Waiter–Client and Client–Waiter games

by

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Abstract

In this thesis, we consider two types of positional games; *Waiter–Client* and *Client–Waiter* games. Each round in a biased $(a : b)$ game begins with Waiter offering $a+b$ free elements of the board to Client. Client claims a elements among these and the remaining b elements are claimed by Waiter. Waiter wins in a Waiter–Client game if he can force Client to fully claim a *winning set*, otherwise Client wins. In a Client–Waiter game, Client wins if he can claim a winning set himself, else Waiter wins.

We estimate the *threshold bias* of four different $(1 : q)$ Waiter–Client and Client–Waiter games. This is the unique value of Waiter's bias q at which the player with a winning strategy changes. We find its asymptotic value for both versions of the complete–minor and non–planarity games and give bounds for both versions of the non–r–colourability and k–SAT games. Our results show that these games exhibit a heuristic called the *probabilistic intuition*.

We also find sharp probability thresholds for the appearance of a graph in the random graph $\mathcal{G}(n, p)$ on which Waiter and Client win the $(1 : q)$ Waiter–Client and Client–Waiter Hamiltonicity games respectively.

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Chapter 1

Introduction

1.1 Positional Games

A *positional game* is a two–player game with no chance moves where, unlike Poker or Bridge, no information is hidden from either player. Such games are known as *perfect information* games. In a positional game $(X, \mathcal{F})$, play occurs on a (usually finite) set X which we call the *board*. X contains special finite subsets called *winning sets* that are defined by the family $\mathcal{F}$. In each round, players take turns to claim some previously unclaimed (*free*) elements of X in an attempt to achieve some goal that involves either claiming or avoiding a winning set, depending on the role of the player and the type of positional game in play. The specific number of elements that a player claims per round is dictated by a fixed quantity known as his *bias*. Often the bias of each player is expressed in the title of the game as a ratio referred to as the *bias of the game*. Once all elements of X have been claimed by some player, the game ends. If a player achieves his goal by the end of the game, we say that he *wins* and his opponent *loses*. However, if neither player achieves his goal by the end of the game, we say it ends in a *draw*.

Given a positional game, we are most interested in its outcome. Does it end in a draw? If not, which player wins? However, we do not care for an instance of a game where players cheat or make mistakes. We only study play between perfect/optimal players *i.e.* players that always choose the best possible valid move per round. Thus, in this setting, any given game has one fixed outcome and a player can only win or draw if he possesses what is known as a *winning* or *drawing* strategy respectively. A *strategy* is a collection of instructions that dictate how a player should play in each round of the game. If a player following some strategy $\mathcal{S}$ is guaranteed to win, no matter how his opponent plays, then $\mathcal{S}$ is a winning strategy. If $\mathcal{S}$ cannot guarantee a win but ensures at least a draw, it is a drawing strategy.

Although one may be tempted to use brute force to find such strategies in order to deduce the outcome of a game, an exhaustive search of all possible sequences of play quickly becomes intractable as the size of the board grows (see Chapter 6). Positional game theory uses combinatorial arguments to find the outcome instead and this has become a widely researched area of combinatorics since the influential papers of Hales and Jewett [57], Lehman [78] and Erdős and Selfridge [44]. For an extensive survey on positional games, the interested reader may refer to the monographs of Beck [13] and Hefetz, Krivelevich, Stojaković and Szabó [64].

Many different types of positional games appear in the literature. We focus on the five main types; strong, Maker–Breaker, Avoider–Enforcer, Waiter–Client and Client–Waiter games.

1.2 Types of Positional Games

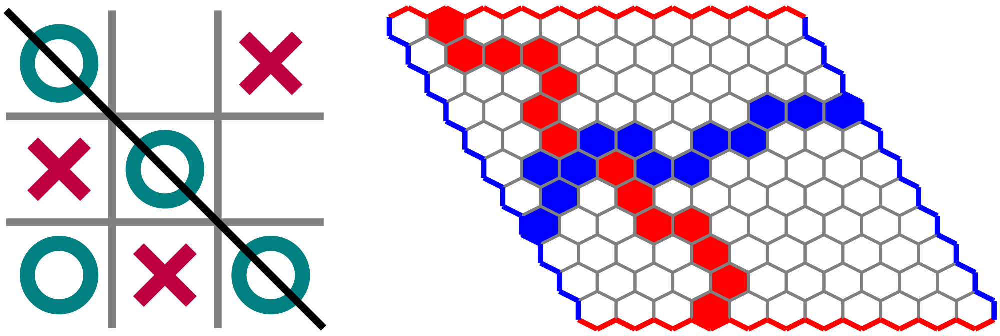

Figure 1.1: Tic–Tac–Toe (left) and Hex (right).

1.2.1 Strong Games

Strong games are often viewed as the most natural type of positional game as the player that claims a winning set *first* is the winner. The beloved *Tic–Tac–Toe* (*a.k.a. Noughts and Crosses*) is a well known example. However, those well versed in this childhood game will know that it is better to play *first* if you want to win. Indeed, a result from classical game theory (see *e.g.* Theorem 1.3.1 in [64]) states that the player that starts playing a strong game *after* his opponent can never have a winning strategy. The best outcome he can hope for is a draw. Since a draw is a feasible outcome and each player effectively has two goals to focus on throughout; to claim a winning set for themselves *and* to prevent their opponent from doing so before them, strong games are notoriously difficult to analyse. In fact, the aforementioned first–player–wins–or–draws result essentially constitutes all that is known about a general strong game. One may wonder if giving each player a goal that complements their opponent's goal makes the analysis easier, since this removes the possibility of a draw. This motivates what are known as *weak* games or *Maker–Breaker* games.

1.2.2 Maker–Breaker Games

In an $(a : b)$ *Maker–Breaker* game, one player takes the role of Maker and the other takes the role of Breaker. Here, a represents Maker's bias and b represents the bias of Breaker. Maker's goal is to claim a winning set by the end of the game. However, in contrast to strong games, Maker does not have to be the first player to do so. Breaker's goal is simply to prevent Maker from achieving this. Therefore, a draw is not possible in a Maker–Breaker game, arguably making it more attractive to play and to study. Indeed, this is demonstrated by the popularity of Maker–Breaker games in the literature (see *e.g.* [15, 21, 62, 61, 74]).

A subtle example of a Maker–Breaker game is the well known *Hex*. The board consists of a block of tessellated hexagons, surrounded by four walls coloured red or blue, such that opposing walls have the same colour. The two players, called Red and Blue, alternately claim one hexagon at a time in an attempt to build a bridge connecting the two walls of their colour. The first player to complete their bridge wins the game.

At first glance, Hex seems to be a strong game. But the winning sets in Hex are different for each player. Blue will not win if he builds a bridge connecting the red walls. He has to connect the walls of his colour to win. In [54], Gale proved that building a bridge between the walls of your colour is equivalent to *blocking* your opponent from building their bridge. Thus, one can view, say, Red as Maker and Blue as Breaker. So Hex is indeed a Maker–Breaker game.

1.2.3 Avoider–Enforcer Games

Another type of positional game is the *Avoider–Enforcer* game (see *e.g.* [70, 65, 31]). Sometimes known as the *miseré* version of Maker–Breaker games, one could view the winning sets in an Avoider–Enforcer game $(X, \mathcal{F})$ as *losing* sets instead. This is because, once Avoider claims a set in $\mathcal{F}$, he loses. Enforcer's goal in this game is to ensure that this occurs. The process of claiming elements is the same as for Maker–Breaker and strong games, with Avoider claiming a elements per round and Enforcer claiming b elements per round in an $(a : b)$ game. Also, as with Maker–Breaker games, Avoider and Enforcer can never draw.

1.2.4 Waiter–Client and Client–Waiter Games

Waiter–Client and *Client–Waiter* games are the final types of positional games that we will discuss here and serve as our main focus for this thesis. These were first introduced by Beck under the names *Picker–Chooser* and *Chooser–Picker* (see *e.g.* [12]). However, in [20], Bednarska–Bzdęga, Hefetz and Łuczak introduced the names *Waiter* and *Client* to replace Picker and Chooser respectively. We adopt this renaming to avoid confusion between the roles of the players.

In a Waiter–Client game, Waiter wins if he can force Client to claim a winning set by the end of the game, otherwise Client wins. However, in a Client–Waiter game, Client wins if he fully claims a winning set, and Waiter wins if he prevents this from happening. Thus, no draw is possible. What distinguishes these games from those discussed previously is the process of claiming elements. In both an $(a : b)$ Waiter–Client game and an $(a : b)$ Client–Waiter game, where a and b represent the bias of Client and Waiter respectively, every round consists of Waiter choosing $a + b$ free elements from the board and offering them to Client. Client then claims a of these offered elements and rejects the remaining b elements, which Waiter then claims. If there are $r < a + b$ free elements left in the last round, Waiter offers all remaining free elements to Client. However, for technical reasons, which will become apparent later, the way that Client claims and rejects elements in the final round differs depending on whether a Waiter–Client or a Client–Waiter game is in play. In the last round of an $(a : b)$ Waiter–Client game, Client rejects $\min\{b, r\}$ elements in Waiter's final offering and claims the rest. Whereas, for the $(a : b)$ Client–Waiter game, Client claims $\min\{a, r\}$ elements from Waiter's final offering *first* and then rejects the rest.

These games are interesting for a number of different reasons. Firstly, when Waiter plays randomly in a Waiter–Client (Client–Waiter) game, it becomes the *avoiding* (*embracing*) Achlioptas process (see *e.g.* [22, 23, 75, 76]). Secondly, these games often obey a fascinating heuristic (discussed in Section 1.4) known as the *probabilistic intuition*. Finally, recent research (see *e.g.* [12, 38, 17, 72]) has revealed interesting connections with Maker–Breaker games.

1.3 Variable Parameters of the Game

As mentioned previously, much of the research on positional games centres around finding the outcome of a given game. Further, we are interested in how the outcome is affected when we vary certain parameters of the game. To illustrate this, let us first consider a $(1:1)$ Waiter–Client game $(X, \mathcal{F})$. As Waiter and Client can never draw, we would like to know which of the two has a winning strategy. Since both Waiter and Client have bias 1, each round removes only two free elements from the board X. Hence, the game lasts many rounds and so Client owns many elements by the end of the game. This makes it very difficult for Client to avoid claiming a winning set and in fact, at least for the interesting choices of X and $\mathcal{F}$, Client will lose this game. A similar argument shows that Waiter loses the $(1:1)$ Client–Waiter game $(X, \mathcal{F})$ in most instances. What parameters can we change to *help* Client and Waiter win in their Waiter–Client and Client–Waiter games respectively? As it appears that a surplus of elements owned by Client at the end of the game is the cause of their loss in their respective games, a natural move would be to vary parameters that can reduce this amount. The first obvious choice is to increase the bias of Waiter. This causes Waiter to remove more free elements from the board per round, making the game shorter and therefore ensuring that Client has less elements at the end of the game. The other parameter we could vary is the board X itself. If we remove some elements from X before play begins, this will also reduce the total number of elements that Client can claim. We study each of these options in turn.

1.3.1 The Bias

We first consider the option of increasing the *bias* of Waiter. If this truly helps Client to win the Waiter–Client game, then the continued increase of Waiter's bias past the value at which Client wins the game for the first time should guarantee that Client keeps winning. The same should be true for Waiter in the Client–Waiter game. This property that we require from Waiter's bias is known as *bias monotonicity.*

Bias Monotonicity

Definition 1.3.1 (Bias Monotonicity) *Consider a positional game $(X, \mathcal{F})$ with players A and B, where a and b denote the bias of A and B respectively in an $(a : b)$ game. $(X, \mathcal{F})$ is said to be* bias monotone *in B's bias if there exists some player $P \in \{A, B\}$ such that the following property holds for any $a \geqslant 1$:*

If P wins the $(a : b)$ game $(X, \mathcal{F})$, then P also wins the $(a : b+1)$ game $(X, \mathcal{F})$.

One can also define a game to be bias monotone in A's bias in an analogous way.

In [29], Chvátal and Erdős observed that Maker–Breaker games are bias monotone in *both* player's biases, with each player helped by the increase of their own bias. But since $(1 : 1)$ Maker–Breaker games are often won by Maker, it is Breaker's bias that is commonly chosen to vary. In contrast, Avoider–Enforcer games are *not* bias monotone in *any* player's bias (see [63] for a counterexample that demonstrates this).

Waiter–Client and Client–Waiter games lie between these two extremes. By simply ignoring one arbitrary element offered to him in each round, Client can use his winning strategy for an $(a : b)$ Waiter–Client game as a winning strategy in the $(a : b+1)$ game too. Hence, Waiter–Client games are bias monotone in Waiter's bias.

Fact 1.3.2 *In a Waiter–Client game, Client wins the* $(a : b+1)$ *game whenever he wins the* $(a : b)$ *game.*

However, in general, Waiter–Client games are *not* bias monotone in Client's bias (see Example B.0.4 in Appendix B).

For Client–Waiter games, the opposite is true. In general, these are *not* bias monotone in Waiter's bias. In fact, increasing Waiter's bias can harm *both* players (see Example B.0.5 in Appendix B). However, Client–Waiter games are bias monotone in Client's bias.

Fact 1.3.3 *In a Client–Waiter game, Client wins the* $(a+1 : b)$ *game whenever he wins the* $(a : b)$ *game.*

This is because Client can use his winning strategy $\mathcal{S}$ for the $(a : b)$ game to win the $(a+1 : b)$ game by simply following $\mathcal{S}$ and claiming an extra arbitrary element in each round. Indeed, the extra elements he claims do not destroy the winning set that following $\mathcal{S}$ guarantees him.

Thus, increasing Waiter's bias is a viable option to truly help Client win in the Waiter–Client game. However, we have seen that Client–Waiter games are *not* bias monotone in Waiter's bias and therefore its increase does not truly help Waiter win. Also, increasing Client's bias is not an option for this game since this helps Client (the winner of the $(1 : 1)$ Client–Waiter game) instead of Waiter. So, to enable the existence of some bias whose increase truly helps Waiter in the Client–Waiter game, we relax its rules in the following way.

Relaxing the Rules

In an $(a : b)$ *monotone* Client–Waiter game, Waiter is allowed to offer *less* elements than the bias of the game specifies. More precisely, in each round he may offer r elements for any r in the range $a \leqslant r \leqslant a+b$, with Client claiming a of these elements as usual. This relaxation makes the game bias monotone in Waiter's bias, whose increase helps the loser (*i.e.* Waiter) of the $(1:1)$ game. Indeed, when playing the $(a:b+1)$ game, Waiter can simply follow his winning strategy for the $(a:b)$ game to prevent Client from claiming a winning set.

Fact 1.3.4 *In a monotone Client–Waiter game, Waiter wins the* $(a:b+1)$ *game whenever he wins the* $(a:b)$ *game.*

Throughout this thesis, we will only study the monotone version of Client–Waiter games.

A similar relaxation can also be performed on Avoider–Enforcer games. In the monotone version, Avoider and Enforcer are allowed to claim more elements than their bias specifies per round, if they choose to. This relaxation makes the game bias monotone in both Enforcer's bias *and* Avoider's bias. However, since Avoider is often the loser of the $(1:1)$ game and the increase of Enforcer's bias helps Avoider, it is Enforcer's bias that is commonly chosen to vary.

The Threshold Bias

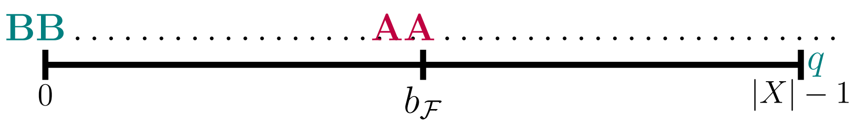

Figure 1.2: The threshold bias of $(X,\mathcal{F})$ played between A and B.

