An Improved Upper Bound for the Universal TSP on the Grid

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Abstract

We study the universal Traveling Salesman Problem in an $n \times n$ grid with the shortest path metric. The goal is to define a (universal) total ordering over the set of grid's vertices, in a way that for any input (subset of vertices), the tour, which visits the points in this ordering, is a good approximation of the optimal tour, i.e. has low competitive ratio.

This problem was first studied by Platzman and Bartholdi [26]. They proposed a heuristic, which was based on the Sierpinski space-filling curve, in order to define a universal ordering of the unit square $[0,1]^2$ under the Euclidean metric. Their heuristic visits the points of the unit square in the order of their appearance along the space-filling curve. They provided a logarithmic upper bound which was shown to be tight up to a constant by Bertsimas and Grigni [3]. Bertsimas and Grigni further showed logarithmic lower bounds for other space-filling curves and they conjectured that any universal ordering has a logarithmic lower bound for the $n \times n$ grid.

In this work, we disprove this conjecture by showing that there exists a universal ordering of the $n \times n$ grid with competitive ratio of $O\left(\frac{\log n}{\log \log n}\right)$. The heuristic we propose defines a universal ordering of the grid's vertices based on a generalization of the Lebesgue space-filling curve. In order to analyze the competitive ratio of our heuristic, we employ techniques from the theory of geometric spanners in Euclidean spaces. We finally show that our analysis is tight up to a constant.

1 Introduction

The *traveling salesman problem (TSP)* is perhaps one the most well-studied and intriguing combinatorial optimization problems: consider a delivery person who needs to deliver parcels to specific places in a city and her intention is to visit all places, by forming a tour which minimizes the total distance. The special case that the visited places are points in the plane and their pairwise distance is the Euclidean, is called the *Euclidean* TSP.

Consider now a different situation where the delivery person has a fixed list of clients and every day has to serve a new subset of them; hence, every day she should solve a new instance of the TSP. Rather than re-optimizing her delivery routes every day, a very simple heuristic would be to define a *universal ordering* or a *master tour* among the whole list of clients and each day to adjust the tour to the clients that need to be served, by following the order that they appear in the master tour. The goal is to define a universal ordering that produces tours that can serve as good approximation of the optimal tour for *any* potential subset of clients, i.e. with *low competitive ratio*. This is known as the *universal traveling salesman problem* (UTSP).

UTSP was first studied in the seminal paper of Platzman and Bartholdi [26] almost 30 years ago. They proposed a space-filling heuristic to define a universal ordering of the unit square $[0,1]^2$ under the Euclidean metric. They used the Sierpinski space-filling curve which is a bijection from the unit interval to the unit square; their heuristic visits the points of the unit square in the order of their appearance along this curve. They showed a logarithmic upper bound on the competitive ratio of that heuristic, which is until today the best known guarantee for the Euclidean space. They also conjectured that this bound could be reduced to a constant, which was later disproved by Bertsimas and Grigni [3]. They provided a counter example showing that the bound of Platzman and Bartholdi was tight up to a constant factor, and they showed similar lower bounds for other space-filling curves, like the Hilbert and the Peano curve. They conjectured that for any space-filling heuristic, or more strongly for any universal ordering, of an $n \times n$ grid, the competitive ratio is $\Omega(\log n)$. More recently, Hajiaghayi, Kleinberg and Leighton [16] showed the first general lower bound of $\Omega\left(\sqrt[6]{\frac{\log n}{\log \log n}}\right)$ for any universal ordering of the $n \times n$ grid, which reinforced the conjecture.

In this work, though, we disprove the conjecture of Bertsimas and Grigni [3].

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THEOREM 1.1. *There exists a universal ordering of the $n \times n$ grid with competitive ratio of* $\Theta\left(\frac{\log n}{\log\log n}\right)$.

Techniques. We define a total ordering on the vertices of the $n \times n$ grid, which we call *generalized Lebesgue space-filling ordering*. Similarly to [26], our ordering is based on a *space-filling curve*[1]. Precisely, it is an extension of the Morton Order, which is based on the Lebesgue space-filling curve[2] (also known as Z-curve).

In order to analyze the competitive ratio of our heuristic, we employ techniques from the theory of geometric spanners in Euclidean spaces[3]. A central tool in our analysis is the so-called *isolation property* which was introduced by Das, Narasimhan and Salowe in [9]. A subset E of edges satisfies the isolation property, if for a constant $c > 0$ and any edge $e \in E$ of weight $w(e)$, it is possible to place a hypercylinder[4] of radius and height $c \cdot w(e)$, such that its axis is a subset of e and it does not intersect any other edge of E. It was shown in [9], that if E satisfies the isolation property, then it has total weight which is only a constant factor away of the weight of the Steiner minimal tree on the endpoints of E.

A crucial part of our proof is the definition of a property which we call *pseudo-isolation* and which is a stronger version of the isolation property. The main difference is that the parameter c of the isolated box around the edge is no longer a constant, but can be a function of n. When a set of edges satisfy pseudo-isolation, we show that their total weight is $O(1/c)$ away[5] from the weight of the minimum Steiner tree. This allow us to obtain the desired competitive ratio, as in our case it is $1/c = O(\log n/\log\log n)$. However, in our way to obtain this stronger bound, we sacrifice in simplicity and as a result both the region and the condition that we define are now less elegant. It is no longer a box, but a much more complicated to define region.

Open Questions. We remark that the the competitive ratio can be analyzed mainly as a function of the number of vertices, n, of the metric space [14, 16, 18], but also as a function of the number of requested vertices, k, [3, 26]. We note that the upper bound of Platzman and Bartholdi [26] is with respect to k, i.e. $O(\log k)$, and trivially a logarithmic bound holds with respect to n. The lower bound of Bertsimas and Grigni [3], regarding the universal ordering suggested in [26], is again with respect to k. However, after a closer look at their lower bound constructions, all space-filling curves examined in [26] have logarithmic lower bounds with respect to n, as well.

Overall, regarding an $n \times n$ grid under the shortest path metric, the competitive ratio of the Sierpinski space-filling ordering is $\Theta(\log n)$ and $\Theta(\log k)$ [3, 26], whereas the competitive ratio of the Generalized Lebesgue space-filling ordering is $\Theta(\log n/\log\log n)$ (Theorem 1.1) and $\Theta(k)$ (Remark 4.1). It is therefore clear that the two universal orderings are *incomparable*. There are two natural questions that are left open:

1. Is there a universal ordering with competitive ratio of $o(\log k)$?

2. Is there a universal ordering with competitive ratio of $O(\min\{\log k, \log n/\log\log n\})$?

Related Work. The traveling salesman problem is one of most fundamental NP-hard optimization problems, which is even NP-hard to approximate with a ratio better than 220/219 [25], unless $P = NP$. It is further well-known that the Euclidean TSP is NP-complete [24].

For the last forty years, the 3/2-approximation algorithm due to Christofides [8] is the best known approximation for the metric TSP, where the pairwise distances form a metric space. During the last years, improvements have been noted regarding the graphic TSP, a special case of the metric TSP, where the underlying metric is defined by shortest path distances in an arbitrary undirected graph. Gharan, Saberi and Singh [11] gave an $(1.5 - \epsilon)$-approximation algorithm, for ϵ of the order 10^{-12}. Mömke and Svensson [21] suggested an algorithm with a better approximation of 1.461, which was later improved to 13/9 by a better analysis of Mucha [22]. The best known approximation algorithm for the graphic TSP is 7/5 due to Sebö and Vygen [30]. Grigni, Koutsoupias and Papadimitriou [13] showed a PTAS for the planar TSP, where the metric is defined by shortest path distances in a planar graph. For weighted planar graphs, Klein [19] improved the

[1]We refer the reader to [2] for more discussion on space filling curves and their properties.

[2]Space-filling curves are continuous mappings from a 1-dimensional space onto a higher-dimensional space. Usually the 1-dimensional space is the $[0, 1]$ interval (like in Hilbert, Peano and Sierpinski curves). Technically, by considering $[0, 1]$, the Lebesgue curve lacks the continuity requirement to be a space-filling curve. However, a standard trick is that if a Cantor set is used instead, the continuity is restored and there is no violation on the definition of Lebesgue curve (see discussion in [2]). Accordingly, the generalized Lebesgue space-filling curve can be defined by considering an appropriate Cantor set.

[3]We refer the reader to [10, 23] for a detailed treatment of the subject.

[4]In this work we are only interested in the two dimensional space, in which case the hypercylinder is just a square.

[5]We remark, that a straight-forward generalization of the techniques that prove the isolation (see also [28]), if one leaves c as a function of n, it would obtain a $O(1/c^2)$ bound.

running time of PTAS to linear. Regarding the Euclidean TSP, a PTAS is also known due to the seminal work of Arora [1] and Mitchell [20].

Jia et al. [18] introduced the notion of *universality* in the context of optimization problems, where the designer seeks of a *single* structure that simultaneously approximates the optimal solution for every possible input. They considered the universal versions of three well-studied combinatorial optimization problems, the traveling salesman, the Steiner tree and the set cover problems, and they provided several upper and lower bounds on their competitive ratio. Regarding the UTSP, they showed an upper bound of $O(\log^4 n/\log\log n)$ for arbitrary metric spaces on n vertices. For the special case of doubling metrics, (which includes both constant-dimensional Euclidean and growth-restricted metrics), their algorithms achieve a bound of $O(\log n)$. Hajiaghayi, Kleinberg and Leighton [16] showed an upper bound of $O(\log^2 n)$ for planar graph metrics and more generally H-minor-free metrics of n points. The best known upper bound so far is due to Gupta, Hajiaghayi and Räcke [14] who showed an $O(\log^2 n)$ bound that holds for general metrics of n vertices.

