# The Undecidability of Arbitrary Arrow Update Logic 

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#### Abstract

Arbitrary Arrow Update Logic is a dynamic modal logic with a modality to quantify over arrow updates. Some properties of this logic have already been established, but until now it remained an open question whether the logic's satisfiability problem is decidable. Here, we show by a reduction of the tiling problem that the satisfiability problem of Arbitrary Arrow Update Logic is coRE hard, and therefore undecidable.


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## 1. Introduction

Update Logics are logics that provide an object language in which one can reason about the effect of changes to a model for that language. Such an underlying model is usually a Kripke model, equipped with a set of states and some relations between them. One of the most prominent examples of updates relate to the incorporation of new information. This field of studies has become popular as Dynamic Epistemic Logic (DEL) [7] in the past decades. In epistemic logic, states in a Kripke model represent a description of the world, and the relations represent 'possibility' (for belief) or 'indistinguishability' (for knowledge) relations. We say that $\square \varphi$ is true in state $s$ in model $\mathcal{M}$, written $\mathcal{M}, s \models \square \varphi$, if for all $t$, if $(s, t) \in R(a)$ then $\mathcal{M}, t \models \varphi$; that is, if in all states that are indistinguishable for agent $a$, formula $\varphi$ holds.

Keeping this epistemic setting in mind for the moment, Public Announcement Logic (PAL) [12, 4], studies updates in which certain states of $\mathcal{M}$ are removed: $[\varphi] \psi$ means that after the announcement $\varphi$ (which is interpreted as the operation in which only the $\varphi$-states are retained in the model), $\psi$ holds. For example, if $\varphi$ means "the door is locked" and $\psi$ means "agent $a$ believes

[^0]she cannot access the room", then $[\varphi] \psi$ means "after it is announced that the door is locked, agent $a$ will believe that she cannot access the room."

In Arrow Update Logic (AUL) [11], updates take the form of removing some access between states: $[(\varphi, a, \chi)] \psi$ denotes that if we only keep connections between two states if they are labelled $a$ and go from a $\varphi$ state to a $\chi$ state, $\psi$ will hold. For example, for the same meaning of $\varphi$ and $\psi$ as above, $[(\varphi, a, \varphi)] \psi$ means "if whenever the door is locked $(\varphi)$ agent $a$ is told so $(\varphi=\chi)$, then she (correctly) believes that she cannot access the room $(\psi)$ ".

Arrow updates are more powerful than public announcements; unlike public announcements, arrow updates can be used to model situations where different agents gain different information. For example, $a$ might be told whether the door is locked while $b$ is left in the dark on the matter. However, arrow updates can only remove arrows, they cannot $a d d$ them. As a result, arrow updates can only be used to model situations where the amount of uncertainty decreases. If we want to model situations where the amount of uncertainty increases we will need to use an even more powerful kind of update. Among these more powerful kinds of updates, the most commonly used are action models [4]. Action models can, for example, be used to model the event where, from agent $b$ 's perspective, it is possible that $a$ is told about whether the door is locked but it is also possible that $a$ is not told.

The logics using public announcements, arrow updates and action models are called Public Announcement Logic $(P A L)$, Arrow Update Logic $(A U L)$ and Action Model Logic $(A M L)^{1}$, respectively.

For each of these logics there is also an "arbitrary" version: for $P A L$ there is Arbitrary Public Announcement Logic (APAL) [3], for $A U L$ there is Arbitrary Arrow Update Logic ( $A A U L$ ) [8] and for $A M L$ there is Arbitrary Action Model Logic $(A A M L)$ [10]. These "arbitrary" logics contain an operator that quantifies over their non-arbitrary counterpart. So in $A P A L$ we have $[!] \psi$ if and only if $[\varphi] \psi$ holds for every $P A L$ formula $\varphi$, in $A A U L$ we have $[\downarrow] \psi$ if and only if $[U] \psi$ for every $A U L$ update $U$ and in $A A M L$ we have $[\times] \psi$ if and only if $[M] \psi$ for every $A M L$ action model $M$.

The logics $P A L, A U L$, and $A M L$ are equally expressive $[4,11]$. The arbitrary versions of the logics are not equally expressive, however. Under reasonable assumptions about the number of agents, the logics $A P A L$ and $A A U L$ are incomparable in expressivity [8], and they are both strictly more expressive than $A A M L[3,8]$, since the latter logic is no more expressive than basic modal logic [10].

Two other logics that are similar to these "arbitrary" logics are Group Announcement Logic ( $G A L$ ) [1] which allows quantification over a specific type of public announcements that are made by a group of agents, and Coalition Announcement Logic (CAL) [2] which allows us to ask whether there is some announcement for a group $G$ such that $\psi$ becomes true regardless of what all

[^1]agents outside of $G$ announce.
It is important to realise that the relevance of this kind of updates goes beyond the realm of epistemic interpretations. In normative reasoning for instance, eliminating (bad) states enables one to reason about deontically 'better' situations, and eliminating (bad) transitions enforces 'better' behaviour. For more on the epistemic and normative interpretations of updates, see [8, Section $2]$.

In this paper, we focus on $A A U L$. So we consider the operator $[\downarrow]$ that quantifies over all arrow updates.

Several technical results regarding $A A U L$ were established in [8]. Specifically, the following results were proven. Expressivity: [8] shows that, under some mild assumptions, $A P A L$ and $A A U L$ are incomparable over the class of all Kripke models. A case in which $A A U L$ is more expressive than $A P A L$ is also identified. Successively, $A A U L$ is compared to a number of other logics: it is established that $A A U L$ is incomparable to epistemic logic with common knowledge, but more expressive than $P A L$. It is known that basic epistemic logic, public announcement logic $P A L$, arbitrary action model logic $A A M L$, and refinement modal logic [6] are all equally expressive. As a corollary of this result we therefore also have that $A A U L$ is more expressive than $A A M L$. Model Checking: [8] shows that the model checking problem for $A A U L$ is PSPACE-complete. Axiomatisation: An (infinitary) proof system for $A A U L$ is introduced in [8] and its soundness and correctness (with respect to the set of intended models) is proven.

The question we address for $A A U L$ in this paper regards its decidability. For some of the 'arbitrary' logics mentioned above, namely $A P A L, G A L$, and $C A L$, the satisfiability problem is undecidable [9, 2]. The satisfiability problem of $A A M L$, on the other hand, is decidable [10]. For $A A U L$, it remained unknown whether the satisfiability problem is decidable. Here, we show that it is not decidable, by demonstrating that $A A U L$ 's satisfiability problem can encode the tiling problem [14]. Because the tiling problem is known to be co-RE complete [5], this shows that the satisfiability problem of $A A U L$ is co-RE hard.

The undecidability result is not surprising, but also not obvious. In $A P A L$, $G A L$, and $C A L$ the undecidability seems to originate in the semantic restriction of quantification: the quantification is only over quantifier-free formulas, not over all formulas; the resulting gaps in the quantification make these logics more expressive than epistemic logic, and this also seems to affect decidability. However, in $A A M L$ it does not matter if we so restrict the semantics of quantifiers: either way, we can eliminate quantifiers from the language by rewriting procedures, and epistemic logic is decidable. As $A A U L$ seems half-way between $A P A L$ and $A A M L$, the scales could have tilted both towards decidability and undecidability.

The undecidability proof presented here is similar to those in [9] and [2] in that they all use the "arbitrary" operators to encode a grid and then reduce the tiling problem to a satisfiability problem on that grid. The similarities between the proofs do not go far beyond that, however.

The structure of this paper is as follows. First, in Section 2 we introduce
the syntax and semantics of $A A U L$. Then, in Section 3 we provide a brief definition of the tiling problem and show that it can be encoded in the satisfiability problem of $A A U L$.

## 2. AAUL Syntax and Semantics

Let $\mathcal{P}$ be a countable set of propositional variables and $\mathcal{A}$ a finite set of agents. We assume that $|\mathcal{A}| \geq 6$.

Definition 1. The language $\mathcal{L}_{A A U L}$ of $A A U L$ is given by the following normal forms:

$$
\begin{aligned}
\varphi & ::=p|\neg \varphi| \varphi \wedge \varphi\left|\square_{a} \varphi\right|[U] \varphi \mid[\downarrow] \varphi \\
U & ::=(\varphi, a, \varphi) \mid U,(\varphi, a, \varphi)
\end{aligned}
$$

where $p \in \mathcal{P}$ and $a \in \mathcal{A}$. The language $\mathcal{L}_{A U L}$ is the fragment of $\mathcal{L}_{A A U L}$ that does not contain $[\uparrow]$.

We use $\vee, \rightarrow, \leftrightarrow, \diamond,\langle U\rangle,\langle\downarrow\rangle, \bigvee$ and $\bigwedge$ in the usual way as abbreviations. Furthermore, we slightly abuse notation by identifying the list $U=\left(\varphi_{1}, a_{1}, \psi_{1}\right), \cdots$, $\left(\varphi_{k}, a_{k}, \psi_{k}\right)$ with the set $U=\left\{\left(\varphi_{1}, a_{1}, \psi_{1}\right), \cdots,\left(\varphi_{k}, a_{k}, \psi_{k}\right)\right\}$. Finally, for $B \subseteq \mathcal{A}$ we use $(\varphi, B, \psi)$ as an abbreviation for $\{(\varphi, a, \psi) \mid a \in B\}$.
$A A U L$ is evaluated on standard multi-agent Kripke models.
Definition 2. A model $\mathcal{M}$ is a triple $\mathcal{M}=(W, R, V)$ where $W$ is a set of states, $R: \mathcal{A} \rightarrow 2^{W \times W}$ assigns to each agent an accessibility relation and $V: \mathcal{P} \rightarrow 2^{W}$ is a valuation.

