Limit Theorems for Sub-Sums of Partial Quotients of Continued Fractions

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Abstract

This paper studies the limit behaviour of sums of the form

$$T_n(x) = \sum_{1 \le j \le n} c_{k_j}(x), \qquad (n = 1, 2, \dots)$$

where $(c_j(x))_{j\ge1}$ is the sequence of partial quotients in the regular continued fraction expansion of the real number x and $(k_j)_{j\ge1}$ is a strictly increasing sequence of natural numbers. Of particular interest is the case where for irrational α, the sequence $(k_j\alpha)_{j\ge1}$ is uniformly distributed modulo one and $(k_j)_{j\ge1}$ is good universal. It was observed by the second author, for this class of sequences $(k_j)_{j\ge1}$ that we have $\lim_{n\to\infty} \frac{T_n(x)}{n} = +\infty$ almost everywhere with respect to Lebesgue measure. The case $k_j = j$ $(j = 1, 2, \dots)$ is classical and due to A. Ya. Khinchine. Building on work of H. Diamond, Khinchin, W. Philipp, L. Heinrich, J. Vaaler and others, in the special case where $k_j = j$ $(j = 1, 2, \dots,)$ we examine the asymptotic behaviour of the sequence $(T_n(x))_{n\ge1}$ in more detail.

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1 Introduction

Let $\mathbb{N} = \{1, 2, \cdots\}$ denote the set of natural numbers. For $x \in (0, 1)$, let $x = [c_1(x), c_2(x), \cdots]$ denote its regular continued fraction expansion. Recall that we say a sequence $(x_n)_{n\ge1}$ is uniformly distributed modulo one if for each interval $I \subseteq [0, 1)$ of length $|I|$ we have

$$\lim_{N\to\infty} \frac{1}{N} \#\{1 \le n \le N : x_n \in I\} = |I|.$$

Here for a finite set F we have used $\#F$ to denote its cardinality. Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T : X \to X$ be a measurable map, that is also measure-preserving. That is, given $A \in \mathcal{B}$, we have $\mu(T^{-1}A) = \mu(A)$, where $T^{-1}A$ denotes the set $\{x \in X : Tx \in A\}$. We call $(X, \mathcal{B}, \mu, T)$ a dynamical system. We say a dynamical system $(X, \mathcal{B}, \mu, T)$ is ergodic if $T^{-1}A = A$ for $A \in \mathcal{B}$ means that either $\mu(A)$ or $\mu(X\backslash A)$ is 0. We say $(k_n)_{n\ge0}$ is L^p good universal if for each dynamical system $(X, \mathcal{B}, \mu, T)$ and for each $f \in L^p(X, \mathcal{B}, \mu)$ the limit

$$\ell_{T,f}(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^{k_n}x),$$

exists μ almost everywhere.

For a real number y let $[y]$ denote the largest integer not greater than y. Also let $\{y\}$ denote the fractional part of y i.e. $y - [y]$. We call

$$G(x) = \begin{cases} \{\frac{1}{x}\}, & \text{if } x \in (0,1) \\ 0 & \text{if } x = 0. \end{cases}$$

the Gauss map. Let ρ_L denote Lebesgue measure on $[0,1)$. Set

$$\rho_G(A) = \tfrac{1}{\log 2} \int_A \tfrac{dx}{1+x}$$

for a ρ_L-measurable set A. We call ρ_G the Gauss measure.

Let $\mathcal{M}$ denote the Lebesgue σ- algebra on $[0,1)$. Applying good universality to the dynamical sysyem $([0,1), \mathcal{M}, \rho_G, G)$, using the fact that

$$c_1(x) = [\frac{1}{x}], \quad c_{k+1}(x) = c_k(G(x)), \qquad (k = 1, 2, \dots)$$

for irrational x in [Nai3], developing ideas in [Doe] and [RN], the following is proved.

Suppose that the function $F : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is continuous and increasing and that for some $p \geq 1$ we have

$$\int_0^1 \frac{|F(c_1(x))|^p}{x+1} dx < \infty.$$

Suppose (i) for each irrational α that $(\{k_j \alpha\})_{j\geq 1}$ is uniformly distributed modulo one, and (ii) that $(k_j)_{\geq 1}$ is L^p good universal. For a finite set of non-negative real numbers $\{a_1, \dots, a_n\}$ we let

$$M_{F,n}(a_1, \dots, a_n) = F^{-1}\left[\frac{F(a_1) + \ldots + F(a_n)}{n}\right].$$

It is shown in [Nai3] that

$$\lim_{n\to\infty} M_{F,n}(c_1(x), \dots, c_n(x)) = F^{-1}\left[\frac{1}{\log 2}\int_0^1 \frac{F(c_1(x))}{x+1} dx\right]$$

almost everywhere with respect to Lebesgue measure. As a corollary it is deduced that if $(k_j)_{j\geq 1}$ satisfies (i) and (ii) and for $(n = 1, 2, \dots)$ we set

$$T_n(x) = \sum_{1\leq j\leq n} c_{k_j}(x), \tag{1}$$

then

$$\lim_{n\to\infty} \frac{T_n(x)}{n} = +\infty,$$

almost everywhere with respect to Lebesgue measure.