Gorodezky, Kleinberg, Shmoys and Spencer [12] showed a lower bound of $\Omega(\log n)$ for general graph metrics (in particular Ramanujan graphs under the shortest path metric). As mentioned earlier, Hajiaghayi, Kleinberg and Leighton [16] showed a lower bound of $\Omega\left(\sqrt[6]{\frac{\log n}{\log\log n}}\right)$ for any universal ordering of the $n \times n$ grid.

The stochastic version of UTSP is known as the *a priori* TSP, in which case there is a fixed probability distribution over the subsets of vertices, and the designer seeks a master tour that minimizes the expected weight of the induced tour [4, 12, 29, 31].

Very related to the notion of the universal solutions, is the concept of *oblivious routing* [14, 15, 17, 27], where routing decision are made without any knowledge of the current state of the network. The concept of universal solutions has also been used in other contexts, notably in the context of hash functions [5] and routing [32]. Finally, there is also some connection with the notion of coordination mechanisms [6], where the task is to design algorithms with good equilibrium solutions, and also with the design of universal ordered protocols which is discussed in [7].

2 Preliminaries

Given a total ordering $\preceq$ of a metric space (V, d), and any finite subset $S \subseteq V$ of vertices, $v_1 \preceq v_2 \preceq \ldots \preceq v_m$, by adopting the convention that $v_{m+1} = v_1$, we define the tour $T_{\preceq}(S)$ on S to be the union of the edges (v_i, v_{i+1}), for $1 \le i \le m$. We further define the weight of an edge $e = (v, u)$ to be $w(e) = d(v, u)$ and the weight of a set of edges, E, to be $w(E) = \sum_{e\in E} w(e)$. This notation naturally extends to a tour T, by viewing T as a set of edges, hence $w(T_{\preceq}(S)) = \sum_{i=1}^{m} d(v_i, v_{i+1}) = \sum_{e\in T} w(e)$. We denote by $T^*(S)$ an optimal tour of S, i.e. if S_m is the permutation group of S, then $w(T^*(S)) = \inf_{\pi\in S_m} w(T_\pi(S))$.

Competitive ratio. The *competitive ratio* of the universal ordering $\preceq$ is defined by

$$\sup_{S\subseteq V} \frac{w(T_{\preceq}(S))}{w(T^*(S))}.$$

Grid G_n. We work on the $n \times n$ grid G_n with the shortest path metric. For convenience we place the vertices of the grid on the plane and use the ℓ_2 metric[6]. In particular, using the standard coordinate system, like in [16], we define G_n as follows

$$G_n = \left\{ \left(\frac{2i-1}{2n}, \frac{2j-1}{2n} \right),\ \text{for } 1 \le i, j \le n \right\}.$$

Any segment parallel to the x or y axis is called *horizontal* or *vertical*, respectively.

3 Generalized Lebesque Space-filling Ordering

We define our universal ordering of the vertices of G_n, by successively subdividing the unit square $[0,1]^2$ into smaller squares. At each level of refinement, we partition each square into smaller squares and we order them lexicographically. The order of the squares induces a partial order among vertices that belong to different squares. We perform sufficiently many iterations, such that the level of refinement guarantees a universal ordering of G_n; at the last level of refinement, each square should contain at most one vertex from G_n.

In particular, let r be the positive integer such that $(r-1)^{r-1} < n \le r^r$. We divide initially $[0,1]^2$ and later each smaller square into a $r \times r$ grid of congruent equal-sized squares, $\{Q_{ij}, 1 \le i, j \le r\}$, such that the position of each Q_{ij}'s center is the point[7] $\left(\frac{2i-1}{2r}, \frac{2j-1}{2r}\right)$. We define an ordering $\preceq$ among Q_{ij}'s that induces an ordering among vertices of different squares, i.e. if $Q_{ij} \preceq Q_{i'j'}$, then for any $v \in Q_{ij}$ and $u \in Q_{i'j'}$, $v \preceq u$.

Ordering of $\mathbf{Q_{ij}}$'s. For any pair of squares Q_{ij} and $Q_{i'j'}$, at the same level of refinement, $Q_{ij} \preceq Q_{i'j'}$ if and only if either $i < i'$, or $i = i'$ and $j < j'$. See Figures 1(a) and 1(b) for an exposition of the ordering for 1 and 2 successive subdivisions and for $r = 3$. By

[6]Note that this affects the competitive ratio by only a constant factor, as this mapping has distortion of $\sqrt{2}$.

[7]Note that indices i and j are associated with x and y coordinates, respectively.

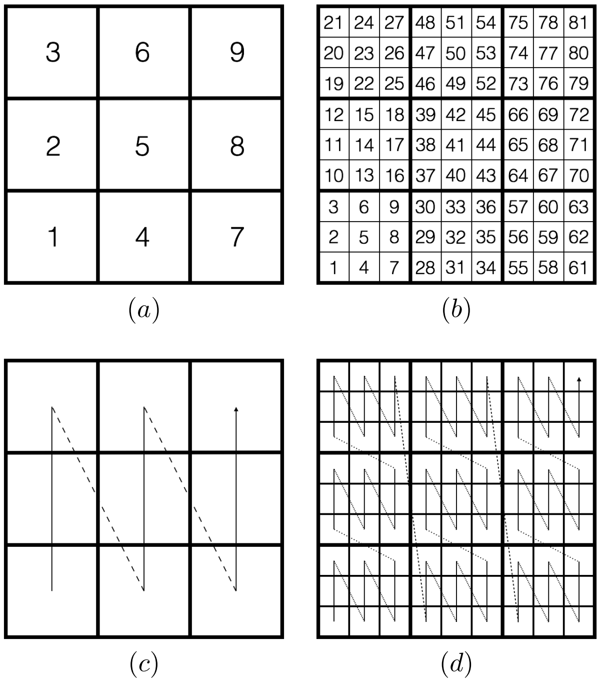


Figure 1: (a) and (b) illustrates the universal ordering of the squares for 1 and 2 successive subdivisions, respectively. (c) and (d) illustrates the first and second iteration of the generalized Lebesgue space-filling curve.

our choice of the parameter r, this recursive procedure will eventually decide an order among every two points in $[0,1]^2$, after r subdivisions [8].

As mentioned earlier, our ordering is based on a generalization of the Lebesgue curve. Figures 1(c) and 1(d) show the generalized Lebesgue space-filling curve for 1 and 2 successive subdivisions.

4 Analysis

In this section, we provide the competitive analysis of the generalized Lebesgue space-filling heuristic. In Section 4.1 we introduce some additional definitions and notation that will be used throughout the analysis. In Section 4.2 we give the general structure of the proof of Theorem 1.1, while in Section 5 we provide the proofs of several technical lemmas that are used in the proof of the main theorem.

4.1 Definitions.

Let $S \subseteq G_n$ be the set of requested vertices, T be the set of edges of the tour produced by the generalized Lebesgue space-filling heuristic, and T^* be an optimal tour on S.

t-division and $\mathcal{Q}$. We call t-division of $[0,1]^2$, for $t \in \{1,\ldots,r\}$, the partition of $[0,1]^2$ into a $r^t \times r^t$ grid of congruent equal-sized squares, $\mathcal{Q}^t = \{Q^t_{ij}, 1 \leq i,j \leq r^t\}$. By convention, $\mathcal{Q}^0 = \{[0,1]^2\}$ and $\mathcal{Q} = \cup_{t\in\{0,\ldots,r\}}\mathcal{Q}^t$. Similarly, we define the t-division of any square $Q \in \mathcal{Q}$.

Level of edge, $\mathbf{E(Q)}$. An edge $e = (v,u)$ belongs to level t for some $t \in \{1,\ldots,r\}$, if and only if:

i) v and u lie in the same square, $Q \in \mathcal{Q}^{t-1}$,

ii) v and u lie in different squares in $\mathcal{Q}^t$.

We further say that e is an edge of Q (or e belongs to Q) if Q is the smallest square of $\mathcal{Q}$ that contains e. We denote by $E(Q)$ the set of all edges of E that belong to Q.

Vertex Coordinates. Let v be a vertex with $v \in Q^t_{ij}$ for some t-division. We define its coordinates w.r.t. the t-division, as $i^t(v) = i, j^t(v) = j$. Similarly, for $v \in Q$, we define the *relative* coordinates $i_Q(v), j_Q(v)$ of v w.r.t. Q, referring to the square Q_{ij} that contains v in the 1-division of Q. When v is the endpoint of some edge of level t, we drop index t, when there is no ambiguity.

When it is clear from the context that i is an integer, we abuse notation and use the shorthand $i \in [i_1,i_2]$ to denote that i is an integer in the interval $[\min\{i_1,i_2\},\max\{i_1,i_2\}]$. We further use the following notation: $(i_1,i_2] = [i_1,i_2] \setminus \{i_1\}$, $[i_1,i_2) = [i_1,i_2] \setminus \{i_2\}$ and $(i_1,i_2) = [i_1,i_2] \setminus \{i_1,i_2\}$.

Edges within S. We denote by $E(S)$ the set of edges within S, i.e. $E(S) = \{(v,u)|v,u \in S\}$. Note that $E(S)$ is a superset of T.

4.1.1 Properties and Grouping of Edges.

Our analysis uses various groupings and transformations of edges. In this section, we collect some basic definitions of properties and groupings of the edges.

Slope Grouping. An edge $e = (v,u)$ is called *column* if $i(v) = i(u)$, *row* if $j(v) = j(u)$, *double-row* if $|j(v)-j(u)| = 1$ and $i(v) \neq i(u)$, and *diagonal* otherwise. We emphasize that for technical reasons, double-row edges are *not* considered as diagonal. See Figure 2.

Length Grouping. An edge $e = (v,u)$ is called *short* if $|i(v)-i(u)| \leq 2$ and $|j(v)-j(u)| \leq 2$, otherwise it is called *long*. See Figure 2.

Well-placed. A set E of edges is called *well-placed* (See Figure 2) if for every pair of edges $e = (u,v), e' = (u',v') \in E$ of the same level:

i) if e, e' are both column edges on the

[8]Note that some vertices may belong to more than one squares, if they lie on the border of some subdivision. In order to define the order consistently, we (arbitrarily) associate each such vertex with exactly one of those squares.

