined from a given Jacobi matrix $\mathbf{J}$. In the following $\mathbf{M}$ and $\mathbf{K}$ are given for the "fixed-free" system as an example by

$$\mathbf{M}=\begin{pmatrix} m_1 & 0 & 0 & \cdots & 0 \\ 0 & m_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & m_n \end{pmatrix},\mathbf{K}=\begin{pmatrix} k_1+k_2 & -k_2 & 0 & \cdots & \cdots & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -k_{n-1} & k_{n-1}+k_n & -k_n \\ 0 & \cdots & \cdots & 0 & -k_n & k_n \end{pmatrix} \quad (8)$$

Let $\mathbf{q}_1=(1,1,\ldots,1)^{\mathrm{T}}$, one has

$$\mathbf{K}\mathbf{q}_1=(k_1,0,\ldots,0)^{\mathrm{T}}. \quad (9)$$

Since $\mathbf{K}=\mathbf{DJD}$, one has

$$\mathbf{DJD}\mathbf{q}_1=\mathbf{DJD}(1,1,\ldots,1)^{\mathrm{T}}=(k_1,0,\ldots,0)^{\mathrm{T}}, \quad (10)$$

where $\mathbf{D}=\mathbf{M}^{1/2}=\mathrm{diag}\left(\sqrt{m_1},\ \sqrt{m_2},\ldots,\ \sqrt{m_n}\right)$. Then Eq.(10) is rewritten as

$$\mathbf{J}\left(\sqrt{m_1},\sqrt{m_2},\ldots,\sqrt{m_n}\right)^{\mathrm{T}}=(k_1/\sqrt{m_1},0,\ldots,0)^{\mathrm{T}}. \quad (11)$$

Since the previously given Jacobi matrix $\mathbf{J}$ is non-singular, it is known that its inverse matrix $\mathbf{J}^{-1}$ is a strictly positive matrix, meaning that each element of $\mathbf{J}^{-1}$ is strictly positive [43]. Therefore, it is guaranteed that $\left(\sqrt{m_1},\sqrt{m_2},\ldots,\sqrt{m_n}\right)^{\mathrm{T}}$ solved from Eq.(11) will be a strictly positive vector.

Now, take the obtained non-singular $\mathbf{J}$ in Section 2.2 and solve $\mathbf{Jq}=(1,0,\ldots,0)^{\mathrm{T}}$ for $\mathbf{q}$. The solution $\mathbf{q}$ is strictly positive. Thus the solution of Eq.(11) can be rewritten as

$$\left(\sqrt{m_1},\sqrt{m_2},\ldots,\sqrt{m_n}\right)^{\mathrm{T}}=c\mathbf{q}, \quad (12)$$

where $c>0$ is to be determined from other considerations, for example, one can assume the total mass of a structure remains unchanged or should not exceed a certain value.

Suppose that the total mass of the system is m. Then

$$m=\sum_{i=1}^{n} m_i=c^2\mathbf{q}\mathbf{q}^{\mathrm{T}}. \quad (13)$$

Thus, with the prescribed m and the obtained $\mathbf{q}$, one can get c from (13), and $\mathbf{D}=$

$\mathrm{diag}\left(\sqrt{m_1},\ \sqrt{m_2}, \dots,\ \sqrt{m_n}\right)$ from (12). Then $\mathbf{K} = \mathbf{DJD}$ completes the reconstruction for the "fixed-free" system. For details of the reconstruction for other types of systems, refer to [43, 44].

The above discussion shows the existence of a meaningful solution of PEVAPSM for simply connected mass-spring systems and how to find it. A numerical example is presented below.

Example 2.1: a five-DOF "fixed-free" type of simply connected mass-spring system, as shown in Fig.1, with $\{\mathbf{M}_0, \mathbf{K}_0\}$ as follows:

$$\mathbf{M}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{K}_0 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Its eigenvalues (or natural frequencies squared) are λ = {0.0810, 0.6903, 1.7154, 2.8308, 3.6825}, respectively. The first two eigenvalues $\mathbf{\Lambda}_1 = \mathrm{diag}(0.0810, 0.6903)$ are required to become $\mathbf{\Sigma}_1 = \mathrm{diag}(0.15, 0.95)$, and the other eigenvalues remain unchanged. It is assumed that the total mass of the system remains unchanged too after the modification. The mass-normalised modal matrix $\mathbf{X}_1$ corresponding to $\mathbf{\Lambda}_1$ is

$$\mathbf{X}_1 = \begin{pmatrix} 0.1699 & -0.4557 \\ 0.3260 & -0.5969 \\ 0.4557 & -0.3260 \\ 0.5485 & 0.1699 \\ 0.5969 & 0.5485 \end{pmatrix}.$$

The obtained matrices $\mathbf{J}_s$ and $\mathbf{J}$ are, respectively,

$$\mathbf{J}_s = \begin{pmatrix} 2.0559 & -0.9255 & 0.0439 & -0.0137 & -0.0579 \\ -0.9255 & 2.0999 & -0.9392 & -0.0140 & -0.0716 \\ 0.0439 & -0.9392 & 2.0419 & -0.9971 & -0.0277 \\ -0.0137 & -0.0140 & -0.9971 & 2.0283 & -0.9532 \\ -0.0579 & -0.0716 & -0.0277 & -0.9532 & 1.1027 \end{pmatrix},$$

$$
\mathbf{J} = \begin{pmatrix} 2.5231 & -1.0909 & 0 & 0 & 0 \\ -1.0909 & 2.4089 & -0.7028 & 0 & 0 \\ 0 & -0.7028 & 1.2693 & -0.7848 & 0 \\ 0 & 0 & -0.7848 & 1.7352 & -1.0278 \\ 0 & 0 & 0 & -1.0278 & 1.3922 \end{pmatrix}
$$