Given some player's bias that is chosen to vary, it is natural to ask which player wins the game generated by each possible value of this variable bias. If the game at hand is bias monotone in the chosen bias, we can obtain this information without the need to study every game generated. Indeed, suppose we consider a $(1:q)$ game $(X,\mathcal{F})$ between players A and B, with the varying bias q belonging to B. If $(X,\mathcal{F})$ is bias monotone in

B's bias, then increasing q helps some player, say A, by definition. Thus, if we know that A wins the $(1 : \overline{q})$ game, for some positive integer $\overline{q}$, then we know that A also wins all $(1 : r)$ games $(X, \mathcal{F})$, where $r > \overline{q}$. Identifying the smallest value $b_{\mathcal{F}}$ for which A wins the $(1 : b_{\mathcal{F}})$ game therefore fully characterises who wins the $(1 : q)$ game $(X, \mathcal{F})$ for every positive integer value of q. $b_{\mathcal{F}}$ is known as the *threshold bias* of $(X, \mathcal{F})$ (see Figure 1.2). Every game that is bias monotone in some player's bias has a unique threshold bias that we aim to locate when we choose the bias as our varying game parameter. In light of our discussion concerning the most common loser of the $(1 : 1)$ Maker–Breaker, Waiter–Client or Client–Waiter game and the resulting choice of variable bias, player A in our definition of the threshold bias represents Breaker, Client and Waiter for these games respectively.

For Avoider–Enforcer games (excluding the aforementioned monotone version) it is not clear whether a threshold bias exists since these are not bias monotone in *any* player's bias. Instead, we look for what are known as the *lower* and *upper* threshold biases, denoted by $f^-_{\mathcal{F}}$ and $f^+_{\mathcal{F}}$ respectively for a game $(X, \mathcal{F})$. These were first defined by Hefetz, Krivelevich and Szabó in [65] as follows. The lower threshold bias $f^-_{\mathcal{F}}$ is the largest integer for which Enforcer wins the $(1 : q)$ game, for every $q \leqslant f^-_{\mathcal{F}}$ and the *upper threshold bias* $f^+_{\mathcal{F}}$ is defined to be the smallest integer for which Avoider wins the $(1 : q)$ game, for every $q \geqslant f^+_{\mathcal{F}}$.

1.3.2 The Board

As mentioned previously, varying the bias of some player need not be our only option. We may also remove some elements of the board before play and observe the effect that this has on who wins, whilst keeping the bias of both players fixed. This process is known as *thinning the board.* In [86], Stojaković and Szabó first introduced the idea that this can be implemented *randomly* by retaining each element independently with some fixed probability p (or equivalently removing each element independently with probability

$1 - p$). Under this random setting, we'd like to know how low our retaining probability p can be such that the winner of the game on the complete board continues to win on the thinner board it generates. We can formulate this aim more precisely in terms of *probability thresholds.*

Probability Thresholds

Let us consider the set/board $X(p)$ generated by the retaining probability p and let $X := X(1)$. Informally, a probability threshold for a monotone increasing property $\mathcal{P}$ is a function $p^*(|X|)$ for which *a.a.s.* (with probability tending to 1 as $|X| \to \infty$) $X(p) \in \mathcal{P}$ when p is larger than $p^*(|X|)$ and *a.a.s.* $X(p) \notin \mathcal{P}$ when p is smaller than $p^*(|X|)$. However, unlike the threshold bias of a game, $p^*(|X|)$ is not unique since there is a window around $p^*(|X|)$ in which $\lim_{|X|\to\infty} \mathbb{P}[X(p) \in \mathcal{P}]$ grows quickly from 0 to 1. Depending on the size of this window, a probability threshold falls into one of two categories; *coarse* and *sharp*. $p^*(|X|)$ is *coarse* if the following statement holds:

$$\lim_{|X|\to\infty} \frac{p(|X|)}{p^*(|X|)} = \begin{cases} \infty & \implies \lim_{|X|\to\infty} \mathbb{P}[X(p(|X|)) \in \mathcal{P}] = 1, \\ 0 & \implies \lim_{|X|\to\infty} \mathbb{P}[X(p(|X|)) \in \mathcal{P}] = 0. \end{cases}$$

On the other hand, we say that $p^*(|X|)$ is *sharp* if, for any fixed $\varepsilon > 0$, the following holds:

$$\lim_{|X|\to\infty} \frac{p(|X|)}{p^*(|X|)} \begin{cases} \geqslant 1 + \varepsilon & \implies \lim_{|X|\to\infty} \mathbb{P}[X(p(|X|)) \in \mathcal{P}] = 1, \\ \leqslant 1 - \varepsilon & \implies \lim_{|X|\to\infty} \mathbb{P}[X(p(|X|)) \in \mathcal{P}] = 0. \end{cases}$$

With the knowledge that A wins some $(a : b)$ game $(X, \mathcal{F})$ against B, we would therefore like to find a sharp probability threshold for the property

$$\mathcal{P}_{\mathcal{F}} = \{X' \subseteq X : A \text{ wins the } (a : b) \text{ game } (X', \mathcal{F})\}.$$

But how can we be sure that a probability threshold exists? In each of the games we have seen (excluding strong games), the claiming of a winning set is desired by one player and unwanted by the other. If our game on a full board is won by the player who does *not* desire a winning set to be claimed, removing elements from the board only makes it easier for this player to continue winning. Thus, no change of winner occurs as the retaining probability decreases. Our interest therefore only lies in the case where the game on the full board is won by the player desiring a winning set. Note that, when playing on a board X, a winning strategy for Maker in a Maker–Breaker game or Waiter in a Waiter–Client game can also be followed in any game whose board contains X, either directly (in the case of Waiter) or by choosing arbitrary free elements within X to replace those claimed outside X by the opponent. This is also true for any winning strategy $\mathcal{S}$ of Client's in a monotone Client–Waiter game on X. Indeed, when playing on some board $X' \supseteq X$, Client may follow $\mathcal{S}$ by ignoring any element offered by Waiter in $X' \setminus X$ and claiming arbitrarily in the rounds where every element offered lies outside X. Therefore, our property $\mathcal{P}_{\mathcal{F}}$ is monotone increasing for these games. Bollobás and Thomason proved in [27] that this is enough to guarantee the existence of a probability threshold. Following this, Friedgut [50] went on to characterise all properties for which this probability threshold is sharp.

1.4 The Probabilistic Intuition

Although the threshold bias and probability threshold of a game are interesting in their own right, our underlying motivation for studying positional games comes from an interest in what is known as the *probabilistic intuition.* This is a heuristic which was first employed by Chvátal and Erdős in 1978 during their study of the Maker–Breaker connectivity game [29]. It states that the player with the highest chance of winning when both players play *randomly* is the player with the winning strategy when both players play *optimally.* In particular, this heuristic provides a predicted outcome of a game based on the typical properties possessed by a random set of board elements.

To illustrate this, let us consider the $(1 : q)$ Maker–Breaker game $(E(K_n), \mathcal{F})$ played on the edge–set $E(K_n)$ of the complete n–vertex graph K_n. Suppose Maker and Breaker play randomly, choosing each edge of their turn uniformly at random out of the free edges available to them. Then, at the end of the game, Maker's graph M is a random graph on n vertices with $m = \lceil \binom{n}{2}/(q+1) \rceil$ edges. The probabilistic intuition predicts that Maker will win the game if *a.a.s.* M contains some winning set $A \in \mathcal{F}$. If *a.a.s.* M does *not* contain a winning set, the probabilistic intuition will predict that Breaker wins. For us to discover what is predicted, we therefore need to know when a winning set appears in the random graph M. Fortunately, the area of random graphs is widely researched (see *e.g.* [25, 69]). However, out of convenience, most results refer to the Erdős–Rényi random graph $\mathcal{G}(n,p)$ obtained by including each edge of K_n independently with probability p. Since it is known that $\mathcal{G}(n,p)$ can model the random graph on n vertices with m edges when $p = m/\binom{n}{2}$ (see *e.g.* [25, 69]), we can model Maker's graph M, in our example, with the random graph $\mathcal{G}(n, 1/(q+1))$. Equipped with the knowledge of a probability threshold $p_{\mathcal{F}}$ for the graph property $\mathcal{P}_{\mathcal{F}} = \{G \subseteq K_n : G \text{ contains some } A \in \mathcal{F}\}$, we are therefore able to decipher what the probabilistic intuition predicts for our $(1 : q)$ Maker–Breaker game $(E(K_n), \mathcal{F})$. Note, however, that the existence of a small interval around $p_{\mathcal{F}}$ (discussed in the previous section) within which $\lim_{n\to\infty} \mathbb{P}[\mathcal{G}(n,p) \in \mathcal{P}_{\mathcal{F}}] \in (0,1)$ means that a prediction is only made for the $(1 : q)$ game whenever q is not too close to $1/p_{\mathcal{F}}$.

There are different levels of success that indicate how well the probabilistic intuition predicts the outcome of the game at hand. Indeed, suppose we are studying the game $(X, \mathcal{F})$ with threshold bias $b_{\mathcal{F}}$ and suppose that the threshold probability of a random $X' \subseteq X$ containing some $A \in \mathcal{F}$ is $p_{\mathcal{F}}$. We say that $(X, \mathcal{F})$ exhibits *strong* probabilistic intuition if $b_{\mathcal{F}} = (1 + o(1))/p_{\mathcal{F}}$. If this is not true, but $b_{\mathcal{F}} = \Theta(1/p_{\mathcal{F}})$, then we say $(X, \mathcal{F})$ exhibits *intermediate* probabilistic intuition. If the order of magnitude of $b_{\mathcal{F}}$ and $1/p_{\mathcal{F}}$

differs, the game $(X, \mathcal{F})$ does *not* exhibit the probabilistic intuition. Alternatively in this case, we may say that the probabilistic intuition *fails*. In general, the strength of the probabilistic intuition exhibited by the game increases with the number of values of bias q for which it gives a correct prediction.

Even within the class of Maker–Breaker games alone, we see examples of every level of probabilistic intuition exhibited in the literature. Many natural Maker–Breaker games exhibit strong probabilistic intuition. For example, this is true for the Maker–Breaker connectivity game $(E(K_n), \mathcal{C})$, where $\mathcal{C}$ consists of the edge–sets of all connected subgraphs of K_n. Indeed, it is well known that the probability threshold for a connected random graph $\mathcal{G}(n,p)$ is $p_{\mathcal{C}} = \log n/n$ (see *e.g.* [25, 69]). In 1978, Chvátal and Erdős [29] proved that the threshold bias $b_{\mathcal{C}}^{MB}$ for the Maker–Breaker connectivity game satisfies $(1/4 - o(1))n/\log n \leqslant b_{\mathcal{C}}^{MB} \leqslant (1 + o(1))n/\log n$. Later, in 1982, Beck [11] improved the constant factor in their result, proving that Maker wins the $(1 : q)$ game whenever $q < (\log(2) - o(1))n/\log n$. Finally, in 2009, Gebauer and Szabó [55] proved that $b_{\mathcal{C}}^{MB} = (1 - o(1))n/\log n = (1 - o(1))/p_{\mathcal{C}}$.

The Maker–Breaker non–planarity game $(E(K_n), \mathcal{NP})$, where Maker's goal is to build a non–planar graph, exhibits intermediate probabilistic intuition. In [59], Hefetz, Krivelevich, Stojaković and Szabó showed that the asymptotic threshold bias for this game is $b_{\mathcal{NP}}^{MB} = (1/2 - o(1))n$. Since $1/n$ is a sharp probability threshold $p_{\mathcal{NP}}$ for planarity (see *e.g.* [25, 69]), it follows that $b_{\mathcal{NP}}^{MB} = \Theta(1/p_{\mathcal{NP}})$.

We also have Maker–Breaker games in which the probabilistic intuition fails. An example is the Maker–Breaker K_3–game, where each triangle in K_n constitutes a winning set. In [29], Chvátal and Erdős found that the threshold bias of this game is $\Theta(n^{1/2})$. However, the threshold probability for having a triangle in the random graph has order $\Theta(1/n)$ (see *e.g.* [25, 69]). From this, the probabilistic intuition predicts that the threshold bias is $\Theta(n)$, which has a different order of magnitude to $\Theta(n^{1/2})$. In fact, upon generalising the result of Chvátal and Erdős, Bednarska and Łuczak [16] found that the probabilistic intuition fails for *every* Maker–Breaker H–game, where H is any fixed, pre–determined graph.

A characterisation of all games that exhibit the probabilistic intuition would be a very powerful tool in the study of positional games. Indeed, this would allow us to correctly predict the threshold bias of any game for which the random setting is understood, without needing to study the game itself. However, as we've just seen, our desired characterisation for all games in which this phenomenon works is non–trivial in the sense that the class of games that exhibit the probabilistic intuition is a non–empty proper subset of the set of all positional games. So far, no sufficient condition for a Maker–Breaker game to exhibit the probabilistic intuition has been found. For Waiter–Client and Client–Waiter games, the same is true. However, despite the existence of games within these classes that do not exhibit the probabilistic intuition, such as the Waiter–Client Hamiltonicity game [18] and the Client–Waiter maximum degree game [40], there also exist many examples of games whose outcomes strongly mimic typical behaviour in the random setting; for example, the Waiter–Client giant component game [18] and the Waiter–Client and Client–Waiter Ramsey games [12]. Hence, it is thought that greater progress towards understanding this phenomenon might be achieved here. Unfortunately, despite our research identifying yet more examples of Waiter–Client and Client–Waiter games that exhibit the probabilistic intuition, we are no closer to understanding it.

1.5 Main Results

In this thesis, we give bounds on the threshold bias for a range of Waiter–Client and Client–Waiter games, played on graphs, hypergraphs and sets of k–clauses. More precisely, we focus on games defined by the properties of containing a complete–minor, being non–planar, being non–r–colourable, and being a satisfiable conjunction of k–clauses. As we shall soon discuss, the large amount of existing research regarding these properties in the random setting makes their corresponding games prime candidates for investigating the probabilistic intuition. Indeed, our results show that all of these games exhibit the probabilistic intuition. The Maker–Breaker and Avoider–Enforcer versions of these games have also been well studied, thereby allowing interesting comparisons with our findings to arise. We additionally give sharp probability thresholds for the $(1:q)$ Waiter–Client and Client–Waiter Hamiltonicity games played on the random graph $\mathcal{G}(n,p)$ when q is fixed. These are more precise than existing analogous results for the Maker–Breaker and Avoider–Enforcer versions. In what follows, we discuss each of the aforementioned games in greater detail and state our results in full.

1.5.1 Complete–Minor Games

Our first game of interest is the K_t–minor game $(E(K_n), \mathcal{M}_t)$ played on the edge–set $E(K_n)$ of the complete graph K_n, where

$$\mathcal{M}_t = \{E(M) : M \text{ is a } K_t\text{–minor admitted by } K_n\}.$$

Much is known about both the Maker–Breaker and Avoider–Enforcer versions of this game. Indeed, in 2005, Bednarska and Pikhurko [14] showed that Breaker can ensure that Maker does not build a cycle when playing any $(1:q)$ Maker–Breaker game on $E(K_n)$ with $q \geqslant n/2$. Hence, Breaker wins the $(1:q)$ Maker–Breaker K_t–minor game, for all $t \geqslant 3$, when $q \geqslant n/2$. In the other direction, Hefetz, Krivelevich, Stojaković and Szabó [59] proved that, for every fixed $\varepsilon > 0$, there exists a constant $c = c(\varepsilon) > 0$ such that Maker wins the $(1:q)$ Maker–Breaker game $(E(K_n), \mathcal{M}_t)$, for every $t \leqslant c\sqrt{n/\log n}$, whenever $q \leqslant (1/2-\varepsilon)n$. Together, these two results show that the asymptotic threshold bias for the Maker–Breaker K_t–minor game is $(1/2 + o(1))n$.

For the Avoider–Enforcer version, bounds for the lower and upper threshold biases,

$f^-_{\mathcal{M}_t}$ and $f^+_{\mathcal{M}_t}$ respectively, are also known. In [59], Hefetz *et. al.* showed that, for every $\varepsilon > 0$, there exists a constant $c = c(\varepsilon) > 0$ such that Enforcer wins the $(1 : q)$ Avoider–Enforcer game $(E(K_n), \mathcal{M}_t)$, for every $t \leqslant n^c$ and every $q \leqslant (1/2 - \varepsilon)n$. Thus, $f^-_{\mathcal{M}_t} \geqslant (1/2 - \varepsilon)n$. On the other hand, Clemens, Ehrenmüller, Person and Tran [31] recently improved a result in [59] by showing that Avoider wins the $(1 : q)$ Avoider–Enforcer K_t–minor game, for every $t \geqslant 4$, whenever $q \geqslant 200n \log n$. Thus, $f^+_{\mathcal{M}_t} \leqslant 200n \log n$.

