BIFURCATION SETS OF FAMILIES OF REFLEXIONS ON SURFACES IN $\mathbb{R}^3$

P.J. GIBLIN AND S.JANECZKO

ABSTRACT. We introduce a new affinely invariant structure on smooth surfaces in $\mathbb{R}^3$, by defining a family of reflexions in all points of the surface. We show that the bifurcation set of this family has a special structure at 'A_2^* points' which are not detected by the flat geometry of the surface. These A_2^* points (without an associated structure on the surface) have also arisen in the study of the centre symmetry set; using our technique we are able to explain how the points are created and annihilated in a generic family of surfaces. We also present the bifurcation set in a global setting.

1. Introduction

Symmetries of various kinds have played an important role in the study of surfaces in euclidean 3-space $\mathbb{R}^3$ and in applications of the geometry of such surfaces. One of the best-known examples is the *(Blum) Medial Axis* or *Medial Axis Transform* [17] which is constructed for a smooth closed surface M by taking the closure of the locus of centres c of spheres S which are entirely contained inside the closed region of $\mathbb{R}^3$ bounded by M and tangent to M in (at least) two places. The connexion here with symmetry is local: the plane π through c perpendicular to the line joining two points $\mathbf{p}, \mathbf{q}$ of contact of S and M is a *local, or first-order plane of symmetry* for M. This means that reflexion in π takes $\mathbf{p}$ to $\mathbf{q}$ and also takes the tangent plane of M at $\mathbf{p}$ to the tangent plane at $\mathbf{q}$. Furthermore π is tangent to the medial axis (at smooth points). Thus the medial axis captures some aspects of the *local reflexional symmetries* of M, where here 'reflexional' means reflexion in a plane. There are close connexions between the singularities of the medial axis and the contact between S and M (or singularities of the distance-squared function on M from points of $\mathbb{R}^3$ [6, 8]). This leads to close relations with the differential geometry of M, such as ridge curves where two contact points $\mathbf{p}$, $\mathbf{q}$ coincide. The bifurcation set of the family of distance-squared functions on M is the union of the focal set of M (locus of centres of principal curvature) and the *symmetry set* of M, which is the closure of the locus of centres of *all* spheres tangent to M in (at least) two points [9, 17]. The medial axis has proved to be a major tool in computer vision [17]. Besides distance-squared functions, which are said to extract the 'spherical geometry' of M, the family of *height functions* on M extracts the affinely invariant 'flat geometry' such as parabolic curves, asymptotic curves and cusps of Gauss (godrons) [5, 7].

A different use of reflexions, also closely connected with differential geometry, was made by Bruce and Wilkinson in [4]. They studied the local reflexional symmetry of a surface in $\mathbb{R}^3$ by considering reflexion in planes containing the normal line to M at a point $\mathbf{p} \in M$ and the family of 'folding maps'. Principal directions on M emerge naturally in this way, but, perhaps more importantly, the bifurcation set of the family of folding maps is the union of the *duals* of the focal set and symmetry set. Thus information is available about the tangential structure of these sets by studying local symmetries. This includes the 'sub-parabolic points' of M (points at which the corresponding point on the focal surface is parabolic), of significance in shape analysis and computational geometry [9].

A number of authors have studied a local version of *central symmetry*: the *centre symmetry set* of a surface M in $\mathbb{R}^3$ is the (affinely invariant) envelope of chords joining distinct points $\mathbf{p}$, $\mathbf{q}$ at which the tangent planes to M are parallel. For a globally centrally symmetric surface this envelope degenerates to a single point which is the centre of all such chords. This subject was initiated for plane curves in [16] and followed up in many articles such as [11, 12]. There are connexions here not only with the differential geometry of M but also with physics, via the 'Wigner caustic' [2, 10] and Finsler geometry [11]. The structure of the centre symmetry set has been shown to give information about M which does not arise from any of the above methods [12].

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In this article we contribute to the above programme of relating differential geometry of surfaces to the theory of singularities and symmetry. We work from a different viewpoint to those above: we study the family of reflexion maps in points of M itself. That is, for each $\mathbf{p} \in M$ we take the map reflecting M in $\mathbf{p}$ and study the contact function between M and the reflected surface M^*. It turns out that the bifurcation set of this family of functions extracts in a very simple way much of the information about the differential geometry of M which comes via the centre symmetry set. In this instance we are able to give a global meaning to the bifurcation set of the family of local reflexion maps, and we are able to determine, in a family of surfaces, how the special features of the bifurcation are created and destroyed. In a subsequent article we plan to carry out a similar investigation for surfaces in $\mathbb{R}^4$.

2. Previous work and plan of this article

In [13] we introduced an affinely invariant family of reflexion maps on a surface M in 3- or 4-dimensional space and studied the relation of this family to the underlying geometry of the surface. Given a point $\mathbf{p}$ of M the reflexion map based at $\mathbf{p}$ takes $\mathbf{m} \in M$ to its reflexion $\mathbf{m}^*$ in $\mathbf{p}$ and as $\mathbf{m}$ traces out a neighbourhood of $\mathbf{p}$, so $\mathbf{m}^*$ traces out a surface M^* whose contact with M at $\mathbf{p}$ is measured by the *contact map*. Varying $\mathbf{p}$ now produces a family of reflexion maps and contact maps which give geometrical information about M distinct from that given by, say, the family of height functions on M which are related to the flat geometry of M. The reflexion maps pick out the parabolic set of M and also special points of the parabolic set, called 'A_2^* points'. These arose in the study of the *affine equidistants* and *centre symmetry set* of M. Both of these depend on chords joining pairs of points of M with parallel tangent planes; an affine equidistant is the locus of points at a fixed ratio of distance from the ends of such chords and the centre symmetry set is the envelope of the chords (when this exists). Then at A_2^* parabolic points of M the structures of the 'halfway equidistant' (ratio of distances = 1) and the centre symmetry set are different from the structures at other parabolic points. See [14, p.68, Def.3.3].