The modified mass and stiffness matrices constructed from $\mathbf{J}$ are, respectively,

$$
\mathbf{M} = \begin{pmatrix} 1.9823 & 0 & 0 & 0 & 0 \\ 0 & 0.8822 & 0 & 0 & 0 \\ 0 & 0 & 1.0688 & 0 & 0 \\ 0 & 0 & 0 & 0.6905 & 0 \\ 0 & 0 & 0 & 0 & 0.3764 \end{pmatrix}
$$

$$
\mathbf{K} = \begin{pmatrix} 5.0014 & -1.4426 & 0 & 0 & 0 \\ -1.4426 & 2.1250 & -0.6824 & 0 & 0 \\ 0 & -0.6824 & 1.3566 & -0.6742 & 0 \\ 0 & 0 & -0.6742 & 1.1982 & -0.5240 \\ 0 & 0 & 0 & -0.5240 & 0.5240 \end{pmatrix}.
$$

The new masses and spring constants that make the partial frequency assignment are given in the brackets in Fig. 1.

Figure 1. A five-DOF “fixed-free” type of original mass-spring system and modified system

3. PEVAPSM solution for multiple-connected mass-spring systems

3.1 Matrix structures of mass and stiffness matrices

For multiple-connected mass-spring systems, the matrix structure of mass matrix $\mathbf{M}$ remains unchanged: it is real, positive and diagonal. However, the matrix structure of stiffness matrix $\mathbf{K}$ varies according to different configurations of the connectivity of masses and springs, except that $\mathbf{K}$ is real symmetric and positive semi-definite. Additionally, stiffness matrix $\mathbf{K} = (k_{ij})$ has the following properties:

(1) $\mathbf{K}$ has positive diagonal elements and non-positive off-diagonal elements, and is at least

weakly diagonally dominant;

(2) If there is a spring, denoted by i_ℓ, between the *a-th* mass and the *b-th* mass, then the entries k_{ab} and k_{ba} of $\mathbf{K}$ are given by $-k_{i_\ell}$, where k_{i_ℓ} is the stiffness of spring i_ℓ. Otherwise $k_{ab} = k_{ba} = 0$. If the *a-th* mass is connected to springs $j_1, \dots, j_h$, then $k_{aa} = \sum_{s=1}^{h} k_{j_s}$.

The mass-normalised stiffness matrix $\mathbf{J} = \mathbf{D}^{-1}\mathbf{K}\mathbf{D}^{-1}$ of a multiple-connected mass-spring system has the same matrix structure as stiffness matrix $\mathbf{K}$, it is real symmetric and positive semi-definite with the same zero entry patterns as $\mathbf{K}$. Here $\mathbf{J}$ is no longer in a Jacobi matrix form as (4) either, and may take the widely populated form, for example, for a general lumped mass-spring system as follows:

$$\mathbf{J} = \begin{pmatrix} k_{11}/m_1 & -k_{12}/\sqrt{m_1 m_2} & -k_{13}/\sqrt{m_1 m_3} & \cdots & -k_{1n}/\sqrt{m_1 m_n} \\ -k_{12}/\sqrt{m_1 m_2} & k_{22}/m_2 & -k_{23}/\sqrt{m_2 m_3} & \cdots & -k_{2n}/\sqrt{m_2 m_n} \\ -k_{13}/\sqrt{m_1 m_3} & -k_{23}/\sqrt{m_2 m_3} & k_{33}/m_3 & \cdots & -k_{3n}/\sqrt{m_3 m_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_{1n}/\sqrt{m_1 m_n} & -k_{2n}/\sqrt{m_2 m_n} & -k_{3n}/\sqrt{m_3 m_n} & \cdots & k_{nn}/m_n \end{pmatrix}. \tag{14}$$

3.2 Inverse eigenvalue problem and the gradient flow method

3.2.1 Problem description

To solve PEVAPSM for a multiple-connected mass-spring system, the first step is to construct a real symmetric matrix $\mathbf{J}_s$, using the formula (6) in Section 2.1, from the original multiple-connected mass-spring system $\{\mathbf{M}_{0,}\mathbf{K}_0\}$, and its partial eigenpairs $\mathbf{\Lambda}_1$ and $\mathbf{X}_1$ and the targeted eigenvalues $\mathbf{\Sigma}_1$. This step is the same as the simply connected mass-spring system. Also, the obtained $\mathbf{J}_s$ usually does not have the same matrix structure form as that of matrix $\mathbf{J}_0 = \mathbf{D}_0^{-1}\mathbf{K}_0\mathbf{D}_0^{-1}$ (the mass-normalised stiffness matrix obtained from the original multiple connected mass-spring system $\{\mathbf{M}_{0,}\mathbf{K}_0\}$). It implies that one cannot physically reconstruct a multiple-connected mass-spring system while keeping the configuration of the structure unchanged in solving PEVAPSM. To overcome this problem, one naturally tries to convert the obtained $\mathbf{J}_s$ into a matrix $\mathbf{J}$ through orthogonal similarity transforms such that $\mathbf{J}$ has the same matrix structure as $\mathbf{J}_0$. Thus the resultant $\mathbf{J}$ can become the mass-normalised stiffness matrix of a new system $\{\mathbf{M}, \mathbf{K}\}$, and one can reconstruct this new multiple-connected mass–spring system from $\mathbf{J}$ with the same configuration of the structure as that of the original system.

Toward this end, a special type of IEP, matrix completion with prescribed eigenvalues [45, 46], is briefly discussed here, and a numerical algorithm, the gradient flow method, which was used to tackle such an IEP, is exploited to achieve the goal mentioned above.