Jointly with Dan Hefetz and Michael Krivelevich, we show that the asymptotic threshold bias of the Waiter–Client K_t–minor game, for every t in the range $4 \leqslant t = \mathcal{O}(\sqrt{n})$, is $(1 + o(1))n$.

Theorem 1.5.1 ([67]) *Let n be a sufficiently large positive integer and let $\varepsilon = \varepsilon(n) \geqslant 4n^{-1/4} > 0$. Also let q and t be positive integers with $t \leqslant \varepsilon^2\sqrt{n}/5$. Consider the $(1 : q)$ Waiter–Client K_t–minor game $(E(K_n), \mathcal{M}_t)$. If $q \leqslant (1 - \varepsilon)n$, then Waiter can force Client to build a graph that admits a K_t–minor. On the other hand, if $q \geqslant n + \eta$, where $\eta = \eta(n) \geqslant n^{2/3} \log n$, then Client can ensure that his graph will be K_4–minor free throughout the game.*

In Theorem 1.5.1, the upper bound $\varepsilon^2\sqrt{n}/5$ on the size of the complete–minor that Waiter can force Client to build is best possible, up to a constant, when q is close to n. This is because Client's graph must have at least $\binom{t}{2}$ edges to admit a K_t–minor. Since $q \approx n$ gives Client $\mathcal{O}(n)$ edges at the end of the game, a complete–minor of size $\mathcal{O}(\sqrt{n})$ is the largest that his graph can contain.

Additionally, Theorem 1.5.1 guarantees a complete–minor of larger size than that achieved by Maker in [59]. Unlike those in [59], the accuracy of our bounds increases as the size of the complete–minor that Waiter is trying to force decreases, due to the dependency of ε on n. We expect this since, intuitively, a smaller minor should be easier to force Client to build than a larger one.

We also show, together with Dan Hefetz and Michael Krivelevich, that the asymptotic threshold bias for the Client–Waiter K_t–minor game matches that of the corresponding Maker–Breaker game with value $(1/2 + o(1))n$.

Theorem 1.5.2 ([67]) *Let n, t and q be positive integers with n sufficiently large and let $0 < \varepsilon = \varepsilon(n) \leqslant 1/2$. Consider the $(1 : q)$ Client–Waiter K_t–minor game $(E(K_n), \mathcal{M}_t)$. If $q \geqslant \lceil n/2 \rceil - 1$, then Waiter has a strategy to keep Client's graph K_3–minor free throughout the game. On the other hand, if $q \leqslant (1/2 - \varepsilon)n$, then Client can build a graph that admits a K_t–minor for $t \geqslant (\varepsilon n)^{c\varepsilon}$, where $c > 0$ is an absolute constant.*

With regards to the probabilistic intuition of the K_t–minor game, it is well known (see *e.g.* [25, 69]) that, for arbitrarily small but fixed $\varepsilon > 0$, *a.a.s.* $\mathcal{G}(n, p)$ has at most one cycle, and is therefore K_4–minor free, when $p \leqslant (1 - \varepsilon)/n$. Also, in 2008, Fountoulakis, Kühn and Osthus [48] found that, for arbitrarily small but fixed $\varepsilon > 0$, $\mathcal{G}(n, p)$ *a.a.s.* admits a K_t–minor for $t = \Theta(\sqrt{n})$ when $p \geqslant (1 + \varepsilon)/n$. Thus, for every $4 \leqslant t = \mathcal{O}(\sqrt{n})$, $1/n$ is a sharp probability threshold for having a K_t–minor. From this, the probabilistic intuition predicts that the threshold bias for the K_t–minor game, when $4 \leqslant t = \mathcal{O}(\sqrt{n})$, is n. Hence, both the Client–Waiter and Maker–Breaker versions of this game exhibit intermediate probabilistic intuition. In contrast, the Waiter–Client K_t–minor game exhibits *strong* probabilistic intuition.

1.5.2 Planarity Games

Kuratowski's Theorem (see, *e.g.* [91]) states that a graph is planar if and only if it does not admit a K_5–minor. Thus, we obtain the asymptotic threshold bias of both the Waiter–Client and Client–Waiter non–planarity games $(E(K_n), \mathcal{NP})$ as simple corollaries of Theorems 1.5.1 and 1.5.2 respectively, where

$$\mathcal{NP} = \{E(H) : H \subseteq G \text{ and } H \text{ is non–planar}\}.$$

These results were also achieved in collaboration with Dan Hefetz and Michael Krivelevich.

Corollary 1.5.3 ([67]) *Let n, q and t be positive integers where n is sufficiently large and consider the $(1:q)$ Waiter–Client non–planarity game $(E(K_n), \mathcal{NP})$. If $q \leqslant (1-\varepsilon)n$, where $\varepsilon = \varepsilon(n) \geqslant 5n^{-1/4}$, then Waiter can force Client to build a non–planar graph. On the other hand, if $q \geqslant n + \eta$, where $\eta = \eta(n) \geqslant n^{2/3} \log n$, then Client can keep his graph planar throughout the game.*

Corollary 1.5.4 ([67]) *Let n, q and t be positive integers where n is sufficiently large and consider the $(1:q)$ Client–Waiter non–planarity game $(E(K_n), \mathcal{NP})$. If $q \geqslant \lceil n/2 \rceil - 1$, then Waiter can keep Client's graph planar throughout the game. On the other hand, there exists a constant $c > 0$ such that Client can build a non–planar graph whenever $q \leqslant n/2 - cn/\log n$.*

Consequently, the probabilistic intuition exhibited by these non–planarity games matches that of the K_t–minor games from which the above results follow.

1.5.3 Colourability Games

The next game of interest is the non–r–colourability game $(E(K_n^{(k)}), \mathcal{NC}_r^{(k)})$, where $k, r \geqslant 2$, played on the edge–set of the complete n–vertex k–uniform hypergraph $K_n^{(k)}$. The winning sets belong to the family

$$\mathcal{NC}_r^{(k)} = \{E(\mathcal{H}) : \mathcal{H} \subseteq K_n^{(k)} \text{ and } \chi(\mathcal{H}) > r\},$$

where $\chi(\mathcal{H})$ denotes the *weak chromatic number* of $\mathcal{H}$ (see Chapter 2). We first consider the case $k = 2$ *i.e.* when the game is played on a graph.

The Maker–Breaker version was studied by Hefetz *et. al.* in [59], where they proved that the threshold bias has order $\Theta(n/r \log r)$. In particular, one can take $c_1 = 2 + o_r(1)$ and $c_2 = \log 2/2 - o_r(1)$ to be the upper and lower bound constant factors respectively as

r tends to infinity. In the same paper, Hefetz *et. al.* also proved that, for an appropriate absolute constant $c > 0$, Enforcer wins the Avoider–Enforcer non–r–colourability game whenever his bias q is at most $cn/(r \log r)$. However, their result regarding Avoider's win was improved by Clemens *et. al.* [31] when they showed that Avoider has a winning strategy, for every $r \geqslant 3$, whenever $q \geqslant 200 n \log n$.

Jointly with Dan Hefetz and Michael Krivelevich, we show that the threshold bias for both the Waiter–Client and Client–Waiter versions of the non–r–colourability game $(E(K_n), \mathcal{NC}_r^{(2)})$ has the same order as that of the aforementioned Maker–Breaker version. Note that these two results refer to games played on graphs.

Theorem 1.5.5 ([67]) *Let r, q and n be positive integers, with n sufficiently large and $r \geqslant 2$ fixed, and consider the $(1 : q)$ Waiter–Client non–r–colourability game $(E(K_n), \mathcal{NC}_r^{(2)})$. There exists a function $\alpha = \alpha(r) = o_r(1) > 0$ such that whenever $q \geqslant (8e + \alpha)n/(r \log r)$, Client can keep his graph r–colourable throughout the game and whenever $q \leqslant (\log 2/4 - \alpha)n/(r \log r)$, Waiter can force Client to build a non–r–colourable graph.*

Theorem 1.5.6 ([67]) *Let r, q and n be positive integers, with n sufficiently large and $r \geqslant 2$ fixed, and consider the $(1 : q)$ Client–Waiter non–r–colourability game $(E(K_n), \mathcal{NC}_r^{(2)})$. There exists a function $\alpha = \alpha(r) = o_r(1) > 0$ such that whenever $q \geqslant (4 + \alpha)n/(r \log r)$, Waiter can keep Client's graph r–colourable throughout the game and whenever $q \leqslant (\log 2/2 - \alpha)n/(r \log r)$, Client can build a non–r–colourable graph.*

The chromatic number of the random graph $\mathcal{G}(n, p)$ has been studied by many (see *e.g.* [43, 56, 26, 83, 28, 90, 7, 5, 34]). Currently, we know that the probability threshold for $\chi(\mathcal{G}(n, p)) \leqslant r$ lies in the interval $[((2r - 1) \log r - 2 \log 2 + o_r(1))/n, ((2r - 1) \log r - 1 + o_r(1))/n]$, with the lower bound due to Coja–Oghlan and Vilenchik [35] and the upper bound due to Coja–Oghlan [32]. Thus, the probabilistic intuition predicts that the threshold bias for the non–r–colourability game, when r is large, should be around $n/(2r\log r)$. This is true, up to a multiplicative constant, for the Maker–Breaker, Waiter–Client and Client–Waiter versions. Hence, all three games exhibit at least intermediate probabilistic intuition.

Generalising to the Hypergraph Setting

By generalising our techniques used to prove Theorems 1.5.5 and 1.5.6, we obtain bounds on the threshold bias for the Waiter–Client and Client–Waiter non–r–colourability games $(E(K_n^{(k)}), \mathcal{NC}_r^{(k)})$, for any $k \geqslant 2$. More precisely, we prove that the threshold bias for the Waiter–Client and Client–Waiter versions is $\frac{1}{n}\binom{n}{k}r^{\mathcal{O}_k(k)}$ and $\frac{1}{n}\binom{n}{k}r^{-k(1+o_k(1))}$ respectively.

Theorem 1.5.7 ([87]) *Let k, q, r and n be positive integers, with n sufficiently large and $k, r \geqslant 2$ fixed, and consider the $(1 : q)$ Waiter–Client non–r–colourability game $(E(K_n^{(k)}), \mathcal{NC}_r^{(k)})$. If $q \leqslant \binom{\lceil n/r\rceil}{k}\frac{\log 2}{2((1+\log r)n+\log 2)}$, then Waiter can force Client to build a non–r–colourable hypergraph. Also, if $q \geqslant 2^{k/r}e^{k/r+1}k\binom{n}{k}/n$, then Client can keep his hypergraph r–colourable throughout the game.*

Theorem 1.5.8 ([87]) *Let k, q, r and n be positive integers, with n sufficiently large and $k, r \geqslant 2$ fixed, and consider the $(1 : q)$ Client–Waiter non–r–colourability game $(E(K_n^{(k)}), \mathcal{NC}_r^{(k)})$. If $q \leqslant \binom{\lceil n/r\rceil}{k}\frac{\log 2}{(1+\log r)n}$, then Client can build a non–r–colourable hypergraph. However, when $q \geqslant k^3r^{-k+5}\binom{n}{k}/n$, Waiter can ensure that Client has an r–colourable hypergraph at the end of the game.*

Thus, for the Waiter–Client and Client–Waiter versions, we have a multiplicative gap of $(1+o(1))(1+\log r)\cdot 2^{k/r+1}e^{k/r+1}r^k k/\log 2$ and $(1+o(1))(1+\log r)r^5k^3/\log 2$ respectively between the upper and lower bounds of q.

For brevity, let us denote by $c_{r,k}$ a threshold (although only conjectured to exist) for c for which a random n–vertex k–uniform hypergraph with $m = cn$ edges is r–colourable. Many results bounding $c_{r,k}$ appear in the literature, particularly for the case

$r = 2$. This began with the bounds $\tilde{c} \cdot 2^k/k^2 < c_{2,k} < 2^{k-1}\log 2 - \log 2/2$, for some small constant $\tilde{c} > 0$, of Alon and Spencer [8]. Together with subsequent improvements (see [1, 3, 36]), this gives an edge threshold of $c_{2,k}n = 2^{k(1+o_k(1))}n$. Consequently, the probabilistic intuition predicts that the threshold bias for the $(1:q)$ non–2–colourability game $(E(K_n^{(k)}), \mathcal{NC}_2^{(k)})$ is $\frac{1}{n}\binom{n}{k}2^{-k(1+o_k(1))}$ which matches the threshold bias (up to the error term in the exponent) given by Theorems 1.5.7 and 1.5.8 when $r = 2$. Research pursuing $c_{r,k}$ for general $r \geqslant 2$ also exists. By generalising a result of Achlioptas and Naor [5] on r–colouring a random graph (2–uniform hypergraph), Dyer, Frieze and Greenhill [41] proved that $(r-1)^{k-1}\log(r-1) \leqslant c_{r,k} \leqslant (r^{k-1}-1/2)\log r$. The lower bound was subsequently improved by Ayre, Coja–Oghlan and Greenhill [10] to $(r^{k-1}-1/2)\log r - \log 2 - 1.01\log r/r$ for sufficiently large r. Thus, for such r, the edge threshold for the r–colourability of a random n–vertex k–uniform hypergraph is $c_{r,k}n = r^{k(1+o_k(1))}n$. Therefore, the probabilistic intuition predicts that the threshold bias for the $(1:q)$ non–r–colourability game $(E(K_n^{(k)}), \mathcal{NC}_r^{(k)})$ is $\frac{1}{n}\binom{n}{k}r^{-k(1+o_k(1))}$ when r is large, which again matches the threshold biases in Theorems 1.5.7 and 1.5.8 (up to the error term in the exponent). Thus, the Waiter–Client and Client–Waiter non–r–colourability games on a k–uniform hypergraph exhibit at least intermediate probabilistic intuition.

1.5.4 k–SAT Games

Given some boolean formula ϕ, the boolean satisfiability problem (SAT) asks whether there exists an assignment of the values 0 or 1 to the boolean variables involved such that ϕ evaluates to 1. If such an assignment exists, ϕ is said to be *satisfiable*. If ϕ is the conjunction of k–clauses, where a k–clause is the disjunction of exactly k literals taken from some set of fixed boolean variables, it is said to be in k conjunctive normal form (k–CNF) and the problem is known as k–SAT. In 1971, Cook proved that SAT is NP–complete [37]. Therefore, not only does SAT lie within the complexity class NP; the class of all problems solvable by a non–deterministic Turing machine in polynomial time, but additionally every problem in NP can be reduced to an instance of SAT in polynomial time. Hence, proving that SAT lies in the complexity class P; the class of all problems solvable by a deterministic Turing machine in polynomial time, is equivalent to resolving the famous P versus NP problem first introduced by Cook in [37].

By viewing literals as vertices, k–clauses as edges of a k–uniform hypergraph, and a satisfying $\{0,1\}$–assignment to the boolean variables as a special 2–colouring of the vertices, it is natural to consider the Waiter–Client and Client–Waiter k–SAT games $(\mathcal{C}_n^{(k)}, \mathcal{F}_{SAT})$ after dealing with non–r–colourability games on k–uniform hypergraphs. The k–SAT game is played on the set $\mathcal{C}_n^{(k)}$ of all $\binom{2n}{k}$ possible *k–clauses*, where each k–clause contains literals taken from n fixed boolean variables $x_1, \ldots, x_n$. By *literal*, we mean a boolean variable x_i or its negation $\neg x_i$. The set $\mathcal{F}_{SAT}$ of winning sets is defined to be

$$\mathcal{F}_{SAT} = \{\mathcal{S} \subseteq \mathcal{C}_n^{(k)} : \bigwedge \mathcal{S} \text{ is not satisfiable}\},$$

where $\bigwedge \mathcal{S}$ denotes the conjunction of all k–clauses in $\mathcal{S}$. To our knowledge, no other research of the k–SAT game appears in the literature. However, the Achlioptas process for k–SAT has been studied (see *e.g.* [84, 81, 39]).

