Birationally rigid complete intersections with a singular point of high multiplicity

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We prove the birational rigidity of Fano complete intersections of index 1 with a singular point of high multiplicity, which can be close to the degree of the variety. In particular, the groups of birational and biregular automorphisms of these varieties are equal, and they are non-rational. The proof is based on the techniques of the method of maximal singularities, the generalized $4n^2$-inequality for complete intersection singularities and the technique of hypertangent divisors.

Bibliography: 19 titles.

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To Yu.I.Manin on the occasion of his 80th birthday

Introduction

0.1. Statement of the main result. The aim of the present paper is to prove the birational superrigidity of generic Fano complete intersections of index 1 with a singular point of high multiplicity. This is a generalization of the result of [1], where a similar fact was shown for hypersurfaces.

Let $k \geqslant 2$ and $M \geqslant 2k+1$ be fixed integers, $\mathbb{P} = \mathbb{P}^{M+k}$ the complex projective space. Fix an ordered integral vector

$$\underline{d} = (d_1, \dots, d_k) \in \mathbb{Z}^k_+,$$

where $2 \leqslant d_1 \leqslant \dots \leqslant d_k$, satisfying the equality

$$|\underline{d}| = d_1 + \dots + d_k = M + k,$$

and an integral vector

$$\underline{\xi} = (\xi_1, \dots, \xi_k) \in \mathbb{Z}^k_+,$$

where $1\leqslant \xi_i\leqslant d_i$ for all $i=1,\dots,k$. Set

$$c_*=\sharp\{i\,|\,\xi_i=d_i,\ i=1,\dots,k\}.$$

Assume that the inequalities

$$\sum_{i=1}^{k}[(d_i+1)(d_i+2)-\xi_i(\xi_i+1)]\geqslant 4M+2d_k+2c_*-2k \tag{1}$$

and

$$M\geqslant 3+\sum_{\xi_i\geqslant 2}(\xi_i+1) \tag{2}$$

hold. Let

$$\mathcal{H}=\prod_{i=1}^{k}H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(d_i))$$

be the space of tuples of homogeneous polynomials $\underline{f}=(f_1,\dots,f_k)$ of degrees $d_1,\dots,d_k$, respectively. Fix a point $o\in\mathbb{P}$. The symbol $\mathcal{H}(\underline{\xi})$ stands for the subset of tuples $\underline{f}\in\mathcal{H}$, such that:

- the set of common zeros

  $$V(\underline{f})=\{f_1=\ldots=f_k=0\}$$

  is an irreducible reduced complete intersection of codimension k in $\mathbb{P}$,

- the subvariety $V(\underline{f})$ is non-singular outside the point o,

- $\mathrm{mult}_o f_i=\xi_i$ for $i=1,\dots,k$.

If $\xi_i\geqslant 2$ for at least one $i\in\{1,\dots,k\}$, then the subvariety $V(\underline{f})$ is singular at the point o. By Grothendieck's theorem [2], the variety $V(\underline{f})$ is factorial, so that $\mathrm{Pic}\,V(\underline{f})\cong\mathbb{Z}H$, where H is the class of a hyperplane section. Note that for a general tuple $\underline{f}\in\mathcal{H}(\xi)$ the unique singular point $o\in V(\underline{f})$ is resolved by one blow up with a non-singular exceptional divisor, the discrepancy of which is positive by the assumption (2). Therefore, the singularity $o\in V(\underline{f})$ is terminal and $V(\underline{f})$ is a Fano variety of index 1 and dimension M.

The main result of the present paper is the following claim.

Theorem 0.1. *There exists a non-empty Zariski open subset*

$$\mathcal{U}\subset\mathcal{H}(\underline{\xi}),$$

such that for any tuple $\underline{f}\in\mathcal{U}$ the variety $V(\underline{f})$ is birationally superrigid.

We remind the definition and the main properties of birationally superrigid varieties below in Subsection 0.3.

Remark 0.1. For a general tuple $\underline{f}\in\mathcal{H}(\underline{\xi})$ the point $o\in V(\underline{f})$ has the multiplicity $\mu=\xi_1\ldots\xi_k$. The anticanonical degree of the variety $V(\underline{f})$ (that is to say, its degree in the projective space $\mathbb{P}$) is $d = d_1 \dots d_k$. It is easy to check that the infimum of the ratios μ/d over all tuples $\underline{d}$, $\underline{\xi}$, satisfying the conditions (1) and (2), is equal to 1, that is, the multiplicity of the point $o \in V(\underline{f})$ can be really high.

0.2. Regular complete intersections. The Zariski open subset $\mathcal{H}_{\rm reg}(\underline{\xi})$ is defined by a set of local conditions, which must hold at every point $p \in V(\underline{f})$. First we consider the conditions for the singular point $o \in V = V(\underline{f})$. Set $l = \sharp\{i \,|\, \xi_i \geqslant 2\}$ and let

$$\underline{\mu} = (\mu_1, \dots, \mu_l)$$

be the ordered (non-decreasing) tuple of multiplicities $\xi_i \geqslant 2$, $i \in \{1, \dots, k\}$, that is, $\mu_1 \leqslant \mu_2 \leqslant \dots \leqslant \mu_l$ and $\mu_\alpha = \xi_{i_\alpha}$, where $\{i_1, \dots, i_l\} = \{i \,|\, \xi_i \geqslant 2\}$. The integral vector $\underline{\mu}$ will be called the *type* of the singularity $o \in V$. Set

$$\mu = \mu_1 \dots \mu_l = \xi_1 \dots \xi_k \quad \text{and} \quad |\underline{\mu}| = \mu_1 + \dots + \mu_l.$$

The following condition is a natural condition of general position for the singular point $o \in V$.

(R0.1) Let $\varphi_{\mathbb{P}}\colon \mathbb{P}^+ \to \mathbb{P}$ be the blow up of the point o, $Q_{\mathbb{P}} = \varphi_{\mathbb{P}}^{-1}(o) \cong \mathbb{P}^{M+k-1}$ the exceptional divisor, $V^+ \subset \mathbb{P}^+$ the strict transform of the complete intersection V, so that $\varphi\colon V^+ \to V$ is the blow up of the point o on V, $Q = \varphi^{-1}(o) = V^+ \cap Q_{\mathbb{P}}$ is the exceptional divisor. The subvariety

$$Q \subset Q_{\mathbb{P}} \cong \mathbb{P}^{M+k-1}$$

is a non-singular complete intersection of codimension l in its linear span

$$\langle Q \rangle \cong \mathbb{P}^{M+l-1}.$$

Let us discuss the condition (R0.1) in more detail. Let $(z_*) = (z_1, \dots, z_{M+k})$ be a system of affine coordinates with the origin at the point o and

$$f_i = q_{i,\xi_i} + \dots + q_{i,d_i}$$

the decomposition of the polynomial f_i (in the non-homogeneous form) into components, homogeneous in z_*. Then $(z_1 : \dots : z_{M+k})$ form a set of homogeneous coordinates on $Q_{\mathbb{P}}$, the linear span $\langle Q \rangle$ is given by the system of equations

$$q_{i,1} = 0$$

for $i \in \{1, \dots, k\}$ such that $\xi_i = 1$, and the complete intersection $Q \subset \langle Q \rangle$ is given by the system of equations

$$q_{i,\xi_i}|_{\langle Q \rangle} = 0$$

for $i \in \{1, \dots, k\}$ such that $\xi_i \geqslant 2$. We will need one more condition of general position for the singular point $o \in V$. We use the coordinate notations introduced above.