The goal of matrix completion is to construct a matrix subject to both the structural constraint of prescribed entries and the spectral constraint of prescribed eigenvalues. This special kind of IEP corresponds to the circumstance that "*a portion of the physical system is known a priori, a portion of the matrix to be constructed has fixed entries. The prescribed entries are used to characterise the underlying structure. The task is to specify values for the remaining entries so that the completed matrix has prescribed eigenvalues*", as indicated in [45]. For the problem of constructing matrix $\mathbf{J}$ such that $\mathbf{J}$ has the same matrix structure as that of $\mathbf{J}_0$ and the same eigenvalues as those of $\mathbf{J}_s$, one can set some non-zeros entries of $\mathbf{J}$ with a general form like (14) to be known, which means that ratios of some spring constants to some masses of the modified system are given *a priori*, or these entries of $\mathbf{J}$ are taken to be the same as those in $\mathbf{J}_0$ at the same locations. Meanwhile, let zero entry patterns of $\mathbf{J}$ be the same as those of $\mathbf{J}_0$. Thus, *the problem is now converted into the completion of matrix* $\mathbf{J}$ *with prescribed eigenvalues of matrix* $\mathbf{J}_s$.

Unfortunately, very few theories or numerical algorithms are available for solving such an IEP. The challenge lies in the intertwining of the cardinality and the locations of the prescribed matrix entries so that the inverse problem is solvable. Chu et al. [45] recast the matrix completion problem as minimising the distance between the isospectral matrices (i.e. those matrices with the same eigenvalues) with the prescribed eigenvalues and the affined matrices with the prescribed entries, and then finding the intersection of them. As the gradient of the objective function can be explicitly calculated, a steepest descent gradient flow therefore can be formulated. By integrating this gradient flow numerically, they developed a way to tackle the matrix completion problem. Additionally, this gradient flow method is general enough that it can be used to explore the question on existence of a solution when the prescribed matrix entries are set at some particular locations with some corresponding cardinalities, such as the case of constructing structured matrix $\mathbf{J}$ discussed in this subsection. In what follows the gradient flow method of matrix completion is outlined.

3.2.2 The gradient flow method

The gradient flow method proposed in [45] is for a general real matrix completion, which is presented in the simplified form for a real symmetric matrix completion as follows.

Let $\mathbf{W} \in \mathbf{P}^{n\times n}$ denotes a real symmetric matrix with distinct eigenvalues $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$. The set

$$\mathfrak{M}(\mathbf{W}) = \{\mathbf{V}\mathbf{W}\mathbf{V}^{\mathrm{T}} \mid \mathbf{V} \text{ is a } n \times n \text{ orthogonal matrix}\} \tag{15}$$

consists of all matrices that are isospectral to $\mathbf{W}$. Given an index subset of locations $\mathcal{K} = \{(i_\nu, j_\nu)\}_{\nu=1}^{\ell}$ and the prescribed values $\mathbf{g} = \{g_1, g_2, \dots, g_\ell\}$, the set

$$\aleph(\mathcal{K}, \mathbf{g}) = \left\{\mathbf{A} \in \mathbf{P}^{n\times n} \mid \mathrm{A}_{i_\nu j_\nu} = g_\nu, \nu = 1, \dots, \ell\right\} \tag{16}$$

contains all matrices with the prescribed entries at the desired locations.

For convenience, split any given matrix $\mathbf{Y}$ in $\mathfrak{M}(\mathbf{W})$ as the sum

$$\mathbf{Y} = \mathbf{Y}_{\mathcal{K}} + \mathbf{Y}_{\mathcal{K}^c}, \tag{17}$$

where entries in $\mathbf{Y}_{\mathcal{K}}$ are the same as $\mathbf{Y}$, except those entries that do not belong to $\mathcal{K}$ are set identically zero; and $\mathcal{K}^c$ is simply the index subset complementary to $\mathcal{K}$. With respect to the Frobenius inner product

$$\langle \mathbf{B}, \mathbf{D} \rangle = \textstyle\sum_{i,j=1}^{n} b_{ij} d_{ij}, \tag{18}$$

the projection $P(\mathbf{Y})$ of any matrix $\mathbf{Y}$ onto the affine subspace $\aleph(\mathcal{K}, \mathbf{g})$ is given by

$$P(\mathbf{Y}) = \mathbf{A}_{\mathcal{K}} + \mathbf{Y}_{\mathcal{K}^c}, \tag{19}$$

where $\mathbf{A}_{\mathcal{K}}$ is a constant matrix in $\aleph(\mathcal{K}, \mathbf{g})$ with zero entries at all locations corresponding to $\mathcal{K}^c$. For each given $\mathbf{Y} \in \mathfrak{M}(\mathbf{W})$, it is intended to minimise the distance between $\mathbf{Y}$ and $\aleph(\mathcal{K}, \mathbf{g})$. Equivalently, it is to minimise the function defined by

$$f(\mathbf{Y}) = \tfrac{1}{2}\langle \mathbf{Y} - P(\mathbf{Y}), \mathbf{Y} - P(\mathbf{Y}) \rangle, \tag{20}$$

where $\mathbf{Y} - P(\mathbf{Y}) = \mathbf{Y}_{\mathcal{K}} - \mathbf{A}_{\mathcal{K}}$.

Let $\mathbf{Y} = \mathbf{V}\mathbf{W}\mathbf{V}^{\mathrm{T}}$. This minimisation with objective function $f(\mathbf{Y})$ can be rewritten as an unconstrained optimisation problem in terms of $\mathbf{V}$ as follows:

$$h(\mathbf{V}) = \tfrac{1}{2}\langle \mathbf{VWV}^{\mathrm{T}} - P(\mathbf{VWV}^{\mathrm{T}}), \mathbf{VWV}^{\mathrm{T}} - P(\mathbf{VWV}^{\mathrm{T}}) \rangle. \tag{21}$$

The gradient ∇_h of objective function h is given by [45]

$$\nabla_h(\mathbf{V}) = \left(\mathbf{VWV}^{\mathrm{T}} - P(\mathbf{VWV}^{\mathrm{T}})\right)\mathbf{VW} - (\mathbf{VWV}^{\mathrm{T}})^{\mathrm{T}}\left(\mathbf{VWV}^{\mathrm{T}} - P(\mathbf{VWV}^{\mathrm{T}})\right)\mathbf{V}. \tag{22}$$