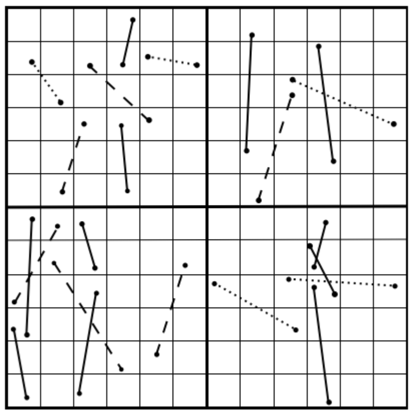


Figure 2: The figure depicts the t-division of some subregion of $[0,1]^2$ and the four bigger squares are derived from the $(t-1)$-division of $[0,1]^2$ (also $r = 6$). It shows an example of well-placed edges of level t, where the solid and dashed edges are, respectively, the column and diagonal edges and the dotted edges are the row and double row edges. Additionally, all the edges in the top left square are short and in the top right square are long.

same column, i.e. $i(v) = i(v')$, then either $\max\{j(v), j(u)\} \leq \min\{j(v'), j(u')\}$ or $\max\{j(v'), j(u')\} \leq \min\{j(v), j(u)\}$,

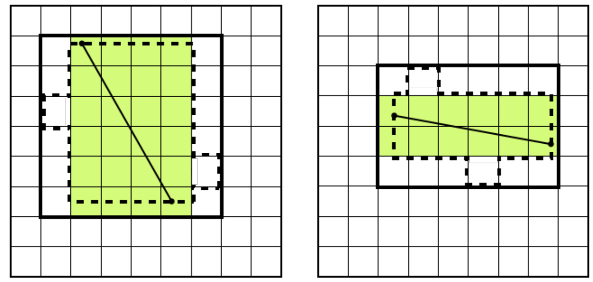
ii) if neither of e, e' is a column edge and $e, e' \in E(Q)$, for some Q, then either $\max\{i(v), i(u)\} \leq \min\{i(v'), i(u')\}$ or $\max\{i(v'), i(u')\} \leq \min\{i(v), i(u)\}$.

We note that the edges produced by our heuristic are well-placed. This is true because for any t-division of $[0,1]^2$, the heuristic orders all the squares of each column in a bottom up fashion; therefore, if the condition for well-placed edges didn't hold for column edges, the universal ordering would be violated. Regarding the rest of the edges, note that for any square Q, the heuristic orders the columns of the 1-division of Q from left to right.

We will next define the stronger property of a *well-distributed* set of edges to bound short edges produced by several transformations of long edges. First, we give the definition of bounded boxes around edges.

Bounded boxes. Let $e = (v, u)$ be an edge of level t. We define the *bounded box* A_e of e as the minimum rectangular region formed by the union of Q^t_{ij}'s that contains e, i.e. A_e is the union of Q^t_{ij}, for all $i \in [i(v), i(u)]$ and $j \in [j(v), j(u)]$.

We now define some extended versions of this notion, that for technical reasons will be defined differently for edges with different slopes. Let first e be a column or diagonal edge with $i(v) \leq i(u)$. We define the *extended bounded box* B_e as the union of A_e and all squares[9] $Q^t_{(i(v)-1)j}$, $Q^t_{(i(u)+1)j}$, for all $j \in [j(v), j(u)]$, (in other words, we extend A_e by one column on the left and one column on the right). Let Γ'_e be the region defined by the intersection of A_e with the region between the horizontal lines that pass through v and u. If there exist $Q^t_{(i(v)-1)j_1}$ and $Q^t_{(i(u)+1)j_2}$ empty of points of S, for some $j_1, j_2 \in (j(v), j(u))$, we define $\Gamma_e = \Gamma'_e \cup Q^t_{(i(v)-1)j_1} \cup Q^t_{(i(u)+1)j_2}$. If any of these two squares doesn't exist, then Γ_e is *undefined*[10].



(a) (b)

Figure 3: The figures show A_e, B_e, Γ_e for a diagonal edge (a) (they are similarly defined for column edges) and for a double-row edge (b) (they are similarly defined for row edges). Specifically, suppose that the edge is of level t, then the squares in the figures are of the t-division; A_e is the shaded rectangle, B_e is the rectangle with bold borders and Γ_e is the region with dashed borders. Note that in both cases the two squares of $\Gamma_e \setminus A_e$ should not intersect S.

For row and double-row edges, B_e and Γ_e are defined accordingly after swapping the axes x, y (e.g. B_e is the extension of A_e by one row above and one row below). See Figure 3 for an illustration of A_e, B_e and Γ_e. By A° we denote the interior of a region A.

Well-distributed. We call a set of edges $E \subseteq E(S)$ *well-distributed* (See Figure 4) if we can assign a value $t_e \in \{1, \ldots, r\}$ to each edge $e = (v, u)$, such that

i) in the t_e-division of $[0,1]^2$, v and u belong to different sub-squares,

[9] For the extreme cases $i(v) = 1$ or/and $i(u) = r^t$, in order to define B_e (and later Γ_e), we introduce some "dummy" squares Q^t_{0j} and $Q^t_{(r^t+1)j}$, for $1 \leq j \leq r^t$.

[10] If there is no index in $(j(v), j(u))$, e.g. e is a column edge with $|j(v) - j(u)| \leq 1$, Γ_e is undefined. However, we use Γ_e only for long edges where it holds that $|j(v) - j(u)| \geq 2$ (for long, column or diagonal edges). Therefore, the reader may safely assume that if Γ_e is undefined, it is because all squares $Q^t_{(i(v)-1)j_1}$ or all squares $Q^t_{(i(u)+1)j_1}$, for $j_1, j_2 \in (j(v), j(u))$, intersect S.

ii) under the convention that e belongs to level t_e, e is short (more formally, $|i^{t_e}(v) - i^{t_e}(u)| \leq 2$ and $|j^{t_e}(v) - j^{t_e}(u)| \leq 2$),

iii) for any pair of edges $e = (v, u)$ and $e' = (v', u')$ of E, with $t_e = t_{e'}$, under the convention that e, e' belong to level t_e, if either both of them are column edges, or neither of them is, it holds that $A_e \not\subseteq A'_e$ (more formally, if either both $i^{t_e}(v) = i^{t_e}(u)$ and $i^{t_e}(v') = i^{t_e}(u')$, or both $i^{t_e}(v) \neq i^{t_e}(u)$ and $i^{t_e}(v') \neq i^{t_e}(u')$, it holds that $A_e \not\subseteq A'_e$, where A_e and $A_{e'}$ are defined for t_e-division).

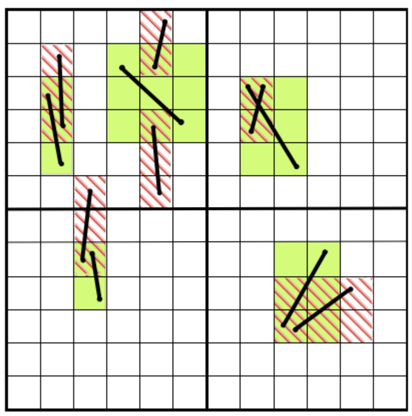


Figure 4: The figure depicts the t-division of some subregion of $[0, 1]^2$ and the four bigger squares are derived from the $(t-1)$-division of $[0, 1]^2$ (also $r = 6$). It shows well-distributed edges, for all of which we assign the value t. The shaded and hatched regions are the bounded boxes A_e.

Weakly and Strongly Detached. Consider an edge $e = (v, u) \in E(Q)$ and let $\{Q_{ij}, 1 \leq i, j \leq r\}$ be the 1-division of Q. We say that e is *weakly detached* if S does not intersect any Q_{ij}, for all $i \in (i_Q(v), i_Q(u))$ and $1 \leq j \leq r$. Furthermore, e is *strongly detached*, if additionally S does not intersect $A_e \setminus \{Q_{i(v)j(v)} \cup Q_{i(u)j(u)}\}$. See Figure 5.

4.2 The Proof of Theorem 1.1

Theorem 1.1 (restated). *There exists a universal ordering of the $n \times n$ grid with competitive ratio of $\Theta\left(\frac{\log n}{\log \log n}\right)$.*

We show that the generalized Lebesgue space-filling heuristic achieves the claimed competitive ratio. Recall that T is the set of edges of the tour produced by the generalized Lebesgue space-filling heuristic on the requested vertices $S \subseteq G_n$ and T^* is an optimal tour on

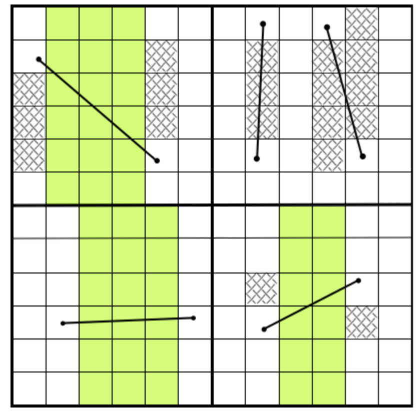


Figure 5: The figure depicts the t-division of some subregion of $[0, 1]^2$ and the four bigger squares are derived from the $(t-1)$-division of $[0, 1]^2$ (also $r = 6$). It illustrates the region that S should not intersect in order for the edges to be weakly and strongly detached; precisely, if there are no points of S in the shaded (green) regions, then the edge is weakly detached and if additionally the are no points of S in the crosshatched regions, then it is strongly detached.

S. Further, recall that $r = \Theta\left(\frac{\log n}{\log \log n}\right)$ and hence we show that $w(T)$ is upper bounded by $O(r)w(T^*)$. Before proceeding to our analysis we give a proof roadmap.

Proof roadmap. Our analysis starts by partitioning and transforming T, into sets of edges that satisfy some special properties, which make it easier to compare their weight with that of the optimal tour. More specifically, we perform the following three steps:

1. We first partition T into short and long edges. We argue that short edges are *well-distributed*. Then, we bound them by $O(r)w(T^*)$ in Lemma 4.1 by using the *Isolation Property* of [9].