(R0.2) The set of homogeneous polynomials $q_{i,j}$, $i=1,\ldots,k$, $j=\xi_i,\ldots,d_i$ forms a regular sequence in $\mathcal{O}_{o,\mathbb{P}}$, that is, the system of homogeneous equations

$$\{q_{i,j}=0\,|\,i=1,\ldots,k,j=\xi_i,\ldots,d_i\}$$

defines a closed set of codimension $\sum\limits_{i=1}^{k}(d_i-\xi_i+1)$ in $\mathbb{C}^{M+k}_{z_*}$.

Now let us consider the conditions for a non-singular point $p\in V$, $p\neq o$. These conditions are identical to the regularity conditions introduced in [3], see also [4, Chapter 3]. Let $u_1,\ldots,u_{M+k}$ be a system of affine coordinates with the origin at the point p and

$$f_i=\Phi_{i,1}+\Phi_{i,2}+\ldots+\Phi_{i,d_i}$$

the decomposition of the (non-homogeneous) equation f_i into components, homogeneous in u_*. (Here we somewhat abuse the notations, using the same symbol f_i both for the homogeneous equation and for its affine presentations in the coordinates (z_*) and (u_*), however this can not lead to any misunderstanding.) The system of linear equations

$$\Phi_{1,1}=\ldots=\Phi_{k,1}=0$$

defines the tangent space $T_pV\cong\mathbb{C}^M$.

(R1) The set of all homogeneous polynomials $\Phi_{i,j}$ but the last one Φ_{k,d_k} forms a regular sequence in $\mathcal{O}_{p,\mathbb{P}}$, that is, the system of homogeneous equations

$$\{\Phi_{i,j}|_{T_pV}=0\,|\,1\leqslant i\leqslant k,2\leqslant j\leqslant d_i,(i,j)\neq(k,d_k)\}$$

defines a finite set of lines in T_pV.

Definition 0.1. The tuple $\underline{f}\in\mathcal{H}(\underline{\xi})$ is *regular*, if the complete intersection $V(\underline{f})$ satisfies the conditions (R0.1,2) at the singular point o and the condition (R1) at every point $p\neq o$.

We denote the set of regular tuples by the symbol $\mathcal{H}_{\rm reg}(\underline{\xi})$. The following fact is true.

Theorem 0.2. *$\mathcal{H}_{\rm reg}(\underline{\xi})\subset\mathcal{H}(\underline{\xi})$ is a non-empty Zariski open subset.*

Note that the openness is obvious, so that we only need to show that it is non-empty.

0.3. Birational rigidity and superrigidity. The modern method of maximal singularities goes back to the classical paper [5]. Recall the definitions of the birational rigidity and superrigidity for the class of varieties, considered in this paper (for the details, see [4, Chapter 2]). Let X be a projective rationally connected variety with $\mathbb{Q}$-factorial terminal singularities, D an effective divisor on X. The *threshold of canonical adjunction* $c(D)=c(D,X)$ is equal to

$$\sup\{t\in\mathbb{Q}_+\,|\,D+tK_X\text{ is pseudoeffective}\}.$$

If Σ is a mobile linear system on X and $D \in \Sigma$, then we set $c(\Sigma) = c(D)$. The *virtual threshold of canonical adjunction* $c_{\rm virt}(\Sigma, X)$ is equal to

$$\inf_{\widetilde{X} \to X} c(\widetilde{\Sigma}, \widetilde{X}),$$

where the infimum is taken over all birational morphisms $\widetilde{X} \to X$ with a non-singular projective $\widetilde{X}$ and $\widetilde{\Sigma}$ is the strict transform of the mobile system Σ on $\widetilde{X}$. The variety X is *birationally superrigid*, if $c(\Sigma) = c_{\rm virt}(\Sigma, X)$ for every mobile linear system Σ, and *birationally rigid*, if for every mobile linear system Σ there exists a birational self-map $\chi \in \mathrm{Bir}(X)$ such that the equality

$$c(\chi_*^{-1}\Sigma) = c_{\rm virt}(\Sigma, X)$$

holds. Now Theorem 0.1 follows from Theorem 0.2 and the following claim.

Theorem 0.3. *For every tuple $\underline{f} \in \mathcal{H}_{\rm reg}(\underline{\xi})$ the Fano variety $V(\underline{f})$ is birationally superrigid.*

Corollary 0.1. *Let $V = V(\underline{f})$ for some tuple $\underline{f} \in \mathcal{H}_{\rm reg}(\underline{\xi})$.*

(i) *Assume that there is a birational map $\chi\colon V \dashrightarrow V'$ onto the total space of the Mori fibre space $V' \to S'$. Then S' is a point and the map χ is a biregular isomorphism.*

(ii) *There is no rational dominant map $\beta\colon V \dashrightarrow S$, where $\dim S \geqslant 1$ and the general fibre is rationally connected. In particular, the variety V is non-rational.*

(iii) *The groups of biregular and birational automorphisms of the variety V coincide:* $\mathrm{Bir}\, V = \mathrm{Aut}\, V$.

Proof of the corollary. These are very well known implications of the birational superrigidity (see, for instance, [4, Chapter 2]), taking into account that V is a primitive Fano variety.

0.4. The structure of the proof. Theorems 0.2 and 0.3 are independent of each other and are shown by different methods. In Section 1 we prove Theorem 0.3. The proof is based on the generalized $4n^2$-inequality that was recently shown in [6]. Assuming that the variety V is not birationally superrigid, we obtain a mobile linear system $\Sigma \subset |nH|$ on V (H is the class of a hyperplane section of the variety $V \subset \mathbb{P}$) with a maximal singularity E (which is an exceptional divisor over V). The centre $B \subset V$ of the maximal singularity E is an irreducible subvariety. If $B \neq o$ is not a singular point, then we obtain a contradiction by word for word the same arguments as in the non-singular case [3]. If $B = o$, then by [6] we obtain the inequality

$$\mathrm{mult}_o Z > 4n^2\mu$$

for the self-intersection $Z = (D_1 \circ D_2)$ of the linear system Σ ($D_1, D_2 \in \Sigma$ are general divisors). Now the contradiction is obtained by means of the technique of hypertangent divisors based on the condition (R0.2). This contradiction completes the proof of Theorem 0.3.