Note that we are using the class of all Kripke models. This is unlike $A P A L$ and $G A L$, which are typically considered on the class of S5 models.

Now, let us consider the semantics of $A A U L$. We start by giving the formal definition, after the definition we briefly discuss the intuition behind some of the operators.

Definition 3. Let $\mathcal{M}=(W, R, V)$ be a model and let $w \in W$. The satisfaction relation $\models$ is given by

$$
\begin{array}{lll}
\mathcal{M}, w \models p & \text { iff } & w \in V(p) \\
\mathcal{M}, w \models \neg \varphi & \text { iff } \quad \mathcal{M}, w \neq \varphi \\
\mathcal{M}, w \models(\varphi \wedge \psi) & \text { iff } & \mathcal{M}, w \models \varphi \text { and } \mathcal{M}, w \models \psi \\
\mathcal{M}, w \models \square_{a} \varphi & \text { iff } & \mathcal{M}, v \models \varphi \text { for each } v \text { such that }(w, v) \in R(a) \\
\mathcal{M}, w \models[U] \varphi & \text { iff } & (\mathcal{M} * U), w \models \varphi \\
\mathcal{M}, w \models[\downarrow] \varphi & \text { iff } & \mathcal{M}, w \models[U] \varphi \text { for each } U \in L_{A U L}
\end{array}
$$

where $(\mathcal{M} * U)$ is given by:

$$
\begin{aligned}
& \mathcal{M} * U=\left(W, R^{U}, V\right) \\
& R^{U}(a)=\left\{\left(v, v^{\prime}\right) \in R(a) \mid \exists\left(\varphi, a, \varphi^{\prime}\right) \in U:\right.
\end{aligned}
$$

$$
\left.\left(\mathcal{M}, v \models \varphi \text { and } \mathcal{M}, v^{\prime} \models \varphi^{\prime}\right)\right\}
$$

A full discussion of the applications of $A A U L$ and of the intuitions behind the semantics of arrow updates and arbitrary arrow updates is outside the scope of this paper. For such a discussion, see [11] and [8]. However, in order to understand the undecidability proof it is important to grasp the semantics of $A A U L$. We therefore do provide a very brief explanation of the intuition behind and the semantics of $A A U L$.

Although our goal is to understand $A A U L$, is is useful to start by considering public announcements. We assume that the reader is familiar with public announcement logic, if not see for example [4]. A public announcement $[\psi]$ informs all agents that $\psi$ is true. As a result, every possible world that the agents previously considered possible that does not satisfy $\psi$ is rejected after the announcement, since it is incompatible with the new information. Semantically, this corresponds to a model $\mathcal{M}$ being transformed into a model $\mathcal{M} * \psi$ where all $\neg \psi$ states of $\mathcal{M}$ have been removed.

Like public announcements, arrow updates provide agents with new information. Unlike with public announcements, however, the new information provided by an arrow update can (i) differ per agent and (ii) differs per state. A typical example is a card game, where cards have been dealt face down. Now, agent $a$ picks up her hand of cards and looks at it. Obviously, the information that $a$ gains from this action is different than the information the other agents gain: $a$ learns what her cards are whereas the other agents only learn that a now knows what her cards are. It is perhaps less obvious that the information that $a$ gains also differs per state. Suppose that $a$ has been dealt the 7 of Hearts. Then by looking at her cards $a$ learns that she has the 7 of Hearts. If, on the other hand, $a$ has been dealt the 8 of Clubs, then she learns that she has the 8 of Clubs. Learning that you have the 7 of Hearts is different from learning that you have the 8 of Clubs, so the information given to $a$ depends on the state of the world.

With arrow updates we formalize the information that the agents gain in such a situation. In principle, we could do this in two ways: we could specify the things that are incompatible with the new information, or the things that are compatible. We choose to follow public announcements in this aspect, so just like $[\psi]$ says that the new information is compatible with $\psi$, we use an arrow update $U$ to specify the information that is compatible with $U$. Since the information gained in an arrow update can depend on the agent and on the current state, we use triples $(\varphi, a, \psi)$. We call such triples clauses; they can be read as "if the current state satisfies $\varphi$, then the information provided to agent $a$ is compatible with $\psi$." Semantically, the effect of a triple $(\varphi, a, \psi)$ is that every transition that is labeled $a$ and that goes from a $\varphi$ state to a $\psi$ state is retained.

An arrow update is a finite set of clauses, $U=\left\{\left(\varphi_{1}, a_{1}, \psi_{1}\right), \cdots,\left(\varphi_{k}, a_{k}, \psi_{k}\right)\right\}$ (where it is possible that $\varphi_{i}=\varphi_{j}, a_{i}=a_{j}$ or $\psi_{i}=\psi_{j}$ for $i \neq j$ ). This still leaves the decision of what to do if a state matches multiple clauses. Suppose, for example, that $\left(\varphi_{1}, a, \psi_{1}\right),\left(\varphi_{2}, a, \psi_{2}\right) \in U$ and that a state satisfies both $\varphi_{1}$ and $\varphi_{2}$. There are several options for how to interpret this situation, we choose to interpret it disjunctively: if a state satisfies $\varphi_{1}$ and $\varphi_{2}$, then any state that satisfies $\psi_{1}$ or $\psi_{2}$ is consistent with the new information.

On the semantical level, this means that $\mathcal{M} * U$ should contain exactly those arrows of $\mathcal{M}$ that match at least one clause of $U$, where we say that $\left(w_{1}, w_{2}\right) \in R(a)$ matches $\left(\varphi_{1}, a_{1}, \psi_{1}\right)$ if and only if $\mathcal{M}, w_{1} \models \varphi_{1}, a=a_{1}$ and $\mathcal{M}, w_{2} \models \psi_{1}$.

Arbitrary arrow updates then quantify over such arrow updates. However, in order to avoid circularity we restrict this quantification to those arrow updates that do not themselves contain an arbitrary arrow update $[\uparrow]$. So $\mathcal{M}, w \models[\uparrow] \varphi$ if and only if $\mathcal{M}, w \models[U] \varphi$ for all $U \in \mathcal{L}_{A U L}$.

## 3. Reducing the Tiling Problem

### 3.1. The Tiling Problem

We will prove the undecidability of $A A U L$ by a reduction of the tiling problem. The tiling problem was introduced in [14] and can be defined as follows.

Definition 4. Let $C$ be a finite set of colors. A tile type is a function $i$ : $\{$ north, south, east, west $\} \rightarrow C$.

An instance of the tiling problem is a finite set types of tile types. A solution to an instance of the tiling problem is a function tiling : $\mathbb{Z} \times \mathbb{Z} \rightarrow$ types such that, for every $\left(z_{1}, z_{2}\right) \in \mathbb{Z} \times \mathbb{Z}$,

$$
\begin{aligned}
\operatorname{tiling}\left(z_{1}, z_{2}\right)(\text { north }) & =\operatorname{tiling}\left(z_{1}, z_{2}+1\right)(\text { south }) \\
\operatorname{tiling}\left(z_{1}, z_{2}\right)(\text { east }) & =\operatorname{tiling}\left(z_{1}+1, z_{2}\right)(\text { west })
\end{aligned}
$$

The tiling problem was shown to be undecidable in [5]. In fact, the tiling problem is co-RE complete. Therefore, by reducing the tiling problem to the satisfiability problem of $A A U L$, we show that the latter problem is co-RE hard. Whether $A A U L$ 's satisfiability problem is co-RE is not currently known.

### 3.2. Encoding the Tiling Problem in AAUL

We want to encode the tiling problem in $A A U L$. So for every instance types of the tiling problem we define a formula $\chi_{\text {types }}$ of $A A U L$ that is satisfiable if and only if types can tile the plane. The strategy for doing this is as follows.

We represent each point of $\mathbb{Z} \times \mathbb{Z}$ by a state $(n, m)$. For every $i \in$ types we then use a propositional variable $p_{i}$ to represent "the current state contains a tile of type $i . "$ For every $c \in C$ we use propositional variables north (resp. south $_{c}$, east $_{c}$, west ${ }_{c}$ ) to represent the northern (resp. southern, eastern, western) edge of the current tile having color $c$. Finally, we use relations up, down, left and right to represent one tile being above, below, to the left and to the right, respectively, of the current tile.

In addition to the states $(n, m)$ that correspond to points in $\mathbb{Z} \times \mathbb{Z}$, we also use an auxiliary state $s_{0}$. This state $s_{0}$ is not part of the grid, and does not contain any tile. Instead, it is the state where $\chi_{\text {types }}$ will be evaluated. We therefore also refer to $s_{0}$ as the origin state. In order to distinguish $s_{0}$ from the states that are part of the grid we use the propositional variable $p$, which holds on $s_{0}$ but not on any $(n, m)$.