By applying the techniques used to prove Theorems 1.5.5 and 1.5.6 in the case $r = 2$, we show that the threshold bias for the $(1 : q)$ Waiter–Client and Client–Waiter versions of $(\mathcal{C}_n^{(k)}, \mathcal{F}_{SAT})$ is $\frac{1}{n}\binom{n}{k}$ up to a factor that is exponential and polynomial in k respectively.

Theorem 1.5.9 ([87]) *Let k, q and n be positive integers, with n sufficiently large and $k \geqslant 2$ fixed, and consider the $(1 : q)$ Waiter–Client game $(\mathcal{C}_n^{(k)}, \mathcal{F}_{SAT})$. When $q \leqslant \binom{n}{k}/(2(n+1))$, Waiter can ensure that the conjunction of all k–clauses claimed by Client by the end of the game is not satisfiable. However, when $q \geqslant 2^{3k/2}e^{k/2+1}k\binom{n}{k}/n$, Client can ensure that the conjunction of all k–clauses he claims remains satisfiable* *throughout the game.*

Theorem 1.5.10 ([87]) *Let k, q and n be positive integers, with n sufficiently large and $k \geqslant 2$ fixed, and consider the $(1:q)$ Client–Waiter game $(\mathcal{C}_n^{(k)}, \mathcal{F}_{SAT})$. When $q < \binom{n}{k}/n$, Client can ensure that the conjunction of all k–clauses he claims by the end of the game is not satisfiable. However, when $q \geqslant 2^9 k^3 \binom{n}{k}/n$, Waiter can ensure that the conjunction of all k–clauses claimed by Client is satisfiable throughout the game.*

Thus, for the Waiter–Client and Client–Waiter versions, we have a multiplicative gap of $(1 + o(1))2^{3k/2+1}e^{k/2+1}k$ and 2^9k^3 respectively between the upper and lower bounds of q.

These games also exhibit at least intermediate probabilistic intuition. Indeed, Coja–Oghlan and Panagiotou [33] found that the threshold for the satisfiability of the conjunction of random k–clauses in $\mathcal{C}_n^{(k)}$ is $(2^k \log 2 - (1 + \log 2)/2 + o_k(1))n$ (see [49, 30, 53, 2, 52, 4, 6] for earlier work). Hence, the probabilistic intuition predicts that the threshold bias for the $(1:q)$ k–SAT game $(\mathcal{C}_n^{(k)}, \mathcal{F}_{SAT})$ is $\frac{1}{n}\binom{n}{k}(\log 2 - o_k(1))^{-1}$. This is matched, up to a constant factor, by the lower bounds for the threshold bias given in Theorem 1.5.9 and 1.5.10 respectively. Since the gap between the upper and lower bounds depends only on k (exponentially in the Waiter–Client game and polynomially in the Client–Waiter game), the threshold bias for both versions of the k–SAT game has the same order of magnitude as that predicted by the probabilistic intuition.

1.5.5 Hamiltonicity Games on the Random Graph

Finally, we consider the Hamiltonicity game $(E(\mathcal{G}(n,p)), \mathcal{HAM})$ played on the binomial random graph $\mathcal{G}(n,p)$, where

$$\mathcal{HAM} = \{E(G) : G \subseteq K_n \text{ is Hamiltonian}\}.$$

As discussed in Section 1.3.2, for this game we are interested in finding probability thresholds for graph properties $\mathcal{W}^q_{\mathcal{HAM}}$ and $\mathcal{C}^q_{\mathcal{HAM}}$, for every positive integer q, where

$$\mathcal{W}^q_{\mathcal{HAM}} = \{G \subseteq K_n : \text{Waiter wins the } (1:q) \text{ Waiter–Client game } (E(G), \mathcal{HAM})\},$$

$$\mathcal{C}^q_{\mathcal{HAM}} = \{G \subseteq K_n : \text{Client wins the } (1:q) \text{ Client–Waiter game } (E(G), \mathcal{HAM})\}.$$

The $(1:1)$ Maker–Breaker version was first considered by Stojaković and Szabó [86] when they proved that *a.a.s.* Maker can build a Hamilton cycle in $\mathcal{G}(n,p)$ whenever $p > 32\log n/\sqrt{n}$. Later, Stojaković improved this lower bound to $5.4\log n/n$ in [85] before a further improvement was subsequently made by Hefetz, Krivelevich, Stojaković and Szabó in [62] to $(\log n + (\log\log n)^\ell)/n$, for some constant $\ell > 0$. This is close to best possible since $\mathcal{G}(n,p)$ *a.a.s.* has at least three vertices of degree at most 3 when $p = (\log n + 3\log\log n - \omega(1))/n$, where $\omega(1)$ is any function tending to infinity with n arbitrarily slowly (see *e.g.* [25, 69]). Because of this, Breaker is able to ensure that Maker has a vertex of degree at most one at the end of the game, thereby preventing Maker from building a Hamilton cycle. A more recent result of Ben–Shimon, Ferber, Hefetz and Krivelevich [21] improved this further still by showing that *a.a.s.* Maker can build a Hamilton cycle in $\mathcal{G}(n,p)$ for every $p \geqslant (\log n + 3\log\log n + \omega(1))/n$. In fact, they proved a stronger result; that a graph on which Maker can build a Hamilton cycle in the $(1:1)$ Maker–Breaker game appears *a.a.s.* at the same time as a vertex of degree at least 4 appears when the edges of K_n are added to the empty graph, one by one, in a uniformly random order. Results regarding the more general $(1:q)$ game are not as accurate. It was conjectured in [86] that, for every $1 \leqslant q \leqslant (1-o(1))n/\log n$, the smallest edge probability p for which *a.a.s.* Maker has a winning strategy in the $(1:q)$ Maker–Breaker Hamiltonicity game is $\Theta(q\log n/n)$. This was proved by Ferber, Glebov, Krivelevich and Naor in [45], where an analogous statement for Avoider–Enforcer games was proved as well. An even stronger result was proved in [45] under the additional assumption that $q = \omega(1)$. In this case, the graph property of being a board on which Maker wins the $(1 : q)$ Maker–Breaker Hamiltonicity game has a sharp threshold at $q \log n/n$.

Jointly with Dan Hefetz and Michael Krivelvich, we show that the $(1 : q)$ Waiter–Client and Client–Waiter versions have sharp probability thresholds for every fixed positive integer q.

Theorem 1.5.11 ([68]) *Let q be a positive integer. Then $\log n/n$ is a sharp threshold for the property $\mathcal{W}^q_{\mathcal{HAM}}$.*

This threshold for property $\mathcal{W}^q_{\mathcal{HAM}}$ coincides with the sharp threshold for the appearance of a Hamilton cycle in $\mathcal{G}(n,p)$, found by Komlós and Szemerédi [73] and independently Bollobás [24].

In contrast to the Waiter–Client game, we find a sharp threshold for the property $\mathcal{C}^q_{\mathcal{HAM}}$ that grows with q, and even for $q = 1$, is already larger than the threshold for the Hamiltonicity of $\mathcal{G}(n,p)$.

Theorem 1.5.12 ([68]) *Let q be a positive integer. Then $(q+1)\log n/n$ is a sharp threshold for the property $\mathcal{C}^q_{\mathcal{HAM}}$.*

The aforementioned sharp threshold of $\log n/n$ for the appearance of a Hamilton cycle in $\mathcal{G}(n,p)$ leads the probabilistic intuition to predict that, for every integer $q \in [1, \binom{n}{2} - 1]$, $(q+1)\log n/n$ should be a sharp threshold for the properties $\mathcal{W}^q_{\mathcal{HAM}}$ and $\mathcal{C}^q_{\mathcal{HAM}}$. Since Theorems 1.5.11 and 1.5.12 refer only to *fixed* values of q, we are in no position to confirm the accuracy of this prediction in full yet, despite the fact that our thresholds are of the predicted order. We will discuss possible thresholds in the case where $q = \omega(1)$ in Chapter 6.

1.6 Summary

In summary, we have introduced the notion of a positional game and looked at a variety of game types within this class; strong, Maker–Breaker, Avoider–Enforcer, Waiter–Client and Client–Waiter. We saw the importance of bias monotonicity in the games we study to enable full characterisation of which player has a winning strategy when we vary one player's bias and fix the other. This led to defining the threshold bias of a game and motivated the need to relax the rules in Client–Waiter games. We also saw how one may choose to fix both player's biases and vary the board they play on instead, by randomly removing elements before play begins and looking for a probability threshold at which the winner of the game played on the full board no longer prevails. The probabilistic intuition was then discussed, with its potential to predict the outcome of a game between optimal players motivating a desire to find a characterisation of games that exhibit it. In the hope of aiding this pursuit, our thesis adds the Waiter–Client and Client–Waiter versions of the K_t–minor, non–planarity, non–r–colourability and k–SAT games to the set of games that exhibit this phenomenon. The sharp probability thresholds we find for both versions of the Hamiltonicity game on the random graph when Waiter's bias is fixed also point to the possibility that this game exhibits the probabilistic intuition. However, there is room for improvement in all of the results we present here. Hence, in Chapter 6, we discuss related open problems that the interested reader may wish to explore.

Chapter 2

Preliminaries

In this chapter, we present notation and basic definitions required to understand the content of the following chapters. We also present and discuss game–theoretic tools/winning criteria that we will use to develop winning strategies for Waiter and Client.

2.1 Notation and Terminology

Most of our results are asymptotic in nature and, whenever necessary, we assume that the number of vertices/boolean variables n is sufficiently large. Throughout this thesis, log stands for the natural logarithm, unless explicitly stated otherwise.

2.1.1 Graphs

Our graph–theoretic notation is standard and follows that of [91]. In particular, we use the following.

A graph G consists of a pair $(V(G), E(G))$ of sets, where $E(G)$ is a set of unordered pairs $\{u, v\}$ of elements $u, v \in V(G)$. We call members of $V(G)$ and $E(G)$ *vertices* and *edges* respectively. For any edge $e = \{u, v\} \in E(G)$, we often refer to the vertices u and v as *endpoints* of e and write $e = uv$ for simplicity. Let $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For a set $A \subseteq V(G)$, let $E_G(A)$ denote the set of edges of G with both endpoints in A and let $e_G(A) = |E_G(A)|$. For disjoint sets $A, B \subseteq V(G)$, let $E_G(A, B)$

denote the set of edges of G with one endpoint in A and one endpoint in B, and let $e_G(A, B) = |E_G(A, B)|$. A graph G' with vertex set $V' \subseteq V(G)$ and edge–set $E' \subseteq E_G(V')$ is a *subgraph* of G and we write $G' \subseteq G$ to denote this. For a set $A \subseteq V(G)$, let $G[A]$ denote the subgraph of G which is *induced* on the set A, *i.e.* with vertex set A and edge–set $\{e \in E(G) : e \subseteq A\}$. Also, let $N_G(A) = \{v \in V(G)\backslash A : \exists u \in A \text{ such that } uv \in E(G)\}$ denote the *outer neighbourhood* of A in G. For a vertex $u \in V(G)$ we abbreviate $N_G(\{u\})$ under $N_G(u)$ and let $d_G(u) = |N_G(u)|$ denote the *degree* of u in G. The *maximum degree* of a graph G is $\Delta(G) = \max\{d_G(u) : u \in V(G)\}$ and the *minimum degree* of a graph G is $\delta(G) = \min\{d_G(u) : u \in V(G)\}$. Often, when there is no risk of confusion, we omit the subscript G from the notation above. Given a pair of subgraphs G_1 and G_2 of a graph G, we write $G_1 \cup G_2$ to denote the subgraph of G with vertex set $V(G_1) \cup V(G_2)$ and edge–set $E(G_1) \cup E(G_2)$.

A *path* is a graph $P = (V, E)$, with $|V| \geqslant 1$, such that there exists an ordering $v_1, v_2, \dots, v_{|V|}$ of the vertices in V where $E = \{v_i v_{i+1} : i \in [|V| - 1]\}$ if $|V| > 1$ and $E = \emptyset$ if $|V| = 1$. If $|V| \geqslant 3$ and $v_1 v_{|V|}$ is also an edge in E then P is a *cycle*. A cycle that visits every vertex of a graph exactly once is a *Hamilton cycle*. If a graph contains a Hamilton cycle, it is said to be *Hamiltonian*. A path that visits every vertex of a graph exactly once is a *Hamiltonian path*. A graph G is *connected* if every pair of its vertices is contained in some path of G. A maximal connected subgraph of a graph G is called a *connected component*. If each connected component of a graph G contains no cycle, G is called a *forest*. A *star* is a connected graph $S = (V, E)$ with some vertex $v \in V$ such that $v \in \bigcap E$. A graph is called a *linear forest* if each of its connected components is a path. The *girth* of a graph G is the number of edges in a shortest cycle of G (if G is a forest, then its girth is infinity). A set $A \subseteq V(G)$ is said to be *independent* in G if $E_G(A) = \emptyset$. The *independence number* of a graph G, denoted by $\alpha(G)$, is the maximum size of an independent set in G. A graph $G = (V, E)$ is a *clique* or *complete* graph if every pair of vertices in V is an edge in E. A clique with t vertices may be referred to as a t–clique. We denote the complete n–vertex graph (i.e. the n–clique) by K_n. The *clique number* of a graph G, denoted by $\omega(G)$, is the largest t such that G contains a t–clique. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the smallest integer k for which $V(G)$ can be partitioned into k independent sets. For a positive integer t and a graph G, we say that G admits a K_t*–minor* if, for every $1 \leqslant i \leqslant t$, there exists a set $B_i \subseteq V(G)$ such that the following three properties hold:

(i) $G[B_i]$ is connected for every $1 \leqslant i \leqslant t$.

(ii) $B_i \cap B_j = \emptyset$ for every $1 \leqslant i < j \leqslant t$.

(iii) $E_G(B_i, B_j) \neq \emptyset$ for every $1 \leqslant i < j \leqslant t$.

A graph G is *planar* if it can be drawn in the plane such that every edge $uv \in E(G)$ only intersects other edges of G at its endpoints u or v.

Assume that some Waiter–Client or Client–Waiter game, played on the edge–set of some graph $H = (V, E)$, is in progress. At any given moment during this game, let E_W denote the set of all edges that were claimed by Waiter up to that moment, let E_C denote the set of all edges that were claimed by Client up to that moment, let $G_W = (V, E_W)$ and let $G_C = (V, E_C)$. Moreover, let $G_F = (V, E_F)$, where $E_F = E \setminus (E_W \cup E_C)$; the edges of E_F are called *free*.

2.1.2 Hypergraphs

For a positive integer k, a k–uniform hypergraph $\mathcal{H}$ consists of a pair $(V(\mathcal{H}), E(\mathcal{H}))$ of sets: vertex set $V(\mathcal{H})$ and edge–set $E(\mathcal{H}) \subseteq 2^{V(\mathcal{H})}$, where each edge $e \in E(\mathcal{H})$ consists of exactly k vertices. For a subset $A \subseteq V(\mathcal{H})$, let $E_{\mathcal{H}}(A)$ denote the set of edges $e \in E(\mathcal{H})$ with $e \subseteq A$. For such A, let $d_{\mathcal{H}}(A)$ denote the number of edges in $\mathcal{H}$ that contain at least one vertex of A but are not contained entirely in A. When $A = \{v\}$ for some $v \in V(\mathcal{H})$, we abuse notation slightly and write $d_{\mathcal{H}}(v)$ instead of $d_{\mathcal{H}}(\{v\})$. Often, when there is no risk of confusion, we omit the subscript $\mathcal{H}$ from the notation above.