In Section 2 we prove Theorem 0.2. The fact that a general smooth complete intersection of the type $\underline{d}$ in $\mathbb{P}$ satisfies the condition (R1) at every point was shown in [3]. However, in our case the variety $V(\underline{f})$ with $\underline{f} \in \mathcal{H}(\xi)$ has a fixed point of high multiplicity, and for that reason we have to prove the condition (R1) for every non-singular point $p \in V(\underline{f})$, $p \neq o$, again. That the conditions (R0.1) and (R0.2) hold for a general tuple $\underline{f} \in \mathcal{H}(\xi)$, is obvious.

The proof of the condition (R1) for a non-singular point $p \in V(\underline{f})$ is elementary but non-trivial: it requires manipulations with coordinate presentations of the equations f_i with respect to the system z_* and the system u_*. A violation of the regularity condition for the tuple of polynomials $\Phi_{i,j}(u_*)$ is translated into conditions for the tuple of polynomials $q_{i,j}(z_*)$ with $\xi_i \leqslant j \leqslant d_i$. In this way we prove that a general tuple $\underline{f} \in \mathcal{H}(\xi)$ satisfies all regularity conditions at all points of the variety $V(\underline{f})$.

0.5. Historical remarks and acknowledgements. The present paper generalizes [1], where Fano hypersurfaces of index 1 with a singular point of high multiplicity were shown to be birationally rigid. Fifteen years ago the local technique of estimating the multiplicity of the self-intersection of a mobile linear system was weaker, and for that reason the proof given in [1] for a point of multiplicity 3 and 4 was very hard. The generalized $4n^2$-inequality, shown in [6], simplifies the argument essentially.

There are also other papers studying singular Fano varieties from the viewpoint of their birational rigidity. After [7] singular three-dimensional quartics became a popular class to investigate, see [8, 9, 10]. Another popular class of singular Fano varieties is formed by weighted three-fold hypersurfaces, see [11, 12, 13]. Certain families of singular higher-dimensional Fano varieties were studied in [14, 15, 16, 17]. The last of them is based on the constructions of [16], some of which are hard to follow. The list given above is far from being complete.

There are also recent papers [18, 19], where an effective estimate is given for the codimension of the complement to the set of birationally superrigid varieties in the given family (of Fano hypersurfaces with quadratic singularities, the rank of which is bounded from below, and of Fano complete intersections of codimension 2 with quadratic and bi-quadratic singularities, the rank of which is bounded from below, respectively).

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1 Proof of birational superrigidity

In this section we prove Theorem 0.3. First (Subsection 1.1) we remind the definition of a maximal singularity and explain, why only maximal singularities with the centre at the singular point o need to be considered; after that (Subsection 1.2) we remind the generalized $4n^2$-inequality that was shown in [6]. Finally, in Subsection 1.3 we exclude the maximal singularity.

1.1. The maximal singularity. Fix a tuple of polynomials $\underline{f} \in \mathcal{H}_{\rm reg}(\underline{\xi})$. Set $V = V(\underline{f})$ to be the corresponding complete intersection. We have:

$$\operatorname{Sing} V = \{o\}, \quad \operatorname{Pic} V = \mathbb{Z}H, \quad K_V = -H,$$

where H is the class of a hyperplane section of the variety $V \subset \mathbb{P}$. Assume that the variety V is not birationally superrigid. This immediately implies (see [4, Chapter 2]) that on V there is a mobile linear system $\Sigma \subset |nH|$, $n \geqslant 1$, with a maximal singularity: for some non-singular projective variety $\widetilde{V}$ and a birational morphism $\varphi\colon \widetilde{V} \to V$ there exists a φ-exceptional prime divisor $E \subset \widetilde{V}$, satisfying the Noether-Fano inequality

$$\operatorname{ord}_E \Sigma = \operatorname{ord}_E \varphi^* \Sigma > na(E),$$

where $a(E) = a(E, V) \geqslant 1$ is the discrepancy of the divisor E with respect to the model V. Let $B = \varphi(E) \subset V$ be the centre of the divisor E on V. This is an irreducible subvariety, satisfying the inequality $\operatorname{mult}_B \Sigma > n$. By the Lefschetz theorem for the numerical Chow group of algebraic cycles of codimension 2 on V we have the equality $A^2 V = \mathbb{Z}H^2$. Now we are able to exclude the simplest case $\operatorname{codim}(B \subset V) = 2$.

Indeed, if $\operatorname{codim}(B \subset V) = 2$, then $B \sim mH^2$ for some $m \geqslant 1$. Consider the self-intersection $Z = (D_1 \circ D_2)$ of the linear system Σ. Obviously, $Z \sim n^2 H^2$. On the other hand, $Z = \beta B + Z_1$, where $\beta > n^2$ and Z_1 is an effective cycle of codimension 2 that does not contain B as a component. Taking the classes of the cycles in $A^2 V$, we obtain the inequality $n^2 \geqslant \beta m > mn^2$. The contradiction excludes the case $\operatorname{codim}(B \subset V) = 2$.

If $\operatorname{codim}(B \subset V) \geqslant 3$ and $B \neq o$, then the inequality $\operatorname{mult}_B Z > 4n^2$ holds (the classical $4n^2$-inequality that goes back to [5], see [4, Chapter 2]). Repeating the arguments of [3, Section 2] word for word, we get a contradiction, excluding the case under consideration. We are able to repeat the arguments word for word, because the proof in [3, Section 2], is based on the regularity condition, which is identical to the condition (R1).

So we are left with the only option: $B = o$. Excluding this option, we complete the proof of Theorem 0.3.

1.2. The generalized $4n^2$-inequality. Recall that $\underline{\mu} = (\mu_1, \dots, \mu_l)$ with $2 \leqslant \mu_1 \leqslant \ldots \leqslant \mu_l$ is the type of the complete intersection singularity $o \in V$, so that by the condition (R0.1)

$$\mu = \mu_1 \cdots \mu_l = \mathrm{mult}_o V.$$

Again, let $Z = (D_1 \circ D_2)$ be the self-intersection of the linear system Σ. The following fact is true.

Theorem 1.1. *The following inequality holds:*

$$\mathrm{mult}_o Z > 4n^2\mu.$$

Proof: this is a particular case of the theorem shown in [6] under essentially weaker assumptions about the generality of the singularity $o \in V$. Q.E.D.

1.3. Hypertangent divisors. Now let us use the technique of hypertangent linear systems. We use the notations of Subsection 0.2 and work in the affine chart $\mathbb{C}^{M+k}$ of the space $\mathbb{P}$ with the coordinates $z_1, \dots, z_{M+k}$. Let $j \geqslant 1$ be an integer.