Now, given any state $(n, m)$, it is relatively easy to check whether the constraints of a tiling are satisfied locally. For example, $\bigvee_{i \in \text { types }} p_{i} \wedge \bigwedge_{i \neq j \in \text { types }} \neg\left(p_{i} \wedge\right.$ $p_{j}$ ) holds if and only if the current state has exactly one type of tile, and $\bigwedge_{c \in C}\left(\right.$ north $_{c} \rightarrow \square_{u p}$ south $\left._{c}\right)$ holds if and only if the northern color of the current tile matches the southern color of the tile above.

Making sure that the global constraints of a tiling are satisfied is harder, though. We do this in the following way. Firstly, we take a relation $R(b)$, and force it to connect between the auxiliary state $s_{0}$ and every state ( $n, m$ ). ${ }^{2}$ So while $\bigvee_{i \in \text { types }} p_{i} \wedge \bigwedge_{i \neq j \in \text { types }} \neg\left(p_{i} \wedge p_{j}\right)$ says that the current state has exactly one tile type, the formula $\square_{b} \bigvee_{i \in \text { types }} p_{i} \wedge \bigwedge_{i \neq j \in \text { types }} \neg\left(p_{i} \wedge p_{j}\right)$ says that all grid states have exactly one tile type. Secondly, we enforce a grid-like structure onto the domain.

We also use another relation $R(a)$ in order to simulate a Boolean variable: every state will have an $a$-arrow to itself (or at least, to a modally indistinguishable state). If an arrow update retains the $a$-arrow departing from a state $s$ we can see this as the variable being true on $s$, and if an arrow update removes the $a$-arrow departing from $s$ we can see this as the variable being false on $s$.

With the above in mind, let us define the formula $\chi_{\text {types }}$.
Definition 5. Let types be an instance of the tiling problem. The formula $\chi_{\text {types }}$ is given by

$$
\chi_{\text {types }}:=\psi_{\text {grid }} \wedge\left[U_{\text {grid }}\right] \psi_{\text {grid }} \wedge \psi_{\text {types }}
$$

where

$$
\begin{aligned}
\psi_{\text {grid }} & :=\psi_{1} \wedge \bigwedge_{x \in D}\left(\psi_{2, x} \wedge \psi_{3, x} \wedge \text { propd }_{x} \wedge \text { return }_{x}\right) \wedge \text { inverse } \wedge \text { commute } \\
U_{\text {grid }} & :=\left(p \rightarrow \psi_{\text {grid }}, a, \top\right),(\top, \mathcal{A} \backslash\{a\}, \top) \\
\psi_{\text {types }} & :=\text { one_tile } \wedge \text { one_color } \wedge \text { tile_colors } \wedge \text { tile_match }
\end{aligned}
$$

and

$$
\begin{aligned}
D & :=\{\text { up, down, left, right }\} \\
\psi_{1} & :=\diamond_{a} \top \wedge[\uparrow]\left(\diamond_{a} \top \rightarrow \square_{b} \square_{b} \diamond_{a} \top\right) \\
\psi_{2, x} & :=p \wedge \diamond_{b} \top \wedge \square_{b}\left(\neg p \wedge \diamond_{b} p \wedge \diamond_{x}\left(\neg p \wedge \diamond_{b} p\right) \wedge \square_{x} \neg p\right) \\
\psi_{3, x} & :=\square_{b}\left(r e f \wedge \square_{x} r e f \wedge[\downarrow]\left(\diamond_{x} \diamond_{a} \top \rightarrow \square_{x} \diamond_{a} \top\right)\right) \\
\text { ref }: & =\diamond_{a} \top \wedge[\downarrow] \square_{a} \diamond_{a} \top \\
\text { propd }_{x} & :=\square_{b}[\downarrow]\left(\left(\square_{a} \perp \wedge \diamond_{x} \diamond_{a} \top \wedge \diamond_{b}\left(\diamond_{b} \top \wedge \square_{b} \diamond_{a} \top\right) \wedge\right.\right. \\
& \left.\left.\langle\downarrow\rangle\left(\diamond_{x} \diamond_{a} \top \wedge \diamond_{b} \diamond_{b} \square_{a} \perp\right)\right) \rightarrow\left[U_{x}\right]\langle\uparrow\rangle\left(\diamond_{x} \diamond_{a} \top \wedge \diamond_{b} \diamond_{b} \square_{a} \perp\right)\right) \\
U_{x}: & =\left(p \vee \square_{a} \perp, b, \top\right),(\top, a, \top),\left(\square_{a} \perp, x, \top\right)
\end{aligned}
$$

[^2]\[

$$
\begin{aligned}
& \text { return }_{x}:=\square_{b}\langle\downarrow\rangle\left(\square_{a} \perp \wedge \diamond_{x} \diamond_{a} \top \wedge \diamond_{b}\left(\diamond_{b} \top \wedge \square_{b} \diamond_{a} \top\right) \wedge\right. \\
& \left.[\downarrow]\left(\diamond_{x} \diamond_{a} \top \rightarrow \square_{b} \square_{b} \diamond_{a} \top\right)\right) \\
& \text { inverse }:=\square_{b}[\uparrow]\left(\square _ { a } \perp \rightarrow \left(\square_{\text {up }} \square_{\text {down }} \square_{a} \perp \wedge \square_{\text {down }} \square_{u p} \square_{a} \perp \wedge\right.\right. \\
& \left.\left.\square_{\text {left }} \square_{\text {right }} \square_{a} \perp \wedge \square_{\text {right }} \square_{\text {left }} \square_{a} \perp\right)\right) \\
& \text { commute }:=\square_{b}[\downarrow] \bigwedge_{(x, y) \in E}\left(\diamond_{x} \diamond_{y} \square_{a} \perp \rightarrow \square_{y} \square_{x} \square_{a} \perp\right) \\
& E:=\{(\text { up }, \text { left }),(\text { up }, \text { right }),(\text { down }, \text { left }),(\text { down }, \text { right }), \\
& \text { (left, up }),(\text { left , down }),(\text { right }, \text { up }),(\text { right }, \text { down })\} \\
& \text { one_tile }:=\square_{b}\left(\bigvee_{i \in \text { tiles }} p_{i} \wedge \bigwedge_{i \neq j \in \text { tiles }} \neg\left(p_{i} \wedge p_{j}\right)\right) \\
& \text { one_color }:=\square_{b} \bigwedge_{c \in C}\left(\text { north }_{c} \rightarrow \bigwedge_{d \in C \backslash\{c\}} \neg \text { north }_{d}\right) \wedge \\
& \square_{b} \bigwedge_{c \in C}\left(\text { south }_{c} \rightarrow \bigwedge_{d \in C \backslash\{c\}} \neg \text { south }_{d}\right) \wedge \\
& \square_{b} \bigwedge_{c \in C}\left(\text { east }_{c} \rightarrow \bigwedge_{d \in C \backslash\{c\}} \neg \text { east }_{d}\right) \wedge \\
& \square_{b} \bigwedge_{c \in C}\left(\text { west }_{c} \rightarrow \bigwedge_{d \in C \backslash\{c\}} \neg \text { west }_{d}\right) \\
& \text { tile_colors }:=\square_{b} \bigwedge_{i \in \text { tiles }}\left(p _ { i } \rightarrow \left(\text { north }_{i(\text { north })} \wedge\right.\right. \\
& \text { south } \left.\left._{i(\text { south })} \wedge \text { east }_{i(\text { east })} \wedge \text { west }_{i(\text { west })}\right)\right) \\
& \text { tile_match }:=\square_{b} \bigwedge_{c \in C}\left(\left(\text { north }_{c} \rightarrow \square_{\text {up }} \text { south }_{c}\right) \wedge\left(\text { west }_{c} \rightarrow \square_{\text {left }} \text { east }_{c}\right)\right)
\end{aligned}
$$
\]

Note that the formulas $\psi_{2, x}, \psi_{3, x}, \operatorname{propd}_{x}$ and return $_{x}$ and the update $U_{x}$ contain a parameter $x$, which ranges over the four directions $D=\{u p$, down, left, right $\}$.

The formula $\psi_{\text {grid }}$, together with $\left[U_{\text {grid }}\right] \psi_{\text {grid }}$, encodes a grid. The formula $\psi_{\text {types }}$ then ensures that the grid is tiled with tiles from types. The formula $\chi_{\text {grid }}$ may look rather intimidating, but we will discuss the various subformulas in detail and explain what they do.

We want to show that $\chi_{\text {types }}$ is satisfiable if and only if types can tile $\mathbb{Z} \times \mathbb{Z}$. We start by showing that if such a tiling exists, then $\chi_{\text {types }}$ is satisfiable.
Lemma 1. Suppose types can tile $\mathbb{Z} \times \mathbb{Z}$. Then $\chi_{\text {types }}$ is satisfiable.
Proof. Let tiling be the tiling, let $p_{n, m} \in \mathcal{P}$ for every $n, m \in \mathbb{Z}$ and let $\mathcal{M}=$ $(S, R, V)$ be the following, quite straightforward, encoding of tiling:

- $S=(\mathbb{Z} \times \mathbb{Z}) \cup s_{0}$
- $R(a)=\{(s, s) \mid s \in S\}$


Figure 1: The model used in Lemma 1.