Post-multiplying Eq.(22) by $\mathbf{V}^{\mathrm{T}}$, one obtains

$$\nabla_h(\mathbf{V})\mathbf{V}^{\mathrm{T}} = [\mathbf{Y} - P(\mathbf{Y}),\ \mathbf{Y}^{\mathrm{T}}], \tag{23}$$

where $[\mathbf{Y} - P(\mathbf{Y}),\ \mathbf{Y}^{\mathrm{T}}]$ denotes the Lie bracket commutator, i.e. $[\mathbf{B}, \mathbf{D}] = \mathbf{BD} - \mathbf{DB}$, and $\mathbf{Y} = \mathbf{VWV}^{\mathrm{T}}$. It follows that the vector field

$$\frac{d\mathbf{V}}{dt} = [\,\mathbf{Y}^{\mathrm{T}}, \mathbf{Y} - P(\mathbf{Y})]\mathbf{V} \tag{24}$$

defines a gradient flow of $h(\mathbf{V})$ in the open set consisting of $n \times n$ orthogonal matrices and moves in the steepest descent direction to reduce the value of $h(\mathbf{V})$ [45]. The system of ordinary differential equations (24) can be readily integrated from a starting point, say, $\mathbf{V}(0) = \mathbf{I}$ (the identity matrix). $\nabla_h\big(\mathbf{V}(t)\big)$ will converge to zero as t goes to infinity, implying that a local minimum for $h(\mathbf{V})$ has been found. The integration stop criterion of Eq.(24) can be chosen as follows:

$$\min\left\{\left\|\mathbf{Y}(t_k) - P\big(\mathbf{Y}(t_k)\big)\right\|_F, \left\|\left[\mathbf{Y}(t_k)^{\mathrm{T}}, \mathbf{Y}(t_k) - P\big(\mathbf{Y}(t_k)\big)\right]\right\|_F\right\} \le 10^{-8}, \tag{25}$$

where $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm of a matrix. It should be noted that, in the event that a solution does not exist, the formulation enables one to find a least-squares solution.

3.3 Numerical examples of PEVAPSM solution

Based on the above discussion, PEVAPSM solutions of two multiple-connected mass-spring systems are presented for the purpose of demonstration in the following. One is a simple 4-DOF system and the other a more complex 10-DOF one. The existing ordinary differential equation solver *ode15s* in Matlab is used to implement the computation in this subsection. To control the integration, local tolerance values of $\mathrm{AbsTol} = 10^{-10}$ and $\mathrm{RelTol} = 10^{-9}$ are set while maintaining all other parameters at the default values in the

Matlab codes.

Example 3.1. a 4-DOF mass-spring system, as shown in Fig.2, with $\{\mathbf{M}_{0,}\mathbf{K}_0\}$ as follows:

$$\mathbf{M}_0=\begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{pmatrix},\mathbf{K}_0=\begin{pmatrix} k_1+k_2+k_5 & -k_2 & -k_5 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 \\ -k_5 & -k_3 & k_3+k_4+k_5 & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{pmatrix}.$$

Let $m_1=m_2=m_3=m_4=1.0$ and $k_1=k_2=k_3=k_4=k_5=1.0$. Its eigenvalues (or natural frequencies squared) are λ = {0.1783, 1.1538, 3.4882, 4.1796}, respectively. The first two eigenvalues $\mathbf{\Lambda}_1=\mathrm{diag}(0.1783{,}1.1538)$ are required to relocate to $\mathbf{\Sigma}_1=\mathrm{diag}(0.5,1.5)$, and the other eigenvalues remain unchanged. $\mathbf{X}_1$ is not listed for the sake of saving space. The mass-normalised stiffness matrix $\mathbf{J}_0=\mathbf{D}_0^{-1}\mathbf{K}_0\mathbf{D}_0^{-1}$ of $\{\mathbf{M}_{0,}\mathbf{K}_0\}$ is given by

$$\mathbf{J}_0=\begin{pmatrix} \frac{k_1+k_2+k_5}{m_1} & -\frac{k_2}{\sqrt{m_1m_2}} & -\frac{k_5}{\sqrt{m_1m_3}} & 0 \\ -\frac{k_2}{\sqrt{m_1m_2}} & \frac{k_2+k_3}{m_2} & -\frac{k_3}{\sqrt{m_2m_3}} & 0 \\ -\frac{k_5}{\sqrt{m_1m_3}} & -\frac{k_3}{\sqrt{m_2m_3}} & \frac{k_3+k_4+k_5}{m_3} & -\frac{k_4}{\sqrt{m_3m_4}} \\ 0 & 0 & -\frac{k_4}{\sqrt{m_3m_4}} & \frac{k_4}{m_4} \end{pmatrix}=\begin{pmatrix} 3 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}. \quad (26)$$

Figure 2. A 4-DOF “fixed-free” type of multiple connected mass-spring system

The constructed real symmetric matrix $\mathbf{J}_s$ from the formula (6) in Section 2.1 is given by

$$\mathbf{J}_s=\begin{pmatrix} 3.0862 & -0.8745 & -0.9280 & -0.0225 \\ -0.8745 & 2.1835 & -0.8998 & -0.0478 \\ -0.9280 & -0.8998 & 3.0890 & -0.9252 \\ -0.0225 & -0.0478 & -0.9252 & 1.3091 \end{pmatrix}. \quad (27)$$

Its eigenvalues satisfy the modification requirement of partial eigenvalues, but its matrix structure is not the same as that of $\mathbf{J}_0$, which means one cannot reconstruct the modified system directly from $\mathbf{J}_s$ with the same configuration structure as $\{\mathbf{M}_{0,}\mathbf{K}_0\}$.