2. It is easy to check that the edges of T are well-placed and all are weakly detached. In this step, we transform the long edges into *strongly detached* edges and a constant number of well-distributed sets; we bound the latter case by using Lemma 4.1. Regarding the strongly detached edges, we argue that there are constant number of sets that are *well-placed*.

3. We then transform each of those sets into *pseudo-isolated* edges (defined later in the proof) and a constant number of well-distributed sets that we bound again by using Lemma 4.1. At the end, we show that the edges remaining to bound form a constant number of sets of *long, pseudo-isolated, well-placed edges* and additionally the diagonal edges are

strongly detached. That kind of edges are handled in Section 5.

4.2.1 First Step.

As a first step we partition T into short and long edges, T^s and T^ℓ, respectively. In Section 4.1 we argued that T is a set of well-placed edges. Observe that the short, well-placed edges are also well-distributed, by setting, for each edge e, the value t_e to be the level that e belongs to. Hence, T^s is well-distributed and we can bound its total weight, $w(T^s)$, by applying the following Lemma 4.1.

LEMMA 4.1. *Let $E \subseteq E(S)$ be a set of well-distributed edges. Then, $w(E) = O(r)w(T^*)$.*

Proof. A crucial element of our proof is the so-called *Isolation Property* which was defined in [9] for d-dimensional spaces. Since our focus is on the plane, we only give here the definition for two dimensions.

Isolation Property [9]. Let $c > 0$ be some constant and F be a set of edges in the plane. If it is possible, for an edge $e \in F$, to place a square, A, of side length $c \cdot w(e)$ around e, such that e passes through the center of A, is parallel to the two sides of A and intersects the other two, and additionally $A \cap (F \setminus \{e\}) = \emptyset$, then e is said to be *isolated*. F satisfies the *isolation property*, if every $e \in F$ is isolated.

Das, Narasimhan and Salowe [9] showed that if a subset of edges satisfy the isolation property, then the weight of these edges is a constant factor away from the weight of the Steiner minimal tree of their endpoints.

Theorem 1.2 of [9]. If a set of line segments F in the plane satisfies the isolation property and SMT is the Steiner minimal tree of the endpoints of F's segments, then $w(F) = O(1)w(SMT)$.

In order to prove our lemma, we will partition E into $O(r)$ subsets such that each of them satisfies the Isolation Property. Then, our lemma follows by applying the theorem of [9] for each subset.

We first partition the edges of E according to their slope into column E_c, row E_r, double-row E_{dr} and diagonal E_d edges. We treat each subset $X \in \{E_c, E_r, E_{dr}, E_d\}$ separately. We partition X into 9 sets X_{ab}, with $a, b \in \{0, 1, 2\}$, such that $e = (v, u) \in X_{ab}$, if $a = (\min\{i(v), i(u)\} \mod 3)$ and $b = (\min\{j(v), j(u)\} \mod 3)$. Finally, we partition the edges of X_{ab} according to their division level t, i.e. $X_{ab} = \cup_t X^t_{ab}$, where X^t_{ab} contains only the subset of edges of level t. We now claim that each of these sets satisfies the Isolation Property for $c \geq 1/3$.

To see this, note that the following statements are true because X^t_{ab} is well-distributed.

- if $X = E_c$, for any $e = (v, u) \in X^t_{ab}$, no other edge of X^t_{ab} intersects the squares Q^t_{ij}, for $\max\{i(v) - 2, 1\} \leq i \leq \min\{i(v) + 2, r^t\}$ and $j \in [j(v), j(u)]$,
- if $X = E_r$, for any $e = (v, u) \in X^t_{ab}$, no other edge of X^t_{ab} intersects the squares Q^t_{ij}, for $\max\{j(v) - 2, 1\} \leq j \leq \min\{j(v) + 2, r^t\}$ and $i \in [i(v), i(u)]$,
- if X is either E_{dr} or E_d, for any $e \in X^t_{ab}$, no other edge of X^t_{ab} intersects the region A_e.

Note that the set of the endpoints of the X^t_{ab}'s edges is a subset of S and therefore the weight of their Steiner minimal tree is upper bounded by $w(T^*)$. Since the number of subsets X^t_{ab}'s is $O(r)$ the lemma follows. □

4.2.2 Second Step.

Then we proceed with T^ℓ. The heuristic produces row or column edges that are strongly detached, while the rest are weakly (but not necessarily strongly) detached. Next we transform the set of edges, $T^\ell_{wd} \subseteq T^\ell$, that are *not* strongly detached, into strongly detached edges. We replace every edge $(v, u) \in T^\ell_{wd}$ of level t by a set of strongly detached edges of the same level with total weight of at least $d(v, u)$.

Transformation to strongly detached edges. Consider an edge $(v, u) \in T^\ell_{wd}$, and w.l.o.g let $i(v) < i(u)$. It must be that $j(u) < j(v)$, otherwise our ordering would imply that (v, u) is strongly detached[11] (see Figure 6 for an illustration). There must be points of S in some squares in the column of v and/or in that of u. Let $j_1 = \min\{j : j \geq j(u), Q^t_{i(v)j} \cap S \neq \emptyset\}$, and let $j_2 = \max\{j : j \leq j_1, Q^t_{i(u)j} \cap S \neq \emptyset\}$. Let also v', u' be two arbitrarily chosen representatives of $Q^t_{i(v)j_1} \cap S$ and $Q^t_{i(u)j_2} \cap S$.

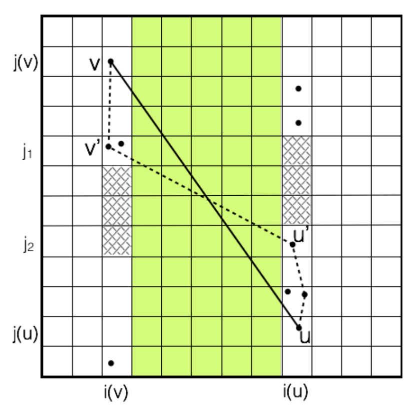


Figure 6: The figure shows the transformation of a weakly detached edge (solid segment) to a set of strongly detached edges (dashed segments).

[11] Moreover, if $i(u) = i(v)$ or $j(v) = j(u)$, the edge (v, u) is either column or row and therefore, strongly detached.

We replace $e = (v, u)$ with a path p_e, that has a vertical component of column edges that connects v to v', the edge (v', u') and finally a vertical path from u' to u (see Figure 6). Note that some of these components may be void (e.g. if $v' = v$ or $u' = u$). Clearly $d(v, u) \leq w(p_e)$. To connect v with v' we choose arbitrary representatives of $S \cap Q^t_{i(v)j}$, if they exist, with $j_1 < j < j(v)$, and we create similarly the path from u' to u. Both components consist of column edges and are trivially strongly detached. From the way we defined indices j_1, j_2 it is easy to check that the edge (v', u') is also strongly detached. Let T^ℓ_{sd} be the set of transformed edges after this step.

Partition into sets of well-placed edges. Recall that T^ℓ_{wd} is well-placed and contains no column edge. Therefore, for each square Q, at most two edges of each $T^\ell_{wd} \cap E(Q)$ have endpoints on the same column (e.g. the solid edges of Figure 7). This means that the column edges of T^ℓ_{sd} (derived by the vertical components) can be partitioned into two sets of well-placed edges. For the rest of the T^ℓ_{sd}'s edges, note we add one edge (v', u') for each edge (v, u) of T^ℓ_{wd}, where v', u' lie on the same columns as v, u, respectively. Since, T^ℓ_{wd} is well-placed, the same should hold for the T^ℓ_{sd}'s edges that are not column. Overall, we can partition T^ℓ_{sd} into two sets[12], of (short or long) strongly detached and well-placed edges, T_{sd1}, T_{sd2}. Figure 7 shows such an example, where the two sets are the dashed and dotted edges, respectively.

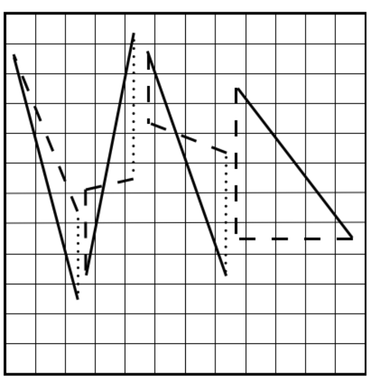


Figure 7: The figure shows an example of the partition of the transformed strongly detached edges T^ℓ_{sd} into two sets of well-placed edges, the dashed segments and the dotted segments, respectively.

[12]We can merge the one set of column edges with the edges that are not column into one set, since in the definition of well-placed edges the conditions posed only between column edges or between the rest.

We partition T_{sd1}, T_{sd2} into short and long edge, $T^s_{sd1}, T^s_{sd2}, T^\ell_{sd1}, T^\ell_{sd2}$, respectively. It is easy to see that any set of well-placed and short edges is also well-distributed by setting, for each edge e, the value t_e to be the level of e. Therefore, we can upper bound the weight of the sets T^s_{sd1}, T^s_{sd2} by Lemma 4.1. Recall that $T^\ell \setminus T^\ell_{wd}$ (by definition of T^ℓ_{wd}) forms one set of long, strongly detached and well-placed edges, which we denote by T^ℓ_{sd0}. It remains to bound the weight of $T^\ell_{sd0}, T^\ell_{sd1}, T^\ell_{sd2}$.

4.2.3 Third Step

We will further partition and transform any set $E \in \{T^\ell_{sd0}, T^\ell_{sd1}, T^\ell_{sd2}\}$, such that we either produce short edges that we again bound their total weight by Lemma 4.1, or the edges are *pseudo-isolated*, a notion that we define next.

Pseudo-isolated edges. An edge e is *pseudo-isolated* if $\overset{\circ}{\Gamma}_e \cap S = \emptyset$ (see Figure 9(a)).