- $R(b)=\left\{\left(s_{0},(n, m)\right) \mid n, m \in \mathbb{Z}\right\} \cup\left\{\left((n, m), s_{0}\right) \mid n, m \in \mathbb{Z}\right\}$
- $R(u p)=\{((n, m),(n, m+1)) \mid n, m \in \mathbb{Z}\}$
- $R($ down $)=\{((n, m),(n, m-1)) \mid n, m \in \mathbb{Z}\}$
- $R($ left $)=\{((n, m),(n-1, m)) \mid n, m \in \mathbb{Z}\}$
- $R($ right $)=\{((n, m),(n+1, m)) \mid n, m \in \mathbb{Z}\}$
- $V(p)=\left\{s_{0}\right\}$
- $V\left(p_{i}\right)=\{(n, m) \mid \operatorname{tiling}(n, m)=i\}$ for $i \in$ tiles
- $V\left(\right.$ north $\left._{c}\right)=\{(n, m) \mid(($ tiling $)(n, m))($ north $)=c\}$ for $c \in C$
- $V\left(\right.$ south $\left._{c}\right)=\{(n, m) \mid(($ tiling $)(n, m))($ south $)=c\}$ for $c \in C$
- $V\left(e^{e a s t}{ }_{c}\right)=\{(n, m) \mid(($ tiling $)(n, m))($ east $)=c\}$ for $c \in C$
- $V\left(\right.$ west $\left._{c}\right)=\{(n, m) \mid(($ tiling $)(n, m))($ west $)=c\}$ for $c \in C$
- $V\left(p_{n, m}\right)=\{(n, m)\}$

The frame of this model (i.e. the model without the valuation) is also drawn in Figure 1.

As mentioned above, the state $s_{0}$ is special: it is the origin state, and the only state that does not have a tile type associated with it. The propositional variable $p$ is used to identify this special state. First, we will show that $\mathcal{M}, s_{0}=\psi_{\text {grid }}$.

There is an $a$-arrow from $s_{0}$ to itself, so $\mathcal{M}, s_{0} \vDash \diamond_{a} \top$. Furthermore, the only $b$ - $b$-successor of $s_{0}$ is $s_{0}$ itself. It follows that every arrow update that retains the $a$-arrow from $s_{0}$ also retains the $a$-arrow from every $b$ - $b$-successor of $s_{0}$. So $\mathcal{M}, s_{0} \vDash[\uparrow]\left(\diamond_{a} \top \rightarrow \square_{b} \square_{b} \diamond_{a} \top\right)$. We have shown that $s_{0}$ satisfies both conjuncts of $\psi_{1}$, so $\mathcal{M}, s_{0}=\psi_{1}$.

The state $s_{0}$ satisfies $p$ and it has at least one $b$-successor, so $\mathcal{M}, s_{0} \models p \wedge \diamond_{b} \top$. Every state $(n, m)$ satisfies $\neg p$ and has a $b$-arrow to the $p$-state $s_{0}$. Furthermore, for every $x \in D$ the state $(n, m)$ has exactly one $x$-successor $\left(n^{\prime}, m^{\prime}\right)$, that also satisfies $\neg p \wedge \diamond_{b} p$. Since every $b$-successor of $s_{0}$ is a state $(n, m)$, it follows that $s_{0}$ satisfies $p \wedge \diamond_{b} \top \wedge \square_{b}\left(\neg p \wedge \diamond_{b} p \wedge \diamond_{x}\left(\neg p \wedge \diamond_{b} p\right) \wedge \square_{x} \neg p\right)$ for every $x, \in D$, so $\mathcal{M}, s_{0}=\psi_{2, x}$.

Now, consider the formula ref. Every state $s$ of $\mathcal{M}$ has exactly one outgoing $a$-arrow, and that $a$-arrow goes to $s$ itself. It is therefore impossible to have an arrow update that retains the $a$-arrow from $s$ to one of its $a$-successors $s^{\prime}$ while removing all $a$-arrows from $s^{\prime}$. It follows that every state of $\mathcal{M}$ satisfies $\diamond_{a} \top \wedge[\uparrow] \square_{a} \diamond_{a} \top$, so all states satisfy ref.

Now, take any direction $x \in D$. From the fact that every state of $\mathcal{M}$ satisfies $r e f$, it follows that every state $(n, m)$ satisfies ref $\wedge \square_{x} r e f$. Furthermore, every state $(n, m)$ has exactly one $x$-successor, so every arrow update that retains the $a$-arrow on one of the $x$-successors of $(n, m)$ retains the arrow on every $x$-successor of $(n, m)$. In other words, we have $\mathcal{M},(n, m) \vDash[\uparrow]\left(\diamond_{x} \diamond_{a} \top \rightarrow\right.$ $\left.\square_{x} \diamond_{a} \top\right)$. Together with the fact that $(n, m)$ satisfies ref $\wedge \square_{x} r e f$, as discussed earlier, this implies that $\mathcal{M},(n, m) \models\left(\operatorname{ref} \wedge \square_{x} r e f \wedge[\uparrow]\left(\diamond_{x} \diamond_{a} \top \rightarrow \square_{x} \diamond_{a} \top\right)\right)$. The above holds for every state $(n, m)$ and every $x \in D$, so $\mathcal{M}, s_{0} \models \psi_{3, x}$ for every $x \in D$.

Let us then consider $\operatorname{propd}_{x}$. For ease of notation we show only that propd $d_{\text {right }}$ holds; the other directions can be proven in the same way. The initial $\square_{b}$ operator of propd $_{\text {right }}$ takes us to any state $(n, m)$. To show is that

$$
\begin{aligned}
\mathcal{M},(n, m) \models & {[\uparrow]\left(\left(\square_{a} \perp \wedge \diamond_{\text {right }} \diamond_{a} \top \wedge \diamond_{b}\left(\diamond_{b} \top \wedge \square_{b} \diamond_{a} \top\right) \wedge\right.\right.} \\
& \left.\left.\langle\uparrow\rangle\left(\diamond_{\text {right }} \diamond_{a} \top \wedge \diamond_{b} \diamond_{b} \square_{a} \perp\right)\right) \rightarrow\left[U_{\text {right }}\right]\langle\uparrow\rangle\left(\diamond_{\text {right }} \diamond_{a} \top \wedge \diamond_{b} \diamond_{b} \square_{a} \perp\right)\right)
\end{aligned}
$$

Let $U_{1}$ be any arrow update such that the antecedent in the above formula is true, i.e. any arrow update such that

$$
\begin{aligned}
\mathcal{M} * U_{1},(n, m)= & \left(\square_{a} \perp \wedge \diamond_{\text {right }} \diamond_{a} \top \wedge \diamond_{b}\left(\diamond_{b} \top \wedge \square_{b} \diamond_{a} \top\right) \wedge\right. \\
& \langle\uparrow\rangle\left(\diamond_{\text {right }} \diamond_{a} \top \wedge \diamond_{b} \diamond_{b} \square_{a} \perp\right) .
\end{aligned}
$$

By $\square_{a} \perp$ the $a$-arrow on $(n, m)$ was removed by $U_{1}$. By $\diamond_{\text {right }} \diamond_{a} \top$ the right-arrow to $(n+1, m)$ and the $a$-arrow on $(n+1, m)$ are retained. By $\diamond_{b}\left(\diamond_{b} \top \wedge \square_{b} \diamond_{a} \top\right)$, the arrow from $(n, m)$ to $s_{0}$ is retained, as well as a $b$-arrow from $s_{0}$ to at least one state $\left(n^{\prime}, m^{\prime}\right)$. Furthermore, every $b$-arrow from $s_{0}$ that is retained, points to a state that still has its $a$-arrow.

The formula $\langle\downarrow\rangle\left(\diamond_{\text {right }} \diamond_{a} \top \wedge \diamond_{b} \diamond_{b} \square_{a} \perp\right)$ then states that there is some update $U_{2}$ such that $\left(\mathcal{M} * U_{1}\right) * U_{2},(n, m) \vDash \diamond_{\text {right }} \diamond_{a} \top \wedge \diamond_{b} \diamond_{b} \square_{a} \perp$. Note that this is impossible if $(n+1, m)$ is the only $b$ - $b$-successor of $(n, m)$ in $\mathcal{M} * U_{1}$, since then
$(n+1, m)$ would need to satisfy $\diamond_{a} \top$ (due to ( $n, m$ ) satisfying $\diamond_{\text {right }} \diamond_{a} \top$ ) as well as $\square_{a} \perp$ (due to ( $n, m$ ) satisfying $\diamond_{b} \diamond_{b} \square_{a} \perp$ ) in $\left(\mathcal{M} * U_{1}\right) * U_{2}$.

To show is that for every such $U_{1}$, we have

$$
\mathcal{M} * U_{1},(n, m) \models\left[U_{\text {right }}\right]\langle\downarrow\rangle\left(\diamond_{\text {right }} \diamond_{a} \top \wedge \diamond_{b} \diamond_{b} \square_{a} \perp\right),
$$

where $U_{\text {right }}=\left(p \vee \square_{a} \perp, b, \top\right),(\top, a, \top),\left(\square_{a} \perp\right.$, right,$\left.\top\right)$. Note that $U_{\text {right }}$ retains all $a$-arrows, the right-arrow from $(n, m)$ to $(n+1, m)$ and the $b$-arrow from $(n, m)$ to $s_{0}$ (because $\mathcal{M} * U_{1},(n, m) \models \square_{a} \perp$ ) as well as all $b$-arrows from $s_{0}$ (because $\mathcal{M} * U_{1}, s_{0} \models p$ ).