Let $\mathcal{H}[A]$ denote the hypergraph with vertex set A and edge–set $E_{\mathcal{H}}(A)$. The *maximum degree* of $\mathcal{H}$ is defined by $\Delta(\mathcal{H}) = \max\{d_{\mathcal{H}}(v) : v \in V(\mathcal{H})\}$ and the *minimum degree* of $\mathcal{H}$ is $\delta(\mathcal{H}) = \min\{d_{\mathcal{H}}(v) : v \in V(\mathcal{H})\}$. We say that A is *independent* in $\mathcal{H}$ if $E_{\mathcal{H}}(A) = \emptyset$. The *independence number* of $\mathcal{H}$, denoted by $\alpha(\mathcal{H})$, is the maximum size of an independent set of vertices in $\mathcal{H}$. A subhypergraph $\mathcal{H}' \subseteq \mathcal{H}$ (*i.e.* a hypergraph $\mathcal{H}'$ with $V(\mathcal{H}') \subseteq V(\mathcal{H})$ and $E(\mathcal{H}') \subseteq E(\mathcal{H})$) is a *clique* in $\mathcal{H}$ if every set of k vertices in $V(\mathcal{H}')$ is an edge of $\mathcal{H}'$. We sometimes refer to a clique with t vertices as a t–clique. The *clique number* of $\mathcal{H}$, denoted by $\omega(\mathcal{H})$, is the largest t such that $\mathcal{H}$ contains a t–clique. The *weak chromatic number* of $\mathcal{H}$, denoted by $\chi(\mathcal{H})$, is the smallest integer r for which $V(\mathcal{H})$ can be partitioned into r independent sets. For a set $F \subseteq E(\mathcal{H})$, we abuse notation slightly by using $\chi(F)$ to denote the chromatic number of the hypergraph with vertex set $V(\mathcal{H})$ and edge–set F. Given some partition $\mathcal{P} = \{V_1, \ldots, V_{|\mathcal{P}|}\}$ of $V(\mathcal{H})$ and an edge $e \in E(\mathcal{H})$, we define $\mathcal{P}(e) = \{V_i \in \mathcal{P} : e \cap V_i \neq \emptyset\}$. We define a *linear forest* in $\mathcal{H}$ with respect to the partition $\mathcal{P}$ to be a sequence $(e_1, \ldots, e_m)$ of edges in $E(\mathcal{H})$ such that $\mathcal{P}(e_i) \cap \mathcal{P}(e_j) \neq \emptyset$ only if $j \in \{i-1, i, i+1\}$. Note that our use of *linear* here does *not* refer to the definition of a linear hypergraph where every pair of edges must intersect in at most one vertex. Two distinct edges $e, e' \in E(\mathcal{H})$ with vertices in parts $V_{i_1}, \ldots, V_{i_k}$ of $\mathcal{P}$ are *complementary* if $e \cap e' = \emptyset$. We also define $\Delta_{\mathcal{P}}(\mathcal{H}) = \max\{d_{\mathcal{H}}(V_i) : i \in [|\mathcal{P}|]\}$.

Let us denote the complete n–vertex k–uniform hypergraph by $K_n^{(k)}$ (*i.e.* $K_n^{(k)}$ is an n–clique). At any given moment in a Waiter–Client or Client–Waiter game, played on $E(K_n^{(k)})$, let E_C denote the set of edges currently owned by Client. We denote the hypergraph with vertex set $V(K_n^{(k)})$ and edge–set E_C by $\mathcal{H}_C$. Moreover, let $\mathcal{H}_F$ be the hypergraph consisting of all edges of $K_n^{(k)}$ that are free at a given moment.

2.2 Tools for Finding a Winning Strategy

2.2.1 A Potential–Type Method

In 1973, Erdős and Selfridge [44] proved a very useful sufficient condition for Breaker to win the $(1:1)$ Maker–Breaker game $(X, \mathcal{F})$.

Theorem 2.2.1 ([44]) *Let X be a set and $\mathcal{F} \subseteq 2^X$. If*

$$\sum_{A \in \mathcal{F}} 2^{-|A|} < 1/2,$$

then Breaker, as the second player, has a winning strategy for the $(1:1)$ weak game $(X, \mathcal{F})$. If Breaker is the first player, the condition can be relaxed to $\sum_{A \in \mathcal{F}} 2^{-|A|} < 1$.

Theorem 2.2.1 is commonly known as the Erdős–Selfridge Theorem and its condition is tight. Indeed, Erdős and Selfridge showed in [44] that, for every integer $n \geqslant 2$, there is an n–uniform hypergraph $\mathcal{F}$ satisfying $\sum_{A \in \mathcal{F}} 2^{-|A|} = 1/2$ such that Maker has a winning strategy for $(X, \mathcal{F})$.

If both players play randomly, the probability that Maker owns a winning set at the end of the game is at most $\sum_{A \in \mathcal{F}} 2^{-|A|}$, which is less than 1 by the hypothesis of Theorem 2.2.1. This tells us that there exists a way for Maker and Breaker to play such that Breaker wins the game. The proof of Theorem 2.2.1 is a *de–randomisation* that converts this existence into a deterministic winning strategy that Breaker can follow. It employs what is known as a *potential–type* or *quasiprobabilistic* method and is, in fact, an instance of the method of conditional probabilities (see [9]). At the end of each player's turn in the $(1:1)$ Maker–Breaker game, each winning set $A \in \mathcal{F}$ is given a *potential* p_A whose value is the probability that every element of A belongs to Maker, given that all remaining free elements of the board X are assigned to Maker or Breaker uniformly at random. The *game potential* at the end of each turn is the sum of potentials p_A

over all $A \in \mathcal{F}$. Note that this is equal to the expected number of winning sets owned by Maker after the described random assignment of remaining free board elements. By always claiming a free element whose occupation minimises the resulting game potential, Breaker ensures that the decrease in game potential caused by his turn outweighs the increase caused by Maker's claim immediately after his. Thus, the game potential at the end of Maker's turn is never greater than its value at the end of Maker's previous turn. Since a winning set $A \in \mathcal{F}$ fully claimed by Maker produces potential $p_A^i = 1$, this strategy is sufficient for Breaker to win the game, provided the game potential just before Breaker's first move is less than 1. This explains the sufficient condition for Breaker's win given in Theorem 2.2.1.

In [13], Beck generalised this argument to create a sufficient condition for Breaker to win the biased $(p : q)$ Maker–Breaker game. Thanks to the similarity between the roles of Maker and Breaker and the roles of Waiter and Client, an analogous result for Waiter–Client and Client–Waiter games is implicit in Beck's proof. We explicitly prove this here.

Theorem 2.2.2 (implicit in [13]) *Let X be a set, let $\mathcal{F} \subseteq 2^X$, and let p and q be positive integers. Suppose Waiter and Client play a $(p : q)$ game $(X, \mathcal{F})$ where Waiter offers r elements per round in the range $q \leqslant r \leqslant p + q$, except for possibly in the last round. Also, in each round suppose that Client rejects* $\min\{r, q\}$ *elements offered to him before he claims any for himself. Then Client has a strategy to avoid fully claiming more than $\sum_{A \in \mathcal{F}} (q+1)^{-|A|/p}$ members of $\mathcal{F}$ for the duration of the game. In particular, if*

$$\sum_{A \in \mathcal{F}} (q+1)^{-|A|/p} < 1,$$

then Client has a strategy to ensure that he does not fully claim any $A \in \mathcal{F}$ by the end of the game.

Proof. We must first introduce some notation. Let M denote the number of rounds for which this game lasts. In the ith round of the game, where $1 \leqslant i \leqslant M$, let $Z_i = \{z_1^i, ..., z_r^i\}$ be the set of free elements of X that Waiter offers Client, let $X_i = \{x_1^i, ..., x_{\min\{r,q\}}^i\}$ be the set of elements in Z_i that Client rejects, and let $Y_i = \{y_1^i, ..., y_{r-q}^i\} = Z_i \setminus X_i$ be the set of elements in Z_i that Client claims. Note that, by hypothesis, $q \leqslant r \leqslant p + q$, except possibly in the final round when r may be less than q. Set $W_0 = \emptyset$ and $C_0 = \emptyset$. For every integer i, where $0 \leqslant i \leqslant M$, let $W_i = \bigcup_{j=1}^i X_j$ and let $C_i = \bigcup_{j=1}^i Y_j$. Let $\mathcal{F}_0 = \mathcal{F}$ and, for every integer i, where $1 \leqslant i \leqslant M$, define $\mathcal{F}_i = \{A \setminus C_i : A \in \mathcal{F} \text{ and } A \cap W_i = \emptyset\}$ as Client's *focus* at the end of round i. Note that $\mathcal{F}_i$ is a multi–family in the sense that it may contain more than one copy of the same set. Let $\mathcal{F}_i^{(0,0)} = \mathcal{F}_i$ for every integer i, where $0 \leqslant i \leqslant M - 1$. For every $1 \leqslant w \leqslant \min\{r, q\}$ and every $0 \leqslant i \leqslant M$, let $\mathcal{F}_i^{(w,0)} = \{A \in \mathcal{F}_i : A \cap \{x_1^{i+1}, ..., x_w^{i+1}\} = \emptyset\}$ be Client's focus immediately after he has rejected w elements in round $i + 1$. For every $0 \leqslant i \leqslant M - 1$, if $r > q$ then let $\mathcal{F}_i^{(q,c)} = \{A \setminus \{y_1^{i+1}, ..., y_c^{i+1}\} : A \in \mathcal{F}_i^{(q,0)}\}$ be Client's focus immediately after he claims c elements in round $i + 1$, for every $1 \leqslant c \leqslant r - q$. For any family $\mathcal{H} \subseteq 2^X$ of sets and any element $v \in X$, let $\mathcal{H}(v) = \{A \in \mathcal{H} : v \in A\}$. Finally, let $\Phi(\mathcal{H}) = \sum_{A \in \mathcal{H}} (q + 1)^{-|A|/p}$ denote the potential of a family of sets $\mathcal{H}$ and set $\lambda = (q + 1)^{1/p}$.

If Waiter succeeds in forcing Client to fully claim $t > \Phi(\mathcal{F})$ members of $\mathcal{F}$ by the end of the game, then there exists an integer $1 \leqslant i \leqslant M$ such that t copies of $\emptyset$ lie in $\mathcal{F}_i$. In particular, this means that

$$\Phi(\mathcal{F}_i) = \sum_{A \in \mathcal{F}_i} \lambda^{-|A|} \geqslant t\lambda^0 = t > \Phi(\mathcal{F}).$$

Therefore, if Client wishes to avoid fully claiming more than $\Phi(\mathcal{F})$ members of $\mathcal{F}$ for the duration of the game, it is enough for him to implement a strategy that ensures that $\Phi(\mathcal{F}_i) \leqslant \Phi(\mathcal{F})$ for every $1 \leqslant i \leqslant M$. Since $\mathcal{F}_{/} = \mathcal{F}$, it therefore suffices to show that

Client has a strategy to ensure that $\Phi(\mathcal{F}_{i+1}) \leqslant \Phi(\mathcal{F}_i)$ for every integer $0 \leqslant i \leqslant M-1$. We claim that this occurs when Client implements the following strategy.

Client's Strategy: For every integer $1 \leqslant i \leqslant M$, after Waiter offers r elements in the ith round, Client first identifies an element $x_1^i \in Z_i$ such that $\Phi(\mathcal{F}_{i-1}(x_1^i)) \geqslant \Phi(\mathcal{F}_{i-1}(z))$ for all $z \in Z_i$. He then rejects x_1^i. Let $1 \leqslant j \leqslant \min\{r, q\}$ and suppose that Client has rejected j elements $x_1^i, ..., x_j^i \in Z_i$ so far. Client then identifies an element $x_{j+1}^i \in Z_i \setminus \{x_1^i, ..., x_j^i\}$ such that $\Phi(\mathcal{F}_{i-1}(x_{j+1}^i)) \geqslant \Phi(\mathcal{F}_{i-1}(z))$ for all $z \in Z_i \setminus \{x_1^i, ..., x_j^i\}$. Once Client has rejected $\min\{r, q\}$ elements $x_1^i, ..., x_{\min\{r,q\}}^i$, he claims the remaining elements, if any, $y_1^i, ..., y_{r-q}^i \in Z_i \setminus \{x_1^i, ..., x_{\min\{r,q\}}^i\}$.

To show that the above strategy ensures that $\Phi(\mathcal{F}_{i+1}) \leqslant \Phi(\mathcal{F}_i)$ for every integer $0 \leqslant i \leqslant M-1$, we fix an arbitrary i in this range and consider when Waiter offers r free elements to Client at the beginning of round $i+1$ via the following cases.

Case 1: $q < r \leqslant p + q$**.** We first show that

$$\Phi(\mathcal{F}_{i+1}) = \Phi(\mathcal{F}_i) - \sum_{k=1}^{q} \Phi(\mathcal{F}_i^{(k-1,0)}(x_k^{i+1})) + \sum_{j=1}^{r-q} (\lambda - 1)\Phi(\mathcal{F}_i^{(q,j-1)}(y_j^{i+1})). \tag{2.2.1}$$

Indeed, suppose the $(i+1)$st round of the game is about to begin. At this point, Client's focus is $\mathcal{F}_i$. After Waiter has offered $r > q$ free elements at the beginning of the $(i+1)$st round, Client first rejects element $x_1^{i+1} \in Z_{i+1}$. Then, every $A \in \mathcal{F}_i$ containing x_1^{i+1} is removed from Client's current focus. So Client's rejection of x_1^{i+1} removes $\Phi(\mathcal{F}_i(x_1^{i+1}))$ from the current potential $\Phi(\mathcal{F}_i)$ and Client's focus is updated to $\mathcal{F}_i^{(1,0)}$. Client then rejects element $x_2^{i+1} \in Z_{i+1} \setminus \{x_1^{i+1}\}$, thereby removing every set $A \in \mathcal{F}_i^{(1,0)}$ containing x_2^{i+1} from Client's current focus. This then removes $\Phi(\mathcal{F}_i^{(1,0)}(x_2^{i+1}))$ from the current potential $\Phi(\mathcal{F}_i^{(1,0)})$. Continuing in this way, once Client has rejected q elements $x_1^{i+1}, ..., x_q^{i+1} \in Z_{i+1}$, the potential of the game has decreased by $\sum_{k=1}^{q} \Phi(\mathcal{F}_i^{(k-1,0)}(x_k^{i+1}))$ since the beginning of the $(i+1)$st round. At this point, Client's focus is $\mathcal{F}_i^{(q,0)}$. Client then claims his first element $y_1^{i+1} \in Z_{i+1}$ of the round. In doing so, Client's focus becomes $\mathcal{F}_i^{(q,1)}$, thereby removing y_1^{i+1} from every set $A \in \mathcal{F}_i^{(q,0)}(y_1^{i+1})$. This increases the current potential by $(\lambda-1)\Phi(\mathcal{F}_i^{(q,0)}(y_1^{i+1}))$. Continuing in this way, once Client has claimed all $r-q$ of his elements for this round, the potential has increased by $\sum_{j=1}^{r-q}(\lambda-1)\Phi(\mathcal{F}_i^{(q,j-1)}(y_j^{i+1}))$ since Client began to claim elements in the $(i+1)$st round. Therefore, we obtain (2.2.1) as claimed.