We first partition E into column and diagonal edges, and the rest (row and double-row edges) and handle each case differently.

Column and diagonal edges. We continue in two steps. We first "chop" the edges, and then we transform only those that are not pseudo-isolated.

Chopping. Let $e = (v, u)$ be an edge of level t, and w.l.o.g. assume $j(v) < j(u)$. Let v' be the vertex of $S \cap Q^t_{i(v)j(v)}$ with the highest y co-ordinate and u' be the vertex of $S \cap Q^t_{i(u)j(u)}$ with the smallest y co-ordinate[13]. We replace e with $e' = (v', u')$ (See Figure 8). Note that $w(e) < 3w(e')$, because e is long.

Note further that the interior of $\Gamma'_{e'}$ (see bounded boxes in Section 4.1) does not contain any point of S, due to the fact that e is strongly detached. Let E_{cd} be the set of chopped edges after this step.

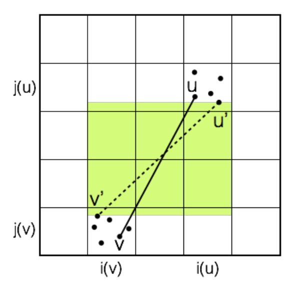
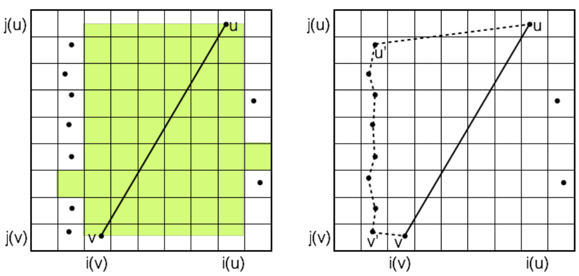


Figure 8: The figure shows the chopping of an edge, the solid segment (v, u) is the original edge and the dashed segment is the chopped edge $e' = (v', u')$; the shaded region is the $\Gamma'_{e'}$.

Transformation. Take a chopped edge $e =$

[13]We clarify that this procedure is not a chopping of the edge itself, but of its projection onto the y axis.

$(v, u) \in E_{cd}$ of level t, and w.l.o.g. let $i(v) \leq i(u)$ and $j(v) < j(u)$ (the other case can be handled similarly). If we can define Γ_e (see bounded boxes in Section 4.1) then e is pseudo-isolated (Figure 9(a)); let E_{cd}^{psI} be the set of the pseudo-isolated edges. Otherwise, either all squares $Q^t_{(i(v)-1)j}$ or all squares $Q^t_{(i(u)+1)j}$, with $j(v) < j < j(u)$ contain points of S. Let's assume that the first happens (see Figure 9(b)), as the other case is similar. We choose a representative vertex of each $S \cap Q^t_{(i(v)-1)j}$, for $j \in (j(v), j(u))$. Additionally, we choose a representative for $j = j(v)$ and $j = j(u)$, if there exists any[14]. Let v' and u' be the representatives with the lowest and highest y co-ordinate, respectively (as in Figure 9(b)). We replace e with a path p_e, that has a vertical component of column short edges that connects v' to u', through the chosen representatives, and the edges (v, v') and (u, u'). Clearly $d(v, u) \leq w(p_e)$.



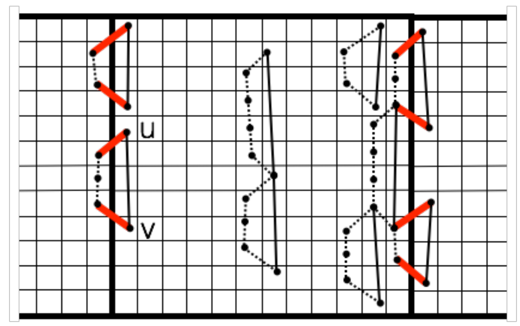
(a) (b)

Figure 9: Figure (a) illustrates the case that the edge is pseudo-isolated; the shaded region is Γ_e. Figure (b) shows the transformation of an edge (solid segment) that is not pseudo-isolated to a set of short edges and (at most) one long edge (dashed segments).

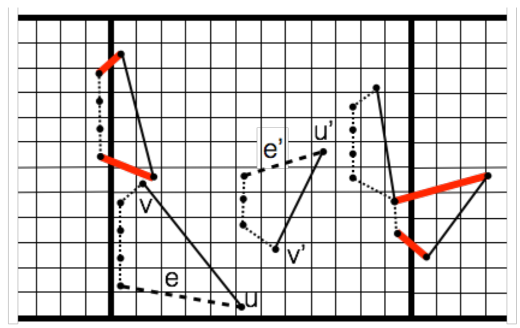
Let E_c and E_d be the column and diagonal edges of E_{cd} and let E'_c and E'_d be the transformed edges of E_c and E_d, respectively.

Paths induced by column edges. We partition E'_c into two well-distributed sets, E'_{c1}, E'_{c2} as follows. For every edge $e \in E_c$, either p_e is on the left or on the right of e[15]. Let E'_{c1} and E'_{c2} be the edges of the paths with respect to the first and the second case, respectively.

We next argue that E'_{c1} and E'_{c2} are well-distributed. For each $e \in E_c$ of level t, we assign the value t to each edge e' of its induced path p_e, i.e. $t_{e'} = t$[16] The first two conditions of well-distributed sets are trivially fulfilled for E'_{c1} and E'_{c2}. Regarding the third one, notice that E_c is well-placed and for each $e = (v, u) \in E_c$, the endpoints of the edges of p_e lie on rows between $j(v)$ and $j(u)$ and only in the one side of e (see Figure 10(a) for an illustration). Then, the weight of each set E'_{c1}, E'_{c2} can be upper bounded by Lemma 4.1.



(a)



(b)

Figure 10: Figure (a) shows a set E'_{c1} and Figure (b) shows a set E'_{d1}. The thick, solid (red) segments depict the edges of p_e that are of different level than e. E.g. in Figure (a), (v, u) is such an edge. Additionally, in Figure (b), the dashed edges e and e' belong to the set F^{ℓ}_{sl}.

Paths induced by diagonal edges. We partition E'_d into two sets, E'_{d1}, E'_{d2}, similarly to the previous case, based on the relative positions of p_e and e. Let F be any of those sets. We further partition F into two sets, F_{dl}, F_{sl} as follows: for every $e' \in p_e$ of F, if e and e' are of different levels (see Figure 10(b) for such examples),

[14] We remark that those representatives are essential only in the case of diagonal edges, where a double-row, long edge may be produced, as in Figure 9(b). That way we guarantee that, for any such edge e of level t, S doesn't intersect the region $A_e \setminus \{Q^t_{i(v)j(v)} \cup Q^t_{i(u)j(u)}\}$. Those edges will be handled along with the rest of row and double-row edges from Step 2.

[15] A path p lies on the left (right) of a line segment e, if for any horizontal line L that intersects both p and e, any point of $p \cap L$ has smaller (larger) x co-ordinate than any point of $e \cap L$.

[16] For any edge $e \in E_c$ of level t, there are two cases regarding any edge e' of p_e: either e' is of level t, or it is of level $t' < t$ (see Figure 10(a)).

then $e' \in F_{dl}$, otherwise $e' \in F_{sl}$.

Handling $\mathbf{F_{dl}}$. We next argue that F_{dl} is well-distributed. For each $e \in E_d$ of level t, we assigning the value $t-1$ to each edge $e' \in F_{dl}$ of its induced path p_e, i.e. $t_{e'} = t-1$. The first condition of the well-distributed sets is satisfied because e' should belong to a level at most $t-1$. The second condition is satisfied because e' should connect neighbouring sub-squares of the $(t-1)$-division. Regarding the third condition, recall that E_d is well-placed and contains only diagonal edges, meaning that there are at most two edges (with assigned value $t-1$) with endpoints at the same squares (see Figure 10(b) for an illustration). Overall, we can bound F_{dl} by Lemma 4.1

Handling $\mathbf{F_{sl}}$. Partition F_{sl} into short, F^s_{sl}, and long, F^ℓ_{sl}, edges. The set F^s_{sl} is well-distributed, since E_d is well-placed, and hence, can be upper bounded by Lemma 4.1.

Claim 4.1. *F^ℓ_{sl} is well-placed.*

Proof. Note that for any $e = (v, u) \in E_d$, the path p_e contains at most one long edge, (the edge (u, u') in Figure 9(b)). For some square Q, consider any two edges of $E(Q)$, $e = (v, u)$ and $e' = (v', u')$, where their induced paths, p_e and $p_{e'}$, produce an edge of F^ℓ_{sl}. W.l.o.g. let e "precedes" e', meaning that $i(v) < i(u) \le i(v') < i(u')$ (see edges (v, u) and (v', u') of Figure 10(b) for an illustration). We argue that $i(u)$ is *strictly* less than $i(v')$ and this would lead to the fact that the two produced long edges are well-placed.

Suppose on the contrary that $i(u) = i(v')$. Recall that either both p_e and $p_{e'}$ are on the left of e and e', respectively, or on the right. Let's assume that the first happens. Then $p_{e'}$ have vertices on column $i(v')-1 = i(u)-1$. Since e is weakly detached, there are no requested vertices in columns (inside Q) between $i(v)$ and $i(u)$, exclusive. The only way that this happens is the case that no such column exists, i.e., $i(v) = i(u)-1$. However, this would mean that p_e contains only short edges, which contradicts the fact that p_e contains a long edge. □

Note that F^ℓ_{sl} contains only row and double-row edges. Recall that there are two such sets, one for the p_e's on the right of e and one for the p_e's on the left of e. We handle those sets next, along with the set of row and double-row edges of E. We denote the three sets as $E^\ell_{r_1}, E^\ell_{r_2}, E^\ell_{r_3}$.