Let

$$S_n(x) = \sum_{j\leq n} c_j(x). \qquad (n = 1, 2, \dots)$$

Evidently $S_n(x) \geq n$ for any irrational $x \in (0,1)$ as $c_j(x) \geq 1$ for any integer $j \geq 1$ and irrational x. One of the implications of [Khi1] is that

$$\lim_{n\to\infty} \frac{S_n(x)}{n} = +\infty,$$

almost everywhere with respect to Lebesgue measure. In fact H. G. Diamond and J. D. Vaaler [DV, Corrollary 1] showed that, there exist $\theta \in [0,1]$ and $n_0(x) \in \mathbb{N}$ such that

$$S_n(x) = \frac{1+o(1)}{\log 2} n \log n + \theta \max_{1\leq j\leq n} c_j(x),$$

if $n > n_0(x)$ almost everywhere with respect to Lebesgue measure. The presence of the term $\theta \max_{1\leq j\leq n} c_j(x)$, here tells us an almost everywhere estimate for $S_n(x)$ is likely to be problematic. Another possibility is to preclude the possibility that c_j is too big. In this situation A. Ya. Khinchin [Khi2] showed that if we let

$$b_j(x) = \begin{cases} c_j(x), & \text{if } c_j(x) < j(\log j)^{4/3} \\ 0, & \text{otherwise,} \end{cases}$$

then

$$\lim_{n\to\infty} \frac{b_1(x) + \ldots + b_n(x)}{n} \to \frac{1}{\log 2}$$

in measure. The same result is not true almost everywhere. This is because as also observed by Khinchin, if $(d_j)_{j\geq 1}$ is a sequence of positive real numbers, then $c_j(x) > d_j$ has finitely many solutions in j if and only if $\sum_{j=1}^{\infty} d_j^{-1}$ is finite [Khi1]. This implies $b_j(x) > j \log j \log\log j$, holds for infinitely many j almost everywhere . The following three theorems in the case $k_j = j$ $(j = 1, 2, \ldots)$ are proved in [Phi1] , case $k_j = j$ for Theorem 1.2 is also proved in [Hei], respectively. Notice that for zero density subsequences of $\mathbb{N}$, for instance $k_j = j^2$ $(j = 1, 2, \ldots)$, the theorems say something fundamentally new not following from the results in [Phi1]. The methods are however adapted from those in [Phi1]. As one of the referees of this paper has suggested, it is possible to obtain our results by considering the ψ-mixing sequence of random variables $(c_{k_j})_{j\geq 1}$ rather than $(c_j)_{j\geq 1}$. We remark that another referee also suggests alternate proofs of our results can be obtained by using the fact that every subsequence of an almost i.i.d. sequence of random variables is exponentially ψ-mixing. See section 2 for a definition of ψ-mixing. See Chapter 4 of [IK] and [GI] for further background . Our results are of greatest interest in the presence of conditions (i) and (ii) to the authors. Whether the result from [Nai3] is true in the absence of conditions (i) and (ii) is unknown. See Section 7 for an extensive list of examples of sequences that satisfy conditions (i) and (ii).

Theorem 1.1. *Suppose $(k_j)_{j\geq 1}$ is a strictly increasing sequence of natural numbers and that T_n defined in (1). Suppose $(\tau_n)_{n\geq 1} \subseteq \mathbb{N}$ satisfies the property that $\frac{\tau_n}{n}$ is non-decreasing as $n \to \infty$. Then*

$$\lim_{n\to\infty}\frac{T_n}{\tau_n}=0 \textit{ or } \limsup_{n\to\infty}\frac{T_n}{\tau_n}=\infty$$

almost everywhere with respect to Lebesgue measure ρ_L depending on whether

$$\sum_{n=1}^{\infty}\frac{1}{\tau_n}<\infty\ .$$

or not.

We also have the following theorem.

Theorem 1.2. *Let*

$$\gamma=\lim_{n\to\infty}\left(\sum_{k\le n/\log 2} k\log(1+\tfrac{1}{k(k+2)})-\log n\right).$$

Also let

$$f(t)=e^{-\frac{\pi}{2}|t|-it\log|t|+i\gamma t}$$

be a complex valued function defined on $\mathbb{R}$. Then for any probability measure ρ on $(0,1)$ absolutely continuous with respect to ρ_L, the distribution functions of the random variables

$$\tfrac{T_n}{n/\log 2}-\log n$$

converge to the distribution function of some random variable with characteristic function $f(t)$ as $n\to\infty$.

Remark 1.2.1. *As a consequence of his mixing random variable techniques L. Heinrich [Hei] obtained bounds on the rates of convergence in the case that $\rho=\rho_G$ when $k_j=j$ $(j=1,2,\ldots)$. One of the referees of this paper informs us that the function $f(t)$ in Theorem 2 is the characteristic function of a stable random variable with characteristic exponent $\alpha=1$ and skewness parameter $\beta=1$. The parameter $\gamma=-c-\log\log 2$, where c is the Euler-Mascheroni constant.*

The final theorem deals with trimmed sums of the sequence $\{c_{k_j}, j\in\mathbb{N}\}$. As in [Phi1] for $n\in\mathbb{N}$ and $1\le j\le n$, define

$$\tau_{n,j}=\#\{i: c_{k_i}>c_{k_j} \text{ for } 1\le i\le n \text{ or } c_{k_i}=c_{k_j} \text{ for } 1\le i\le j\}.$$

One can see that $\{\tau_{n,j}:1\le j\le n\}=\{1,\cdots,n\}$. For $\tau_{n,j}=k$, let $c_n^{(k)}=c_{k_j}$. Then $c_n^{(k)}$ is a non-increasing finite sequence with respect to k for fixed n. That is, we are re-arranging the sequence $\{c_{k_1},\cdots,c_{k_n}\}$ from large to small. Let $\{p_n, n\in\mathbb{N}\}$ and $\{\xi_n, n\in\mathbb{N}\}$ be two sequences of integers such that

$$\lim_{n\to\infty}p_n=\infty,\ \xi_n\ge 7.$$

Also let I_S be the indicator function of the set S. Then let

$$T_n^*=T_n-\Sigma_{j<p_n\xi_n}c_n^{(j)}I_{[c_n^{(j)}>n/p_n]}.$$

We have

Theorem 1.3. *Let*

$$U(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}t^2} dt$$

be a function from $\mathbb{R}$ to $\mathbb{R}$. Then for any probability measure ρ on $(0,1)$ absolutely continuous with respect to ρ_L, the distribution functions of the random variables

$$\frac{T_n^*}{n}(p_n \log 2)^{1/2} - (\frac{p_n}{\log 2})^{1/2} \log \frac{n}{p_n}$$

converge to $U(y)$ as $n \to \infty$.