Now, let $U_{2}:=\left(p_{n+1, m}, a, \top\right),(\top,\{b$, right $\}, \top)$. This update retains all $b$ - and right-arrows as well as the $a$-arrow on $(n+1, m)$ while removing all other $a$-arrows. Since $(n, m)$ had at least one $b$ - $b$-successor $\left(n^{\prime}, m^{\prime}\right) \neq(n, m)$ in $\left(\mathcal{M} * U_{1}\right) * U_{\text {right }}$, it follows that, in $\left(\left(\mathcal{M} * U_{1}\right) * U_{\text {right }}\right) * U_{2}$ the state $s_{0}$ has a rightsuccessor that satisfies $\nabla_{a} \top$ (namely $(n, m)$ ) and a $b$ - $b$-successor that satisfies $\square_{a} \perp\left(\right.$ namely $\left.\left(n^{\prime}, m^{\prime}\right)\right)$. We therefore have $\left(\left(\mathcal{M} * U_{1}\right) * U_{\text {right }}\right) * U_{2},(n, m) \models$ $\diamond_{\text {right }} \diamond_{a} \top \wedge \diamond_{b} \diamond_{b} \square_{a} \perp$ and therefore $\left.\mathcal{M} * U_{1},(n, m) \vDash\left[U_{\text {right }}\right]\langle\uparrow\rangle\right\rangle\left(\diamond_{\text {right }} \diamond_{a} \top \wedge\right.$ $\diamond_{b} \diamond_{b} \square_{a} \perp$ ). We have now shown that $\mathcal{M}, s_{0}=$ propd $_{\text {right }}$.

We continue with return $n_{x}$. Once again, we consider the case $x=$ right, the other directions can be proven in a similar way. The formula return right starts with a $\square_{b}$ operator, so take any $b$-successor $(n, m)$ of $s_{0}$. Furthermore, let $U_{1}:=\left(p_{n+1, m}, a, \top\right),\left(\top, b, p \vee p_{n+1, m}\right),(\top$, right,$\top)$. So in $\mathcal{M} * U_{1}$ the state $(n+1, m)$ is the only one to still have its $a$-arrow, and it is also the only rightand $b$ - $b$-successor of $(n, m)$. Hence $\mathcal{M} * U_{1},(n, m) \vDash \square_{a} \perp \wedge \diamond_{\text {right }} \diamond_{a} \top \wedge \diamond_{b}\left(\diamond_{b} \top \wedge\right.$ $\square_{b} \nabla_{a} \top$ ). Furthermore, any arrow update that removes the $a$-arrow from the right-successor of $(n, m)$ also removes the $a$-arrow from the $b$ - $b$-successor of $(n, m)$, since those successors are the same state $(n+1, m)$. Therefore, $\mathcal{M} *$ $U_{1},(n, m) \vDash[\downarrow]\left(\diamond_{\text {right }} \diamond_{a} \top \rightarrow \square_{b} \square_{b} \diamond_{a} \top\right)$. Putting these things together, we obtain

$$
\begin{aligned}
\mathcal{M},(n, m)= & {\left[U_{1}\right]\left(\square_{a} \perp \wedge \diamond_{\text {right }} \diamond_{a} \top \wedge \diamond_{b}\left(\diamond_{b} \top \wedge \square_{b} \diamond_{a} \top\right) \wedge\right.} \\
& {\left.[\downarrow]\left(\diamond_{\text {right }} \diamond_{a} \top \rightarrow \square_{b} \square_{b} \diamond_{a} \top\right)\right) }
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\mathcal{M},(n, m) \vDash & \langle\uparrow\rangle\left(\square_{a} \perp \wedge \diamond_{\text {right }} \diamond_{a} \top \wedge \diamond_{b}\left(\diamond_{b} \top \wedge \square_{b} \diamond_{a} \top\right) \wedge\right. \\
& {\left.[\uparrow]\left(\diamond_{\text {right }} \diamond_{a} \top \rightarrow \square_{b} \square_{b} \diamond_{a} \top\right)\right) . }
\end{aligned}
$$

Since this holds for any $b$-successor $(n, m)$ of $s_{0}$, it follows that return right holds in $s_{0}$.

We continue with inverse. In $\mathcal{M}$, the relations $u p$ and down are each others inverses, as are left and right. Furthermore, all four direction relations are functions. It follows immediately that, for every $(n, m)$, we have $\mathcal{M},(n, m) \models[\downarrow]\left(\square_{a} \perp \rightarrow \square_{\text {right }} \square_{\text {left }} \square_{a} \perp\right)$, and similarly for the other combinations of directions. So we have $\mathcal{M}, s_{0}=$ inverse.

Similarly, in $\mathcal{M}$ we have $R($ right $) \circ R(u p)=R(u p) \circ R($ right $)$, and the same for the other directions. It follows that $\mathcal{M}, s_{0} \models \square_{b}[\uparrow] \bigwedge_{(x, y) \in E}\left(\diamond_{x} \diamond_{y} \square_{a} \perp \rightarrow\right.$ $\left.\square_{y} \square_{x} \square_{a} \perp\right)$.

We have now considered all the conjuncts of $\psi_{\text {grid }}$, so we have shown that $\mathcal{M}, s_{0} \vDash \psi_{\text {grid }}$. Furthermore, the only $p$ state in $\mathcal{M}$ is the state $s_{0}$, and $s_{0}$ satisfies $\psi_{\text {grid }}$. The update $U_{\text {grid }}=\left(p \rightarrow \psi_{\text {grid }}, a, \top\right),(\top, \mathcal{A} \backslash\{a\}, \top)$ therefore retains all arrows. So $\psi_{\text {grid }}$ remains true after this update, which gives us $\mathcal{M}, s_{0}=\left[U_{\text {grid }}\right] \psi_{\text {grid }}$.

This only leaves the formula $\psi_{\text {types }}$. This formula simply encodes that tiling is a tiling on $\mathbb{Z} \times \mathbb{Z}$, so it is straightforward to verify that $\mathcal{M}, s_{0} \models \psi_{\text {types }}$.

We have now shown that all the conjuncts of $\chi_{\text {types }}$ are satisfied in $\mathcal{M}, s_{0}$, so $\mathcal{M}, s_{0} \models \chi_{\text {types }}$, which was to be shown.

We have shown that if types can tile the plane, then $\chi_{\text {types }}$ is satisfiable. Left to show is that if $\chi_{\text {types }}$ is satisfiable, then types can tile the plane. The main strategy that we use in this proof is to show that the subformulas $\psi_{\text {grid }}$ and $\left[U_{\text {grid }}\right] \psi_{\text {grid }}$ of $\chi_{\text {types }}$ only hold in models that resemble the grid-like model shown in Figure 1. The subformula $\psi_{\text {types }}$ of $\chi_{\text {types }}$ then only holds if the grid can be tiled with types.

Unfortunately, there is one significant complication. The language of $A A U L$ is not expressive enough to guarantee uniqueness of states. So, for example, a state ( $n, m$ ) may have two (or more) different right-successors, $(n+1, m)$ and $(n+1, m)^{\prime}$. We can, however, use $\psi_{\text {grid }}$ to show that if $(n+1, m)$ and $(n+1, m)^{\prime}$ are both right-successors of $(n, m)$, then $(n+1, m)$ and $(n+1, m)^{\prime}$ are modally indistinguishable. So a pointed model where $\chi_{\text {types }}$ is satisfied resembles the model from Figure 1 modulo modal indistinguishability. This suffices to show that $\chi_{\text {types }}$ is only satisfiable if types can tile $\mathbb{Z} \times \mathbb{Z}$.

In Lemma 5 we will prove that satisfiability of $\chi_{\text {types }}$ implies that types can tile the plane. Before doing so, however, it is useful to consider a few auxiliary definitions and lemmas.

Definition 6. Fix a state $s_{0}$, and let $\mathcal{M}=(S, R, V)$ be any model that has $s_{0}$ as one of its states. The set $\left[s_{0}\right]_{\mathcal{M}}$ is the smallest set of states of $\mathcal{M}$ such that

- $s_{0} \in\left[s_{0}\right]_{\mathcal{M}}$ and
- if $s \in\left[s_{0}\right]_{\mathcal{M}}$ and $\left(s, s^{\prime}\right) \in R_{b} \circ R_{b}$ then $s^{\prime} \in\left[s_{0}\right]_{\mathcal{M}}$.

Lemma 2. Suppose $\mathcal{M}, s_{0} \models \psi_{\text {grid }}$, and let $s$ be any b-b-successor of $s_{0}$. Then $s_{0}$ and $s$ are modally indistinguishable.

Proof. Suppose towards a contradiction that there is a modal formula $\delta$ such that $\mathcal{M}, s_{0}=\delta$ and $\mathcal{M}, s \not \vDash \delta$.