Let $m \in Z_{i+1} \setminus \{x_1^{i+1}, ..., x_q^{i+1}\}$ be such that $\Phi(\mathcal{F}_i^{(q,0)}(m)) \geqslant \Phi(\mathcal{F}_i^{(q,0)}(z))$ for every $z \in Z_{i+1} \setminus \{x_1^{i+1}, ..., x_q^{i+1}\}$. By definition, for every $1 \leqslant k \leqslant q$ we have $\mathcal{F}_i^{(q,0)} \subseteq \mathcal{F}_i^{(k-1,0)}$. In particular, $\mathcal{F}_i^{(q,0)}(m) \subseteq \mathcal{F}_i^{(k-1,0)}(m)$ and therefore $\Phi(\mathcal{F}_i^{(q,0)}(m)) \leqslant \Phi(\mathcal{F}_i^{(k-1,0)}(m))$. So by Client's choice of x_k^{i+1}, we obtain

$$\Phi(\mathcal{F}_i^{(k-1,0)}(x_k^{i+1})) \geqslant \Phi(\mathcal{F}_i^{(k-1,0)}(m)) \geqslant \Phi(\mathcal{F}_i^{(q,0)}(m)). \tag{2.2.2}$$

Observe that, for every $1 \leqslant j \leqslant r-q \leqslant p$, we have

$$\begin{aligned}
\Phi(\mathcal{F}_i^{(q,j-1)}(y_j^{i+1})) &= \sum_{A \in \mathcal{F}_i^{(q,j-1)}(y_j^{i+1})} \lambda^{-|A|} = \sum_{A \in \mathcal{F}_i^{(q,0)}(y_j^{i+1})} \lambda^{-|A \setminus \{y_1^{i+1}, \dots, y_{j-1}^{i+1}\}|} \\
&\leqslant \sum_{A \in \mathcal{F}_i^{(q,0)}(y_j^{i+1})} \lambda^{-|A|+j-1} = \lambda^{j-1}\Phi(\mathcal{F}_i^{(q,0)}(y_j^{i+1})) \\
&\leqslant \lambda^{j-1}\Phi(\mathcal{F}_i^{(q,0)}(m)). \qquad (2.2.3)
\end{aligned}$$

Therefore, via (2.2.1), (2.2.2) and (2.2.3), we obtain

$$\begin{aligned}\Phi(\mathcal{F}_{i+1}) &= \Phi(\mathcal{F}_i) - \sum_{k=1}^{q} \Phi(\mathcal{F}_i^{(k-1,0)}(x_k^{i+1})) + \sum_{j=1}^{r-q} (\lambda - 1)\Phi(\mathcal{F}_i^{(q,j-1)}(y_j^{i+1})) \\ &\leqslant \Phi(\mathcal{F}_i) - q\Phi(\mathcal{F}_i^{(q,0)}(m)) + (\lambda - 1)\Phi(\mathcal{F}_i^{(q,0)}(m)) \sum_{j=1}^{p} \lambda^{j-1} \\ &= \Phi(\mathcal{F}_i) + \Phi(\mathcal{F}_i^{(q,0)}(m)) \left((\lambda - 1)\frac{\lambda^p - 1}{\lambda - 1} - q \right) = \Phi(\mathcal{F}_i),\end{aligned}$$

where the penultimate equality follows from the evaluation of the geometric series

$$\sum_{r=0}^{N} x^r = \frac{x^{N+1} - 1}{x - 1},$$

and the last equality follows from the definition of λ.

Case 2: $r \leqslant q$**.** Then Client rejects all r elements in Z_{i+1}. Thus, Client's focus is updated to $\mathcal{F}_i^{(r,0)} = \mathcal{F}_{i+1}$. Hence, the potential at the end of the round is

$$\Phi(\mathcal{F}_{i+1}) = \Phi(\mathcal{F}_i^{(r,0)}) = \Phi(\mathcal{F}_i) - \sum_{k=1}^{r} \Phi(\mathcal{F}_i^{(k-1,0)}(x_k^{i+1})) \leqslant \Phi(\mathcal{F}_i).$$

Hence, $\Phi(\mathcal{F}_{i+1}) \leqslant \Phi(\mathcal{F}_i)$ for every $0 \leqslant i \leqslant M - 1$. □

By the definition of a Waiter–Client game, the following result follows immediately from Theorem 2.2.2.

Theorem 2.2.3 (implicit in [13]) *Let q be a positive integer, let X be a finite set and let $\mathcal{F}$ be a family of (not necessarily distinct) subsets of X. Then, when playing the $(p : q)$ Waiter–Client game $(X, \mathcal{F})$, Client has a strategy to avoid fully claiming more than $\sum_{A \in \mathcal{F}} (q+1)^{-|A|/p}$ sets in $\mathcal{F}$. In particular, if*

$$\sum_{A \in \mathcal{F}} (q+1)^{-|A|/p} < 1,$$

then Client has a strategy to avoid fully claiming any $A \in \mathcal{F}$.

Sometimes it may benefit Client in a Client–Waiter game to avoid members of some family $\mathcal{F}$ of subsets of the board; for example in the proof of Theorem 1.5.2. Yet, since the way we define Client–Waiter games allows Waiter to offer fewer board elements per round than his bias specifies (see Section 1.3.1 in Chapter 1), Client cannot have a strategy that guarantees the avoidance of every $A \in \mathcal{F}$. However, the following result of Dean and Krivelevich [40] shows that Client can ensure that he does not claim too many forbidden sets of $\mathcal{F}$.

Theorem 2.2.4 ([40]) *Let q be a positive integer, let X be a finite set, let $\mathcal{F}$ be a family of (not necessarily distinct) subsets of X and let $\Phi(\mathcal{F}) = \sum_{A \in \mathcal{F}} (q+1)^{-|A|}$. Then, when playing the $(1:q)$ Client–Waiter game $(X, \mathcal{F})$, Client has a strategy to claim the elements of a set $X_C \subseteq X$ of size $|X_C| \geqslant \lfloor |X|/(q+1) \rfloor$ which fully contains at most $2\Phi(\mathcal{F})$ sets in $\mathcal{F}$.*

2.2.2 Transversal Games

For a finite set X and a family $\mathcal{F}$ of subsets of X, the *transversal* family of $\mathcal{F}$ is defined to be $\mathcal{F}^* := \{A \subseteq X : A \cap B \neq \emptyset \text{ for every } B \in \mathcal{F}\}$. We refer to the positional game $(X, \mathcal{F}^*)$ as a *transversal game*. It can be beneficial for Waiter or Client to focus on winning some transversal game to aid them in winning the game at hand. For example, if having large minimum degree helps Client win a game played on $E(K_n)$, he can focus on claiming an edge in every large star of K_n to achieve this. By using an alternative perspective of Client's role in a Waiter–Client or Client–Waiter game, we can obtain a winning criteria for Client in a Client–Waiter transversal game directly from Theorem 2.2.2.

A Different Perspective

In reality, each $(p:q)$ positional game $(X, \mathcal{F})$ begins with two empty bins $B_1(p)$ and $B_2(q)$, with $B_1(p)$ and $B_2(q)$ only able to accept at most p and q elements of X per round respectively. The interaction of the players with the elements of X partitions X between the bins and each player has a different aim in mind regarding what these bins should contain at the end of the game. When Waiter and Client play, Client's job is to decide how the elements Waiter gives him in a round should be split amongst $B_1(p)$ and $B_2(q)$. With this perspective of the game, Theorem 2.2.2 says that, if $\sum_{A\in\mathcal{F}}(q+1)^{-|A|/p} < 1$ and Waiter offers r elements in each round for some r in the range $q \leqslant r \leqslant p+q$ (except for possibly the final round), then Client has a strategy $\mathcal{S}$ to sort the elements of X, round by round, such that $B_1(p)$ contains no $A \in \mathcal{F}$ at the end of the game. Additionally, $\mathcal{S}$ ensures he places exactly q elements per round in $B_2(q)$, except possibly in the final round. Since each member of X must be in one of the two bins at this point, $B_2(q)$ must contain at least one member of each $A \in \mathcal{F}$. In other words, $B_2(q)$ must be a member of the transversal family $\mathcal{F}^*$.

Suppose we swap variables p and q in the discussion above and set $p = 1$. Then Theorem 2.2.2 says that, if $\sum_{A\in\mathcal{F}} 2^{-|A|/q} < 1$, then $\mathcal{S}$ sorts the elements of X, round by round, so that $B_2(1)$ contains a member of $\mathcal{F}^*$ at the end of the game. Since $\mathcal{S}$ achieves this whilst allowing Waiter to offer less than $q+1$ elements per round (but at least one) if he wishes and instructing Client to place an element in $B_2(1)$ before he places any in $B_1(q)$, we may view this game as a $(1:q)$ Client–Waiter game $(X, \mathcal{F}^*)$. In particular, we have the following result.

Theorem 2.2.5 (implicit in [13]) *Let X be a set, let $\mathcal{F} \subseteq 2^X$, and let q be a positive integer. If*

$$\sum_{A\in\mathcal{F}} 2^{-|A|/q} < 1,$$

then Client has a winning strategy for the $(1:q)$ Client–Waiter game $(X, \mathcal{F}^*)$.

In fact, the sufficient condition in Theorem 2.2.5 can be improved as the following theorem demonstrates.

Theorem 2.2.6 ([68]) *Let q be a positive integer, let X be a finite set and let $\mathcal{F}$ be a family of subsets of X. If*

$$\sum_{A \in \mathcal{F}} \left(\frac{q}{q+1}\right)^{|A|} < 1,$$

then Client has a winning strategy for the $(1:q)$ Client–Waiter game $(X, \mathcal{F}^*)$.

Proof. Client will play randomly, that is, in each round he will choose one of the elements Waiter offers him uniformly at random, independently of all previous choices. Since Client–Waiter games are finite, perfect information games with no chance moves and no draws, in order to prove that Client has a winning strategy, it suffices to show that, given any fixed strategy $\mathcal{S}_W$ of Waiter,

$$\mathbb{P}[\text{Client loses } (X, \mathcal{F}^*) \mid \text{Waiter follows } \mathcal{S}_W] < 1.$$

Fix some strategy $\mathcal{S}_W$ of Waiter and a set $A \in \mathcal{F}$. Given that Waiter plays according to $\mathcal{S}_W$, let r denote the total number of rounds played in the game and, for every $1 \leqslant i \leqslant r$, let Z_i denote the set of elements Waiter offers Client in the ith round, let $z_i = |Z_i|$ and let $a_i = |A \cap Z_i|$. Note that r, z_i and a_i might depend on Client's random choices. For every $1 \leqslant i \leqslant r$, given z_i and a_i, the probability that Client claims an element of A in the ith round is a_i/z_i, independently of his previous choices. Hence, the probability that Client does not claim any element of A throughout the game is

$$\prod_{i=1}^{r} \left(1 - \frac{a_i}{z_i}\right) \leqslant \prod_{i=1}^{r} \left(1 - \frac{a_i}{q+1}\right) \leqslant \prod_{i=1}^{r} \left(1 - \frac{1}{q+1}\right)^{a_i} = \left(\frac{q}{q+1}\right)^{|A|},$$

where the second inequality holds by Bernoulli's inequality.

Taking a union bound over the elements of $\mathcal{F}$, we conclude that

$$\mathbb{P}[\text{Client loses } (X, \mathcal{F}^*) \mid \text{Waiter follows } \mathcal{S}_W] \leqslant \sum_{A \in \mathcal{F}} \left(\frac{q}{q+1}\right)^{|A|} < 1,$$

as claimed. $\square$

The following rephrasing of Corollary 1.5 in [17] also gives a sufficient condition for Waiter to win a $(1 : q)$ Waiter–Client transversal game.

Theorem 2.2.7 ([17]) *Let q be a positive integer, let X be a finite set and let $\mathcal{F}$ be a family of subsets of X. If*

$$\sum_{A \in \mathcal{F}} 2^{-|A|/(2q-1)} < 1/2\,,$$

then Waiter has a winning strategy for the $(1 : q)$ Waiter–Client game $(X, \mathcal{F}^*)$.

Chapter 3

Complete–Minor and Planarity Games

3.1 Results

In this chapter, we focus on the Waiter–Client and Client–Waiter complete–minor games played on the edge–set $E(K_n)$ of the complete n–vertex graph K_n. We also discuss both versions of the non–planarity game played on the same board.

3.1.1 Complete–Minor Games

For each positive integer t, the K_t–minor game $(E(K_n), \mathcal{M}_t)$ has its winning sets defined by

$$\mathcal{M}_t = \{E(M) : M \text{ is a } K_t\text{–minor admitted by } K_n\}.$$

In Section 3.3.1 we show that the asymptotic threshold bias of the $(1 : q)$ Waiter–Client K_t–minor game has order $(1 + o(1))n$, for every t in the range $4 \leqslant t = \mathcal{O}(\sqrt{n})$, by presenting the proof of Theorem 1.5.1, restated here for convenience.

Theorem 1.5.1 ([67]) *Let n be a sufficiently large positive integer and let $\varepsilon = \varepsilon(n) \geqslant 4n^{-1/4} > 0$. Also let q and t be positive integers with $t \leqslant \varepsilon^2\sqrt{n}/5$.*

Consider the $(1:q)$ *Waiter–Client* K_t*–minor game* $(E(K_n), \mathcal{M}_t)$*. If* $q \leqslant (1-\varepsilon)n$*, then Waiter can force Client to build a graph that admits a* K_t*–minor. On the other hand, if* $q \geqslant n + \eta$*, where* $\eta = \eta(n) \geqslant n^{2/3} \log n$*, then Client can ensure that his graph will be* K_4*–minor free throughout the game.*

Proof Overview.

Waiter's Strategy: Waiter forces Client to build a K_t–minor by partitioning $V(K_n)$ into two sets A and B. He then performs the following three steps:

1. Only offering edges in $E_{K_n}(B)$, Waiter forces Client to build a long path P.
2. Offering only edges of $E_{K_n}(A, V(P))$, Waiter forces Client to build a large matching M.
3. Waiter partitions P into t consecutive vertex–disjoint paths $P_1, \ldots, P_t$, each containing many endpoints of the matching M. We use D_i to denote the set of vertices in A adjacent to some vertex in $V(P_i)$ via an edge of M, for each $i \in [t]$. Offering only edges of $E_{K_n}(A)$, Waiter then forces Client to claim an edge between each pair of sets D_i and D_j for $i, j \in [t]$, $i \neq j$.

Once the above steps are completed, one may contract the edges of the matching M and paths P_i to see that Client's graph admits a K_t–minor (see Figure 3.1).

Client's Strategy: Client avoids building a K_4–minor throughout the game by using Theorem 2.2.3 to avoid building pairs of intersecting cycles. It is clear that a graph with no intersecting cycles cannot contain a K_4–minor.

□

Since the threshold bias is defined to be a unique integer, Theorem 1.5.1 seems to suggest that the Waiter–Client K_t–minor game has threshold bias n. However, the following theorem demonstrates that the threshold bias drops below n when t is large.

Theorem 3.1.1 ([67]) *Let n, α and t be positive integers where n is sufficiently large, $0 < \alpha < ct \log t$ for some sufficiently small constant $c > 0$, $\alpha = o(n)$ and $t \geqslant \frac{\bar{c} \log \log n}{\log \log \log n}$ for some sufficiently large constant $\bar{c}$. Then Client can avoid building a K_t–minor when playing the $(1 : q)$ Waiter–Client game $(E(K_n), \mathcal{M}_t)$ for every $q \geqslant n - \alpha$.*

Proof Overview.

Client's Strategy: Using Theorem 2.2.3, Client can ensure that he builds fewer cycles than are required to contain a K_t–minor. □

We also prove Theorem 3.1.1 in Section 3.3.1. In Section 3.3.1, we show that the asymptotic threshold bias for the $(1 : q)$ Client–Waiter K_t–minor game is $(1/2 + o(1))n$ by presenting the proof of Theorem 1.5.2, restated here for convenience.

Theorem 1.5.2 ([67]) *Let n, t and q be positive integers with n sufficiently large and let $0 < \varepsilon = \varepsilon(n) \leqslant 1/2$. Consider the $(1 : q)$ Client–Waiter K_t–minor game $(E(K_n), \mathcal{M}_t)$. If $q \geqslant \lceil n/2 \rceil - 1$, then Waiter has a strategy to keep Client's graph K_3–minor free throughout the game. On the other hand, if $q \leqslant (1/2 - \varepsilon)n$, then Client can build a graph that admits a K_t–minor for $t \geqslant (\varepsilon n)^{c\varepsilon}$, where $c > 0$ is an absolute constant.*

Proof Overview.

Waiter's Strategy: Waiter forces Client to have a connected graph by the end of the game via a result by Bednarska–Bzdęga, Hefetz, Krivelevich and Łuczak [18] (see Theorem 3.2.3). For large q, Client has too few edges at the end of the game to have both a connected graph and a cycle. His endgame graph is therefore a spanning tree which contains no cycle and therefore no K_3–minor.

Client's Strategy: Using Theorem 2.2.4, Client ensures his graph contains few short cycles. By deleting an edge from each of the short cycles present, we obtain a graph with average degree and girth large enough to satisfy the conditions of a result by Hefetz, Krivelevich, Stojaković and Szabó [59] which guarantees a K_t–minor. □

3.1.2 Planarity Games

Thanks to Kuratowski's Theorem (see, *e.g.* [91]), which states that a graph is planar if and only if it does not admit a K_5–minor, we obtain the asymptotic threshold bias of the Waiter–Client and Client–Waiter non–planarity games $(E(K_n), \mathcal{NP})$, where

$$\mathcal{NP} = \{E(H) : H \subseteq G \text{ and } H \text{ is non–planar}\},$$

directly from our results for the K_t–minor game. More precisely, Corollaries 1.5.3 and 1.5.4 show that the asymptotic threshold bias of the Waiter–Client and Client–Waiter non–planarity games is $(1 + o(1))n$ and $(1/2 + o(1))n$ respectively. We restate these results here for convenience and prove them in Section 3.3.2.