In the present article we consider in detail the affinely invariant bifurcation set of the family of reflexion maps, for a smooth surface M in $\mathbb{R}^3$. We identify this bifurcation set with the set of critical values of a *symmetric* map $\mathbb{R}^2 \to \mathbb{R}^2$, and use the classification of projections of surfaces with boundary in [3] to identify it. (For an alternative approach to this classification, see [15].) The role of the 'boundary' is taken by the parabolic set of M and the bifurcation set in a neighbourhood of a parabolic point of M always contains the parabolic set. At special parabolic points it has an extra branch arising from a 'semifold' singularity in the terminology of [3]. These points are the 'A_2^*' points referred to above, and they are affinely invariant points of M. There is no special structure at a cusp of Gauss (godron): the reflexion maps do not 'recognize' these points which affect the flat geometry of M.

Our work provides a geometrical interpretation of the A_2^* points, as well as an explanation of how they are created or disappear in a generic 1-parameter family of surfaces, something that was left unsolved in [18]. In fact these transitions occur at singularities of the semilips or semibeaks type in the language of [3]. We show that another codimension 1 singularity, the semicusp, cannot occur in the present context.

The bifurcation set which we study appears to be most meaningful in a neighbourhood of the parabolic set, but it does extend over the surface, in both elliptic and hyperbolic regions. Apart from in §7 we shall work locally, parametrizing M as a graph.

In §3 we study the family of reflexions. In §4 we study the bifurcation set $\mathcal{B}_F$, splitting into three cases: Case 1, a nonsingular parabolic point which is not a cusp of Gauss (godron); Case 2, a cusp of Gauss; and Case 3, a singular parabolic point. These split into subcases according to additional geometric conditions on M. In §5 we study in more detail families of surfaces in order to discover how the bifurcation set evolves when the surface M is perturbed in a generic way. The same three cases are considered. In §6 we give some explicit examples and in §7 we place our investigation in the setting of global surfaces, given by equations of the form $g(x, y, z) = 0$ in $\mathbb{R}^3$.

Here are the main results of this article, with references to the sections where details can be found.

Theorem 2.1. *Let $\mathbf{p} \in M$. The structure of the bifurcation set of the family of reflexion maps in a neighbourhood of $\mathbf{p}$ is (as a subset of M) as follows:*

the parabolic curve, if $\mathbf{p}$ *is not an* A_2^* *point of* M*. See §4, Case 1a (also Case 2a for* $\mathbf{p}$ *a cusp of Gauss);*

the parabolic curve and a branch tangent to it and terminating at $\mathbf{p}$*, if* $\mathbf{p}$ *is an* A_2^* *point of* M*. The branch can lie locally in either the elliptic or hyperbolic region of* M*, giving two types, elliptic and hyperbolic* A_2^* *points. See §4, Case 1b and Figure 1(i).*

Furthermore,

For a generic 1-parameter family of surfaces, A_2^* *points of the same type are created or destroyed through 'semilips' or 'semibeaks' transition. See §§4,5, Case 1c.*

The projection of surfaces with boundary singularity 'semi-cusp' cannot occur in the present context. See §4 Case 1d.

There is no creation of A_2^* *points at Morse-type transitions of the parabolic curve. See §§4,5 Case 3.* $\square$

3. The family of reflexions

For the local version of the family we take a surface M in $\mathbb{R}^3$ in Monge form, that is as the graph $z = f(x, y)$ of a smooth function f which we write in the form

$$f(x,y) = f_{20}x^2 + f_{11}xy + f_{02}y^2 + \ldots + f_{ij}x^iy^j + \ldots, \tag{1}$$

in order to perform calculations. For the most part we take the origin to be a parabolic point of M, with $f_{11} = f_{02} = 0$, and we can scale to make $f_{20} = 1$. All our constructions are invariant under affine transformations in the ambient $\mathbb{R}^3$.

Let $\mathbf{p} = (p, q, r)$, where $r = f(p, q)$, be a point of M and consider the map which reflects M in this point. That is, for any other point $\mathbf{m} = (p+x, q+y, r+z)$ of M, where $z = f(p+x, q+y) - f(p, q)$, we reflect in $\mathbf{p}$ to obtain $\mathbf{m}^* = (p-x, q-y, r-z)$ which will of course not in general lie on M. As x and y vary, $\mathbf{m}^*$ traces another surface M^* through $\mathbf{p}$ and we can measure the contact between M and M^* at $\mathbf{p}$. The contact function is the composite F given by parametrizing M^* and following by the function whose zero-set is M:

$$\begin{aligned}(x,y) &\mapsto (p-x, q-y, r-z) \mapsto f(p-x, q-y) - (r-z) \\ &= f(p+x, q+y) + f(p-x, q-y) - 2f(p,q) = F(x,y,p,q).\end{aligned} \tag{2}$$

Note that F is symmetric in x and y: we always have $F(x, y, p, q) = F(-x, -y, p, q)$. It can be regarded as a 2-parameter unfolding, with parameters p, q of the function $F_0(x, y) = F(x, y, 0, 0) = f(x, y) + f(-x, -y)$, which is twice the even part of f, and the bifurcation set we study in this article is in this sense.

Our object in this article is to calculate the bifurcation set of F, namely (using suffices to denote partial derivatives)

$$\mathcal{B}_F = \{(p,q) : \text{ there exist } x, y \text{ with } F_x = F_y = F_{xx}F_{yy} - F_{xy}^2 = 0\}.$$

This lies in the parameter plane of M but can also be considered as a subset of M itself. If the origin is not a parabolic point then the equation $F_{xx}F_{yy} - F_{xy}^2 = 0$ contains a constant term, namely a multiple of $4f_{20}f_{02} - f_{11}^2$, so that small solutions for x, y are not possible. We shall assume unless otherwise stated that the origin is parabolic, since we are chiefly interested in the germ of $\mathcal{B}_F$ at a point of the parabolic set of M. Nevertheless the bifurcation set does have a global structure, away from the parabolic curve, and we give a 'semi-global' example in Figure 3.