From $\mathcal{M}, s_{0} \models \psi_{\text {grid }}$ it follows that, in particular, $\mathcal{M}, s_{0} \models \psi_{1}$ and therefore (by definition) $\mathcal{M}, s_{0} \vDash \diamond_{a} \top \wedge[\uparrow]\left(\diamond_{a} \top \rightarrow \square_{b} \square_{b} \diamond_{a} \top\right)$. The $\diamond_{a} \top$ subformula implies that $s_{0}$ has at least one $a$-successor $s_{0}^{\prime}$.

Consider the update $U=(\delta, a, \top),(\top, b, \top)$. This $U$ retains all $b$-arrows, so $s$ is still a $b$ - $b$-successor of $s_{0}$ in the updated model $\mathcal{M} * U$. Furthermore, since $U$ retains exactly those $a$-arrows that depart from a $\delta$-world, we have $\mathcal{M} * U, s_{0} \models$ $\diamond_{a} \top$ and $\mathcal{M} * U, s \not \vDash \diamond_{a} \top$. It follows that $\mathcal{M}, s_{0} \vDash \neg[U]\left(\diamond_{a} \top \rightarrow \square_{b} \square_{b} \diamond_{a} \top\right)$, contradicting $\mathcal{M}, s_{0} \models[\uparrow]\left(\diamond_{a} \top \rightarrow \square_{b} \square_{b} \diamond_{a} \top\right)$.

Our assumption that a distinguishing modal formula $\delta$ exists must therefore have been false, so $s_{0}$ and $s$ are modally indistinguishable.

Lemma 3. If $\mathcal{M}, s_{0} \vDash \psi_{\text {grid }}$, then all elements of $\left[s_{0}\right]_{\mathcal{M}}$ are modally indistinguishable from $s_{0}$.

Proof. Let $\mathcal{M}, s_{0}$ be any pointed model such that $\mathcal{M}, s_{0} \models \psi_{\text {grid }}$, and let $s$ be any element of $\left[s_{0}\right]_{\mathcal{M}}$. Then there is a sequence $s_{0}, s_{1}, \cdots, s_{n}$ of states such that $s=s_{n}$ and, for every $0 \leq i<n$, the state $s_{i+1}$ is a $b$ - $b$-successor of $s_{i}$.

We show that $s$ is modally indistinguishable from $s_{0}$, by induction on $n$. As base case, suppose $n=1$. Then it follows immediately from Lemma 2 that $s$ and $s_{0}$ are modally indistinguishable. Assume then as induction hypothesis that $n>1$ and that $s_{0}, s_{1}, \cdots, s_{n-1}$ are modally indistinguishable from $s_{0}$.

If $s$ is modally distinguishable from $s_{0}$, then there is some modal formula $\delta$ that holds on $s_{0}$ but not on $s$. Since $s_{0}$ is modally indistinguishable from its $b$ - $b$-successors, we also have $\mathcal{M}, s_{0}=\square_{b} \square_{b} \delta$. However, since $s$ is a $b$ - $b$-successor of $s_{n-1}$ and $\delta$ does not hold on $s$, we have $\mathcal{M}, s_{n-1} \not \vDash \square_{b} \square_{b} \delta$. This implies that there is a modal formula that distinguishes between $s_{0}$ and $s_{n-1}$, contradicting the induction hypothesis.

It follows that there can be no modal $\delta$ that holds on $s_{0}$ but not on $s$. This completes the induction step and thereby the proof.

Lemma 4. If $\mathcal{M}, s_{0} \models \psi_{\text {grid }} \wedge\left[U_{\text {grid }}\right] \psi_{\text {grid }}$, then all elements of $\left[s_{0}\right]_{\mathcal{M}}$ satisfy $\psi_{\text {grid }}$.
Proof. First, note that $\mathcal{M}, s_{0} \models p$, because $\mathcal{M}, s_{0} \models \psi_{\text {grid }}$ and therefore $\mathcal{M}, s_{0} \models$ $\psi_{2, x}$. By Lemma 3, all elements of $\left[s_{0}\right]_{\mathcal{M}}$ are modally indistinguishable, so all of them satisfy $p$.

Now, take any $s \in\left[s_{0}\right]_{\mathcal{M}}$. Then there is a finite sequence $s_{0}, s_{1}, \cdots, s_{n}$ of states such that $s=s_{n}$ and for every $0 \leq i<n$ the state $s_{i+1}$ is a, $b$ - $b$-successor of $s_{i}$.

Recall that $U_{\text {grid }}=\left(p \rightarrow \psi_{\text {grid }}, a, \top\right),(\top, \mathcal{A} \backslash\{a\}, \top)$. The $b$-arrows on the path from $s_{0}$ to $s$ are retained by $U_{\text {grid }}$ since that update retains all $b$-arrows. This implies that $s \in\left[s_{0}\right]_{\mathcal{M} * U_{\text {grid }}}$. Furthermore, $\mathcal{M} * U_{\text {grid }}, s_{0} \models \psi_{\text {grid }}$ because, by the assumptions of the lemma, $\mathcal{M}, s_{0} \models\left[U_{\text {grid }}\right] \psi_{\text {grid }}$. Lemma 3 therefore implies that $s_{0}$ and $s$ are modally indistinguishable in $\mathcal{M} * U_{\text {grid }}$.

By the assumptions of the lemma, $\mathcal{M}, s_{0} \models \psi_{\text {grid }}$. This implies that $s_{0}$ has at least one $a$-successor in $\mathcal{M}$, and that the $a$-arrow to this successor is retained by the update. So $\mathcal{M} * U_{\text {grid }}, s_{0} \models \diamond_{a} \top$.

Suppose towards a contradiction that $\mathcal{M}, s \not \vDash \psi_{\text {grid }}$. Then $\mathcal{M}, s \not \vDash p \rightarrow$ $\psi_{\text {grid }}$, so the update $U_{\text {grid }}$ would remove all $a$-arrows from $s$ and we would have $\mathcal{M} * U_{\text {grid }}, s \not \vDash \diamond_{a} \top$. This would contradict the modal indistinguishability of $s_{0}$ and $s$ in $\mathcal{M} * U_{\text {grid }}$. It follows that $\mathcal{M}, s \models \psi_{\text {grid }}$, which is what was to be shown.

Having dealt with these preliminaries, we can show that satisfiability of $\chi_{\text {types }}$ implies that types can tile $\mathbb{Z} \times \mathbb{Z}$.

Lemma 5. If $\chi_{\text {types }}$ is satisfiable, then types can tile $\mathbb{Z} \times \mathbb{Z}$.
Proof. Let $\mathcal{M}, s_{0}$ be any pointed model such that $\mathcal{M}, s_{0} \vDash \chi_{\text {types }}$. Then, in particular, $\mathcal{M}, s_{0} \models \psi_{\text {grid }} \wedge\left[U_{\text {grid }}\right] \psi_{\text {grid }}$ and therefore all elements of $\left[s_{0}\right]_{\mathcal{M}}$ satisfy $\psi_{\text {grid }}$ and are modally indistinguishable from each other.

We will now explain that the fact that $\psi_{\text {grid }}$ holds on all of $\left[s_{0}\right]_{\mathcal{M}}$ implies that $\mathcal{M}$ is "grid-like." During this explanation, it is useful to draw diagrams of the model $\mathcal{M}$. Because $\mathcal{M}$ may be infinitely large, it is not very practical to draw the entire model, so we will only draw the parts that are relevant to the part of the proof they are intended to illustrate.

Take any $s \in\left[s_{0}\right]_{\mathcal{M}}$. We start by considering the formula $\psi_{2, x}$, that holds in $s$ for every $x \in D$. So, by the definition of $\psi_{2, x}$, we have

$$
\mathcal{M}, s \models p \wedge \diamond_{b} \top \wedge \square_{b}\left(\neg p \wedge \diamond_{b} p \wedge \diamond_{x}\left(\neg p \wedge \diamond_{b} p\right) \wedge \square_{x} \neg p\right)
$$

This implies that $s$ satisfies $p$, that $s$ has at least one $b$-successor and that every $b$-successor $s_{1}$ of $s$ satisfies $\neg p$. So far, this can be drawn as follows.


Furthermore, this $s_{1}$ has at least one $b$-successor $s^{\prime}$ that satisfies $p$, and one $x$-successor $s_{2}$ that satisfies $\neg p \wedge \diamond_{b} p$. Our diagram becomes the following.


Now, consider $\psi_{3, x}$, which is also one of the conjuncts of $\psi_{\text {grid }}$ and therefore holds on $s$. It states that

$$
\mathcal{M}, s \models \square_{b}\left(\operatorname{ref} \wedge \square_{x} \operatorname{ref} \wedge[\mathcal{\uparrow}]\left(\diamond_{x} \diamond_{a} \top \rightarrow \square_{x} \diamond_{a} \top\right)\right)
$$

So $s_{1}$ and $s_{2}$ both satisfy ref. This implies that $s_{1}$ and $s_{2}$ both have at least one outgoing $a$-arrow. Suppose now, towards a contradiction, that $s_{1}$ has an $a$-successor $s_{1}^{\prime}$ that is modally distinguishable from $s_{1}$. Then there is some modal formula $\delta$ that holds on $s_{1}$ but not on $s_{1}^{\prime}$, so we would have $\mathcal{M}, s_{1} \models$
$[(\delta, a, \top)] \neg \square_{a} \diamond_{a} \top$, contradicting the $[\downarrow] \square_{a} \diamond_{a} \top$ part of ref. It follows that $s_{1}$ is modally indistinguishable from all its $a$-successors. The same reasoning shows that $s_{2}$ is also modally indistinguishable from its $a$-successors.