Now, the mass-normalised stiffness matrix $\mathbf{J}$ of the modified system with the same matrix structure as $\mathbf{J}_0$ and the same eigenvalues as $\mathbf{J}_s$ is to be constructed using the gradient flow method of the matrix completion discussed in subsection 3.2.2. Let some non-zero entries of $\mathbf{J}$ be set to be known *a priori*. For example, suppose modified values of m_3, m_4 and k_4 are prescribed *a priori*, or for convenience, their values are left unchanged, i.e. $\widetilde{m}_3 = \widetilde{m}_4 = 1.0$, $\tilde{k}_4 = 1.0$, which means $\mathbf{J}(3,4) = \mathbf{J}(4,3) = -1$, $\mathbf{J}(4,4) = 1$. Additionally, let zero entry pattern of $\mathbf{J}$ be the same as that of $\mathbf{J}_0$, which means $\mathbf{J}(1,4) = \mathbf{J}(2,4) = \mathbf{J}(4,1) = \mathbf{J}(4,2) = 0$. At this point, using notation in subsection 3.2.2, one has

$$\mathbf{A}_{\mathcal{K}} = \begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad \mathbf{W} = \mathbf{J}_s,$$

where the stars in $\mathbf{A}_{\mathcal{K}}$ indicate unknown entries to be determined. Set $\mathbf{V}(0) = \mathbf{I}$ (a 4×4 identity matrix), and start with $\mathbf{Y}_0 = \mathbf{V}(0)\mathbf{J}_s\mathbf{V}(0)^{\mathrm{T}} = \mathbf{J}_s$.

The gradient flow method gives $\mathbf{J} = \mathbf{Y} = \mathbf{V}\mathbf{J}_s\mathbf{V}^{\mathrm{T}}$ as follows:

$$\mathbf{J} = \begin{pmatrix} 3.0933 & -0.8264 & -0.7768 & 0 \\ -0.8264 & 2.2711 & -0.6801 & 0 \\ -0.7768 & -0.6801 & 3.3034 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}. \tag{28}$$

It is easily verified that the obtained $\mathbf{J}$ has eigenvalues $\{0.5, 1.5, 3.4882, 4.1796\}$.

Now, one can reconstruct the modified mass-spring system with the same configuration structure as $\{\mathbf{M}_{0,}\mathbf{K}_0\}$, shown in Fig.2, from $\mathbf{J}$. Note that the physical parameters of masses and springs of the modified system constitute the entries of $\mathbf{J}$, just like $\mathbf{J}_0$ shown in (26).

Take $\mathbf{q} = (1, 1, \dots, 1)^{\mathrm{T}}$ as the static displacements of all the masses, one has

$$\mathbf{Kq} = (\tilde{k}_1, 0, \dots, 0)^{\mathrm{T}} \tag{29}$$

$$\mathbf{u} = \mathbf{Dq} = \mathbf{M}^{1/2}\mathbf{q} = (\tilde{m}_1^{1/2}, \tilde{m}_2^{1/2}, \tilde{m}_3^{1/2}, \tilde{m}_4^{1/2})^{\mathrm{T}} \tag{30}$$

$$\mathbf{Ju} = \mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}\mathbf{M}^{1/2}\mathbf{q} = \mathbf{M}^{-1/2}\mathbf{Kq} = (\tilde{k}_1/\sqrt{\tilde{m}_1}, 0,0,0)^{\mathrm{T}} \tag{31}$$

Substituting $\mathbf{J}$ of (28) into (31), expanding the first three equations of (31), one has

$$\begin{cases} 3.0933\tilde{m}_1^{1/2} - 0.8264\tilde{m}_2^{1/2} - 0.7768\tilde{m}_3^{1/2} = \tilde{k}_1 \big/ \sqrt{\tilde{m}_1} \\ -0.8264\tilde{m}_1^{1/2} + 2.2711\tilde{m}_2^{1/2} - 0.6801\tilde{m}_3^{1/2} = 0 \\ -0.7768\tilde{m}_1^{1/2} - 0.6801\tilde{m}_2^{1/2} + 3.3034\tilde{m}_3^{1/2} - \tilde{m}_4^{1/2} = 0 \end{cases}. \tag{32}$$

Simultaneously solving the second and third equation of (32) in terms of $\tilde{m}_3 = \tilde{m}_4 = 1.0$, one gets $\tilde{m}_1$ and $\tilde{m}_2$. Substituting them into the first equation of (32), one gets $\tilde{k}_1$. Because entries $\mathbf{J}(1,2) = -\tilde{k}_2 \big/ \sqrt{\tilde{m}_1\tilde{m}_2} = -0.8264$, $\mathbf{J}(1,3) = -\tilde{k}_5 \big/ \sqrt{\tilde{m}_1\tilde{m}_3} = -0.7768$, $\mathbf{J}(2,2) = (\tilde{k}_2 + \tilde{k}_3)\big/\tilde{m}_2 = 2.2711$, one obtains $\tilde{k}_2$, $\tilde{k}_3$, and $\tilde{k}_5$. Thus one has entire physical parameters of the modified system, as shown in Table 1.

Table 1. Masses and spring constants of the original and modified structures $(\tilde{m}_{1-4}$ and $\tilde{k}_{1-5})$

m_1	m_2	m_3	m_4	k_1	k_2	k_3	k_4	k_5
1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$\tilde{m}_1$	$\tilde{m}_2$	$\tilde{m}_3$	$\tilde{m}_4$	$\tilde{k}_1$	$\tilde{k}_2$	$\tilde{k}_3$	$\tilde{k}_4$	$\tilde{k}_5$
4.2024	1.0929	1.0	1.0	9.6359	1.7710	0.7110	1.0	1.5924

It is worthwhile to note that (1) if $\mathbf{V}(0)$ is chosen to be an arbitrary orthogonal matrix, the gradient flow method would ends at a different limit point, such as

$$\mathbf{J} = \begin{pmatrix} 1.6600 & -0.1625 & -0.6931 & 0 \\ -0.1625 & 4.0368 & -0.3498 & 0 \\ -0.6931 & -0.3498 & 2.9710 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix},$$

which means one can reconstruct another PEVAPSM solution of the original system from $\mathbf{J}$ above; (2) if other non-zero entries of $\mathbf{J}$ are set to be known *a priori*, one can also reconstruct different PEVAPSM solutions.