Row and double-row edges. We handle these edges similarly to the way we handled column edges by swapping the x, y axes. Either an edge is pseudo-isolated, or we transform it to a set of short edges. Similarly to the case of column edges, we partition the short edges into constant number of well-distributed sets that we bound by Lemma 4.1. Let E^{psI}_{r1}, E^{psI}_{r2} and E^{psI}_{r3} be the sets of pseudo-isolated edges from each $E^\ell_{r_1}, E^\ell_{r_2}, E^\ell_{r_3}$, respectively.

4.2.4 Summarizing.

Overall, for every $E \in \{T^\ell_{sd0}, T^\ell_{sd1}, T^\ell_{sd2}\}$, it remains to bound the weight of a set of long, pseudo-isolated, strongly detached and well-placed edges, E^{psI}_{cd}, (these are the column and diagonal edges of Step 3 that are pseudo-isolated) and three sets of row and double-row edges that are long, pseudo-isolated and well-placed, E^{psI}_{r1}, E^{psI}_{r2} and E^{psI}_{r3}. Lemma 5.1 (see Section 5) concludes the proof.

4.2.5 Lower bound.

At last, we show in the next claim that our analysis is tight up to a constant.

Claim 4.2. *There exists a set $S \subseteq G_n$ of requested vertices, such that $\frac{w(T_\pi(S))}{w(T^*(S))} = \Omega\left(\frac{\log n}{\log\log n}\right)$, where π is the generalized Lebesgue space-filling ordering.*

Proof. Let S be the set of $2r$ vertices, $v_1, \ldots, v_r$ and $u_1, \ldots u_r$, such that v_j, u_j are the vertices of G_n that belong to Q^1_{1j}, Q^1_{rj}, respectively, with the smallest x and y co-ordinates.

By adopting the convention that $v_{r+1} = v_1$, the tour induced by the heuristic is $T_\pi(S) = \{(v_j, u_j), (u_j, v_{j+1}) | 1 \le j \le r\}$. Note that $d(v_j, u_j)$ and $d(u_j, v_{j+1})$ equal to $\Omega(1)$ and therefore, $w(T_\pi(S)) = \Omega(r)$.

The tour formed by the horizontal segments $(v_1, u_1), (v_r, u_r)$ and two vertical segments, one connecting the v_j's vertices and the other connecting the u_j's vertices, has weight $O(1)$. Therefore, $w(T^*(S)) \le O(1)$.

Overall, $\frac{w(T_\pi(S))}{w(T^*(S))} = \Omega(r) = \Omega\left(\frac{\log n}{\log\log n}\right)$. □

Remark 4.1. *It is easy to see that the instance in the proof of Claim 4.2 provides a lower bound $\Omega(k)$, where $k = \Theta\left(\frac{\log n}{\log\log n}\right)$ is the number of requested vertices. An upper bound of $O(k)$ for any universal ordering can be easily derived by observing that the cost of each edge of the tour is upper bounded by the weight of the optimum tour. Therefore, the competitive ratio of the generalized Lebesgue space-filling ordering w.r.t. the requested vertices, k, is $\Theta(k)$.*

5 Long Edges

In this section we show how to upper bound long, pseudo-isolated and well-placed edges, supposing that the diagonal edges are also strongly detached. As a first step we partition the edges into constant number of sets that satisfy the following two properties:

(1) **Nested Property.** A set of edges F satisfies the *Nested Property*, if for any pair of edges $e, e' \in F$, their extended bounded boxes $B_e, B_{e'}$ are either disjoint or one is completely contained within the other, i.e. it is either $B_e \subset B_{e'}$, or $B_{e'} \subset B_e$, or $\overset{o}{B_e} \cap \overset{o}{B_{e'}} = \emptyset$.

(2) **Border-Nested Property.** A set of edges F satisfies the *Border-Nested Property*, if for any pair $e, e' \in F$, such that $A_{e'} \subset B_e \setminus A_e$, then $\overset{o}{B_{e'}} \cap \overset{o}{A_e} = \emptyset$.

Then, we show that, for any such set, the interiors of the Γ regions are disjoint (Claim 5.3). At last, in Lemma 5.2, we prove that the *Nested Property* and the *disjoint* Γ *interiors* are sufficient in order to upper bound the weight of those edges by $O(r)w(T^*)$.

Lemma 5.1. *If $E \subseteq E(S)$ is a set of long, pseudo-isolated and well-placed edges, and additionally the diagonal edges are strongly detached, then $w(E) = O(r) \cdot w(T^*)$.*

Proof. We partition E into sets that satisfy (1) and (2). The following claim suggests that there are only a constant number of such sets.

Claim 5.1. *There is a partition of $E = E_1 \cup \ldots \cup E_{O(1)}$ such that each E_i satisfies (1) and (2).*

Proof. We partition E according to their slope into column edges E_c, diagonal edges E_d, and row or double-row edges E_r. We next show that we can partition each of E_c, E_d, E_r into at most $10, 9$ and 18 sets, respectively, such that each of them satisfies the above nested properties[17].

Column Edges. Since the set is well-placed, the Nested Property can be restated as follows.

Nested Property. A set of edges, F, satisfies the Nested Property if:

(1a) For any two edges, $e = (v, u)$ and $e' = (v', u')$, of F that are of the same level, if $i(v) = i(v')$, then $\overset{o}{B_e} \cap \overset{o}{B_{e'}} = \emptyset$.

(1b) For any two edges, $e = (v, u)$ and $e' = (v', u')$, of F that are of the same level, if $i(v) \neq i(v')$, then $\overset{o}{B_e} \cap \overset{o}{B_{e'}} = \emptyset$.

(1c) For any two edges $e, e' \in F$ of levels t and t', respectively, with $t < t'$, either $B_{e'} \subset B_e$ or $\overset{o}{B_e} \cap \overset{o}{B_{e'}} = \emptyset$.

We partition E_c into 10 sets that satisfy (1a), (1b), (1c) and (2) in two steps.

[17] This holds for $r \geq 2$, which is clearly in the range of interest.

First partition. We first partition E_c into 2 sets, E_{c1}, E_{c2}, each of which satisfies (1a) of the Nested Property. This partition can be easily done, since E_{c1} and E_{c2} are well-placed, as subsets of E. Particularly, we order the edges, $e = (v, u)$, of E_c that are of the same level and has the same $i(v)$, according to the value of $\min\{j(v), j(u)\}$. By following that order, we alternately add each edge to the sets E_{c1} and E_{c2}.

Second partition. In the second step we partition each of the E_{c1}, E_{c2} into 5 sets, $E_{c11} \ldots E_{c15}$ and $E_{c21} \ldots E_{c25}$, for which we show that (1b) and (1c) of the Nested Property and the Border-Nested Property are satisfied. Let F be any of the sets E_{c1}, E_{c2}.

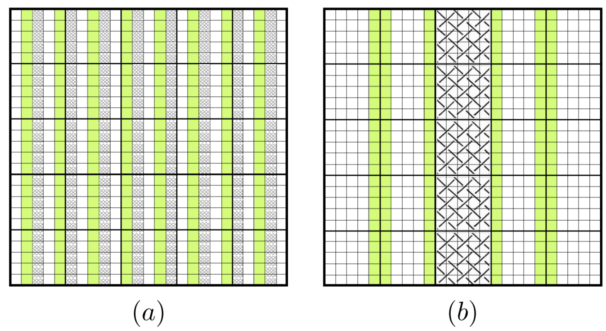


Figure 11: Figure (a) shows a column colouring that satisfies Property (1b). Figure (b) illustrates the restrictions in order to satisfy Properties (1c) and (2); the crosshatched column should be differently painted from the shaded columns.

For every $Q \in \mathcal{Q}$, let $C(Q) = \{C_1(Q), \ldots, C_r(Q)\}$ be the set of its columns w.r.t. the 1-division of Q, i.e. if $\{Q_{ij}, 1 \leq i, j \leq r\}$ is the 1-division of Q, then $C_i(Q) = \cup_{j \in \{1,\ldots,r\}} Q_{ij}$. Further, let $C = \cup_{Q \in \mathcal{Q}} C(Q)$ be the set of all columns. We define a column colouring $\chi : C \to \{\chi_1, \ldots, \chi_5\}$, which induces an edge colouring $\chi^* : F \to \{\chi_1, \ldots, \chi_5\}$, where each edge inherits the colour of the column that it belongs to. More specifically, for each edge, $e = (v, u)$ that belongs to some $E(Q)$, we define $\chi^*(e) = \chi(C_{i_Q(v)}(Q))$. We then define the partitions as $F_a = \{e \in F | \chi^*(e) = \chi_a\}$, for $1 \leq a \leq 5$.

In order to paint the columns we construct a graph H and associate each column to a unique vertex of H (we though allow more than one columns to be associated with the same vertex); a vertex colouring of H will correspond to the column colouring. H will be carefully constructed such that any vertex colouring of H will guarantee (1b), (1c) and (2) for each F_a.

Sufficient conditions for satisfying (1b), (1c), (2). In order to satisfy (1b), it is sufficient that, for any $Q \in \mathcal{Q}$, any three consecutive columns of Q are differently coloured and moreover, if $Q = Q^t_{ij}$ and

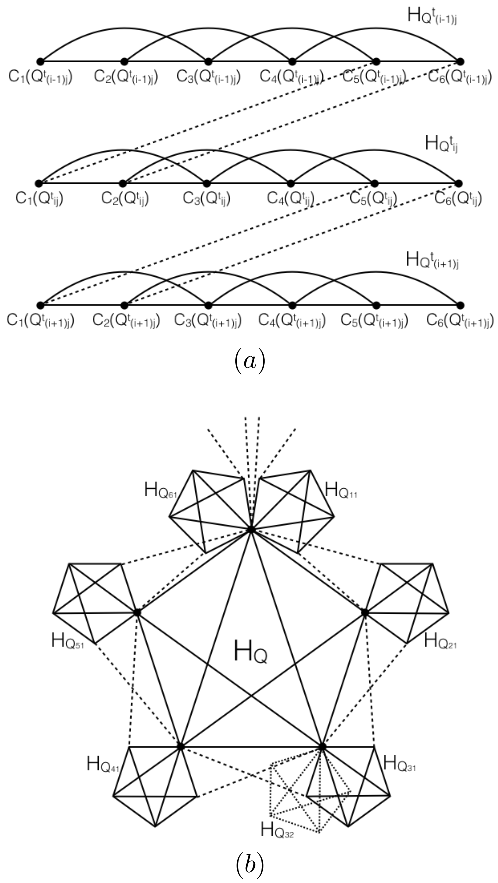


Figure 12: Figure (a) shows the first two steps of the H's construction for column edges, where solid edges are added in step 1 and the dashed one in step 2. Figure (b) illustrates the contraction of the vertices in step 3.