Additionally, $s_{1}$ satisfies $[\uparrow]\left(\diamond_{x} \diamond_{a} \top \rightarrow \square_{x} \diamond_{a} \top\right)$. This implies that all $x$ successors of $s_{1}$ are modally indistinguishable from one another, since otherwise there would be some arrow update that retains the $a$-arrow from $s_{2}$ while removing all $a$-arrows from at least one of the other $x$-successors of $s_{1}$.

Now we should consider the two most complex conjuncts of $\psi_{\text {grid }}: \operatorname{propd}_{x}$ and return $n_{x}$. The formula propd $_{x}$ states that

$$
\begin{aligned}
\mathcal{M}, s= & \square_{b}[\uparrow]\left(\left(\square_{a} \perp \wedge \diamond_{x} \diamond_{a} \top \wedge \diamond_{b}\left(\diamond_{b} \top \wedge \square_{b} \diamond_{a} \top\right) \wedge\right.\right. \\
& \left.\left.\langle\mathfrak{\imath}\rangle\left(\diamond_{x} \diamond_{a} \top \wedge \diamond_{b} \diamond_{b} \square_{a} \perp\right)\right) \rightarrow\left[U_{x}\right]\langle\uparrow\rangle\left(\diamond_{x} \diamond_{a} \top \wedge \diamond_{b} \diamond_{b} \square_{a} \perp\right)\right),
\end{aligned}
$$

where

$$
U_{x}=\left(p \vee \square_{a} \perp, b, \top\right),(\top, a, \top),\left(\square_{a} \perp, x, \top\right)
$$

The initial $\square_{b}$ of propd $_{x}$ takes us to any $b$-successor of $s$, so to $s_{1}$ in the above diagram. The remainder of $\operatorname{propd}_{x}$ now has the form $[\downarrow]\left(\varphi_{1} \rightarrow \varphi_{2}\right)$. So for any arrow update $U_{1} \in \mathcal{L}_{A U L}$, if the antecedent $\varphi_{1}$ holds in $\mathcal{M} * U_{1}, s_{1}$ then the consequent $\varphi_{2}$ should hold there as well. Let us take a closer look at what it means for $\varphi_{1}$ to hold in $\mathcal{M} * U_{1}, s_{1}$.

We have $\varphi_{1}=\left(\square_{a} \perp \wedge \diamond_{x} \diamond_{a} \top \wedge \diamond_{b}\left(\diamond_{b} \top \wedge \square_{b} \diamond_{a} \top\right) \wedge\langle\downarrow\rangle\left(\diamond_{x} \diamond_{a} \top \wedge \diamond_{b} \diamond_{b} \square_{a} \perp\right)\right.$. So $s_{1}$ satisfies $\square_{a} \perp$. Furthermore, at least one $x$-successor of $s_{1}$ satisfies $\diamond_{a} \top$. Since the $x$-successors of $s_{1}$ are modally indistinguishable (in $\mathcal{M}$, and therefore also in $\mathcal{M} * U_{1}$ ) this implies that all $x$-successors of $s_{1}$ retain their $a$-arrow.

The $b$-successor $s^{\prime}$ of $s_{1}$ (which is also unique up to modal indistinguishability) satisfies $\diamond_{b} \top \wedge \square_{b} \diamond_{a} \top$. So the situation we are in can be represented by the diagram drawn below.


Note that $s_{2}^{\prime}$ is a $b$-successor of a state $s^{\prime} \in\left[s_{0}\right]_{\mathcal{M}}$. Above we concluded that any $b$-successor $s_{1}$ of any state $s \in\left[s_{0}\right]_{\mathcal{M}}$ must be modally indistinguishable from all its $a$-successors, so $s_{2}^{\prime}$ is also modally indistinguishable from its $a$-successors.

The final conjunct of the antecedent $\varphi_{1}$ now states that there is some arrow update $U_{2}$ that retains the $a$-arrow from $s_{2}$, while removing the $a$-arrow from $s_{2}^{\prime}$. This is the case if and only if $s_{2}$ and $s_{2}^{\prime}$ are modally distinguishable.

The consequent $\varphi_{2}$ then states that $\langle\downarrow\rangle\left(\diamond_{x} \diamond_{a} \top \wedge \diamond_{b} \diamond_{b} \square_{a} \perp\right)$ still holds after the application of $U_{x}$, so $\left(\mathcal{M} * U_{1}\right) * U_{x}, s_{1} \models\langle\downarrow\rangle\left(\diamond_{x} \diamond_{a} \top \wedge \diamond_{b} \diamond_{b} \square_{a} \perp\right)$. This implies that, in $\left(\mathcal{M} * U_{1}\right) * U_{x}$, the states $s_{2}$ and $s_{2}^{\prime}$ are still modally distinguishable.

The update $U_{x}$ removes all $\mathcal{A} \backslash\{a\}$-arrows from $s_{2}$ and $s_{2}^{\prime}$. Furthermore, the $a$-arrows go from $s_{2}$ to a state modally indistinguishable from $s_{2}$ and from $s_{2}^{\prime}$ to a state modally indistinguishable from $s_{2}^{\prime}$. This implies that, after the update $U_{x}$, the states $s_{2}$ and $s_{2}^{\prime}$ can only be modally distinguishable from each other if they are propositionally distinguishable.

In summary: $\operatorname{propd}_{x}$ guarantees that if $\mathcal{M} * U_{1}$ matches the above diagram (i.e. $\mathcal{M} * U_{1}, s_{1} \models \square_{a} \perp \wedge \diamond_{x} \diamond_{a} \top \wedge \diamond_{b}\left(\diamond_{b} \top \wedge \square_{b} \diamond_{a} \top\right)$ ) and $s_{2}^{\prime}$ is modally distinguishable from $s_{2}\left(\right.$ in $\left.\mathcal{M} * U_{1}\right)$ then $s_{2}^{\prime}$ is propositionally distinguishable from $s_{2}$.

Furthermore, we can show that $s_{2}$ and $s_{2}^{\prime}$ also have to be propositionally distinguishable if they are modally distinguishable in $\mathcal{M}$ (as opposed to $\mathcal{M} *$ $U_{1}$ ). Suppose that $\mathcal{M} * U_{1}$ matches the above diagram, and that $s_{2}$ is modally distinguishable from $s_{2}^{\prime}$ in $\mathcal{M}$. Let $\delta$ be a modal formula that distinguishes between $s_{2}$ and $s_{2}^{\prime}($ in $\mathcal{M})$, and assume without loss of generality that $\delta$ holds on $s_{1}$. Note that we previously concluded that $\mathcal{M}, s_{2} \models \diamond_{b} \top$ and that, since $s_{2}^{\prime}$ is a $b$-successor of a state $s^{\prime} \in\left[s_{0}\right]_{\mathcal{M}}$, also $\mathcal{M}, s_{2}^{\prime} \models \diamond_{b} \top$.

We now distinguish between 3 cases.

- Suppose $\mathcal{M} * U_{1}, s_{2} \vDash \square_{b} \perp$ and $\mathcal{M} * U_{1}, s_{2}^{\prime} \vDash \square_{b} \perp$. Then let $U_{1}^{\prime}:=$ $U_{1} \cup\{\chi \wedge \neg p, b, \top\}$.
- Suppose one of $s_{2}$ and $s_{2}^{\prime}$ satisfies $\square_{b} \perp$ while the other satisfies $\diamond_{b} \top$. Then let $U_{1}^{\prime}:=U_{1}$.
- Suppose $\mathcal{M} * U_{1}, s_{2} \models \diamond_{b} \top$ and $\mathcal{M} * U_{1}, s_{2}^{\prime} \models \diamond_{b} \top$. Then let $U_{1}^{\prime}$ be the update obtained by replacing every clause $(\varphi, b, \psi) \in U_{1}$ by $(\varphi \wedge(p \vee$ $\delta), b, \psi)$.

In any of the three cases, $\mathcal{M} * U_{1}^{\prime}$ matches the diagram, and $s_{2}$ is distinguishable from $s_{2}^{\prime}$ in $\mathcal{M} * U_{1}^{\prime}$ by the formula $\diamond_{b} \top$. So $s_{2}$ and $s_{2}^{\prime}$ are propositionally distinguishable.

Summarizing again: $\operatorname{propd}_{x}$ guarantees that if $\mathcal{M} * U_{1}$ matches the above diagram (i.e. $\mathcal{M} * U_{1}, s_{1} \models \square_{a} \perp \wedge \diamond_{x} \diamond_{a} \top \wedge \diamond_{b}\left(\diamond_{b} \top \wedge \square_{b} \diamond_{a} \top\right)$ ) and $s_{2}$ is modally distinguishable from $s_{2}^{\prime}($ in $\mathcal{M})$, then $s_{2}$ is propositionally indistinguishable from $s_{2}^{\prime}$.