In order to study $\mathcal{B}_F$ we shall consider the critical set

$$\Sigma_F = \{(x, y, p, q) : F_x = F_y = 0.\} \tag{3}$$

A significant source of difficulty is that, from the definition in (2), all points $(0, 0, p, q)$ belong to Σ_F. These points, the 'trivial component' of Σ_F, prevent Σ_F from being smooth; they have to be eliminated before we can compute the structure of $\mathcal{B}_F$, as the set of critical values of a projection $\pi : \Sigma_F^0 \to \mathbb{R}^2$,

$(x, y, p, q) \mapsto (p, q)$, where Σ_F^0 is the 'non-trivial' component of Σ_F. This is defined precisely in Lemma 4.1 below.

4. The bifurcation set of F for a generic surface

In this section we show how to eliminate the 'trivial' component of Σ_F, yielding a smooth surface in (x, y, p, q)-space; this allows us to reduce the problem to the study of a symmetric map from the plane to the plane. We take the classification up to codimension 1, that is singularities which we expect to occur on a generic surface or in a generic 1-parameter family of surfaces.

Case 1: the origin is a parabolic point of M but not a cusp of Gauss. At an ordinary parabolic point of M the tangent plane has A_2 contact with M, while at a cusp of Gauss (godron) it has A_3 contact. With M in Monge form the contact at the origin with the tangent plane is measured by the function f itself. Thus in (1) we take $f_{20} = 1, f_{11} = 0, f_{02} = 0$, and also $f_{03} \neq 0$ to avoid A_3 contact. Using this, it is easy to check that, by an affine transformation, we can also assume $f_{12} = 0$ and we shall do this since it simplifies the formulae which occur later. Thus, for Case 1, f has the form

$$f(x, y) = x^2 + f_{30}x^3 + f_{21}x^2y + f_{03}y^3 + \text{ higher terms, } f_{03} \neq 0.$$

Lemma 4.1. *Let f be as above. Then the component Σ_F^0 other than $\{(0, 0, p, q)\}$ of Σ_F is locally a smooth 2-manifold parametrized locally in the form $(y, p) \mapsto (X(y, p, Q(y, p)), y, p, Q(y, p))$ where Q is in fact a smooth function of y^2 and p; in fact $Q_y = yV(y, p)$ where $V(0, 0) = -\frac{4}{3}f_{04}$.*

Proof. We break the proof up into several steps, (i)-(v) below. The coefficient of x in F_x is 2, and hence $F_x = 0$ can be solved locally for $x = X(y, p, q)$, say.

(i) *$X(y, p, q) = -X(-y, p, q)$ for all y, p, q; in particular $X(0, p, q) \equiv 0$.* *Proof:* $x = X$ is the unique solution to $F_x(x, y, p, q) = 0$, and $F_x(-x, -y, p, q) = -F_x(x, y, p, q)$ from the definition of F. Thus $F_x(-X(-y, p, q), -y, p, q) = -F_x(X(y, p, q), y, p, q) \equiv 0$, and $x = -X(-y, p, q)$ is the unique solution to $F_x(x, y, p, q) = 0$ and hence equals $X(y, p, q)$. □

Now substitute $x = X$ into $F_y = 0$; then $F_y(X(0, p, q), 0, p, q) = F_y(0, 0, p, q) \equiv 0$, so that by Hadamard's lemma $F_y(X(y, p, q), y, p, q) = yU(y, p, q)$ for a smooth function U. Note that the solution $y = 0$ to $F_y = 0$ leads to $x = X(0, p, q) = 0$ so it is the other solution, $U(y, p, q) = 0$, which we wish to pursue.

(ii) *$U(y, p, q) = U(-y, p, q)$; therefore U is a function of y^2, p and q. Also the only term in U of degree ≤ 1 is $3q$, so $U(0, 0, 0) = 0$.* *Proof:* $F_y(X(-y, p, q), -y, p, q) = -yU(-y, p, q)$. But the left-hand side of this is $U(-X(y, p, q), -y, p, q)$ by (i), and $F_y(-x, -y, p, q) = -F_y(x, y, p, q)$ for any x, y, from the definition of F. So $-yU(y, p, q) = -F_y(X(y, p, q), y, p, q) = -yU(-y, p, q)$ for all values of y, p, q and this gives the first result. The last sentence is a direct calculation. □

Calculation shows that $U_q(0, 0, 0) = F_{yyq}(0, 0, 0, 0) = 12 \neq 0$, so that $U = 0$ can be solved for $q = Q(y, p)$, say, so that $U(y, p, Q(y, p)) \equiv 0$.

(iii) *$Q(y, p) = Q(-y, p)$, so that Q is a function of y^2 and p (and $Q(0, 0) = 0$).* *Proof:* $q = Q(y, p)$ is the unique solution to $U(y, p, q) = 0$, so $U(y, p, Q(y, p)) \equiv 0$ and replacing y by $-y$ we have $U(-y, p, Q(-y, p)) \equiv 0$. But the left-hand side is $U(y, p, Q(-y, p))$ by (ii), so by uniqueness $Q(-y, p) = Q(y, p)$. □

(iv) *$Q_y(-y, p) = -Q_y(y, p)$ for all y, p and in particular $Q_y(0, p) = 0$ for all p, so that $Q_y = yV(y, p)$ for a smooth function V, and the critical set of π away from the parabolic curve is given by $V(y, p) = 0$* *Proof:* This is immediate from (iii). □

(v) *$V(0, 0) = -\frac{4}{3}f_{04}$.* *Proof:* direct calculation. This completes the proof of the lemma. □

Corollary 4.2. *The bifurcation set $\mathcal{B}_F$ which we wish to study is the set of critical values of the projection $\pi : \Sigma_F^0 \to \mathbb{R}^2$, $\pi(y, p) = (p, Q(y, p))$.* □

Lemma 4.3. *The set of points $\pi(0, p) = (p, Q(0, p))$ is locally the parabolic set of M, and this forms part of the bifurcation set.*

Note that when f has 2-jet x^2 the parabolic set on $M : z = f(x, y)$ is smooth at the origin unless $f_{12} = f_{03} = 0$, and parametrized by x provided $f_{03} \neq 0$.