Remark 1.3.1. *As in proving [Phi2, Theorem 4], it can be seen that Theorem 1.3 follows if one can prove the result in the case for $\rho = \rho_G$. This is because we can use strong stationarity and mixing properties to prove the result for the measure ρ_G and the fact that ρ is absolutely continuous with respect to ρ_G together with the fact that the density $\frac{d\rho}{d\rho_g}$ is bounded in L^1 to deduce properties of ρ from those of ρ_G.*

2 Mixing

We refer to [GI] for general background on the various mixing concepts in this section. Consider a sequence of random vectors $\{Y_j : j \in \mathbb{N}\}$ defined on a common probability space $(\Omega, \mathcal{M}', \mu)$. Let

$$\mathcal{M}'_{k,l} = \sigma(Y_j : k \le j \le l), 1 \le k < l \le \infty,$$

be the σ-algebra generated by the indicated sets of the random vectors. Define the dependence coefficients of the sequence $\{Y_j : j \in \mathbb{N}^+\}$ to be:

$$\phi(h) = \sup_{j\in\mathbb{N}} \sup \Big\{ |\frac{\mu(A \cap B)}{\mu(A)} - \mu(B)| : A \in \mathcal{M}'_{1,j}, B \in \mathcal{M}'_{j+h,\infty}, \mu(A) > 0 \Big\}$$

for $h \in \mathbb{N}$. The function $\phi(h)$ is non-decreasing and $\phi(1) \le 1$. We say the sequence of random vectors $\{Y_j(x), j \in \mathbb{N}^+\}$ is *ϕ-mixing* if $\lim_{h\to\infty} \phi(h) = 0$. Set

$$\psi(h) = \sup_{j\in\mathbb{N}} \sup \Big\{ |\frac{\mu(A \cap B)}{\mu(A)\mu(B)} - 1| : A \in \mathcal{M}'_{1,j}, B \in \mathcal{M}'_{j+h,\infty}, \mu(A)\mu(B) > 0 \Big\}$$

for $h \in \mathbb{N}$. The function $\psi(h)$ is non-decreasing and $\psi(1) \le \infty$. We say the sequence of random vectors $\{Y_j(x), j \in \mathbb{N}^+\}$ is *ψ-mixing* if $\lim_{h\to\infty} \psi(h) = 0$. Finally we set

$$\psi^* = \sup_{j\in\mathbb{N}} \sup \Big\{ \frac{\mu(A \cap B)}{\mu(A)\mu(B)} : A \in \mathcal{M}'_{1,j}, B \in \mathcal{M}'_{j+1,\infty}, \mu(A)\mu(B) > 0 \Big\}.$$

Obviously $\phi(h) \leq \psi(h)$ and $\psi^* \leq 1 + \psi(1)$.

Remark 2.0.1. *Our proofs of Theorem 1.2 and 1.3 rely on mixing and J. Samur's results [Sam, Corollary 4.6, Corollary 5.10].*

We now consider the case of partial quotients of continued fractions. For a strictly increasing sequence $\{k_j, j \in \mathbb{N}\}$ of natural numbers $\{X_j = c_{k_j}, j \in \mathbb{N}\}$ is a sequence of random variables defined on the probability space $((0,1), \mathcal{M}, \rho_G)$, where $\mathcal{M}$ is the σ-algebra of the ρ_L-measurable sets.

Let $\mathcal{M}^*_{1,j}$ be the σ-algebra generated by the rank j fundamental intervals

$$\{x \in (0,1) : c_1(x) = r_1, \cdots, c_j(x) = r_j\}$$

for fixed $r_i \in \mathbb{N}, 1 \leq i \leq j$. Let $\mathcal{M}^*_{j+h,\infty}$ be the σ-algebra generated by the sets

$$\{x \in (0,1) : c_{j+h}(x) = r_{j+h}, c_{j+h+1}(x) = r_{j+h+1}\}$$

for fixed $r_i, h \in \mathbb{N}, j+h \leq i < \infty$. Let $\mathcal{M}_{1,j}$ be the σ-algebra generated by the sets

$$\{x \in (0,1) : c_{k_1}(x) = r_1, \cdots, c_{k_j}(x) = r_j\}$$

for fixed $r_i \in \mathbb{N}, 1 \leq i \leq j$. Let $\mathcal{M}_{j+h,\infty}$ be the σ-algebra generated by the sets

$$\{x \in (0,1) : c_{k_{j+h}}(x) = r_{j+h}, c_{k_{j+h+1}}(x) = r_{j+h+1}, \ldots\}$$

for fixed $r_i, h \in \mathbb{N}, j+h \leq i < \infty$. It is easy to see that $\mathcal{M}_{1,j}$ and $\mathcal{M}_{j+h,\infty}$ are sub-algebras of $\mathcal{M}^*_{1,k_j}$ and $\mathcal{M}^*_{k_{j+h},\infty}$.