$Q_{+1} = Q^t_{(i+1)j}$ (i.e. Q_{+1} is Q's right neighbour), $C_{r-1}(Q)$ has different colour from $C_1(Q_{+1})$, and $C_r(Q)$ has different colour from both $C_1(Q_{+1})$ and $C_2(Q_{+1})$ (see Figure 11(a)). In order to satisfy (1c) and (2), it is sufficient that, for any square $Q \in \mathcal{Q}$ and any column $C_i(Q)$, any "smaller" columns (derived from any t-division of Q) on the vertical borders of $C_{i-1}(Q)$ and $C_{i+1}(G)$ and on the right vertical border of $C_{i-2}(Q)$ and on the left vertical border of $C_{i+2}(Q)$ have different colour from $C_i(Q)$ (see Figure 11(b)). If $i \in \{1, 2, r^t - 1, r^t\}$, we can accordingly define the same restrictions by using further the t-division of Q's left and right neighbours.

Graph construction. Bearing that in mind, we construct H as follows, in three steps.

1. For every $Q \in \mathcal{Q}$, let $V(Q) = \{v_1(Q), \ldots, v_r(Q)\}$ be a set of ordered vertices. We associate each $C_i(Q)$ with vertex $v_i(Q)$. We construct a graph H_Q on $V(Q)$ by adding an edge between $v_i(Q)$ and $v_{i+1}(Q)$, for $i < r$, and an edge between $v_i(Q)$ and $v_{i+2}(Q)$, for $i < r - 1$ (these are the solid edges of Figure 12(a)). Those edges help to satisfy (1b) between columns of the same square.

2. For every t-division of $[0, 1]^2$, we add an edge between $v_{r-1}(Q^t_{ij})$ and $v_1(Q^t_{(i+1)j})$ and an edge between $v_r(Q^t_{ij})$ and $v_2(Q^t_{(i+1)j})$ for every $i < r$ and j (these are the dashed edges of Figure 12(a)). Those edges help partially to satisfy (1b) between columns of different squares. To completely satisfy (1b), we further need an edge between $v_r(Q^t_{ij})$ and $v_1(Q^t_{(i+1)j})$ which will clearly appear in the next step.

3. We derive H by connecting H_Q's via contracting vertices[18]. For every square $Q \in \mathcal{Q}$, with $\{Q_{ij}, 1 \leq i, j \leq r\}$ being its 1-division, we contract $v_1(Q_{ij})$ and $v_r(Q_{ij})$ with $v_i(Q)$, for all $i < r$ and j (see Figure 12(b)). For convenience, we keep the vertices in the sets $V(Q)$ and $V(Q_{ij})$. Notice that $v_r(Q_{ij})$ and $v_1(Q_{(i+1)j})$ are connected via an edge by the construction of H_Q; this completes (1b).

   By the above contractions, $C_i(Q)$ is painted by the same colour with the columns on the borders of $C_i(Q)$. Therefore, if $C_i(Q)$ is differently painted from the columns $C_{i-1}(Q)$, $C_{i-2}(Q)$, $C_{i+1}(Q)$, $C_{i+2}(Q)$, it is also differently painted from the columns on their borders. Hence, any vertex colouring of H would satisfy (1c) and (2)

   Figures 13(a) and 13(b) show an example of column colouring of Q and $\{Q_{ij}, 1 \leq i, j \leq r\}$, respectively, induced by some vertex colouring of H.

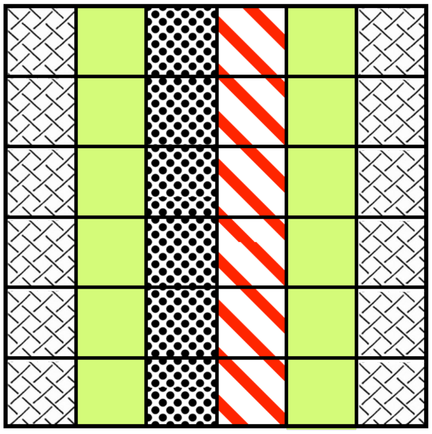
Claim 5.2. *The chromatic number of H is at most* 5.

Proof. We paint the vertex of H by using a greedy colouring. This means that we consider the vertices in sequence and assign to each vertex some available colour which is different from its painted neighbours' colours.

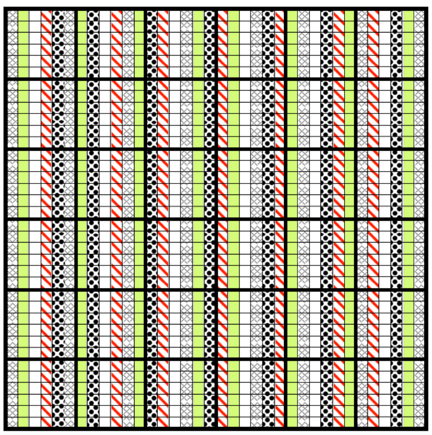
The first vertex in the order is $v_1([0, 1]^2)$. We order the rest of the vertices based on the $V(Q)$'s. For any squares $Q \in \mathcal{Q}_t, Q' \in \mathcal{Q}_{t'}$, such that $t < t'$, all vertices of $V(Q)$ precede all vertices of $V(Q') \setminus V(Q)$. This defines a partial order of the vertices. We order the rest of the vertices arbitrarily.

We next argue that by following such an order, while painting a vertex, we have painted at most 4 of

[18] The contraction of two vertices v_1 and v_2 is the replacement of those vertices with a single vertex v, such that v is adjacent to the union of all edges to which v_1 and v_2 were originally adjacent.



(a)



(b)

Figure 13: The Figures show an example of column colouring of Q, Figure (a), and $\{Q_{ij}, 1 \le i, j \le r\}$, Figure (b). This particular coloring could have been induced by a vertex colouring of H.

its neighbours, meaning that there is always a colour available to paint that vertex. To see this, suppose that we paint the vertices of some $V(Q)$. The vertex $v_1(Q)$ has already been painted as a vertex of some $V(Q')$, with Q' of lower division than Q. $v_2(Q)$ has at most four neighbours that are already painted: $v_1(Q)$, $v_3(Q)$, $v_4(Q)$ and the neighbour introduced in step 2. The same holds for $v_{r-1}(Q)$. For any other vertex $v \in V(Q)$, the only neighbours that may have been painted before v belong to $V(Q)$, hence, by the construction of H_Q, there are at most 4. $\square$

Diagonal Edges. The idea of the proof is similar to the case of column edges. The main difference is that the vertices of H do not represent all columns but only the ones that intersect S. Recall, that the diagonal edges are strongly detached, resulting in the fact that the endpoints of each edge belong to consecutive "activated" columns. A slightly more generalized analysis than the one for the column edges results in the desired partition.

Row and double-row Edges. This case should be handled separately because the B_e is defined differently. However, by combining the above ideas for column and diagonal edges, we can easily prove this case. In fact a more careful analysis may reduce the number of groups needed. $\square$

Claim 5.3. *Let E' be a subset obtained by Claim 5.1. Then, for every $e, e' \in E'$, it holds $\Gamma^{o}_{e} \cap \Gamma^{o}_{e'} = \emptyset$.*

Proof. Note that the edges of E' are pseudo-isolated (i.e. for any $e \in E'$, $\Gamma^{o}_{e} \cap S = \emptyset$) and either all are column, or all diagonal, or all row and double-row, meaning that their B_e and Γ_e have the same orientation.

For any $e, e' \in E'$, if B^{o}_{e} and $B^{o}_{e'}$ are disjoint, then trivially Γ^{o}_{e} and $\Gamma^{o}_{e'}$ are disjoint. Consider the case that $B_{e'} \subset B_e$, and suppose that e is of level t. Then $A_{e'}$ is entirely contained within one square of the t-division of $[0,1]^2$. This means that either $A_{e'} \subset A_e$ or $A_{e'} \subset B_e \backslash A_e$ (see Figure 14 for examples of diagonal edges).

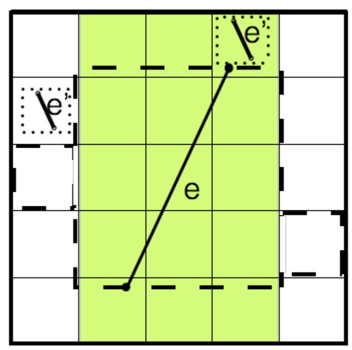


Figure 14: The figure shows the B_e region, where the region with dashed borders is the Γ_e and the shaded region is the A_e. The regions with dotted borders are possible $B_{e'}$'s for the case that $B_{e'} \subset B_e$.