Corollary 1.5.3 ([67]) *Let n, q and t be positive integers where n is sufficiently large and consider the $(1 : q)$ Waiter–Client non–planarity game $(E(K_n), \mathcal{NP})$. If $q \leqslant (1-\varepsilon)n$, where $\varepsilon = \varepsilon(n) \geqslant 5n^{-1/4}$, then Waiter can force Client to build a non–planar graph. On the other hand, if $q \geqslant n + \eta$, where $\eta = \eta(n) \geqslant n^{2/3} \log n$, then Client can keep his graph planar throughout the game.*

Corollary 1.5.4 ([67]) *Let n, q and t be positive integers where n is sufficiently large and consider the $(1 : q)$ Client–Waiter non–planarity game $(E(K_n), \mathcal{NP})$. If $q \geqslant \lceil n/2 \rceil - 1$,* *then Waiter can keep Client's graph planar throughout the game. On the other hand, there exists a constant $c > 0$ such that Client can build a non–planar graph whenever $q \leqslant n/2 - cn/\log n$.*

3.2 Useful Tools

Along with winning criteria from Section 2.2 of Chapter 2, our proofs will make use of the following results. The first will be used in our proof of Theorem 1.5.1.

Claim 3.2.1 *Playing a $(1 : q)$ Waiter–Client game on $E(K_m)$, Waiter can force Client to build a path on at least $m - q$ vertices.*

Proof. Waiter first chooses an arbitrary vertex $x_1 \in V(K_m)$ and sets $P_1 = x_1$. Then he offers $q + 1$ edges, each with x_1 as an endpoint. By claiming one of these edges, say x_1x_2, Client creates a path $P_2 = x_1x_2$. Waiter continues in a similar way, responding to the creation of path $P_i = x_1 \dots x_i$ in Client's graph, for a positive integer i, by offering the edges of $\{x_iy_j : 1 \leqslant j \leqslant q+1\}$, where $y_1, \dots, y_{q+1}$ are arbitrary vertices of $V(K_m) \setminus V(P_i)$. By claiming any one of these edges, Client extends P_i to a path $P_{i+1} = x_1x_2 \dots x_ix_{i+1}$. Once Waiter can no longer offer in this way, we must have $m-i = |V(K_m)\setminus V(P_i)| < q+1$, entailing $i \geqslant m - q$. □

In the proof of Theorem 1.5.2, we will make use of the following two results; the first by Hefetz, Krivelevich, Stojaković and Szabó [59] and the second by Bednarska–Bzdęga, Hefetz, Krivelevich and Łuczak [18].

Proposition 3.2.2 ([59], Lemma 4.8) *Let G be a graph with average degree $2 + \alpha$, for some $\alpha > 0$, and girth $g \geqslant (1 + 2/\alpha)(4\log_2 t + 2\log_2\log_2 t + c')$, where c' is an absolute constant (*i.e. *independent of t and α). Then G admits a K_t–minor.*

Theorem 3.2.3 ([18], Theorem 1.3) *For every integer $n \geqslant 4$, when playing a $(1 : q)$ Waiter–Client game on $E(K_n)$, Waiter can force Client to have a connected graph at the end of the game if and only if $q \leqslant \lfloor n/2 \rfloor - 1$.*

3.3 Main Proofs

Here we present the proofs of our results in Section 3.1.

3.3.1 The K_t–Minor Game $(E(K_n), \mathcal{M}_t)$

The Waiter–Client K_t–Minor Game

Proof of Theorem 1.5.1. Let n be sufficiently large.

Waiter's Strategy: We describe a strategy for Waiter to force a K_t–minor in Client's graph when $q \leqslant (1-\varepsilon)n$; it is divided into the following three stages (see also Figure 3.1):

Stage I: Let $A \subseteq V(K_n)$ be an arbitrary set of size $\varepsilon n/2$ and let $B = V(K_n) \setminus A$. Offering only edges of $E_{K_n}(B)$, Waiter forces Client to build a path P on at least $\varepsilon n/2$ vertices.

Stage II: Offering only edges of $E_{K_n}(A, V(P))$, Waiter forces Client to build a matching M of size at least $\varepsilon^2 n/5$.

Stage III: Let P be split into t consecutive vertex–disjoint paths $P_1, \ldots, P_t$, each containing at least $\lfloor \sqrt{n} \rfloor$ endpoints of the matching M. For every $1 \leqslant i \leqslant t$, let

$$D_i = \{u \in A : \exists v \in V(P_i) \text{ such that } uv \in M\}.$$

For as long as there exist indices $1 \leqslant i < j \leqslant t$ such that $E_{G_C}(D_i, D_j) = \emptyset$, Waiter chooses such indices arbitrarily and offers Client $q+1$ arbitrary edges of $E_{K_n}(D_i, D_j)$. Once $E_{G_C}(D_i, D_j) \neq \emptyset$ for all $1 \leqslant i < j \leqslant t$, Waiter plays arbitrarily until the end of the game.

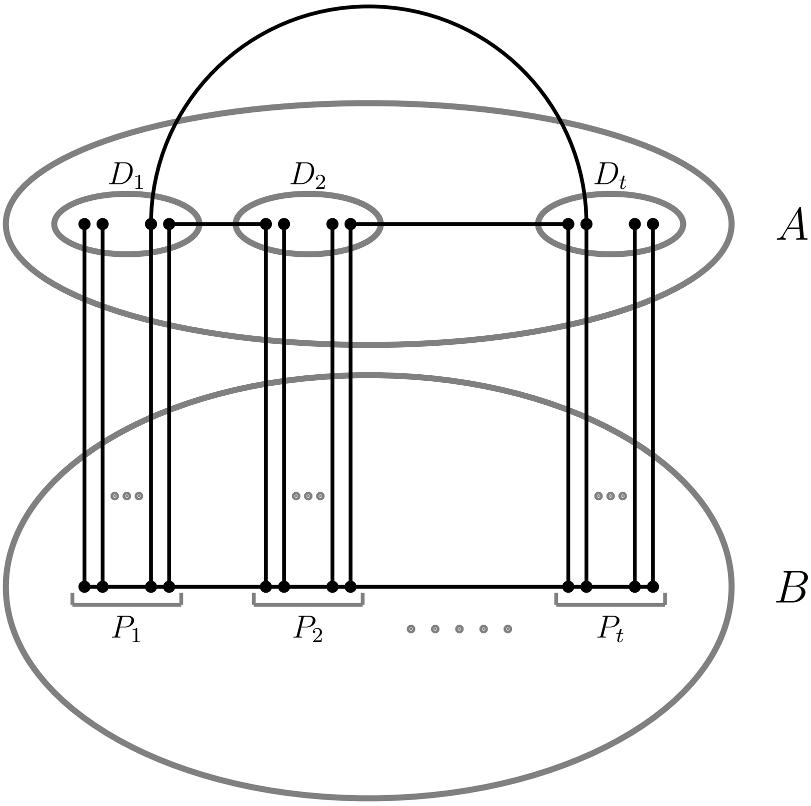

Figure 3.1: An illustration of the graph that Waiter forces Client to build.

Assuming that Waiter can follow the proposed strategy, by contracting every edge with both endpoints in $V(P_i) \cup D_i$ for every $1 \leqslant i \leqslant t$, we obtain the graph K_t. Hence, Client's graph admits a K_t–minor as claimed. It thus remains to prove that Waiter can indeed play according to the proposed strategy; we do so for each stage separately.

Since $|B| - q \geqslant n - \varepsilon n/2 - (1-\varepsilon)n = \varepsilon n/2$, the fact that Waiter can follow Stage I of the proposed strategy is an immediate corollary of Claim 3.2.1.

Next, we prove that Waiter can follow Stage II of the proposed strategy. Note that, by the description of Stage I, all edges of $E_{K_n}(A, V(P))$ are free at the beginning of Stage II. For as long as possible, in each round of this stage, Waiter offers Client $q+1$ arbitrary edges, which are disjoint from any edge Client has previously claimed in Stage II. It is thus evident that the graph Client builds in this stage is a matching; it remains to prove that it contains at least $\varepsilon^2 n/5$ edges. Suppose for a contradiction that, when following this strategy, Waiter can only force a matching of size $r < \varepsilon^2 n/5$ in Client's graph. Since Waiter cannot further enlarge Client's matching, it follows that he does not have enough edges to offer in accordance with Stage II of the proposed strategy. In particular, since there are at least $(\varepsilon n/2 - r)^2$ edges in $E_{K_n}(A, V(P))$ that do not share an endpoint with Client's matching and at most qr of these were claimed by Waiter during the rounds where this matching was built, the number of free edges that Waiter can offer according to Stage II is at least $(\varepsilon n/2 - r)^2 - qr$. Thus, if Waiter cannot continue to obey the proposed strategy of Stage II, we must have

$$(\varepsilon n/2 - r)^2 - qr < q + 1\,. \tag{3.3.1}$$

However, by the assumed lower bound on ε we have

$$(\varepsilon n/2 - r)^2 - qr > (\varepsilon n/2 - \varepsilon^2 n/5)^2 - \varepsilon^2 n^2 (1-\varepsilon)/5 \geqslant \varepsilon^2 n^2/20 \geqslant (1-\varepsilon)n + 1 \geqslant q + 1\,,$$

contrary to (3.3.1).

Finally, we prove that Waiter can play according to Stage III of the proposed strategy. It follows by Stage II and by the assumed upper bound on t that $|M| \geqslant \varepsilon^2 n/5 \geqslant t\sqrt{n}$. Therefore, P can indeed be split into t consecutive vertex disjoint paths $P_1, \ldots, P_t$, each containing at least $\lfloor \sqrt{n} \rfloor$ endpoints of M. By definition, $|D_i| \geqslant \lfloor \sqrt{n} \rfloor$ holds for every $1 \leqslant i \leqslant t$. Therefore, $|D_i||D_j| \geqslant \lfloor \sqrt{n} \rfloor^2 \geqslant (1-\varepsilon)n + 1 \geqslant q + 1$ holds for all $1 \leqslant i < j \leqslant t$. Since, by the description of Stages I and II, all edges of $E_{K_n}(D_i, D_j)$ are free at the beginning of Stage III, it follows that Waiter can ensure that Client will claim an edge of $E_{K_n}(D_i, D_j)$ for all $1 \leqslant i < j \leqslant t$.

Client's Strategy: Next, assume that $q \geqslant n + \eta$. Let $\mathcal{F}_1$ denote the family of edge–sets of cycles of K_n of length at least $\sqrt[3]{n}/2$. Then

$$\begin{aligned}
\Phi(\mathcal{F}_1) &= \sum_{A\in\mathcal{F}_1}(q+1)^{-|A|} = \sum_{k=\sqrt[3]{n}/2}^{n}\binom{n}{k}\frac{(k-1)!}{2}(q+1)^{-k} < \frac{1}{2}\sum_{k=\sqrt[3]{n}/2}^{\infty}\frac{1}{k}\left(\frac{n}{q}\right)^k \\
&< \left(\frac{n}{q}\right)^{\sqrt[3]{n}/2-1}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{n}{q}\right)^k = \left(\frac{n}{q}\right)^{\sqrt[3]{n}/2-1}\log\left(\frac{q}{q-n}\right) \\
&\leqslant \left(\frac{n}{n+n^{2/3}\log n}\right)^{\sqrt[3]{n}/2-1}\log\left(\frac{n+n^{2/3}\log n}{n^{2/3}\log n}\right) \\
&\leqslant \exp\left\{-\frac{(\sqrt[3]{n}/2-1)n^{2/3}\log n}{n+n^{2/3}\log n}\right\}\cdot\log n = o(1)\,, \qquad (3.3.2)
\end{aligned}$$

where the third equality follows from the Taylor expansion $-\log(1-x) = \sum_{k=1}^{\infty} x^k/k$.

Let $\mathcal{F}_2$ denote the family of edge–sets of all pairs of cycles (C_1, C_2) of K_n, such that $|C_1| = \ell_1$, $|C_2| = \ell_2$, $\ell_2 \leqslant \ell_1 \leqslant \sqrt[3]{n}/2$, and $C_1 \cap C_2$ is a path on $s \geqslant 1$ vertices. Then

$$\begin{aligned}
\Phi(\mathcal{F}_2) &= \sum_{A\in\mathcal{F}_2}(q+1)^{-|A|} \leqslant \sum_{\ell_1=3}^{\sqrt[3]{n}/2}\sum_{\ell_2=3}^{\ell_1}\sum_{s=1}^{\ell_2}\binom{n}{\ell_1}\frac{(\ell_1-1)!}{2}\cdot\ell_1\cdot(n)_{\ell_2-s}\cdot(n+\eta)^{-(\ell_1+\ell_2-s+1)} \\
&\leqslant \sum_{\ell_1=3}^{\sqrt[3]{n}/2}\sum_{\ell_2=3}^{\ell_1}\sum_{s=1}^{\ell_2} 1/n \leqslant (\sqrt[3]{n}/2)^3\cdot n^{-1} = 1/8\,. \qquad (3.3.3)
\end{aligned}$$

Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. Combining (3.3.2) and (3.3.3) we conclude that $\Phi(\mathcal{F}) < 1$. It thus follows by Theorem 2.2.3 that Client has a strategy to build a graph G_C such that, if C_1 and C_2 are cycles of G_C, then $V(C_1) \cap V(C_2) = \emptyset$. It is easy to see that a graph with no pair of intersecting cycles is K_4–minor free. $\square$

Proof of Theorem 3.1.1. Let n, t and α be as in the statement of the theorem. If a graph admits a K_t–minor, it must contain at least

$$\sum_{k=3}^{t}\binom{t}{k}\frac{(k-1)!}{2} \geqslant \frac{(t-1)!}{2} \geqslant (c_1 t)^t \geqslant e^{c_2 t\log t}$$

cycles, where c_1 and c_2 are positive constants. It is therefore sufficient to show that Client has a strategy to avoid building $e^{c_2 t \log t}$ cycles.

Let $\mathcal{F} = \{E(C) : C \text{ is a cycle of } K_n\}$. Then

$$\begin{aligned}\Phi(\mathcal{F}) &= \sum_{A \in \mathcal{F}} (q+1)^{-|A|} = \sum_{k=3}^{n} \binom{n}{k} \frac{(k-1)!}{2} (q+1)^{-k} \leqslant \sum_{k=3}^{n} \frac{1}{k} \left(\frac{n}{n-\alpha} \right)^k \\ &\leqslant \sum_{k=1}^{n} \frac{e^{\alpha k/(n-\alpha)}}{k} \leqslant e^{(1+o(1))\alpha} \sum_{k=1}^{n} \frac{1}{k} \leqslant e^{(1+o(1))\alpha} (\log n + 1) < e^{c_2 t \log t},\end{aligned}$$

where the third inequality holds since $\alpha = o(n)$ and the last inequality holds by the assumed bounds on t and α.

It thus follows by Theorem 2.2.3 that Client has a strategy to avoid fully claiming $e^{c_2 t \log t}$ cycles. This concludes the proof of the theorem. □

The Client–Waiter K_t–Minor Game

Proof of Theorem 1.5.2. Let n be sufficiently large.

Waiter's Strategy: Assume first that $q \geqslant n/2 - 1$. We consider two cases according to the parity of n. If n is even, then by monotonicity, and since $n/2$ is an integer, we may assume that $q = n/2 - 1$. Note that $\binom{n}{2}/(q+1) = n - 1$; in particular, Waiter offers exactly $q + 1$ edges in each round of the game. By Theorem 3.2.3, Waiter has a strategy to force Client to build a connected graph. Since, moreover, $e(G_C) = n - 1$ at the end of the game, it must be a spanning tree which is K_3–minor free.