Definition 1.1. The linear system

$$\Lambda(j) = \left\{ \left(\sum_{i=1}^{k} \sum_{\alpha=\xi_i}^{d_i-1} f_{i,\alpha} s_{i,j-\alpha} \right) \Bigg|_V = 0 \right\},$$

where $s_{i,j-\alpha}$ independently run through the set of homogeneous polynomials of degree $j - \alpha$ in the variables z_* (if $j - \alpha < 0$, then $s_{j-\alpha} = 0$), and

$$f_{i,\alpha} = q_{i,\xi_i} + \ldots + q_{i,\alpha}$$

is the truncated equation f_i, is called the *j-th hypertangent system at the point o.*

For $j \geqslant 1$ set

$$c(j) = \sharp\{(i,\alpha) \,|\, i = 1, \dots, k, \xi_i \leqslant \alpha \leqslant \min\{j, d_i - 1\}\,\}.$$

Set also $a = \min\{j \geqslant 1 \,|\, c(j) \geqslant 1\}$. Obviously, the system $\Lambda(j)$ is non-empty only for $j \geqslant a$. Moreover, the equality

$$\mathrm{codim}_o(\mathrm{Bs}\,\Lambda(j) \subset V) = c(j)$$

holds. Now set $m(j) = c(j) - c(j-1)$ and choose in every non-empty hypertangent system $\Lambda(j)$ precisely $m(j)$ general divisors

$$D_{j,1}, \dots, D_{j,m(j)}$$

(if $m(j) = 0$, then we do not choose any divisors in the system $\Lambda(j)$), for $j = a, \dots, d_k - 1$. It is easy to see that we obtained a tuple consisting of

$$m = \sum_{i=1}^{k} (d_i - \xi_i)$$

effective divisors on V. Denoting by the symbol $|D_{j,\alpha}|$ the support of the divisor $D_{j,\alpha}$, we conclude: by the condition (R0.2)

$$\mathrm{codim}_o\left(\left(\bigcap_{j=a}^{d_k-1}\bigcap_{\alpha=1}^{m(j)}|D_{j,\alpha}|\right)\subset V\right)=m$$

(the symbol codim_o stands for the codimension in a neighborhood of the point o). Let us place the divisors $D_{j,\alpha}$ in the *standard order*: D_{j_1,α_1} precedes D_{j_2,α_2}, if $j_1<j_2$ or $j_1=j_2$, but $\alpha_1<\alpha_2$. We obtain a sequence

$$R_1,\dots,R_m$$

of effective divisors on V. Now let us consider the effective cycle of the self-intersection $Z=(D_1\circ D_2)$ of the mobile system Σ. We denote its support by the symbol $|Z|$. By the condition (R0.2) for every $i\in\{3,\dots,m\}$ we have the equality

$$\mathrm{codim}_o\left(\bigcap_{j=3}^{i}|R_j|\cap|Z|\right)=i.$$

Let $Y\subset|Z|$ be an irreducible component of the support $|Z|$, which has the maximal value of the ratio $\mathrm{mult}_o/\deg$ of the multiplicity at the point o to the degree (in the space $\mathbb{P}$). By Theorem 1.1

$$\frac{\mathrm{mult}_o}{\deg}Y>\frac{4n^2}{n^2d}\mu=\frac{4}{d}\mu,$$

where $d=d_1\cdots d_k=\deg V$ and $\mu=\mu_1\cdots\mu_l=\xi_1\cdots\xi_k$. Now let us construct in the usual way (see [4, Chapter 3]) a sequence of irreducible subvarieties $Y_2=Y,Y_3,\dots,Y_m$, satisfying the following properties:

- $\mathrm{codim}(Y_i\subset V)=i$,
- Y_{i+1} is an irreducible component of the algebraic cycle of the scheme-theoretic intersection $(Y_i\circ R_{i+1})$ with the maximal value of $\mathrm{mult}_o/\deg$.

Such construction is possible, because by the condition (R0.2) and the genericity of the divisors $D_{j,\alpha}$ in the linear system $\Lambda(j)$ we have

$$Y_i\not\subset|R_{i+1}|.$$

The irreducible subvariety Y_m is of positive dimension and contains the point o. Let us see how the value of the ratio $\mathrm{mult}_o/\deg$ changes when we make the step from Y_i to Y_{i+1}.

Let $R_i=D_{b,\alpha}$ for some $b\in\{a,\dots,d_k-1\}$ and $\alpha\in\{1,\dots,m(b)\}$, in particular, $R_i\in\Lambda(b)$. The number

$$\beta_i=\beta(R_i)=\frac{b+1}{b}$$

is called the *slope* of the divisor R_i.

Proposition 1.1. *For $i=3,\dots,m$ the following inequality holds:*

$$\frac{\mathrm{mult}_o}{\deg}Y_i \geqslant \beta_i\,\frac{\mathrm{mult}_o}{\deg}Y_{i-1}.$$

Proof. By construction, $D_{b,\alpha}\sim bH$. For the strict transform $D^+_{b,\alpha}$ on V^+ we have $D^+_{b,\alpha}\sim bH-(b+1)Q$, since

$$f_{j,b}|_V=(q_{j,\xi_j}+\ldots+q_{j,b})|_V=(-q_{j,b+1}+\ldots)|_V$$

for every $j\in\{1,\dots,k\}$ such that $\xi_j\leqslant b\leqslant d_j-1$. Q.E.D. for the proposition.

The maximal possible value of the slope β_i is 2 (for $b=1$). The hypertangent divisors are ordered in such way that their slopes do not increase: $\beta_1\geqslant\beta_2\geqslant\ldots\geqslant\beta_m$. Therefore,

$$\frac{\mathrm{mult}_o}{\deg}Y_m>4\frac{\xi_1\ldots\xi_k}{d_1\ldots d_k}\prod_{i=3}^{m}\beta_i\geqslant\frac{4}{\beta_1\beta_2}\frac{\xi_1\ldots\xi_k}{d_1\ldots d_k}\prod_{i=1}^{m}\beta_i\geqslant\frac{\xi_1\ldots\xi_k}{d_1\ldots d_k}\prod_{i=1}^{k}\prod_{j=\xi_i}^{d_i-1}\frac{j+1}{j}=1.$$

Thus $\mathrm{mult}_o Y_m>\deg Y_m$, which is, of course, impossible. We obtained a contradiction which completes the proof of Theorem 0.3.