In the first case, $A^{o}_{e'} \cap \Gamma^{o}_{e} = \emptyset$ since e is pseudo-isolated. Then, trivially, $\Gamma^{o}_{e} \cap \Gamma^{o}_{e'} = \emptyset$. In the second case, by Property (2), it holds that $B^{o}_{e'}$ and A^{o}_{e} are disjoint. Then, again the fact that e is pseudo-isolated implies that $B^{o}_{e'}$ and Γ^{o}_{e} are disjoint and the claim follows. $\square$

Finally, the following lemma (Lemma 5.2) shows that the Nested Property in combination with the fact that, for every $e, e' \in E'$, it holds that Γ^{o}_{e} and $\Gamma^{o}_{e'}$ are disjoint, is sufficient in order to upper bound the weight of each set, E', by $O(r) \cdot w(T^*)$. This concludes the proof of the lemma. $\square$

Lemma 5.2. *Let $E \subseteq E(S)$ be a set of edges that satisfies the Nested Property and additionally, for every $e, e' \in E$, $\Gamma^{o}_{e} \cap \Gamma^{o}_{e'} = \emptyset$. Then $w(E) \leq O(r) \cdot w(T^{*})$.*

Proof. For this proof we employ techniques from the theory of geometric spanners in Euclidean spaces. We partition the set E and for each subset, we either associate each edge e with a distinct optimum's segment of weight $O(1/r)w(e)$, or we process the edges one by one and at each step we transform the optimum such that its *horizontal* weight reduces by $O(1/r)w(e)$. Either way, we bound the subset's weight by $O(r)$ times the optimum.

For analysis purpose, we bound the weight of each subset by using a constant approximation of the optimum instead of the optimum itself. We proceed by partitioning the edges and transforming the optimum.

First partition. We partition E into two sets E_{cd} and E_r; E_{cd} contains all the column and diagonal edges and E_r contains all the row and double-row edges. We prove the statement only for the set E_{cd}. The statement can be similarly proved for the set E_r after swapping the axes x, y.

Transformation of the optimum. We first transform the Steiner minimal tree (SMT) that connects all the endpoints of the edges of E_{cd}, by replacing any non-vertical and non-horizontal edge (v, u) of the SMT with two edges (v, z) and (z, u), such that (v, z) is vertical and (z, v) is horizontal. We denote this graph by Φ and clearly $w(\Phi) = O(1)w(SMT) = O(1)w(T^{*})$. Hence, it is sufficient to bound the weight of E_{cd} by $O(r)w(\Phi)$.

Second partition. We partition E_{cd} into two groups of edges, E_{cd1} and E_{cd2}, based on the intersection of Φ with Γ^{o}_{e}. More specifically, if $w(\Phi \cap \Gamma^{o}_{e}) < w(e)/\sqrt{2}r$ then $e \in E_{cd1}$, otherwise $e \in E_{cd2}$. Due to the fact that the Γ^{o}_{e} 's are disjoint, $w(E_{cd2}) \leq O(r)w(\Phi)$.

Handling $\mathbf{E_{cd1}}$. We next show that $w(E_{cd1}) \leq O(r)w(\Phi)$, which concludes the lemma. More precisely, if $hw(\Phi)$ is the weight of the horizontal segments of Φ, we will show that $w(E_{cd1}) \leq O(r)hw(\Phi)$, which is clearly stronger. We are going to process the edges of E_{cd1} one by one and while processing edge e, we transform Φ into Φ', such that:

i) Φ' connects all the endpoints of the unprocessed edges and contains only vertical and horizontal segments,

ii) $hw(\Phi) - hw(\Phi') = O(1/r)w(e)$ (which will lead to the desired bound),

iii) for every unprocessed e' of E_{cd1}, $w(\Phi \cap \Gamma^{o}_{e'}) < \frac{w(e)}{\sqrt{2}r}$.

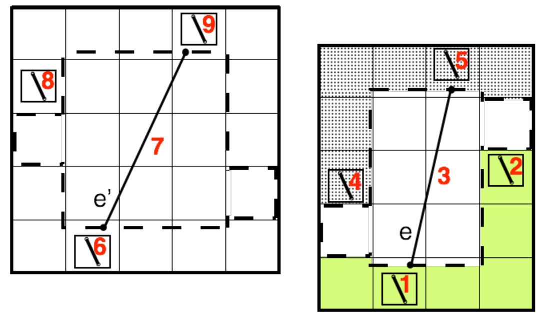


Figure 15: The figure shows the processing order of the edges. R^b_e is the shaded region and R^a_e is the dotted region.

For the rest of the proof, for simplicity, the reader may consider that any vertical segment of Φ and Φ' has zero weight. In order to define the processing order of the edges of E_{cd1} we need some more definitions.

Nested tree. We next define a nested tree (NT) to be a tree capturing the hierarchy induced by the Nested Property. For every edge $e \in E_{cd1}$ we introduce a vertex v_e; we further consider a vertex v_{e^*} to be the root of the NT that we associate with a virtual edge e^*, such that B_{e^*} is the region $[0,1]^2$. Then, NT can be defined by the following property: v_e is an ancestor of $v_{e'}$ in NT, if and only if $B_{e'} \subseteq B_e$.

For every edge $e \in E_{cd1}$, $B_e \setminus \Gamma_e$ forms two disjoint regions, R^b_e and R^a_e, below and above[19] of Γ_e, respectively (See Figure 15). For every non-leaf vertex, v_e, of the NT, we partition the children of v_e into two sets C^b_e and C^a_e, such that for any children $v_{e'}$ of v_e, if $B_{e'}$ intersects R^b_e, then e' belongs to C^b_e, otherwise e' belongs to C^a_e. If v_e is the root of NT, we set all children to belong to C^b_e.

Processing order. We now define a total order among the vertices of NT that will serve as the processing order of the edges. For every non-leaf vertex v_e, all the vertices of the C^b_e 's subtrees precede v_e in the order and all the vertices of C^a_e 's subtrees follow v_e in the order. Consider any two vertices v_e and $v_{e'}$ that both belong to $C^b_{e''}$ or both belong to $C^a_{e''}$, for some vertex $v_{e''}$ of NT. Suppose that the bottom[20] horizontal border of B_e has smaller y co-ordinate than the bottom horizontal border of $B_{e'}$, then all the vertices of the subtree of v_e precedes all the vertices of the subtree of $v_{e'}$. See Figure 15 for an illustration.

[19] A region A lies below (above) of a region B, if for any vertical line L that intersects both A and B, any point of $A \cap L$ has smaller (larger) y co-ordinate than any point of $B \cap L$.

[20] The bottom horizontal border of B_e is the horizontal border with the smallest y co-ordinate.

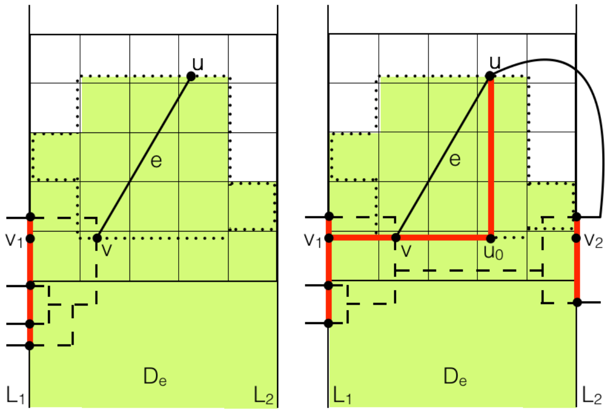


Figure 16: The figure shows the transformation of Φ^j for the cases of $V_2 = \emptyset$ and $V_1, V_2 \neq \emptyset$. The shaded region is the D_e and the dashed component represents Φ_v^j. We transform Φ^j by removing Φ_v^j and adding the thick, solid (red) segments.

Transforming Φ. For every edge e, extend both vertical borders of B_e to infinity, and let L_1 and L_2 be the lines with the smaller and larger x co-ordinate, respectively. We call D_e the region between L_1 and L_2 that lies below the R_e^a. It is not hard to check that any e', such that $\Gamma_{e'}^{\mathrm{o}}$ intersects D_e, precedes e in the processing order.

Suppose now that we are going to process $e = (v, u) \in E_{cd1}$ and let Φ^j be the transformed Φ so far. Then Φ^j is a graph connecting all the endpoints of the unprocessed edges, containing only vertical and horizontal edges and for every unprocessed $e' \in E_{cd1}$, $w(\Phi^j \cap \Gamma_{e'}^{\mathrm{o}}) < w(e)/\sqrt{2}r$. We will transform only the part of Φ^j that is inside D_e and therefore the transformation will not affect the portion of Φ^j inside $\Gamma_{e'}^{\mathrm{o}}$, for any other unprocessed edge e', meaning that the third requirement above is guaranteed.

Let Φ_v^j be the connected component of Φ^j that contains v and is entirely between L_1 and L_2. Since $w(\Phi^j \cap \Gamma_e^{\mathrm{o}}) < w(e)/\sqrt{2}r$, we easily infer that Φ_v^j does not intersect the borders of R_e^a, meaning that Φ_v^j is entirely in the interior of D_e and on its borders defined by L_1 and L_2. Let V_1 and V_2 be the intersection points of Φ_v^j with L_1 and L_2, respectively; if their intersection includes some intervals, we represent them by infinitely many points. Additionally, let v_1 and v_2 be the points of L_1 and L_2, respectively, that have the same y co-ordinate with v. We next derive Φ^{j+1} from Φ^j; for an illustration see Figure 16.

We first remove Φ_v^j from Φ^j and we add two minimal vertical segments, the one connecting the points of $V_1 \cup \{v_1\}$ and the other connecting the points of $V_2 \cup \{v_2\}$. If either V_1 or V_2 is empty, the connectivity of the endpoints of all remaining unprocessed edges is retained and also $hw(\Phi^j) - hw(\Phi^{j+1}) \leq w(e)/\sqrt{2}r$.

Only if both $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$, we add the horizontal segment (v_1, v_2) and the vertical segment (u, u_0), where u_0 is the point of the segment (v_1, v_2) that has the same x co-ordinate with u. Notice that so far $hw(\Phi^{j+1}) \leq hw(\Phi^j)$. Finally, suppose that the path from v to u in the current Φ^{j+1}, passes through v_i, for some $i \in \{1, 2\}$, then we remove the horizontal segment (u_0, v_i). The connectivity of the endpoints of all remaining unprocessed edges is retained and also $hw(\Phi^j) - hw(\Phi^{j+1}) \leq w(e)/\sqrt{2}r$.

Overall, after processing all edges of E_{cd1}, we can conclude that $w(E_{cd1}) \leq O(r)hw(\Phi)$. □

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