Now, consider the formula return ${ }_{x}$. It states that

$$
\begin{aligned}
\mathcal{M}, s \models & \square_{b}\langle\downarrow\rangle\left(\square_{a} \perp \wedge \diamond_{x} \diamond_{a} \top \wedge \diamond_{b}\left(\diamond_{b} \top \wedge \square_{b} \diamond_{a} \top\right) \wedge\right. \\
& {\left.[\uparrow]\left(\diamond_{x} \diamond_{a} \top \rightarrow \square_{b} \square_{b} \diamond_{a} \top\right)\right) }
\end{aligned}
$$

Once again, the initial $\square_{b}$ operator takes us to any $b$-successor $s_{1}$ of $s$. Then, there is some update $U_{1}$ such that

$$
\begin{aligned}
\mathcal{M} * U_{1}, s_{1} \models & \square_{a} \perp \wedge \diamond_{x} \diamond_{a} \top \wedge \diamond_{b}\left(\diamond_{b} \top \wedge \square_{b} \diamond_{a} \top\right) \wedge \\
& {[\uparrow]\left(\diamond_{x} \diamond_{a} \top \rightarrow \square_{b} \square_{b} \diamond_{a} \top\right) }
\end{aligned}
$$

The first four conjuncts state that $\mathcal{M} * U_{1}$ matches the diagram drawn above. So if $s_{2}$ is modally distinguishable from some $b$ - $b$-successor $s_{2}^{\prime}$ of $s_{1}$ in the model $\mathcal{M}$,
then those two worlds are propositionally distinguishable. The final conjunct states that every arrow update $U_{2}$ that retains the $x$-arrow from $s_{1}$ to $s_{2}$ and the $a$-arrow from $s_{2}$ either removes one of the $b$-arrows between $s_{1}$ and $s_{2}^{\prime}$ or retains the $a$-arrow on $s_{2}^{\prime}$.

An update $U_{2}$ that removes the $b$-arrows between $s_{1}$ and $s_{2}^{\prime}$ can be modified to an update $U_{2}^{\prime}:=U_{2} \cup\{(\top, b, \top)\}$ that retains the same $a$ - and $x$-arrows as $U_{2}$, but retains all $b$-arrows. It follows that every $U_{2}$ that retains the $x$-arrow from $s_{1}$ to $s_{2}$ and the $a$-arrow from $s_{2}$ must also retain the $a$-arrow on $s_{2}^{\prime}$. The state $s_{2}$ and $s_{2}^{\prime}$ must therefore be modally indistinguishable (in $\mathcal{M} * U_{1}$ ). In particular, this implies that $s_{2}$ and $s_{2}^{\prime}$ are propositionally indistinguishable. If $s_{2}$ and $s_{2}^{\prime}$ had been modally distinguishable in $\mathcal{M}$ they would have been propositionally distinguishable, so it follows that $s_{2}$ and $s_{2}^{\prime}$ are modally indistinguishable in $\mathcal{M}$.

In summary: return $n_{x}$ guarantees that (in $\mathcal{M}$ ) there is a $b$ - $b$-successor $s_{2}^{\prime}$ of $s_{1}$ that is modally indistinguishable from $s_{2}$.

The state $s^{\prime}$ is a member of $\left[s_{0}\right]_{\mathcal{M}}$, so $s_{2}^{\prime}$ a $b$-successor of a state that satisfies $\psi_{\text {grid }}$. All the conclusions that we drew about $s_{1}$ therefore also apply to $s_{2}^{\prime}$ : there is a unique, up to modal indistinguishability (utmi), $x$-successor $s_{3}^{\prime}$ of $s_{2}^{\prime}$, and this $s_{3}^{\prime}$ is modally indistinguishable from a state $s_{3}^{\prime \prime}$ that is $b$ - $b$-accessible from $s_{2}^{\prime}$. The same procedure can be used on $s_{3}^{\prime \prime}$, and so on.

All in all, we get the following diagram,

where states with the same subscript are modally indistinguishable from one another.

Now, because $s_{2}$ is modally indistinguishable from $s_{2}^{\prime}$, and $s_{2}^{\prime}$ has a unique (utmi) $x$-successor $s_{3}^{\prime}$, it follows that $s_{2}$ must have a unique (utmi) $x$-successor $s_{3}$ that is modally indistinguishable from $s_{3}^{\prime}$, and therefore also from $s_{3}^{\prime \prime}$. Furthermore, $s_{3}^{\prime \prime}$ has a unique (utmi) $x$-successor $s_{4}^{\prime \prime}$, so $s_{3}^{\prime}$ and $s_{3}$ have unique (utmi) $x$-successors $s_{4}$ and $s_{4}^{\prime}$, respectively, that are modally indistinguishable from $s_{4}^{\prime \prime}$. The diagram can therefore be completed to the following:


The important thing to note is that $s_{1}, s_{2}, s_{3}, \cdots$ form an infinite sequence of states that each have a unique (utmi) $x$-successor and that each of them is modeally indistinguishable from a $b$-successor of a state that is a member of $\left[s_{0}\right]_{\mathcal{M}}$.

Furthermore, this holds for any direction $x$. So $s_{1}$ has of four such successor sequences, one for each direction. Every state in these successor sequences is modally indistinguishable from a $b$-successor of some state $s^{(n)} \in\left[s_{0}\right]_{\mathcal{M}}$, so every such state has four successor sequences of its own.

Now, consider inverse and commute. The formula inverse guarantees that, for opposite directions $x$ and $y$, the $x-y$-successor of $s_{n}$ is indistinguishable from $s_{n}$. The formula commute, guarantees that, for perpendicular directions $x$ and $y$, the $x-y$-successor of $s_{n}$ is indistinguishable from its $y$-x-successor. The successor sequences of $s_{1}$ therefore form a $\mathbb{Z} \times \mathbb{Z}$ grid.

Now, finally, consider $\psi_{\text {types }}$. This is a purely modal formula and it holds on $s_{0}$, so it holds on every $s \in\left[s_{0}\right]_{\mathcal{M}}$. The formula $\psi_{\text {types }}$ guarantees that the conditions of a tiling are locally satisfied on the immediate $b$-successors of $s$. Let $s_{1}$ be any $b$-successor of $s \in\left[s_{0}\right]_{\mathcal{M}}$. Then one_tile guarantees that $s_{1}$ has exactly one tile type, one_color guarantees that that every side of $s_{1}$ has at most one color, tile_colors guarantees that every side of $s_{1}$ has the appropriate color for a tile of its type and tile_match guarantees that the tile edges match,; i.e. that the north side of the current tile is the same as the south side of its up-successor, and similarly for the other directions.

Since every state in the grid is modally indistinguishable from a $b$-successor of some $\left[s_{0}\right]_{\mathcal{M}}$ state, the conditions of a tiling are locally satisfied in every world of the grid, so they are globally satisfied. So if $\mathcal{M}, s_{0} \models \chi_{\text {types }}$, then $\mathbb{Z} \times \mathbb{Z}$ can be tiled using types.

Theorem 1. The satisfiability problem of AAUL is co-RE hard.
Proof. Given an instance types of the tiling problem, the formula $\chi_{\text {types }}$ is computable. Furthermore, Lemmas 1 and 5 show that $\chi_{\text {types }}$ is satisfiable if and only
if types can tile the plane. The tiling problem is known to be co-RE complete [5], therefore the satisfiability problem of $A A U L$ is co-RE hard.

## 4. Conclusion

We have shown that the satisfiability of $A A U L$ is uncomputable, like that of similar logics such as $A P A L$ [9], $G A L$ and $C A L$ [2]. It is not currently known whether the satisfiability problems of $A A U L, A P A L$ and $G A L$ are co-RE. Typically, one would show that a satisfiability problem is co-RE by providing an axiomatization for the logic, thereby showing the validities of the logic to be RE. However, while there are known axiomatizations for $A A U L, A P A L$ and $G A L$, these axiomatizations are infinitary and therefore cannot be used to enumerate the valid formulas of the logics in question. ${ }^{3}$

One interesting direction for future research is therefore to determine whether the satisfiability problems of $A A U L, A P A L$ and $G A L$ are co-RE complete, and whether these logics admit finitary axiomatizations.

In principle, the proof that we gave for the undecidability of $A A U L$ applies only to the satisfiability problem when considered over the class of all Kripke models. We consider this to be the most important version of the satisfiability problem for $A A U L$, since the class of all Kripke models is the "natural habitat" of $A A U L$, see [8] for details. Still, the satisfiability problem for $A A U L$ with respect to smaller classes of models could be formulated. We believe that, with only minor modifications, the proof presented in this paper would also show that satisfiability of $A A U L$ is undecidable with respect to other common classes of models such as KD45 and S5.

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[^1]:    ${ }^{1} A M L$ is also sometimes referred to as Dynamic Epistemic Logic ( $D E L$ ), but here we reserve that name for the family of update logics of which $A M L$ is one.

[^2]:    ${ }^{2}$ This is far easier said then done, we will spend several pages proving that $R(b)$ connects to every relevant state.

[^3]:    ${ }^{3}$ Finitary axiomatizations for $A P A L$ and $G A L$ were proposed, in [3] and [1] respectively, but these were later shown to be unsound.