2 Regular complete intersections

In this section we prove Theorem 0.2. In Subsection 2.1 we consider a convenient system of coordinates for the points o and $p\neq o$ and give the formulas that relate the local presentations of the polynomials f_i at those points. In Subsection 2.2 we explain what impact the position of the line connecting the points o and p has for the regularity conditions. In Subsection 2.3 we reduce the claim of Theorem 0.2 to a certain fact about tuples of polynomials $q_{i,j}$ (the homogeneous components of the equations f_i). In Subsection 2.4 we consider the special case of violation of the regularity conditions when the polynomials $\Phi_{i,j}$ vanish on a line. Finally, in Subsection 2.5 we consider the general case of violation of the regularity conditions and complete the proof of Theorem 0.2.

2.1. Preliminary constructions. We fix the system of affine coordinates $(z_*)=(z_1,\dots,z_{M+k})$ with the origin at the point o. The symbol $\mathcal{P}_{l,M+k}$ stands for the linear space of homogeneous polynomials of degree l in $M+k$ variables (z_*). Set

$$\mathcal{P}_{[a,b],M+k}=\prod_{l=a}^{b}\mathcal{P}_{l,M+k}$$

to be the space of polynomials of the form $q_a+\ldots+q_b$, where $q_l\in\mathcal{P}_{l,M+k}$. Finally, let

$$\mathcal{P}=\prod_{i=1}^{k}\mathcal{P}_{[\xi_i,d_i],M+k}$$

be the space of tuples $\underline{f} = (f_1, \ldots, f_k)$. Obviously, for a general tuple $\underline{f} \in \mathcal{P}$ the conditions (R0.1) and (R0.2) are satisfied.

Let $p \in \mathbb{P}$, $p \neq o$, be an arbitrary point. We will assume that p lies in the affine chart with coordinates (z_*) and, moreover, has in that chart the coordinates $(1, 0, \ldots, 0)$. Let

$$u_1 = z_1 - 1, u_2 = z_2, \ldots, u_{M+k} = z_{M+k}$$

be a system of coordinates with the origin at the point p. It can be written as $u_* = (u_1, \ldots, u_{M+k}) = (u_1, z_2, \ldots, z_{M+k})$. Recall that the equation f_i can be naturally written in the form

$$f_i = q_{i,\xi_i} + \ldots + q_{i,d_i}.$$

Set $q_{i,j} = q_{i,j,j} + z_1 q_{i,j,j-1} + \ldots + z_1^j q_{i,j,0}$, where $q_{i,j,l}$ is a homogeneous polynomial of degree $l \leqslant j$ in the variables $z_2, \ldots, z_{M+k}$. In the new coordinates (u_*) the polynomial f_i takes the form

$$f_i = \sum_{e=0}^{d_i} \left[\sum_{\alpha=0}^{e} u_1^{e-\alpha} \left(\sum_{j=\max(\xi_i, e)}^{d_i} \binom{j-\alpha}{e-\alpha} q_{i,j,\alpha} \right) \right]. \tag{3}$$

The expression in the square brackets is the homogeneous polynomial $\Phi_{i,e}$ in the notations of Subsection 0.2. Therefore, the shift of the first coordinate $z_1 = 1 + u_1$, with the other coordinates being the same, defines a linear map

$$\tau \colon \mathcal{P} \to \prod_{i=1}^{k} \mathcal{P}_{[0,d_i],M+k}(u_*).$$

The following claim is true.

Theorem 2.1. *The set of tuples $\underline{f} \in \mathcal{P}$, such that*

$$\tau(\underline{f}) \in \prod_{i=1}^{k} \mathcal{P}_{[1,d_i],M+k}(u_*)$$

(that is, $p \in V(\underline{f})$) and $\tau(\underline{f})$ does not satisfy the condition (R1), is of codimension at least $M + k + 1$ in the space $\mathcal{P}$.

Proof of Theorem 0.2. Theorem 2.1 implies that the set of tuples $\underline{f} \in \mathcal{H}(\underline{\xi})$ (here we again abuse the notations, using the same symbol f_i both for a homogeneous polynomial of degree d_i on $\mathbb{P}$ and for its non-homogeneous presentation in the coordinates (z_*)) such that the variety $V(f)$ is not regular at at least one point $p \in V(\underline{f})$, $p \neq o$, has a positive codimension in $\mathcal{H}(\underline{\xi})$. Therefore, the set $\mathcal{H}_{\rm reg}(\underline{\xi})$ is non-empty. It is obviously open. Q.E.D. for Theorem 0.2.

The rest of this section is a proof of Theorem 2.1.

Note that $\Phi_{i,0} = q_{i,\xi_i,0} + \ldots + q_{i,d_i,0}$ and the equalities $\Phi_{i,0} = 0$, $i = 1, \ldots, k$, give k independent linear conditions for the polynomials f_i (expressing the fact that $p \in V(\underline{f})$). Furthermore, for the linear terms we get

$$\Phi_{i,1} = (\xi_i q_{i,\xi_i,0} + \ldots + d_i q_{i,d_i,0}) u_1 + q_{i,\xi_i,1} + \ldots + q_{i,d_i,1}$$

and the linear dependence of these linear forms gives in addition $M+1$ independent conditions for the coefficients of the polynomials f_i. It is for this reason that the open set $\mathcal{H}(\underline{\xi})$ is non-empty.

For k fixed linearly independent linear forms $L_1,\dots,L_k$ in the variables u_* we denote by the symbol

$$\mathcal{P}[p;L_1,\dots,L_k]\subset\mathcal{P}$$

the affine subspace of tuples $\underline{f}$ such that $\Phi_{1,0}=\ldots=\Phi_{k,0}=0$ and $\Phi_{1,1}=L_1,\dots,$ $\Phi_{k,1}=L_k$.

2.2. The line connecting the points o **and** p**.** Let us denote this line by the symbol $[o,p]$. We say that we are in the *non-special case*, if $[o,p]\not\subset T_pV$, and in the *special case*, if $[o,p]\subset T_pV$. Consider first the non-special case.

In the coordinates u_* on the space $T_p\mathbb{P}\cong\mathbb{C}^{M+k}$ the line $[o,p]$ is given by the equations