From now on we specialise the symbols of $\phi(h), \psi(h), \psi^*$ to our sequence $\{X_j, j \in \mathbb{N}\}$, that is,

$$\phi(h) =$$

$$\sup_{j\in\mathbb{N}} \sup \Big\{ \Big|\frac{\rho_G(A \cap B)}{\rho_G(A)} - \rho_G(B)\Big| : A \in \mathcal{M}_{1,j}, B \in \mathcal{M}_{j+h,\infty}, \rho_G(A) > 0 \Big\}$$

for $h \in \mathbb{N}$,

$$\psi(h) =$$

$$\sup_{j\in\mathbb{N}} \sup \Big\{ \Big|\frac{\rho_G(A \cap B)}{\rho_G(A)\rho_G(B)} - 1\Big| : A \in \mathcal{M}_{1,j}, B \in \mathcal{M}_{j+h,\infty}, \rho_G(A)\rho_G(B) > 0 \Big\}$$

for $h \in \mathbb{N}$, and

$$\psi^* = \sup_{j\in\mathbb{N}} \sup \Big\{ \frac{\rho_G(A \cap B)}{\rho_G(A)\rho_G(B)} : A \in \mathcal{M}_{1,j}, B \in \mathcal{M}_{j+1,\infty}, \rho_G(A)\rho_G(B) > 0 \Big\}.$$

A sequence of random vectors $\{Y_j(x), j \in \mathbb{N}\}$ with joint distribution $\mathcal{L}$ is said to be stationary if

$$\mathcal{L}(Y_1, \cdots, Y_n) = \mathcal{L}(Y_{1+k}, \cdots, Y_{n+k})$$

for any $1 \le n < \infty, 1 \le k < \infty$. We refer to [VW, 1.3] by Aad W. van der Vaart and Jon A. Wellner for a definition in terms of measures.

We will show that $\{X_j\}$ is a ϕ-mixing sequence in Section 3. It is well known that the sequence of random variables $\{c_j, j \in \mathbb{N}\}$ is stationary as the Gauss map $G(x)$ is measure-preserving with respect to ρ_G. From this we can deduce easily that the sequence $\{X_j, j \in \mathbb{N}\}$ is also stationary.

3 Some preliminary lemmas

For a sequence of positive integers $j_1, j_2, \cdots, j_n, j_{n+1}$, let

$$J_n = \{x \in (0,1) : c_i(x) = j_i, 1 \le i \le n\}$$

and let

$$J_{n+1} = \{x \in (0,1) : c_i(x) = j_i, 1 \le i \le n+1\}$$

be the fundamental intervals of of rank n and $n+1$ respectively. Let $\rho_L(E)$ be the Lebesgue measure of a measurable set $E \subset (0,1)$. Then as observed on line (57) in [Khi1] we know that

$$\frac{1}{3j_{n+1}} \le \frac{\rho_L(J_{n+1})}{\rho_L(J_n)} \le \frac{2}{j_{n+1}^2}. \tag{2}$$

The following lemma generalises [Khi1, Theorem 30].

Lemma 3.1. *For any strictly increasing sequence of integers $k_n \in \mathbb{N}$ and any sequence of positive integers $\tau_n \in \mathbb{N}$, $n \in \mathbb{N}$, the system of inequalities*

$$c_{k_n}(x) \ge \tau_n \tag{3}$$

is satisfied by infinitely many n, for almost all real numbers $x \in (0,1)$, if $\sum_{n=1}^{\infty} \frac{1}{\tau_n} = \infty$. The same system of inequalities is satified for only finitely many n for almost all real numbers in $(0,1)$ if $\sum_{n=1}^{\infty} \frac{1}{\tau_n} < \infty$.

The second assertion in Lemma 3.1 follows directly from second assertion of [Khi1, Theorem 30] by applying [Khi1, Theorem 30] to the sequence

$$\sigma_j = \begin{cases} \tau_n & j = k_n \\ j^2 & k_n < j < k_{n+1} \end{cases} \quad \text{for any } n, j \in \mathbb{N}$$

with $\sum_{j=1}^{\infty} \frac{1}{\sigma_j} < \infty$. Now we give a proof of the first assertion in Lemma 3.1.

Proof. Let

$$A_{k_n} = \{x \in (0,1) : c_{k_{n+j}} < \tau_{n+j} \text{ for any integer } j \in \mathbb{N}\},$$

and let

$$A_{k_n,l} = \{x \in (0,1) : c_{k_{n+j}} < \tau_{n+j} \text{ for any integer } 1 \leq j \leq l\}$$

for fixed $n, l \in \mathbb{N}$. Then

$$A_{k_n} = \cap_{l=1}^{\infty} A_{k_n,l} \tag{4}$$

for fixed $n \in \mathbb{N}$. Let $J_{k_{n+l}} \subset A_{k_n,l}$ be an interval of rank k_{n+l}. Let $J_{k_{n+l+1}} = J_{k_{n+l}} \cap A_{k_n,l+1}$. Note that $J_{k_{n+l+1}}$ is not an interval. We claim that

$$\rho_L(J_{k_{n+l+1}}) < (1 - \frac{1}{3(1+\tau_{n+l+1})})\rho_L(J_{k_{n+l}}). \tag{5}$$

To show this, let

$$J^j_{k_{n+l+1}} = \{x \in J_{k_{n+l}} : c_{k_{n+l+1}}(x) = j\}$$

for fixed $j \in \mathbb{N}$. Let $I^j_{k_{n+l+1}} \subset J^j_{k_{n+l+1}}$ be an interval of rank k_{n+l+1}. Denote by $I_{k_{n+l+1}-1}$ the interval of rank $k_{n+l+1}-1$ containing $I^j_{k_{n+l+1}}$. There is one and only one such rank $k_{n+l+1}-1$ interval. Then by (2),

$$\rho_L(I^j_{k_{n+l+1}}) > \frac{1}{3j^2}\rho_L(I_{k_{n+l+1}-1}). \tag{6}$$

Summing inequality (6) over all rank k_{n+l+1} intervals $I^j_{k_{n+l+1}} \subset J^j_{k_{n+l+1}}$ gives

$$\rho_L(J^j_{k_{n+l+1}}) > \frac{1}{3j^2}\rho_L(J_{k_{n+l}}) \tag{7}$$

for any $j \in \mathbb{N}$, as any rank $k_{n+l+1}-1$ interval in $J_{k_{n+l}}$ contains one and only one interval $I^j_{k_{n+l+1}} \subset J^j_{k_{n+l+1}}$. Then

$$\rho_L(\cup_{j \geq \tau_{n+l+1}} J^j_{k_{n+l+1}}) > \frac{1}{3}\rho_L(J_{k_{n+l}}) \textstyle\sum_{j \geq \tau_{n+l+1}} \frac{1}{j^2} >$$
$$\frac{1}{3}\rho_L(J_{k_{n+l}}) \textstyle\sum_{j=1}^{\infty} \frac{1}{(\tau_{n+l+1}+j)^2} > \frac{1}{3}\rho_L(J_{k_{n+l}}) \int_{\tau_{n+l+1}+1}^{\infty} \frac{dy}{y^2} =$$
$$\frac{1}{3(\tau_{n+l+1}+1)}\rho_L(J_{k_{n+l}}).$$