Proof. This is a matter of using the information in (i)-(iv) above, together with $F_{xx}(0, 0, p, q) = 2f_{xx}(p, q)$ and similarly for F_{xy}, F_{yy}. Thus differentiating $F_x(X(y, p, q), y, p, q) \equiv 0$ with respect to y, and $F_y(X(y, p, q), y, p, q) = yU(y, p, q)$ with respect to y, putting $y = 0$ and using the properties above we deduce that $F_{xx}(0, 0, p, Q(0, p))F_{yy}(0, 0, p, Q(0, p)) = F_{xy}(0, 0, p, Q(0, p))^2$ for all p. The last sentence follows from (iv) above. $\square$

Since Q is a function of y^2 and p by Lemma 4.1, say $Q(y, p) = K(y^2, p)$, the map π in the corollary will be classified according to the classification of symmetric maps, which coincides with that of singularities of projections of surfaces with boundary. We use the inductive classification in [3, §3] and for this we need the expansion of the function K about $(0, 0)$. In our situation the parabolic set takes the role of the boundary $y = 0$, by Lemma 4.3. We write Y for y^2, so that K is a function of Y and p and the singularity which is relevant for us is the corank 1 singularity of $\tilde{\pi} : (Y, p) \mapsto (p, K(Y, p))$ so that changes of coordinates in the source must preserve the 'boundary' $Y = 0$.

Case 1a. *An ordinary ('A_2') point of the parabolic set.* Recall that $f_{03} \neq 0$; if also $f_{04} \neq 0$ then $K(Y, p) = -\frac{2}{3}\frac{f_{04}}{f_{03}}Y+$ higher terms, so that as a boundary singularity $\tilde{\pi}$ is equivalent to $(Y, p) \mapsto (p, Y)$:

if $f_{03} \neq 0, f_{04} \neq 0$ then the bifurcation set consists locally of the (smooth) parabolic set.

Thus by an ordinary A_2 point we mean one for which $f_{04} \neq 0$. As noted above, by A_2 we refer to the contact of M with its tangent plane. The boundary singularity is of type local diffeomorphism.

Case 1b. *A parabolic point at which $f_{04} = 0$ (an 'A_2^* point'.* Such points were called A_2^* points in [14] where they were related to the structure of certain equidistants. When $f_{04} = 0$ the function K has 2-jet, ignoring terms depending only on p and using c_{ij} to denote the coefficient of $Y^i p^j$,

$$c_{20}Y^2 + c_{11}Yp = \frac{f_{13}^2 - 4f_{06}}{4f_{03}}Y^2 + \frac{2(f_{21}f_{13} - f_{14})}{3f_{03}}Yp.$$

The classification in [3, §3] is according to which of these coefficients is nonzero.

Suppose that *both are nonzero*; then the boundary singularity is equivalent to $(Y, p) \mapsto (p, Yp + Y^2)$ which is 2-determined. This is the stable 'semifold' singularity:

if $f_{03} \neq 0, f_{04} = 0, f_{13}^2 - 4f_{06} \neq 0, f_{21}f_{13} - f_{14} \neq 0$ then the bifurcation set consists locally of the (smooth) parabolic set and another smooth curve tangent to it at the origin, and ending there.

See Figure 1(i) for a schematic representation.

There are two distinct kinds of semifold, since the curve tangent to the parabolic set can lie locally in the elliptic or the hyperbolic region of M; we can refer to these as *elliptic A_2^* points* or *hyperbolic A_2^* points*. See Figure 3 for an example with several hyperbolic A_2^* points.

Lemma 4.4. *Assume as above that $f_{13}^2 - 4f_{06}$ and $f_{21}f_{13} - f_{14}$ are nonzero. Then the second branch of the bifurcation set $\mathcal{B}_F$ at an A_2^* point is locally in the hyperbolic (resp. elliptic) region of M if and only if $f_{13}^2 - 4f_{06} > 0$ (resp. $f_{13}^2 - 4f_{06} < 0$). This holds if and only if the singularity of $F_0(x, y) = F(x, y, 0, 0)$ is of type A_5^+ (resp. A_5^-), that is right-equivalent to $x^2 + y^6$ (resp. $x^2 - y^6$).*

Proof. Following through the calculations up to (iv) above we find that the critical set of π, away from the parabolic set, is given by $V(y, p) = 0$ where

$$V(y, p) = \frac{4(f_{21}f_{13} - f_{14})}{3f_{03}}p + \frac{f_{13}^2 - f_{06}}{f_{03}}y^2 + \ldots,$$

and solving this for p as a function of y, say $p = A(y)$, the branch of the critical set is given by $(A(y), Q(y, A(y)))$. Substituting these into $f_{xx}f_{yy} - f_{xy}^2$ gives $3(4f_{06} - f_{13}^2)y^4 + \ldots$. When this is negative the branch lies in the hyperbolic region.