$$u_2=\ldots=u_{M+k}=0.$$

If $[o,p]\not\subset T_pV$, then

$$\dim\langle u_2|_{T_pV},\dots,u_{M+k}|_{T_pV}\rangle=M,$$

so that for every $j\geqslant 1$ the space

$$\{q_j|_{T_pV}\mid q_j\in\mathcal{P}_{j,M+k-1}\}$$

(where $\mathcal{P}_{j,M+k-1}$ is the linear space of homogeneous polynomials of degree j in $u_2,\dots,u_{M+k}$) is the whole space of homogeneous polynomials of degree j on T_pV. It follows from (3) that $\Phi_{i,e}=q_{i,d_i,e}+(*)$, where $(*)$ is a linear combination of terms $u_1^{e-\alpha}q_{i,j,\alpha}$ with either $j<d_i$ or $j=d_i$ and $\alpha<e$. As the polynomials $q_{i,j,\alpha}$ are arbitrary homogeneous polynomials of degree α in $u_2,\dots,u_{M+k}$, we conclude, that when $\underline{f}$ runs through the space $\mathcal{P}[p;L_1,\dots,L_k]$, the homogeneous polynomials $\Phi_{i,e}|_{T_pV}$, $e\geqslant 2$, run through the whole spaces of homogeneous polynomials of degree e on T_pV, independently of each other. For that reason, by [3, Section 3] the set of tuples $\underline{f}\in\mathcal{P}[p;L_*]$, for which the condition (R1) is violated at the point p, has in $\mathcal{P}[p,L_*]$ the codimension at least $M+1$. Since for different tuples $(L_1,\dots,L_k)\neq(L'_1,\dots,L'_k)$ of linearly independent linear forms the affine subspaces $\mathcal{P}[p;L_*]$ and $\mathcal{P}[p;L'_*]$ are disjoint, the reference to [3, Section 3] completes the proof of Theorem 2.1 in the non-special case.

Starting from this moment, we assume that we are in the special case, that is, for all $i=1,\dots,k$ we have $L_i|_{[o,p]}\equiv 0$. Explicitly, this means that for all $i=1,\dots,k$ the equality

$$\xi_iq_{i,\xi_i,0}+\ldots+d_iq_{i,d_i,0}=0$$

holds. If $\xi_i\leqslant d_i-1$, that is, the hypersurface $\{f_i=0\}$ is not a cone with the vertex at the point o, we obtain a new independent condition for $\underline{f}$. Therefore, the set

$$\bigsqcup_{\text{special}}\mathcal{P}[p;L_1,\dots,L_k],$$

where the union is taken over all tuples (L_*), consisting of k linearly independent forms, vanishing on the line $[o,p]$, has in $\mathcal{P}$ the codimension

$$k+\sharp\{i\,|\,\xi_i\leqslant d_i-1,\ i=1,\dots,k\}.$$

Set $\mathbb{T}=\mathbb{P}(T_pV)\cong\mathbb{P}^{M-1}$. The point, corresponding to the line $[o,p]$, we denote by the symbol ω. Theorem 2.1 is implied by the following claim.

Proposition 2.1. *In the special case the set of tuples* $\underline{f}\in\mathcal{P}[p;L_*]$ *such that the system of equations*

$$\Phi_{i,j}|_{\mathbb{T}}=0,\quad 1\leqslant i\leqslant k,\ 2\leqslant j\leqslant d_i,\ (i,j)\neq(k,d_k)\tag{4}$$

has in $\mathbb{T}$ *a positive-dimensional set of solutions, is of codimension at least*

$$M+1-\sharp\{i\,|\,\xi_i\leqslant d_i-1,\ i=1,\dots,k\}$$

in the space $\mathcal{P}[p;L_*]$.

2.3. Plan of the proof of Proposition 2.1. The linear forms $\Phi_{i,1}=L_i$ and the projective space $\mathbb{T}$ are fixed. Placing the polynomials $\Phi_{i,j}|_{\mathbb{T}}$ in the standard order (which means that $(i_1,j_1)<(i_2,j_2)$, if $i_1<i_2$ or $i_1=i_2$, but $j_1<j_2$), we get $M-1$ polynomials on $\mathbb{P}^{M-1}$:

$$p_1,p_2,\dots,p_{M-1},$$

$\deg p_{i+1}\geqslant\deg p_i$. In the special case it is not true that p_i run through the corresponding spaces of polynomials independently of each other, and for that reason the estimates that were obtained in [3, Section 3] can not be applied directly. Let us consider the following subsets of the affine space $\mathcal{A}=\mathcal{P}[p;L_*]$.

Let $\mathcal{B}_{\rm line}\subset\mathcal{A}$ be the set of tuples $\underline{f}\in\mathcal{A}$ such that $p_i|_R\equiv 0$ for some line $R\subset\mathbb{T}$ for all $i=1,\dots,M-1$. Furthermore, set $\mathcal{B}_i\subset\mathcal{A}\backslash\mathcal{B}_{\rm line}$, where $i=1,\dots,M-1$, to be the set of tuples $\underline{f}\in\mathcal{A}$ such that

$$\mathrm{codim}(\{p_1=\ldots=p_{i-1}=0\subset\mathbb{T}\})=i-1,$$

but for some irreducible component B of the set $\{p_1=\ldots=p_{i-1}=0\}$ we have $p_i|_B\equiv 0$. (For $i=1$ this condition means that $p_1\equiv 0$.) Recall that

$$c_*=\sharp\{i\mid\xi_i=d_i,\ i=1,\dots,k\}.$$

Proposition 2.1 is implied by the following two facts.

Proposition 2.2. *The following inequality holds:*

$$\mathrm{codim}(\mathcal{B}_{\rm line}\subset\mathcal{A})\geqslant M+1+c_*-k.$$

Proposition 2.3. *For all* $i=1,\dots,M-1$ *the following inequality holds:*

$$\mathrm{codim}(\mathcal{B}_i\subset\mathcal{A})\geqslant M+1.$$

Remark 2.1. Let $(v_0 : v_1 : \ldots : v_{M-1})$ be a system of homogeneous coordinates on $\mathbb{T}$, in which the point ω is the point $(1 : 0 : \ldots : 0)$. The formula (3) implies that for fixed polynomials $p_1, \ldots, p_{i-1}$ the set, which the polynomial p_i runs through, is a disjoint union of affine subspaces of the form

$$p_i^{\circ} + \mathcal{P}_{\deg p_i, M-1}(v_1, \ldots, v_{M-1}),$$

where p_i° is some polynomial. Applying the method of linear projections (see [4, Chapter 3, Section 1]), we obtain the inequality

$$\operatorname{codim}(\mathcal{B}_i \subset \mathcal{A}) \geqslant \binom{M - i - 1 + \deg p_i}{\deg p_i}.$$

For high values of i (those close to $M-1$) this estimate can be not strong enough. However, for $i = 1, \ldots, k$ it gives more than we need:

$$\operatorname{codim}(\mathcal{B}_i \subset \mathcal{A}) \geqslant \binom{M - k + 1}{2}.$$

Therefore it is sufficient to prove Proposition 2.3 for $i \geqslant k+1$, so that $\deg p_i \geqslant 3$.

The proof of Proposition 2.2 is given in Subsection 2.4, of Proposition 2.3 in Subsection 2.5.

2.4. The case of a line. First of all, let us break the set $\mathcal{B}_{\text{line}}$ into two (overlapping) subsets, corresponding to the case when the line R contains the point ω (the subset $\mathcal{B}^+_{\text{line}}$) and when it does not (the subset $\mathcal{B}^-_{\text{line}}$). Let us estimate the codimensions of these sets in the space $\mathcal{A}$ separately.