Example 3.2. a 10-DOF mass-spring system [48], as shown in Fig.3, with $\{\mathbf{M}_{0,}\mathbf{K}_0\}$ as follows:

$$\mathbf{M}_0 = \mathrm{diag}(m_1, m_2, \ldots, m_{10}),$$

$$\mathbf{K}_0 = \begin{pmatrix} k_1+k_{13}+k_{14} & 0 & -k_{13} & -k_{14} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2+k_{11}+k_{12} & -k_{12} & -k_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ -k_{13} & -k_{12} & k_3+k_{12}+k_{13}+k_{16} & -k_{16} & 0 & 0 & 0 & 0 & 0 & 0 \\ -k_{14} & -k_{11} & -k_{16} & k_4+k_{11}+k_{14}+k_{15}+k_{16}+k_{22} & -k_{15} & 0 & -k_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_{15} & k_5+k_{15}+k_{17}+k_{18} & 0 & -k_{18} & -k_{17} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_6+k_{19}+k_{20} & -k_{19} & 0 & -k_{20} & 0 \\ 0 & 0 & 0 & -k_{22} & -k_{18} & -k_{19} & k_7+k_{18}+k_{19}+k_{21}+k_{22}+k_{24} & -k_{21} & 0 & -k_{24} \\ 0 & 0 & 0 & 0 & -k_{17} & 0 & -k_{21} & k_8+k_{17}+k_{21}+k_{23} & 0 & -k_{23} \\ 0 & 0 & 0 & 0 & 0 & -k_{20} & 0 & 0 & k_9+k_{20}+k_{25} & -k_{25} \\ 0 & 0 & 0 & 0 & 0 & 0 & -k_{24} & -k_{23} & -k_{25} & k_{10}+k_{23}+k_{24}+k_{25} \end{pmatrix}$$

All spring constants are 2.4×10^5 N/m, and $m_1 = 30\mathrm{kg}, m_2 = 35\mathrm{kg}, m_3 = 40\mathrm{kg}, m_4 = 45\mathrm{kg}, m_5 = 45\mathrm{kg}, m_6 = 45\mathrm{kg}, m_7 = 40\mathrm{kg}$, $m_8 = 35\mathrm{kg}$, $m_9 = 30\mathrm{kg}, m_{10} = 25\mathrm{kg}$. Its eigenvalues (or natural frequencies squared) are λ = {6298.12，9628.31，14109.22, 22117.92, 22733.69, 27718.30, 32139.94, 35557.23, 42219.32, 49077.96}, respectively. The first two eigenvalues $\mathbf{\Lambda}_1 = \mathrm{diag}(6298.12，9628.31)$ are required to relocate to $\mathbf{\Sigma}_1 = \mathrm{diag}(9012，12118)$, and the other eigenvalues remain unchanged. $\mathbf{X}_1$, $\mathbf{J}_0$ and $\mathbf{J}_s$ are not listed for the sake of saving space.

Figure 3. A 10-DOF “fixed-fixed” type of multiple connected mass-spring system

The matrix structure of the obtained $\mathbf{J}_s$ is not the same as that of $\mathbf{J}_0$ either, which means one cannot reconstruct the modified system directly from $\mathbf{J}_s$ with the same configuration structure as $\{\mathbf{M}_{0,}\mathbf{K}_0\}$. Now, the mass-normalised stiffness matrix $\mathbf{J}$ of the modified system with the same matrix structure as $\mathbf{J}_0$ and the same eigenvalues as $\mathbf{J}_s$ is to be constructed using the gradient flow method of matrix completion. Sometimes it is convenient to allow some entries of $\mathbf{J}$, for example, $\mathbf{J}(9,9) = (\tilde{k}_9 + \tilde{k}_{20} + \tilde{k}_{25})/\widetilde{m}_9$, $\mathbf{J}(9,10) = \mathbf{J}(10,9) = -\tilde{k}_{25}/\sqrt{\widetilde{m}_9\widetilde{m}_{10}}$, $\mathbf{J}(8,10) = \mathbf{J}(10,8) = -\tilde{k}_{23}/\sqrt{\widetilde{m}_8\widetilde{m}_{10}}$, $\mathbf{J}(7,10) = \mathbf{J}(10,7) = -\tilde{k}_{24}/\sqrt{\widetilde{m}_7\widetilde{m}_{10}}$, and $\mathbf{J}(10,10) = (\tilde{k}_{10} + \tilde{k}_{23} + \tilde{k}_{24} + \tilde{k}_{25})/\widetilde{m}_{10}$, to be equal to the corresponding entries of $\mathbf{J}_0$. The mathematical expressions of these entries of $\mathbf{J}$ are explicitly given to aid understanding of their physical meanings . Meanwhile, let zero entry pattern of $\mathbf{J}$ be the same as that of $\mathbf{J}_0$. Thus one has

$$\mathbf{A}_{\mathcal{K}} = \begin{pmatrix} * & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * & 0 & * & 0 \\ 0 & 0 & 0 & * & * & * & * & * & 0 & \mathrm{J}_0(7,10) \\ 0 & 0 & 0 & 0 & * & 0 & * & * & 0 & \mathrm{J}_0(8,10) \\ 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & \mathrm{J}_0(9,9) & \mathrm{J}_0(9,10) \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{J}_0(10,7) & \mathrm{J}_0(10,8) & \mathrm{J}_0(10,9) & \mathrm{J}_0(10,10) \end{pmatrix}.$$