The last sentence follows because $F_0(x,y)$ is twice the even part of f, using a standard argument with right-equivalence of functions. □

Remark 4.5. We remark that the A_2^* points are not related to the 'goose' points on the parabolic set, as described in for example [1]. These are parabolic points such that projection of the surface in the unique asymptotic direction to a plane yields a rhamphoid cusp in the image. A necessary condition for this is, in the notation of (1), and assuming $f_{11}=f_{02}=0$, that $f_{12}^2=3f_{21}f_{03}$, a condition unrelated to that for an A_2^* point.

Case 1c. $f_{40}=0, c_{11}=0$, i.e. $f_{21}f_{13}-f_{14}=0; c_{20}\neq 0$, i.e. $f_{13}^2-4f_{06}\neq 0$. Following the classification in [3, §3] we examine the cubic terms of K; a change of variable of the form $Y\to Y(1+\alpha p+\beta Y)$ for constants α and β (this change of variable preserving $Y=0$) removes the cubic terms in Y^2p and Y^3, leaving the important term $c_{12}Yp^2$ where

$$c_{12}=\frac{3f_{03}(3f_{31}f_{13}+4f_{21}f_{23}-2f_{24}-2f_{21}^2f_{22}-6f_{30}f_{21}f_{13})+10f_{05}(f_{22}-f_{21}^2)}{9f_{03}^2}.$$

If $c_{12}\neq 0$ then from the classification $\widetilde{\pi}$ is equivalent to $(Y,p)\mapsto(p,Y^2\pm Yp^2)$, where the sign is that of $c_{20}c_{12}$. This is called a *semilips* or *semibeaks* according as the sign is $+$ or $-$. In this case the critical set of $\widetilde{\pi}$ is tangent to the 'boundary' $Y=0$ but transverse to the kernel line of the map $\widetilde{\pi}$. This case will be relevant in the next section, where we discuss 1-parameter families of surfaces. We sum it up as follows.

Lemma 4.6. *With the conditions* $f_{40}=0,\ f_{21}f_{13}-f_{14}=0,\ f_{13}^2-4f_{06}\neq 0$, *the boundary singularity of* $\widetilde{\pi}$ *is of semilips or semibeaks type according as* $c_{20}c_{12}$ *is* >0 *or* <0. *It is hyperbolic or elliptic according as* $c_{20}>0$ *or* <0, *where* $c_{20}=(f_{13}^2-4f_{06})/4f_{03}$ *as above.* □

For additional information on Case 1c, see §5.

In the case of the singularity, called semigoose in [3], which is a degeneration of the semilips/beaks discussed here, we have $c_{20}=c_{11}=0$ and the vanishing of the complicated expression c_{12} above means in geometrical terms that the critical set of $\widetilde{\pi}$ and the 'boundary' $Y=0$ have inflexional contact. This is in line with the classification of [3].

Case 1d. $f_{40}=0, c_{11}\neq 0$, i.e. $f_{21}f_{13}-f_{14}\neq 0; c_{20}=0$, i.e. $f_{13}^2-4f_{06}=0$. In this case a change of variable in Y, as in Case 1c, removes the cubic terms except for Y^3, but the coefficient c_{30} of Y^3 turns out to be a multiple of c_{20}, which we are assuming is zero. So the 'semicusp case', equivalent to $(Y,p)\mapsto(p,Yp+Y^3)$, cannot occur in our situation.

Case 2. The origin is a cusp of Gauss (godron). Here, in (1) we take $f_{20}=1, f_{11}=0, f_{02}=0$, and also $f_{03}=0$, but assume $f_{12}\neq 0$ so that the parabolic set of M is not singular at the origin. (So in this case f_{12} cannot be made zero by an affine transformation, as it could in Case 1.) In fact there is very little to say here, since cusps of Gauss are no different from parabolic points as in Case 1a above: the reflexion maps do not distinguish cusps of Gauss but they do distinguish quite different points, the A_2^* points of the parabolic curve. We shall not give so much detail as above; in the present case the parametrization of the 'nontrivial' part of the critical set Σ_F is different since p is a smooth function of q rather than the other way round. We reduce to a map $\pi:(y,q)\mapsto(q,P(y,q))$ where as before P is a function of $Y=y^2$. Thus as a boundary singularity we consider the map $\overline{\pi}:(Y,q)\mapsto L(Y,q)$ where $L(Y,q)=P(y,q)$. Then we find the following.

Case 2a. $f_{04}\neq 0$. Then the 1-jet of $L(Y,q)$ is $-2f_{04}Y/f_{12}$ so that L is equivalent to the boundary singularity $(Y,q)\mapsto(q,Y)$ and the bifurcation set consists locally of just the parabolic curve. This is the same as Case 1a, an ordinary point of the parabolic curve.

Case 2b. $f_{04}=0$. This means that the origin is a cusp of Gauss which is 'also an A_2^* point'. It gives a semifold singularity provided we also have $f_{12}f_{13}-5f_{05}\neq 0$ and $f_{13}^2-4f_{06}\neq 0$. For additional information on this case see §5.

Case 3. The parabolic curve is singular.

Case 3a. *The quadratic terms of f are not identically zero.* This amounts to $f_{20} = 1, f_{11} = 0, f_{02} = 0, f_{12} = 0, f_{03} = 0$ and occurs generically only in a 1-parameter family of surfaces (see §5). We shall assume by genericity that $f_{04} \neq 0$ (vanishing of f_{04} would give an 'A_2^* point at which the parabolic curve is singular' and require a 2-parameter family of surfaces). The first steps (i) and (ii) of the proof of Lemma 4.1 still apply, so that we can solve $F_x = 0$ for x as a function of y, p, q and substitute in $F_y = 0$, divide by y and obtain a function U which depends on y^2, p, q. But now the 2-jet of U is (dividing by 4)

$$2f_{04}y^2 + (f_{22} - f_{21}^2)p^2 + 3f_{13}pq + 8f_{04}q^2,$$

with no linear term in p or q. However we can replace y^2 by Y in U and solve for Y to give

$$y^2 = Y = \frac{f_{21}^2 - f_{22}}{2f_{04}}p^2 - \frac{3f_{12}}{2f_{04}}pq - 3q^2 + \text{ higher terms.}$$