For $\underline{f} \in \mathcal{B}^-_{\text{line}}$ the conditions $p_i|_R \equiv 0$, $i = 1, \ldots, M-1$, similarly to the non-special case, give $\sum\limits_{i=1}^{M-1} (\deg p_i + 1)$ independent conditions for $\underline{f}$. Since the line R varies in a $2(M-2)$-dimensional family, we obtain the estimate

$$\operatorname{codim}(\mathcal{B}^-_{\text{line}} \subset \mathcal{A}) \geqslant \sum_{i=1}^{M-1} \deg p_i - M + 3.$$

Lemma 2.1. *The following inequality holds:*

$$\sum_{i=1}^{M-1} \deg p_i \geqslant 2M - 2.$$

Proof. It is easy to check that

$$\sum_{i=1}^{M-1} \deg p_i = \sum_{i=1}^{k-1} \frac{d_i(d_i+1)}{2} + \frac{(d_k - 1)d_k}{2} - k. \tag{5}$$

Furthermore, it is easy to see that if $d_{i-1} \leqslant d_i - 2$, then, replacing d_{i-1} by $d_{i-1}+1$, and d_i by $d_i - 1$, we will not increase the value of the expression (5). Therefore, the minimum of that expression for the fixed k and M is attained at

$$d_1 = \ldots = d_{k-l} = a+1, \ d_{k-l+1} = \ldots = d_k = a+2,$$

where $M = ka + l$, $l \in \{0, 1, \ldots, k-1\}$. Now elementary computations complete the proof of the lemma. Q.E.D.

Lemma 2.1 implies the estimate

$$\mathrm{codim}(\mathcal{B}^-_{\mathrm{line}} \subset \mathcal{A}) \geqslant M + 1,$$

which is stronger than the inequality of Proposition 2.2. For that reason, Proposition 2.2 is implied by the following claim.

Proposition 2.4. *The following inequality is true:*

$$\mathrm{codim}(\mathcal{B}^+_{\mathrm{line}} \subset \mathcal{A}) \geqslant M + 1 + c_* - k.$$

Proof. Let $R \ni \omega$ be a line. In the notations of Remark 2.1 let

$$\lambda = (0 : a_1 : \ldots : a_{M-1}) = R \cap \{v_0 = 0\}.$$

The conditions $p_i|_R \equiv 0$, $i = 1, \ldots, M-1$, give a smaller codimension than in the case $R \not\ni \omega$, considered above. As a compensation, the lines $R \ni \omega$ vary in a $(M-2)$-dimensional family. Let us fix the line R and the point λ.

Lemma 2.2. *For every $i = 1, \ldots, k-1$ the conditions*

$$\Phi_{i,2}|_R \equiv \ldots \equiv \Phi_{i,d_i}|_R \equiv 0$$

are equivalent to the conditions

$$q_{i,j,\alpha}(\lambda) = 0,$$

$\xi_i \leqslant j \leqslant d_i$, $\alpha = 0, 1, \ldots, j$.

Proof. For the homogeneous polynomial

$$\Phi(v_*) = v_0^l q_0 + v_0^{l-1} q_1 + \ldots + v_0 q_{l-1} + q_l,$$

where $q_i(v_1, \ldots, v_{M-1})$ is a homogeneous polynomial of degree i, the condition $\Phi|_R \equiv 0$ means that

$$q_0 = q_1(\lambda) = \ldots = q_l(\lambda) = 0.$$

The formula (3) implies that if all polynomials $\Phi_{i,j}$ (for a fixed $i \leqslant k-1$) vanish identically on the line R, then the equalities

$$\sum_{j=\max(\xi_i, e)}^{d_i} \binom{j-\alpha}{e-\alpha} q_{i,j,\alpha}(\lambda) = 0$$

hold for all $e = 0, \dots, d_i$ and $\alpha = 0, \dots, e$. Setting $e = d_i$, we obtan the system of equalities

$$q_{i,d_i,\alpha}(\lambda) = 0,\ \ \alpha = 0, \dots, d_i.$$

If $\xi_i = d_i$, then the claim of the lemma is shown.

Assume that $\xi_i \leqslant d_i - 1$. Setting $e = d_i - 1$, we obtain the system of equalities

$$q_{i,d_i-1,\alpha}(\lambda) + \binom{d_i - \alpha}{d_i - 1 - \alpha} q_{i,d_i,\alpha}(\lambda) = 0$$

for $\alpha = 0, \dots, d_i - 1$, whence, taking into account the previous equalities, we conclude that

$$q_{i,d_i-1,\alpha}(\lambda) = 0,\ \ \alpha = 0, \dots, d_i - 1.$$

Proceeding in the same spirit, we consider the values $e = d_i - 2, \dots, \xi_i$ and complete the proof of the lemma. Q.E.D.

Lemma 2.3. *The conditions*

$$\Phi_{k,2}|_R \equiv \dots \equiv \Phi_{k,d_k-1}|_R \equiv 0$$

define a linear subspace of codimension

$$\frac{1}{2}[d_k(d_k + 1) - \xi_k(\xi_k + 1)]$$

in the space of tuples of homogeneous polynomials $q_{k,j,\alpha}$, $\xi_k \leqslant j \leqslant d_k - 1$, $\alpha = 0, 1, \dots, j$.

Proof. Adding the condition

$$\Phi_{k,d_k}|_R \equiv 0$$

and applying the previous lemma, we obtain $\frac{1}{2}[(d_k+1)(d_k+2) - \xi_k(\xi_k+1)]$ independent linear conditions $q_{k,j,\alpha}(\lambda) = 0$. Vanishing of the form Φ_{k,d_k} on the line R adds precisely $d_k + 1$ linear conditions. Q.E.D. for the lemma.

Combining Lemmas 2.2 and 2.3, we see that Proposition 2.4 follows from the inequality

$$\frac{1}{2}\sum_{i=1}^{k}[(d_i + 1)(d_i + 2) - \xi_i(\xi_i + 1)] - (d_k + 1) - (M - 2) \geqslant M + 1 + c_* - k.$$

The last inequality is precisely (1). Q.E.D. for Propositions 2.4 and 2.2.

2.5. Estimating the codimension in the general case. Let us show Proposition 2.3. By Remark 2.1 we assume that $i \geqslant k + 1$, so that $\deg p_i \geqslant 3$. We use the method of good sequences and associated subvarieties, developed in [3, Section 3], see also [4, Chapter 3, Section 3].