Set $\mathbf{V}(0) = \mathbf{I}$ (a 10×10 identity matrix), and start with $\mathbf{Y}_0 = \mathbf{V}(0)\mathbf{J}_s\mathbf{V}(0)^{\mathrm{T}} = \mathbf{J}_s$, the gradient flow method gives $\mathbf{J} = \mathbf{Y} = \mathbf{V}\mathbf{J}_s\mathbf{V}^{\mathrm{T}}$ as follows:

$\mathbf{J} =$

$$
\begin{pmatrix}
2.4090e+4 & 0 & -6.4017e+3 & -5.8373e+3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2.0638e+4 & -5.5792e+3 & -4.7904e+3 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6.4017e+3 & -5.5792e+3 & 2.5478e+4 & -4.1434e+3 & 0 & 0 & 0 & 0 & 0 & 0 \\
-5.8373e+3 & -4.7904e+3 & -4.1434e+3 & 3.3804e+4 & -4.6084e+3 & 0 & -4.3752e+3 & 0 & 0 & 0 \\
0 & 0 & 0 & -4.6084e+3 & 1.8417e+4 & 0 & -2.0021e+3 & -7.3117e+3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.7552e+4 & -3.0753e+3 & 0 & -5.9524e+3 & 0 \\
0 & 0 & 0 & -4.3752e+3 & -2.0021e+3 & -3.0753e+3 & 3.8423e+4 & -5.6127e+3 & 0 & -7.5895e+3 \\
0 & 0 & 0 & 0 & -7.3117e+3 & 0 & -5.6127e+3 & 2.6001e+4 & 0 & -8.1135e+3 \\
0 & 0 & 0 & 0 & 0 & -5.9524e+3 & 0 & 0 & 2.4000e+4 & -8.7636e+3 \\
0 & 0 & 0 & 0 & 0 & 0 & -7.5895e+3 & -8.1135e+3 & -8.7636e+3 & 3.8400e+4
\end{pmatrix}
$$

It should be noted that the entries with numerical values in the orders of 10^{-11} or below in the above matrix $\mathbf{J}$ are already set to be zero.

Now, one can reconstruct the modified mass-spring system with the same configuration structure as $\{\mathbf{M}_{0,}\mathbf{K}_0\}$, shown in Fig.3, from $\mathbf{J}$. Similar to (29)-(31), one has

$$\mathbf{J}\left(\tilde{m}_1^{1/2}, \tilde{m}_2^{1/2}, \dots, \tilde{m}_{10}^{1/2}\right)^{\mathrm{T}} = \left(\tilde{k}_1/\sqrt{\tilde{m}_1}, \tilde{k}_2/\sqrt{\tilde{m}_2}, \dots, \tilde{k}_{10}/\sqrt{\tilde{m}_{10}}\right)^{\mathrm{T}}. \qquad (33)$$

At this point, a different reconstruction procedure for physical parameters is used. One can prescribe values of the entries of the right-hand vector in Eq.(33). Here the ratios of $\tilde{k}_1/\sqrt{\tilde{m}_1}, \tilde{k}_2/\sqrt{\tilde{m}_2}, \dots, \tilde{k}_{10}/\sqrt{\tilde{m}_{10}}$ are taken to be the same as that of the original system. Then solving Eq.(33), one has $\mathbf{D} = \mathbf{M}^{1/2} = \mathrm{diag}\left(\tilde{m}_1^{1/2}, \tilde{m}_2^{1/2}, \dots, \tilde{m}_{10}^{1/2}\right)$, and subsequently $\mathbf{K} = \mathbf{DJD}$. The results are listed as follows:

$$\mathbf{M} = \mathrm{diag}(14.620, 15.591, 15.641, 15.492, 27.244, 17.019, 12.720, 25.479, 18.588, 16.043),$$

$\mathbf{K} =$

$$
\begin{pmatrix}
3.5220e+5 & 0 & -9.6806e+4 & -8.7850e+4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3.2175e+5 & -8.7124e+4 & -7.4449e+4 & 0 & 0 & 0 & 0 & 0 & 0 \\
-9.6806e+4 & -8.7124e+4 & 3.9851e+5 & -6.4498e+4 & 0 & 0 & 0 & 0 & 0 & 0 \\
-8.7850e+4 & -7.4449e+4 & -6.4498e+4 & 5.2371e+5 & -9.4676e+4 & 0 & -6.1418e+4 & 0 & 0 & 0 \\
0 & 0 & 0 & -9.4676e+4 & 5.0174e+5 & 0 & -3.7269e+4 & -1.9264e+5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2.9871e+5 & -4.5246e+4 & 0 & -1.0587e+5 & 0 \\
0 & 0 & 0 & -6.1418e+4 & -3.7269e+4 & -4.5246e+4 & 4.8873e+5 & -1.0104e+5 & 0 & -1.0841e+5 \\
0 & 0 & 0 & 0 & -1.9264e+5 & 0 & -1.0104e+5 & 6.6248e+5 & 0 & -1.6403e+5 \\
0 & 0 & 0 & 0 & 0 & -1.0587e+5 & 0 & 0 & 4.4612e+5 & -1.5134e+5 \\
0 & 0 & 0 & 0 & 0 & 0 & -1.0841e+5 & -1.6403e+5 & -1.5134e+5 & 6.1604e+5
\end{pmatrix}
$$

This modified system $\{\mathbf{M}, \mathbf{K}\}$ accurately assigns the first two eigenvalues to $(9012,\ 12118)$, and keeps the remaining eigenvalues of $\{\mathbf{M}_{0,}\mathbf{K}_0\}$ unchanged. The absolute errors of the remaining eigenvalues between $\{\mathbf{M},\mathbf{K}\}$ and $\{\mathbf{M}_0,\mathbf{K}_0\}$ are listed in Table 2.