It is now clear that the projection map from Σ_F to the (p, q)-plane has no singularities except those corresponding to singularities of Σ_F itself, which occur when Y, or y, is zero, and this is the parabolic set of M. Hence:

Lemma 4.7. *At a generic singular point of the parabolic curve, where the quadratic terms of the surface $f = 0$ are not identically zero, the bifurcation set $\mathcal{B}_F$ consists locally of the parabolic curve only.*

Case 3b. *The quadratic terms of f are identically zero (an elliptic or hyperbolic umbilic).* Again this occurs only in a generic family of surfaces. We find again that the bifurcation set remains throughout just the parabolic curve, that is no A_2^* points are involved. Because the calculations in this case are quite different from those in Case 3a we give an example in §6.

For additional information about Case 3, see §5.

5. The bifurcation set of F for a generic 1-parameter family of surfaces

In this section we shall interpret the Cases 1c, 2b and 3 from the previous section as occurring in generic 1-parameter families of surfaces, say $z = f(x, y, \varepsilon)$. In order to discover the evolution of the bifurcation set as ε passes through 0 we need to add some conditions which allow us to determine the structure of the 'big bifurcation set' in (p, q, ε)-space, and to determine the level sets $\varepsilon =$ constant of the parameter ε.

Case 1c. *Semilips and semibeaks.* This transition explains the way in which A_2^* points are formed in an evolution of a surface. The first examples of this are in [18, Ch.7,§5]. Suppose that $z = \widetilde{f}(x, y, \varepsilon)$ is a family of surfaces in Monge form, so that ε only enters the terms of degree 2 or higher, and suppose that for $\varepsilon = 0$ the surface $z = \widetilde{f}(x, y, 0) = f(x, y)$ satisfies the conditions above, namely $f_{20} = 1,\ f_{11} = f_{02} = f_{04} = 0,\ f_{03} \neq 0,\ f_{13}^2 - 4f_{06} \neq 0,\ f_{21}f_{13} - f_{14} \neq 0$. Then the boundary singularity $(Y, p) \mapsto (p, K(Y, p))$ is of type semilips or semibeaks (with normal form $(p, Y^2 \pm Yp^2)$), provided the additional condition $c_{12} \neq 0$ above holds. According to [3, p.410] for the ε terms to give a versal unfolding we need a term εY in K. For this we need to include a term εy^2 in the family of surfaces, that is ensure that $\widetilde{f}_{yy\varepsilon}(0, 0, 0) \neq 0$, and then the coefficient of εY works out as $10f_{05}/9f_{03}^2$. Thus we have the following.

Lemma 5.1. *Assume that all the conditions of Case 1c above are satisfied, and that in addition $f_{05} \neq 0$ and that a family of surfaces $z = \widetilde{f}(x, y, \varepsilon)$ satisfies $\widetilde{f}_{yy\varepsilon}(0, 0, 0) \neq 0$. Then the bifurcation set passes through a semilips or semibeaks transition according as the sign of $c_{20}c_{12}$ (in Case 1c) is positive or negative.*

Note that this represents the way in which A_2^* points are created or destroyed in a family of surfaces. Because of the nature of the transitions, both A_2^* points must be elliptic, or both hyperbolic, in the terminology of Lemma 4.4. There is an example in the next section. See Figures 1 and 2.

Case 2b. *An A_2^* point which is also a cusp of Gauss.* This case is much less interesting since it simply means that the conditions of Case 2a are satisfied, and that in the family of surfaces the coefficient f_{04} passes through 0. There is no change in the bifurcation set: it is a semifold throughout and at the moment when $f_{04} = 0$ the basepoint is a cusp of Gauss. The additional branch characteristic of an A_2^* point simply 'slides along the parabolic curve' through the cusp of Gauss.

Case 3. *Singular parabolic set.* The result of the discussion in Case 3 in §4 above is that throughout the transition the local bifurcation set remains simply the parabolic set. The key consequence is that A_2^* points are not created or destroyed by the evolution of the parabolic set through the standard Morse transitions (as described in, for example, [5, 7]). This was suggested by examples in [18, Ch.7]. The creation and destructon of A_2^* points takes place only through the transition of Case 1c, in a generic 1-parameter family of surfaces. We give an example of Case 3b in §6 below since the calculations are rather different from those encountered above.

6. Examples

Figure 1 shows in schematic form the cases (i) 1b (semifold), (ii) 1c (semilips) and (iii) 1c (semibeaks). The horizontal line represents the boundary, which in our case is the parabolic set of M, and the curved lines the additional component of the bifurcation set $\mathcal{B}_F$. The dots are A_2^* points, in the middle diagrams of (ii) and (iii) these are degenerate A_2^* points at which $f_{21}f_{13} = f_{14}$.

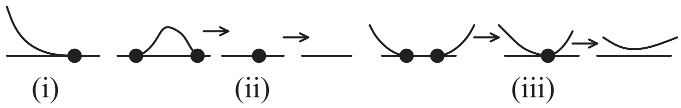

Figure 1. Schematic diagrams of (i) Case 1b, (ii) and (iii) Case 1c. In all cases the region above the horizontal line (which represents the parabolic set of M) can be the elliptic or the hyperbolic region.

Example 1. The most interesting case is 1c, which can be realized by the 1-parameter family of surfaces

$$z = \widetilde{f}(x, y, \varepsilon) = x^2 + \varepsilon y^2 + y^3 \pm x^3 y + xy^3 - y^5, \tag{4}$$

where $+$ realizes semilips and $-$ realizes semibeaks, at the origin. Figure 2 shows the boundary between the pairs (ε, p) which give positive or negative values of Y, and hence real or complex values for y, other than the solution $y = 0$, when the equation $Q_y = 0$ is solved for the critical set of Q. The figure also shows the actual bifurcation sets which pass through a semilips or semibeaks transition, causing the creation or destruction of two A_2^* points of the same kind, elliptic or hyperbolic.