So

$$\rho_L(J_{k_{n+l+1}}) = \rho_L(\cup_{j<\tau_{n+l+1}} J^j_{k_{n+l+1}}) = \rho_L((\cup_{j=1}^{\infty} J^j_{k_{n+l+1}}) \backslash (\cup_{j \geq \tau_{n+l+1}} J^j_{k_{n+l+1}})) =$$
$$\rho_L(J_{k_{n+l}} \setminus (\cup_{j \geq \tau_{n+l+1}} J^j_{k_{n+l+1}})) < (1 - \frac{1}{3(\tau_{n+l+1}+1)})\rho_L(J_{k_{n+l}}),$$

which shows our claim (5).

Now summing inequality (5) over all rank k_{n+l} intervals $J_{k_{n+l}}$ in $A_{k_n,l}$ gives

$$\rho_L(A_{k_n,l+1}) < (1 - \frac{1}{3(\tau_{n+l+1}+1)})\rho_L(A_{k_n,l}). \quad (8)$$

By induction we have

$$\rho_L(A_{k_n,l}) < \rho_L(A_{k_n,1}) \textstyle\prod_{i=2}^{l}(1 - \frac{1}{3(\tau_{n+i}+1)}).$$

If $\sum_{n=1}^{\infty} \frac{1}{\tau_n} = \infty$, then

$$\textstyle\sum_{i=2}^{\infty} \frac{1}{3(\tau_{n+i}+1)} = \infty,$$

so

$$\textstyle\prod_{i=2}^{\infty}(1 - \frac{1}{3(\tau_{n+i}+1)}) = 0.$$

Considering (4) we have $\rho_L(A_{k_n}) = 0$. Now let $A = \cup_{n=1}^{\infty} A_{k_n}$. Therefore $\rho_L(A) = 0$.

If $x \in (0,1)$ is a number such that (3) is only satisfied finitely many times, then there must be some $n \in \mathbb{N}$ such that $x \in A_{k_n} \subset A$, so the first assertion of the lemma is proved.

□

Lemma 3.2. *For any sets $A \in \mathcal{M}_{1,j}, B \in \mathcal{M}_{j+h,\infty}$, we have*

$$|\rho_G(A \cap B) - \rho_G(A)\rho_G(B)| \le \rho_G(A)\rho_G(B)\lambda^{k_{j+h}-k_j} \le \rho_G(A)\rho_G(B)\lambda^h$$

with $\lambda < 0.8$. That is, $\{X_j\}$ is a stationary ϕ-mixing sequence with $\phi(1) < 0.8$. Moreover, $\psi^* < \infty$.

Proof. For the set $A \in \mathcal{M}_{1,j}$, let

$$A(a_1, a_2, \cdots, a_{k_j}) = A \cap \{x \in (0,1) : c_l(x) = a_l \text{ for } 1 \le l \le k_j\}$$

for $a_l \in \mathbb{N}, 1 \le l \le k_j$ and $l \notin \{k_1, k_2, \cdots, k_j\}$. Obviously

$$A(a_1, a_2, \cdots, \cdots, a_{k_j}) \subset A$$

is a set in $\mathcal{M}^*_{1,k_j}$. Then A can be partitioned into disjoint unions of finite numbers of sets

$$A = \cup_{a_1=1}^{\infty} \cup_{a_2=1}^{\infty} \cdots \cup_{a_{k_j}=1}^{\infty} A(a_1, a_2, \cdots, a_{k_j})$$

in $\mathcal{M}^*_{1,k_j}$. Similarly, let

$$B(b_{k_{j+h}}, b_{k_{j+h}+1}, \cdots) = B \cap \{x \in (0,1) : c_l(x) = b_l \text{ for } j+h \le l < \infty\}.$$

Then B can be partitioned into disjoint unions of sets

$$B = \cup_{b_{k_{j+h}}=1}^{\infty} \cup_{b_{k_{j+h}+1}=1}^{\infty} \cdots B(b_{k_{j+h}}, b_{k_{j+h}+1}, \cdots)$$

in $\mathcal{M}^*_{k_{j+h},\infty}$. In the sequel, for ease of expression, we simplify some notation on indexation as one can easily understand the range of the indices. Then

$$\begin{array}{rl}
 & |\rho_G(A \cap B) - \rho_G(A)\rho_G(B)| \\
= & |\rho_G((\cup_{a_1=1,\cdots,a_{k_j-1}=1}^{\infty} A(a_1,\cdots,a_{k_j-1})) \cap (\cup_{a_{j+h+1}=1,\cdots}^{\infty} B(b_{k_{j+h}+1},\cdots))) \\
 & -\rho_G((\cup_{a_1=1,\cdots,a_{k_j-1}=1}^{\infty} A(a_1,\cdots,a_{k_j-1}))\rho_G((\cup_{a_{j+h+1}=1,\cdots}^{\infty} B(b_{k_{j+h}+1},\cdots))| \\
= & |\Sigma\Sigma\cdots\rho_G(A(a_1,\cdots,a_{k_j-1}) \cap B(b_{k_{j+h}+1},\cdots)) \\
 & -\Sigma\Sigma\cdots\rho_G(A(a_1,\cdots,a_{k_j-1}))\rho_G(B(b_{k_{j+h}+1},\cdots))| \\
\leq & \Sigma\Sigma\cdots|\rho_G(A(a_1,\cdots,a_{k_j-1}) \cap B(b_{k_{j+h}+1},\cdots)) \\
 & -\rho_G(A(a_1,\cdots,a_{k_j-1}))\rho_G(B(b_{k_{j+h}+1},\cdots))| \\
\leq_{①} & \Sigma\Sigma\cdots\rho_G(A(a_1,\cdots,a_{k_j-1}))\rho_G(B(b_{k_{j+h}+1},\cdots))\lambda^{k_{j+h}-k_j} \\
= & \Sigma\Sigma\cdots\rho_G(A(a_1,\cdots,a_{k_j-1}))\Sigma\Sigma\cdots\rho_G(B(b_{k_{j+h}+1},\cdots))\lambda^{k_{j+h}-k_j} \\
= & \rho_G(A)\rho_G(B)\lambda^{k_{j+h}-k_j},
\end{array}$$