Table 2. The absolute errors of the remaining eigenvalues

$\lvert\mu_3-\lambda_3\rvert$	$\lvert\mu_4-\lambda_4\rvert$	$\lvert\mu_5-\lambda_5\rvert$	$\lvert\mu_6-\lambda_6\rvert$	$\lvert\mu_7-\lambda_7\rvert$	$\lvert\mu_8-\lambda_8\rvert$	$\lvert\mu_9-\lambda_9\rvert$	$\lvert\mu_{10}-\lambda_{10}\rvert$
1.1879e-5	1.5775e-6	6.7383e-5	6.2182e-6	2.2771e-6	3.8391e-5	1.0212e-5	1.0869e-5

which indicates an excellent assignment.

Clearly, the PEVAPSM solution of this 10-DOF multiple-connected mass-spring system is not unique either. Additionally, it should be pointed out firstly that according to Eq. (5) (i.e. a real symmetric matrix J_s constructed), the spectral orders of the eigenvalues of the original system to be assigned before assignment must be in the same spectral orders of the modified system after assignment. Secondly, the first method based on Lanczos algorithm is just applicable to the simply connected systems and is computationally effective; while the second method based on the gradient flow algorithm is applicable to both systems, but is computationally more expensive for large systems.

4. Conclusions

Passive structural modifications for partial assignment of natural frequencies of lumped mass-spring systems are successfully made. For two kinds of mass-spring systems, i.e. simply connected in-line mass-spring systems and multiple-connected mass-spring systems, two numerical solution procedures are proposed to construct the required mass-normalised stiffness matrix, which satisfies the partial assignment requirement of natural frequencies and at the same time keeps the structural configuration of the original system, that is, the structure of the mass and stiffness matrices remains unchanged after modifications. The methods only need information of those few eigenpairs to be assigned and the analytical mass and stiffness matrices of the original system. Their solutions are not unique and dependent on the prescribed conditions on the physical parameters of masses and springs of the modified system.

For continuous structures (or distributed systems), quite often lumped mass matrices are used in the finite element discretisation, and the methods put forward in this paper are also applicable.

The methods also allow other design constraints to be considered, for example, maintenance of the total mass. Structural optimisation with partial eigenvalue assignment can be carried out.

It will be a challenge to be able to deal with non-diagonal mass matrices (for example, consistent mass matrices in the FEM). This will be the authors' next research topic.

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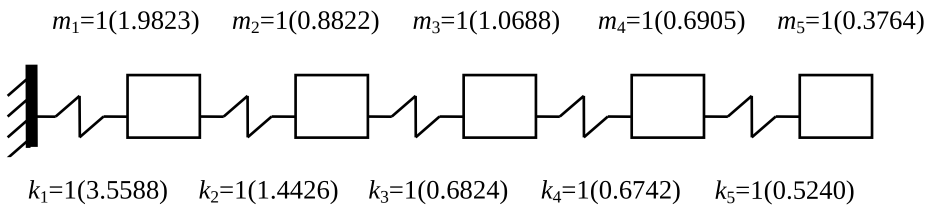

Figure 1. A five-DOF “fixed-free” type of original mass-spring system and modified system

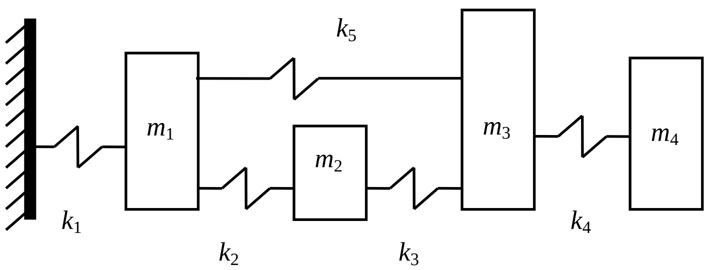

Figure 2. A 4-DOF “fixed-free” type of multiple connected mass-spring system

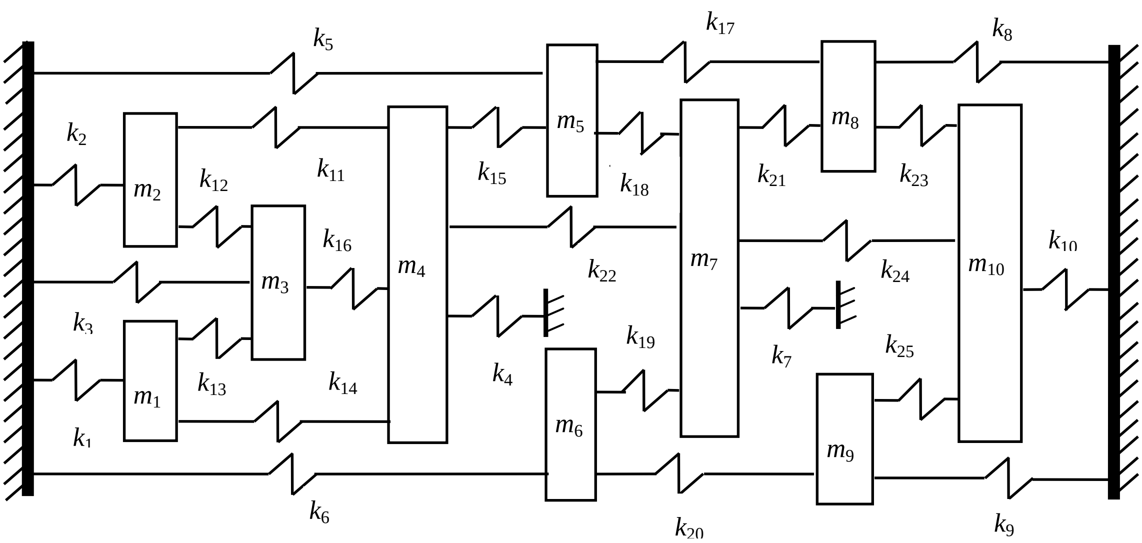

Figure 3. A 10-DOF “fixed-fixed” type of multiple connected mass-spring system