Example 2. Figure 3 shows an example of the bifurcation set $\mathcal{B}_F$ where there are two A_2^* points within the range shown. The equation of the surface in this example, which satisfies all the conditions of Lemma 4.4, is sufficiently simple that the bifurcation set can be calculated directly.

Example 3. To illustrate Case 3b in §4 consider the elliptic/hyperbolic umbilic at the origin on the surface $z = f(x, y) = x^2y + y^3 + y^4$, placed in the family of surfaces given by

$$z = tx^2 + \varepsilon x^2 y + y^3 + f_{04}y^4$$

where t is small, $f_{04} \neq 0$ and ε is 1 for a hyperbolic umbilic and -1 for an elliptic umbilic (see [5, 7]). Writing down F_x and F_y we can eliminate x to obtain an equation for y^2 which solves to give

$$x = -\frac{\varepsilon p y}{t + \varepsilon q}, \; y^2 = \frac{-3tq + p^2 - 3(\varepsilon + 2f_{04}t)q^2 - 6\varepsilon f_{04}q^3}{2f_{04}(t + \varepsilon c)}.$$

But substituting this in the third defining equation $F_{xx}F_{yy} - F_{xy}^2 = 0$ we find exactly that the numerator of the expression for y^2 is zero, giving $x = y = 0$ which shows that the bifurcation set consists entirely of the parabolic set. When $t = 0$ the parabolic set is locally empty for $\varepsilon = 1$ and has two smooth transverse branches when $\varepsilon = -1$.

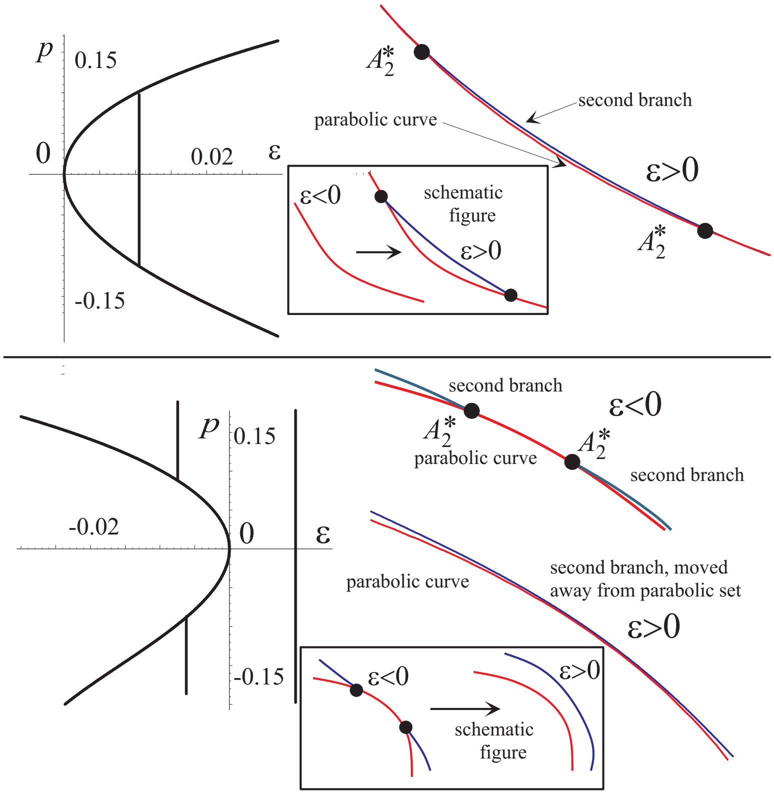

FIGURE 2. The values of p for which there are real points on $\mathcal{B}_F$, besides the parabolic set, for the family (4), are shown on the left of each figure (upper/lower) by thick vertical lines. The upper figure represents semilips, where real values of p exist only for $\varepsilon \geq 0$, and the lower figure represents semibeaks, where for $\varepsilon < 0$ there is a gap in the values of p. The right of each figure shows the actual bifurcation sets during the transition. These are all hyperbolic A_2^* points in this example, that is the branches all lie in the hyperbolic region of M. The boxed schematic figures show the transition with the curves more separated than in the actual example.

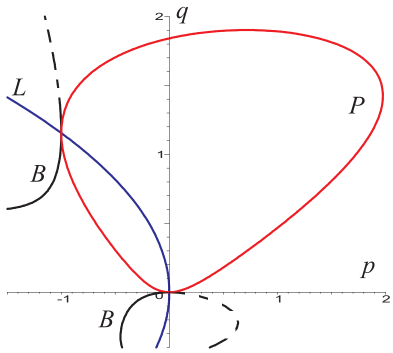

FIGURE 3. The parameter (p, q) plane of the surface $z = x^2 + y^3 + 2x^2y + xy^3$, which has an A_2^* point at the origin. One component of the parabolic set is marked P. The curve marked L determines, together with P, whether the solutions for Y in the bifurcation set are positive, and hence whether the point (p, q) is a real point or not. The solid lines marked B are the real bifurcation set and the dashed continuations are those excluded because $Y < 0$. Thus the figure contains two A_2^* points, the endpoints of B. The A_2^* points in this example are both hyperbolic. The diagram is cut off above $q = -\frac{1}{2}$ since this example becomes highly degenerate at $(p, q) = \left(-\frac{3}{16}, -\frac{1}{2}\right)$, with an entire line $(x, 0, p, q)$ projecting to a point.