Let $\mathcal{B}_{i,b} \subset \mathcal{B}_i$ be the subset of tuples $\underline{f} \in \mathcal{A}$ such that for some irreducible component B of the set $\{p_1 = \dots = p_{i-1} = 0\}$ (which has codimension $i - 1$ in $\mathbb{T}$, since $\underline{f} \in \mathcal{B}_i$), such that $\mathrm{codim}(\langle B \rangle \subset \mathbb{T}) = b$, we have $p_i|_B \equiv 0$. The parameter b runs through the set of values $\{0, 1, \ldots, i-1\}$ for $i \leqslant M-2$, and through the set $\{0, \ldots, M-3\}$ for $i = M-1$. When $b = i-1$, the component B is a linear subspace in $\mathbb{T}$, and the codimension $\mathrm{codim}(\mathcal{B}_{i,i-1} \subset \mathcal{A})$ can be calculated explicitly in the same way as the codimension of the subset $\mathcal{B}_{\mathrm{line}}$, but we do not need that.

In order to estimate the codimension of the subsets $\mathcal{B}_{i,b}$, we need a small modification of the technique of [3, Section 3]. Let P be a linear subspace of codimension b in $\mathbb{T}$. By the symbol $\mathcal{B}_{i,b}(P)$ we denote the subset of tuples $\underline{f} \in \mathcal{B}_{i,b}$ such that the linear span of the irreducible subvariety B (see the definition of the set $\mathcal{B}_{i,b}$) is P. Obviously,

$$\mathrm{codim}(\mathcal{B}_{i,b} \subset \mathcal{A}) \geqslant \mathrm{codim}(\mathcal{B}_{i,b}(P) \subset \mathcal{A}) - b(M-b).$$

Furthermore, for a subset of indices

$$I = \{j_1 < \ldots < j_{i-1-b}\} \subset \{1, \ldots, i-1\}$$

let $\mathcal{B}_{i,b,I}(P) \subset \mathcal{B}_{i,b}(P)$ be the subset of tuples $\underline{f} \in \mathcal{B}_{i,b}(P)$ such that there exists a sequence of irreducible subvarieties

$$Y_0 = P,\ Y_1,\ \ldots,\ Y_{i-1-b} = B,$$

satisfying the following properties:

- for every $l \in \{1, \ldots, i-1-b\}$ and every index $j_{l-1} < j < j_l$ (where $j_0 = 0$) the polynomial p_j vanishes identically on Y_{l-1},
- for every $l \in \{1, \ldots, i-1-b\}$ we have $p_{j_l}|_{Y_{l-1}} \not\equiv 0$ and $Y_l \subset Y_{l-1}$ is an irreducible component of the closed set $\{p_{j_l}|_{Y_{l-1}} = 0\}$, containing the subvariety B.

In the terminology of [3, Section 3] the polynomials $p_{j_l}|_P$, $l = 1, \ldots, i-1-b$, form a good sequence and B is one of its associated subvarieties. Obviously,

$$\mathcal{B}_{i,b}(P) = \bigcup_I \mathcal{B}_{i,b,I}(P).$$

Lemma 2.4. *The following inequality holds:*

$$\mathrm{codim}(\mathcal{B}_{i,b,I}(P) \subset \mathcal{A}) \geqslant (2b+3)(M-1-b) - 2.$$

Proof repeats the arguments in [3, Section 3] (the proof of Proposition 4) or in [4, Chapter 3, Section 3], but $(M-1)$ needs to be replaced by $(M-2)$, since as we have already mentioned, $\Phi_{i,e} = q_{i,d_i,e} + \ldots$, where $q_{i,d_i,e}$ is an arbitrary homogeneous polynomial of degree e in the variables $u_2, \ldots, u_{M+k}$. We check the polynomials p_j that were not included into the good sequence, one by one. When the polynomials p_γ with $\gamma < j$ are fixed, the condition $p_j|_{Y_{l-1}} \equiv 0$ imposes on the coefficients of the polynomial p_j at least

$$\deg p_j (M-2-b) + 1 \geqslant 2(M-2-b) + 1$$

independent conditions, since $\langle Y_{l-1}\rangle = P$ (recall that $Y_{l-1} \supset B$). The condition $p_i|_B \equiv 0$ gives (with $p_1, \dots, p_{i-1}$ fixed) at least

$$\deg p_i(M-2-b)+1 \geqslant 3(M-2-b)+1$$

independent conditions. Putting together, we complete the proof of the lemma. Q.E.D.

Now we complete the proof of Proposition 2.3. Let us look first at the values $i \leqslant M-2$, when the parameter b takes the values $0, 1, \dots, i-1$. Let us consider the quadratic function

$$\varphi_1(t) = (2t+3)(M-1-t) - t(M-t).$$

Since $\varphi_1''(t) = -1 < 0$, its minimum on the set $[0, i-1]$ is attained either at $t=0$, or at $t = i-1$. Therefore, for $i = k+1, \dots, M-2$ we get

$$\operatorname{codim}(\mathcal{B}_i \subset \mathcal{A}) \geqslant \min\{3M-5, (M-i-1)(i+2)+1\}.$$

Since $3M-5 \geqslant M+1$, which is what we need, let us consider the quadratic function

$$\varphi_2(t) = (M-t-1)(t+2)+1.$$

Again, $\varphi_2''(t) = -1 < 0$, so that its minimum on the set $[k+1, M-2]$ is attained either at $t = k+1$, or at $t = M-2$. In the latter case we get $\varphi_2(M-2) = M+1$, as required. Therefore, in order to prove Proposition 2.3 for $i \leqslant M-2$, it is sufficient to show the inequality

$$(k+3)(M-k-2)+1 \geqslant M+1.$$

For M fixed, the left hand side of the last inequality is a quadratic function in k with $-k^2$ as the senior term. Therefore, its minimum is attained either when $k=2$, when the value of the left hand side is $5M-19$ (which is not less than $M+1$), or for the maximal possible value of k (for the given M). If M is odd, then we need to check the value $k = \frac{1}{2}(M-1)$ or, equivalently, substitute $M = 2k+1$ into both left and right hand sides of the last displayed inequality which gives

$$(k+3)(k-1)+1 \geqslant 2k+2,$$

which is true for $k \geqslant 2$. If M is even, we substitute $M = 2k+2$ and get

$$(k+3)k+1 \geqslant 2k+3,$$

which is true, either. Thus the claim of Proposition 2.3 is shown for $i \leqslant M-2$.

Finally, in the case $i = M-1$ the parameter b takes the values $0, 1, \dots, M-3$ (it is precisely to treat the option $b = M-2$ that we considered the case of a line separately in Proposition 2.2). If $b=0$, we get $\varphi_1(0) = 3M-3$ as above which is fine. Substituting $b = M-3$, we get the value $\varphi_1(M-3) = M+3$ which leads to the estimate

$$\operatorname{codim}(\mathcal{B}_{M-1} \subset \mathcal{A}) \geqslant M+1,$$

as required. Now the proof of Proposition 2.3 is complete.

Q.E.D. for Theorem 0.2.

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