with $\lambda < 0.8$. The inequality ① is obtained by applying [Phi1, Lemma 2.1]. Note that the difference between the indexation is $k_{j+h} - k_j$ instead of h now. So by the definition,

$$\phi(h) = \sup_{j\in\mathbb{N}^+} \sup\Big\{|\frac{\rho_G(A\cap B)}{\rho_G(A)} - \rho_G(B)| : A \in \mathcal{M}_{1,j}, B \in \mathcal{M}_{j+h,\infty}, \rho_G(A) > 0\Big\} \leq \lambda^h.$$

Then

$$\Sigma_{h=1}^{\infty}\phi(h)^{\frac{1}{2}} \leq \Sigma_{h=1}^{\infty}\lambda^{\frac{h}{2}} < \infty$$

and

$$\psi^* \leq \lambda + 1 < \infty.$$

□

Remark 3.2.1. *P. Billingsley [Bil, (4.28)] obtained that for $A \in \mathcal{M}^*_{1,j}$ and $B \in \mathcal{M}^*_{j+h,\infty}$, we have*

$$|\rho_G(A \cap B) - \rho_G(A)\rho_G(B)| \leq \theta\rho_G(A)\rho_G(B)\lambda_*^h$$

with $\theta > 0$ and $0 < \lambda_* < 1$. The sharper estimate $\lambda_* < \frac{1}{2}$ was given by P. Szüsz,[Szu, (2.3)].

Lemma 3.3. *For a large enough real number y and $t > 0$, we have*

$$\lim_{y\to+\infty} \frac{\rho_G(\{x : c_{k_1}(x) > y\})}{\rho_G(\{x : c_{k_1}(x) > ty\})} = t.$$

Proof. We know that

$$\rho_G(\{x : c_1(x) > y\}) = \frac{1}{\log 2}\Sigma_{k>[y]}\int_{\frac{1}{k+1}}^{\frac{1}{k}} \frac{1}{x+1}dx$$

$$= \frac{1}{\log 2}\log(\frac{1}{[y]+1}+1)$$

$$= \frac{1}{\log 2}\frac{1}{[y]+1}+o(\frac{1}{y})$$

$$= \frac{1}{\log 2}\frac{1}{y}+o(\frac{1}{y})$$

as $y \to \infty$. Similarly,

$$\rho_G(\{x : c_{k_1}(x) > ty\}) = \frac{1}{\log 2}\frac{1}{ty}+O(\frac{1}{y}),$$

so

$$\lim_{y\to+\infty}\frac{\rho_G(\{x : c_1(x) > y\})}{\rho_G(\{x : c_1(x) > ty\})} = t.$$

As $\{c_j\}$ is a stationary sequence, the lemma holds for c_{k_1}. □

4 Proof of Theorem 1.1

We first begin with some notation. For a random variable X, let $E(X)$ denote its expectation and let $Var(X)$ denote its variance. By Lemma 3.1, if $\Sigma_{n\in\mathbb{N}}\frac{1}{\tau_n} < \infty$, then $c_{k_n}(x) > \tau_n$ is satisfied by only finitely many $n \in \mathbb{N}$ for almost all $x \in (0, 1)$. Let

$$c_n^*(x) = c_{k_n}(x)I_{[c_{k_n}(x)\le\tau_n]}.$$

Then we have

Lemma 4.1. *If $\Sigma_{j\in\mathbb{N}}\frac{1}{\tau_j} < \infty$, the sequence*

$$\tfrac{1}{\tau_n}\Sigma_{j=1}^n(c_j(x)^* - E(c_j^*(x))) \to 0 \text{ a.e.}$$

as $n \to \infty$.

Proof. We have

$$E(c_j^*(x)) = \int_{(0,1)}\frac{c_j^*(x)}{1+x}dx$$

$$= \int_{(0,1)}\frac{c_{k_j}(x)}{1+x}I_{[c_{k_j}(x)\le\tau_j]}dx$$

$$= \int_{(0,1)}\frac{c_1(x)}{1+x}I_{[c_1(x)\le\tau_j]}dx$$

$$\leq \int_{(0,1)} c_1(x) I_{[c_1(x)\leq \tau_j]} dx$$

$$\leq \Sigma_{i\leq \tau_j} i \frac{1}{i(i+1)} < \log \tau_j$$

for j large enough. Similar calculations show

$$E((c_j^*(x))^2) = \int_{(0,1)} \frac{(c_j^*(x))^2}{1+x} dx < \tau_j.$$

Then

$$\Sigma_{j\in\mathbb{N}} \frac{Var(c_j^*)}{\tau_j^2} = \Sigma_{j\in\mathbb{N}} \frac{E((c_j^*)^2) - (E(c_j^*))^2}{\tau_j^2}$$

$$\leq \Sigma_{j\in\mathbb{N}} \frac{\tau_j + (\log \tau_j)^2}{\tau_j^2} < \infty.$$

The last inequality holds because $\Sigma_{j\in\mathbb{N}^+} \frac{1}{\tau_j} < \infty$. Now the lemma follows by an application of [IT, Theorem 1.1.15]. The first two assumptions of the theorem are guaranteed by Lemma 3.2.