7. Global surfaces, and a Lagrangian interpretation

Consider $\mathbb{R}^3 \times \mathbb{R}^3$, with coordinates $(p, q, r; u, v, w)$; the surface M in $\mathbb{R}^3$ is now given by an equation: $M : g(p, q, r) = 0, \nabla g \mid_M \neq 0$. The surface obtained by reflecting M in the point (p, q, r) has equation

$$g(2p - u, 2q - v, 2r - w) = 0,$$

and the function $g(2p - u, 2q - v, 2r - w)$ is also the contact function of the reflected surface with M at (p, q, r).

Thus the contact Lagrangian submanifold $\tilde{L}_M$ for the family of reflexions:

$$\tilde{L}_M \subset T^*(\mathbb{R}^3 \times \mathbb{R}^3) = T^*\mathbb{R}^3 \times T^*\mathbb{R}^3, \Omega = \omega \oplus \omega,$$

ω being the canonical 2-form on $T^*\mathbb{R}^3$, is defined by the generating function of contact

$$G(p, q, r; u, v, w, \lambda) = g(2p - u, 2q - v, 2r - w) + \lambda g(u, v, w).$$

It is a smooth constrained Lagrangian submanifold in $T^*(\mathbb{R}^3 \times \mathbb{R}^3)$ with λ being a Morse parameter. The contact structure of the surface M with the family of its reflexions is given by the singular set (caustic) of the reduced Lagrangian submanifold over $\mathbb{R}^3$ with coordinates (p, q, r). Reduction is given by the coisotropic submanifold

$$N = \{(\mu_u, \mu_u, \mu_v) = 0\}$$

where $(\mu_p, \mu_q, \mu_r; p, q, r, \mu_u, \mu_v, \mu_w; u, v, w)$ are coordinates on $T^*\mathbb{R}^3 \times T^*\mathbb{R}^3$. Furthermore

$$\nu : N \to T^*\mathbb{R}^3, (\mu_p, \mu_q, \mu_r; p, q, r)$$

is the canonical projection such that $\Omega \mid_N = \nu^*\omega$.

Now the Lagrangian submanifold of contact of M with its reflexions is given by reduction

$$L_M = \mu(N \cap \tilde{L}_M) \subset T^*\mathbb{R}^3, (\mu_p, \mu_q, \mu_r; p, q, r).$$

Its generating Morse family is

$$h(u, v, w, \lambda; p, q, r) = g(2p - u, 2q - v, 2r - w) + \lambda g(u, v, w),$$

with four Morse parameters u, v, w, λ.

We use subscripts u, v, w to denote derivatives with respect to first, second, third, variables of $g(u, v, w)$ and write $\overline{\mathbf{p}}$ for $(2p - u, 2q - v, 2r - w)$ and $\mathbf{u}$ for (u, v, w). Then L_M is given by equations:

$$\begin{aligned}
\mu_p &= \frac{\partial h}{\partial p} = 2g_u(\overline{\mathbf{p}}) \\
\mu_q &= \frac{\partial h}{\partial q} = 2g_v(\overline{\mathbf{p}}) \\
\mu_r &= \frac{\partial h}{\partial r} = 2g_w(\overline{\mathbf{p}}) \\
0 &= \frac{\partial h}{\partial u} = -g_u(\overline{\mathbf{p}}) + \lambda g_u(\mathbf{u}) \\
0 &= \frac{\partial h}{\partial v} = -g_v(\overline{\mathbf{p}}) + \lambda g_v(\mathbf{u}) \\
0 &= \frac{\partial h}{\partial w} = -g_w(\overline{\mathbf{p}}) + \lambda g_w(\mathbf{u}) \\
0 &= \frac{\partial h}{\partial \lambda} = g(\mathbf{u})
\end{aligned}$$

The additional equation for the bifurcation set ΣL_M is

$$\det \begin{pmatrix} g_{uu}(\overline{\mathbf{p}}) + \lambda g_{uu}(\mathbf{u}) & g_{uv}(\overline{\mathbf{p}}) + \lambda g_{uv}(\mathbf{u}) & g_{uw}(\overline{\mathbf{p}}) + \lambda g_{uw}(\mathbf{u}) & g_u(\mathbf{u}) \\ g_{uv}(\overline{\mathbf{p}}) + \lambda g_{uv}(\mathbf{u}) & g_{vv}(\overline{\mathbf{p}}) + \lambda g_{vv}(\mathbf{u}) & g_{vw}(\overline{\mathbf{p}}) + \lambda g_{vw}(\mathbf{u}) & g_v(\mathbf{u}) \\ g_{uw}(\overline{\mathbf{p}}) + \lambda g_{uw}(\mathbf{u}) & g_{vw}(\overline{\mathbf{p}}) + \lambda g_{vw}(\mathbf{u}) & g_{ww}(\overline{\mathbf{p}}) + \lambda g_{ww}(\mathbf{u}) & g_w(\mathbf{u}) \\ g_u(\mathbf{u}) & g_v(\mathbf{u}) & g_w(\mathbf{u}) & 0 \end{pmatrix} = 0.$$

These are identical with the conditions $F_x = F_y = F_{xx}F_{yy} - F_{xy}^2 = 0$ in §3, when M is given as a graph.

Thus, for the Lagrangian formulation, u, v, w are treated globally, not as a tubular neighbourhood of M but as a coordinates of the extended ambient space $\mathbb{R}^3 \times \mathbb{R}^3$.

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Department of Mathematical Sciences, The University of Liverpool, Liverpool L69 7ZL, U.K.
E-mail address: pjgiblin@liv.ac.uk

Instytut Matematyczny PAN, ul. Sniadeckich 8, 00-950 Warszawa, Poland and Wydzial Matematyki i Nauk Informacyjnych, Politechnika Warszawska, Pl. Politechniki 1, 00-661 warszawa, poland
E-mail address: janeczko@mini.pw.edu.pl