□

Now we prove Theorem 1.1.

Proof. We first deal with the first assertion. By assumption $\Sigma_j \frac{2^j}{\tau_{2^j}} < \infty$, so

$$\lim_{j\to\infty} \frac{j \cdot 2^j}{\tau_{2^j}} = 0.$$

That is,

$$\lim_{n\to\infty} \frac{n \log n}{\tau_n} = 0.$$

Then

$$\frac{1}{\tau_n} \Sigma_{j=1}^n E(c_j^*(x)) = \frac{1}{\tau_n} \Sigma_{j=1}^n \int_{(0,1)} \frac{c_j^*(x)}{1+x} dx$$

$$< \frac{1}{\tau_n} \Sigma_{j=1}^n \log \tau_j < \frac{n \log \tau_n}{\tau_n} \to 0$$

as $n \to \infty$. Combining Lemma 4.1 this implies

$$\lim_{n\to\infty} \frac{T_n}{\tau_n} = 0 \text{ a.e.}$$

as $c_{k_j}(x) \neq c_j^*(x)$ for only finitely many $j \in \mathbb{N}$ and almost all $x \in (0,1)$. This shows the first assertion of the theorem.

For the second assertion, let M be a positive number. By Lemma 3.1, the inequality

$$c_n(x) \geq M\tau_n$$

are satisfied for infinitely many $n \in \mathbb{N}$ for almost all $x \in (0,1)$. Then

$$\limsup_{n\to\infty} \frac{T_n(x)}{\tau_n} \geq \limsup_{n\to\infty} \frac{c_n(x)}{\tau_n} \geq M$$

for almost all $x \in (0,1)$. As M can be arbitrarily large, this shows

$$\limsup_{n\to\infty} \frac{T_n(x)}{\tau_n} = \infty$$

for almost all $x \in (0,1)$.

□

5 Proof of Theorem 1.2

Proof. Combining Lemma 3.2 and 3.3, we know that our sequence of random variables $\{X_n = c_{k_n}\}$ satisfies the assumptions of [Sam, Corollary 5.10] with $l_1 = 0, l_2 = 1, \alpha = 1$. Proof of the theorem goes similarly to the proof of Theorem 2 in [Phi1].

□

6 Proof of Theorem 1.3

In view of the proof of Theorem 3 in [Phi1], our strategy is to prove a subsequence-version result of Lemma 4.1 in [Phi1] and Lemma 4.2 in [Phi1]. We first split the sum T_n into two, depending on whether the terms in it exceed n/p_n or do not. For $1 \leq j \leq n$, let

$$c_{j,+} = c_{k_j} I_{[c_{k_j} > n/p_n]}, \quad c_{j,-} = c_{k_j} I_{[c_{k_j} \leq n/p_n]},$$

and let

$$T_{n,+} = \Sigma_{j\leq n} c_{j,+}, \quad T_{n,-} = \Sigma_{j\leq n} c_{j,-}.$$

One can see that $T_n = T_{n,+} + T_{n,-}$. First we show the following lemma.

Lemma 6.1. *The sequence of random variables*

$$T^*_{n,+} = T_{n,+} - \Sigma_{j\leq p_n \xi_n} c_n^{(j)} I_{[c_n^{(j)} > n/p_n]} \to 0,$$

in distribution as $n \to \infty$.

Proof. Let $\Gamma_n = \{1, \cdots, n\}$. Γ_n has $\binom{n}{k}$ subsets such that each one contains a total element of $k \leq n$ elements. Denote these $\binom{n}{k}$ sets by $\Gamma^j_{n,k}, 1 \leq j \leq \binom{n}{k}$. By Lemma 3.2, we have

$$\rho_G(\{x \in (0,1) : c_{k_i}(x) > n/p_n, i \in \Gamma^j_{n,k}\}) \leq (p_n/n)^k (1+\lambda)^k$$

for any $1 \leq j \leq \binom{n}{k}$. So

$$
\begin{aligned}
& \rho_G(\{x \in (0,1) : T^*_{n,+}(x) > 0\}) \\
= \;& \Sigma^n_{k=p_n\xi_n+1} \Sigma^{\binom{n}{k}}_{j=1} \; \rho_G(\{x \in (0,1) : c_{k_i}(x) > n/p_n \text{ for } i \in \Gamma^j_{n,k} \\
& \text{and } c_{k_i}(x) \leq n/p_n \text{ for } i \in \Gamma_n \setminus \Gamma^j_{n,k}\}) \\
\leq \;& \Sigma^n_{k=p_n\xi_n+1} \; \binom{n}{k}(p_n/n)^k(1+\lambda)^k \\
\leq \;& \Sigma^n_{k=p_n\xi_n+1} \; \frac{p_n^k}{k!} 1.8^k.
\end{aligned}
$$

The last sum tends to 0 as $n \to \infty$ as the tail of an exponential series. □

Now we proceed to show that

Lemma 6.2. *The distribution functions of the random variables*

$$\frac{T^*_{n,-}}{n}(p_n \log 2)^{1/2} - \left(\frac{p_n}{\log 2}\right)^{1/2} \log \frac{n}{p_n}$$

converge to $U(y)$ *as* $n \to \infty$.

Proof. This is a direct result by application of [Sam, Corollary 4.6] to our ϕ-mixing triangular array

$$c_{n,j} = c_{k_j}(Var(T_{n,-}))^{-1/2}$$

for $j \leq n$. The four conditions (the last one is empty) of the corollary are satisfied by our triangular array, as one can imitate similar calculations as in proof of Lemma 4.1 of [Phi1] using our Lemma 3.2 and the fact that the sequence $\{c_j, j \in \mathbb{N}\}$ is stationary. □

Now combining Lemma 6.1 and 6.2, Theorem 1.3 follows immediately.

7 Sequences satisfying conditions (i) and (ii)

The following is a list of sequences $(k_j)_{j\geq 1}$ that are L^p good universal for some $p \geq 1$, such that for each irrational α it is also true that the sequence $(k_j\alpha)_{j\geq 1}$ is uniformly distributed modulo one.

1. $k_j = j$ $(j = 1, 2, \ldots)$. This is Birkhoff's pointwise ergodic theorem.

2. $k_j = \phi(j)$ $(j = 1, 2, \ldots)$, where ϕ is a polynomial mapping the natural numbers to themselves [Bou] .

3. $k_j = \phi(p_j)$ $(j = 1, 2, \ldots)$, where ϕ is a polynomial mapping the natural numbers to themselves and $(p_j)_{j\geq 1}$ is the sequence of rational primes [Nai1].

4. Set $k_j = [g(j)]$ $(j = 1, \ldots)$ where $g : [1, \infty) \to [1, \infty)$ is a differentiable function whose derivative increases with its argument. Let A_n denote the cardinality of the set $\{j : k_j \leq n\}$, and suppose for some function $a : [1, \infty) \to [1, \infty)$ increasing to infinity as its argument does, that we set

$$b_M = \sup_{\{z\} \in [\frac{1}{a(M)}, \frac{1}{2})} \left| \sum_{j : a_j \leq M} e(zk_j) \right|.$$

(Here $e(x) = e^{2\pi i x}$ for real x.) Suppose also for some decreasing function $c : [1, \infty) \to [1, \infty)$ and some positive constant $C > 0$ that

$$\frac{b_M + A_{[a(M)]} + \frac{M}{a(M)}}{A_M} \le Cc(M).$$

Then, if we have

$$\sum_{s=1}^{\infty} c(\theta^s) < \infty$$

for $\theta > 1$ we say that $\underline{k} = (k_n)_{n=1}^{\infty}$ satisfies conditions H.

Specific sequences of integers that satisfy conditions H include $k_n = [g(n)]$ $(n = 1, 2, \ldots)$ where

I. $g(x) = x^{\omega}$ if $\omega > 1$ and $\omega \notin \mathbb{N}$.

II. $g(x) = e^{\log^{\gamma} x}$ for $\gamma \in (1, \frac{3}{2})$.

III. $g(x) = P(x) = b_k x^k + \ldots + b_1 x + b_0$ for $b_k, \ldots, b_1$ not all rational multiples of the same real number.

IV. *Hardy fields:* By a Hardy field, we mean a closed subfield (under differentiation) of the ring of germs at $+\infty$ of continuous real-valued functions with addition and multiplication taken to be pointwise. Let $\mathcal{H}$ denote the union of all Hardy fields. Conditions for $(a_n)_{n=1}^{\infty} = ([\psi(n)])_{n=1}^{\infty}$, where $\psi \in \mathcal{H}$ to satisfy condition H are given by the hypotheses of Theorems 3.4, 3.5 and 3.8. in [BKQW]. Note the term ergodic is used in this paper in place of the older term Hartman uniformly distributed.

5. A random example: (i) Suppose $S = (n_k)_{n=1}^{\infty} \subseteq \mathbb{N}$ is a strictly increasing sequence of natural numbers. By identifying S with its characteristic function I_S, we may view it as a point in $\Lambda = \{0, 1\}^{\mathbb{N}}$, the set of maps from $\mathbb{N}$ to $\{0, 1\}$. We may endow Λ with a probability measure by viewing it as a Cartesian product $\Lambda = \prod_{n=1}^{\infty} X_n$ where for each natural number n we have $X_n = \{0, 1\}$, and specify the probability π_n on X_n by $\pi_n(\{1\}) = q_n$, with $0 \le q_n \le 1$ and $\pi_n(\{0\}) = 1 - q_n$ such that $\lim_{n\to\infty} q_n n = \infty$. The desired probability measure on Λ is the corresponding product measure $\pi = \prod_{n=1}^{\infty} \pi_n$. The underlying σ-algebra β is that generated by the "cylinders"

$$\{\lambda = (\lambda_n)_{n=1}^{\infty} \in \Lambda : \lambda_{i_1} = \alpha_{i_1}, \ldots, \lambda_{i_r} = \alpha_{i_r}\}$$

for all possible choices of $i_1, \ldots, i_r$ and $\alpha_{i_1}, \ldots, \alpha_{i_r}$. Then almost any $(k_j)_{j=1}^{\infty}$ in Λ with respect to the measure π is L^p good universal for all $p > 1$[Bou].

6. Block L^1 good universal sequences: If $(k_n)_{n\ge 1} = \cup_{k=1}^{\infty}[d_k, d_k + e_k]$ ordered by absolute value for disjoint $([d_k, d_k + e_k])_{k\ge 1}$ with $d_{k-1} = O(e_k)$ as k tends to infinity. Note that if $d_{k-1} = o(e_k)$ the sequence $(k_n)_{n=}^{\infty}$ is zero density [BL].

7. A random perturbation : Suppose $(k_j)_{j\ge 1}$ is L^2-good universal and $(k_j\alpha)_{j\ge 1}$ is uniformly distributed modulo one for any non-integer α . Suppose $\theta = \{\theta_n, n \ge 1\}$ denotes a sequence of $\mathbb{N}$-valued independent, identically distributed random variables with basic probability space $(\Omega, \mathcal{A}, \mathcal{P})$, and a $\mathcal{P}$-complete σ-field $\mathcal{A}$. We assume that there exist $0 < \beta < 1$ and $B > 1/\beta$, such

that

$$k_j = O(e^{j^\beta}),$$

and we have

$$\mathbb{E}(\log_+^B |\theta_1|) < \infty.$$

Then $(k_j + \theta_j(\omega))_{j\geq 1}$ also satisfies conditions (i) and (ii) [NW].

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