Assume then that n is odd. By monotonicity, and since q is an integer, we may assume that $q = (n+1)/2 - 1$. By Theorem 3.2.3, Waiter has a strategy to force Client to build a connected graph when playing on $E(K_{n+1})$; let $\mathcal{S}$ be such a strategy. We present a strategy $\mathcal{S}'$ for Waiter to force Client to build a K_3–minor free graph when playing on $E(K_n)$. Waiter pretends the board is $E(K_{n+1})$, *i.e.* in his mind he adds an imaginary vertex and n imaginary edges, and follows $\mathcal{S}$. If in some round he is instructed by $\mathcal{S}$ to offer only imaginary edges, then he pretends that he did, and then he chooses one of these edges arbitrarily and pretends that Client claimed it. If in some round he is instructed by $\mathcal{S}$ to offer at least one imaginary edge and at least one real edge (*i.e.* an edge which is actually on the board $E(K_n)$), then he offers only the real edges (recall that in a Client–Waiter game, Waiter is allowed to offer fewer board elements than his bias specifies) but pretends he offered all edges $\mathcal{S}$ instructed him to claim. In every other round he plays precisely as $\mathcal{S}$ instructs him to. Thus, in his mind, Waiter follows $\mathcal{S}$ exactly. Since $\mathcal{S}$ is a winning strategy for the game on $E(K_{n+1})$, this means that Client's graph is a subgraph of a spanning tree of K_{n+1} at the end of the game. Hence, G_C is a forest which is K_3–minor free.

Client's Strategy: Now, suppose that $q \leqslant (1/2 - \varepsilon)n$; by monotonicity we can in fact assume that $q = \lfloor (1/2 - \varepsilon)n \rfloor$. Assume first that $\varepsilon \leqslant 1/7$. Let α be a constant satisfying

$$\left\lfloor \frac{\binom{n}{2}}{\lfloor (1/2-\varepsilon)n \rfloor + 1} \right\rfloor \geqslant (1+\alpha)n.$$

Note that $\alpha \geqslant \frac{\varepsilon}{1-\varepsilon}$. Let $k = \lfloor \log_3(\alpha n/4) \rfloor$ and let $\mathcal{F}_k$ denote the family of edge–sets of all cycles of K_n whose length is strictly smaller than k. Then

$$\begin{aligned}\Phi(\mathcal{F}_k) &= \sum_{A \in \mathcal{F}_k} (q+1)^{-|A|} = \sum_{s=3}^{k-1} \binom{n}{s} \frac{(s-1)!}{2} (q+1)^{-s} < \sum_{s=3}^{k-1} \left(\frac{n}{\lfloor (1/2-\varepsilon)n \rfloor} \right)^s \\ &< \sum_{s=3}^{k-1} 3^s < 3^k \leqslant \alpha n/4\,,\end{aligned}$$

where the second inequality holds by our assumption that $\varepsilon \leqslant 1/7$.

Using Theorem 2.2.4 we infer that Client has a strategy to build a graph G_C which contains a subgraph H_C with at least $(1+\alpha)n$ edges and fewer than $\alpha n/2$ cycles of length at most $k-1$. Deleting one edge from each such cycle results in a graph H with average degree at least $2+\alpha$ and with girth at least k. Let t be the largest integer for which $(1+2/\alpha)(4\log_2 t + 2\log_2\log_2 t + c') \leqslant k$; it is easy to see that there exists a constant $c > 0$ such that $t \geqslant (\varepsilon n)^{c\varepsilon}$. It follows from Proposition 3.2.2 that H admits a K_t–minor. Clearly, G_C admits the same minor.

Finally, if $\varepsilon > 1/7$, then, by monotonicity, Client can build a graph which admits a K_t–minor for $t = (n/7)^{c/7} \geqslant (\varepsilon n)^{c'\varepsilon}$ for appropriate positive constants c and c'. □

3.3.2 The Non–Planarity Game $(E(K_n), \mathcal{NP})$

The Waiter–Client Non–Planarity Game

Proof of Corollary 1.5.3. Let n be sufficiently large.

Waiter's Strategy: Assume first that $q \leqslant (1-\varepsilon)n$. If $\varepsilon \geqslant 5n^{-1/4}$, then $\varepsilon^2\sqrt{n}/5 \geqslant 5$ and thus it follows by Theorem 1.5.1 that Waiter can force Client's graph to admit a K_5–minor; Client's graph is then non–planar.

Client's Strategy: Assume then that $q \geqslant n+\eta$, where $\eta = \eta(n) \geqslant n^{2/3}\log n$. It follows by Theorem 1.5.1 that Client has a strategy to keep his graph K_4–minor free. It is easy to see that both K_5 and $K_{3,3}$ admit a K_4–minor and thus Client's graph is planar by Kuratowski's Theorem (see, e.g. [91]). □

The Client–Waiter Non–Planarity Game

Proof of Corollary 1.5.4. Let n be sufficiently large.

Waiter's Strategy: Assume first that $q \geqslant \lceil n/2 \rceil - 1$. It follows by Theorem 1.5.2 that Waiter has a strategy to force Client to build a K_3–minor free graph; in particular, such a graph is planar.

Client's Strategy: Assume then that $q \leqslant n/2 - cn/\log n$. Then $q \leqslant (1/2 - \varepsilon)n$, where $\varepsilon = c/\log n$. For a sufficiently large constant c, it follows by Theorem 1.5.2 that Client has a strategy to build a graph which admits a K_5–minor and is thus non–planar. □

Chapter 4

Colourability and k–SAT Games

4.1 Results

The following chapter is devoted to Waiter–Client and Client–Waiter colourability games played on the edge–set of the complete k–uniform hypergraph $K_n^{(k)}$, for every integer $k \geqslant 2$. The closely related k–SAT games will also be discussed.

4.1.1 Colourability Games

For every pair of integers $r, k \geqslant 2$, the winning sets of the non–r–colourability game $(E(K_n^{(k)}), \mathcal{NC}_r^{(k)})$ are defined by the set

$$\mathcal{NC}_r^{(k)} = \{E(\mathcal{H}) : \mathcal{H} \subseteq K_n^{(k)} \text{ and } \chi(\mathcal{H}) > r\},$$

where $\chi(\mathcal{H})$ denotes the *weak chromatic number* of $\mathcal{H}$ (see Chapter 2).

Playing on a Graph

We first focus on the case $k = 2$, *i.e.* the non–r–colourability game played on the edge–set of the complete graph K_n. Indeed, in Section 4.3.1, we show that the Waiter–Client version has threshold bias of order $\Theta(n/r \log r)$ by proving Theorem 1.5.5, restated here for convenience.

Theorem 1.5.5 ([67]) *Let r, q and n be positive integers, with n sufficiently large and $r \geqslant 2$ fixed, and consider the $(1 : q)$ Waiter–Client non–r–colourability game $(E(K_n), \mathcal{NC}_r^{(2)})$. There exists a function $\alpha = \alpha(r) = o_r(1) > 0$ such that whenever $q \geqslant (8e + \alpha)n/(r \log r)$, Client can keep his graph r–colourable throughout the game and whenever $q \leqslant (\log 2/4 - \alpha)n/(r \log r)$, Waiter can force Client to build a non–r–colourable graph.*

Proof Overview.

Client's Strategy: For large r, using Theorem 2.2.3, Client avoids building intersecting pairs of cycles of length 3 and 4, avoids having too large a proportion of the edges among any set of vertices, and avoids having a high proportion of the available edges both in some set S of vertices and between S and $V(K_n) \setminus S$. This produces a graph G_C at the end of the game with the following properties:

(a) No two cycles of length at most 4 have intersecting vertex sets.

(b) Amongst every set S of vertices, G_C does not contain too many edges from $E_{K_n}(S)$.

(c) If G_C has a large proportion of the edges on a set S of vertices, G_C has few edges between S and $V(K_n) \setminus S$.

By partitioning $V(K_n)$ into two parts; a set X containing all low degree vertices in G_C and a set $Y = V(K_n) \setminus X$, we observe that (a) guarantees the existence of a partition (X_1, X_2) of X where both $G_C[X_1]$ and $G_C[X_2]$ have girth at least 5. The definition of X also ensures that $\Delta(G_C[X_i])$ is small for $i = 1, 2$. A result of Kim [71] (see Theorem 4.2.1) then ensures that $\chi(G_C[X]) \leqslant \chi(G_C[X_1]) + \chi(G_C[X_2]) \leqslant 2r/3$. The definition of Y, together with (b) and (c), ensure that all subsets Z of Y have low minimum degree in G_C. This ensures that $\chi(G_C[Y]) \leqslant r/3$ which gives $\chi(G_C) \leqslant \chi(G_C[X]) + \chi(G_C[Y]) \leqslant r$.

For small r, it is enough for Client to ensure that (b) holds. With an appropriate definition of *few*, this gives a graph G_C where each of its subgraphs contains a vertex of degree at most $r-1$.

Waiter's Strategy: Using Theorem 2.2.7, Waiter forces Client to claim an edge in each $\lceil n/r \rceil$–clique of K_n, thereby ensuring that $\alpha(G_C) < \lceil n/r \rceil$. This suffices since $\alpha(G_C)\chi(G_C) \geqslant n$. □

Our bounds on the threshold bias for the Waiter–Client game can be tightened in the case $r = 2$. In particular, we have the following theorem.

Theorem 4.1.1 ([67]) *Let q and n be positive integers, with n sufficiently large, and consider the $(1:q)$ Waiter–Client non–bipartite game $(E(K_n), \mathcal{NC}_2^{(2)})$. If $q \leqslant (1/2-\varepsilon)n$, where $\varepsilon = \varepsilon(n) \geqslant \sqrt{3/n}$, Waiter can force Client to build a non–bipartite graph. However, if $q \geqslant n+\alpha$, where $\alpha \geqslant (1-\tanh(2))n/\tanh(2) \approx 0.04n$, Client can keep his graph bipartite throughout the game.*

Proof Overview.

Waiter's Strategy: Waiter forces Client to build an odd cycle by first forcing him to build a long path P. He then partitions $V(P)$ into two parts V_1, V_2 such that the edges of P lie between them. By offering only edges from $E_{K_n}(V_1) \cup E_{K_n}(V_2)$ in the following round, Waiter forces Client to close an odd cycle in his graph.

Client's Strategy: Using Theorem 2.2.3, Client can avoid building an odd cycle. □

Theorem 4.1.1 is proved in Section 4.3.2. Additionally, we show that the threshold bias of the Client–Waiter non–r–colourability game has the same order as the aforementioned Waiter–Client version. More precisely, we prove Theorem 1.5.6 in Section 4.3.1, restated here for convenience.

Theorem 1.5.6 ([67]) *Let r, q and n be positive integers, with n sufficiently large and $r \geqslant 2$ fixed, and consider the $(1 : q)$ Client–Waiter non–r–colourability game $(E(K_n), \mathcal{NC}_r^{(2)})$. There exists a function $\alpha = \alpha(r) = o_r(1) > 0$ such that whenever $q \geqslant (4+\alpha)n/(r \log r)$, Waiter can keep Client's graph r–colourable throughout the game and whenever $q \leqslant (\log 2/2 - \alpha)n/(r \log r)$, Client can build a non–r–colourable graph.*

Proof Overview.

Waiter's Strategy: For large r, Waiter's strategy consists of two stages. In Stage I, he forces Client to build a graph H_1 with small maximum degree and girth at least 5, leaving few free edges at each vertex of K_n. He does this by offering $q+1$ arbitrary edges in the first round and, in each subsequent round of this stage, he offers as many edges touching the last edge claimed by Client as possible (up to $(q+1)/2$ at each endpoint), whilst never offering an edge that closes a 3 or 4–cycle. If every free edge touching Client's most recent edge closes a 3 or 4–cycle, Waiter performs this procedure on any viable edge previously claimed by Client. If no such edge exists, the first stage ends.

In Stage II, Waiter forces Client to build a linear forest H_2 in the following way. In the first round of this stage, he offers all edges touching a chosen arbitrary vertex. In each subsequent round i, he offers all free edges that share an endpoint with the edge claimed by Client in round $i-1$. If no such free edges exist, Waiter plays as in the first round of Stage II.

A result of Kim [71] (see Theorem 4.2.1) guarantees that $\chi(H_1) \leqslant r/2$. Also, since H_2 is a linear forest, it has chromatic number at most 2. Hence, $\chi(G_C) \leqslant \chi(H_1)\chi(H_2) \leqslant r$.

For small r, Waiter can keep $\Delta(G_C) \leqslant r-1$ using a method similar to that in Stage I described above. Clearly this also gives $\chi(G_C) \leqslant r$.

Client's Strategy: Using Theorem 2.2.6, Client claims an edge in each $\lceil n/r \rceil$–clique to ensure that $\alpha(G_C) < \lceil n/r \rceil$. This suffices since $\alpha(G_C)\chi(G_C) \geqslant n$. □

As with the Waiter–Client version, we also obtain tighter bounds for the Client–Waiter game in the case $r = 2$. However, the gap is greater here than in Theorem 4.1.1. Our proof of this result is proved in Section 4.3.2.

Theorem 4.1.2 ([67]) *Let q and n be positive integers, with n sufficiently large, and consider the* $(1 : q)$ *Client–Waiter non–bipartite game* $(E(K_n), \mathcal{NC}_2^{(2)})$. *Whenever* $q \geqslant \lceil n/2 \rceil - 1$, *Waiter can keep Client's graph acyclic and therefore bipartite. However, whenever* $q \leqslant (1/4 - o(1))n$, *Client can build an odd cycle* i.e. *a non–bipartite graph.*

Proof Overview.

Waiter's Strategy: Waiter simply follows the strategy described in the proof of Theorem 1.5.2 to keep Client's graph acyclic.

Client's Strategy: Using Theorem 2.2.6, Client prevents Waiter from claiming a pair of cliques whose collective number of vertices is n by claiming an edge in each such pair. Since Waiter must own such a pair of cliques if Client's graph is bipartite at the end of the game, this is sufficient. □

Playing on a Hypergraph

We also generalise the techniques used in Theorems 1.5.5 and 1.5.6 to obtain bounds on the Waiter–Client and Client–Waiter $(E(K_n^{(k)}), \mathcal{NC}_r^{(k)})$ threshold biases for every $k, r \geqslant 2$. Indeed, in Section 4.3.3 we show that the threshold bias for the Waiter–Client version is $\frac{1}{n}\binom{n}{k} r^{\mathcal{O}_k(k)}$ by proving Theorem 1.5.7, restated here for convenience.

Theorem 1.5.7 ([87]) *Let k, q, r and n be positive integers, with n sufficiently large and $k, r \geqslant 2$ fixed, and consider the $(1 : q)$ Waiter–Client non–r–colourability game $(E(K_n^{(k)}), \mathcal{NC}_r^{(k)})$. If $q \leqslant \binom{\lceil n/r \rceil}{k} \frac{\log 2}{2((1+\log r)n+\log 2)}$, then Waiter can force Client to build a non–r–colourable hypergraph. Also, if $q \geqslant 2^{k/r} e^{k/r+1} k \binom{n}{k}/n$, then Client can keep his hypergraph r–colourable throughout the game.*

We also prove Theorem 1.5.8 in Section 4.3.3, thereby showing that the threshold bias for the Client–Waiter game $(E(K_n^{(k)}), \mathcal{NC}_r^{(k)})$ is $\frac{1}{n}\binom{n}{k} r^{-k(1+o_k(1))}$. Here, Theorem 1.5.8 is restated for convenience.

Theorem 1.5.8 ([87]) *Let k, q, r and n be positive integers, with n sufficiently large and $k, r \geqslant 2$ fixed, and consider the $(1 : q)$ Client–Waiter non–r–colourability game $(E(K_n^{(k)}), \mathcal{NC}_r^{(k)})$. If $q \leqslant \binom{\lceil n/r \rceil}{k} \frac{\log 2}{(1+\log r)n}$, then Client can build a non–r–colourable hypergraph. However, when $q \geqslant k^3 r^{-k+5} \binom{n}{k}/n$, Waiter can ensure that Client has an r–colourable hypergraph at the end of the game.*

4.1.2 k–SAT Games

Finally, we consider the k–SAT game $(\mathcal{C}_n^{(k)}, \mathcal{F}_{SAT})$, played on the set $\mathcal{C}_n^{(k)}$ of all $\binom{2n}{k}$ possible k–clauses, with winning sets defined by

$$\mathcal{F}_{SAT} = \{\mathcal{S} \subseteq \mathcal{C}_n^{(k)} : \bigwedge \mathcal{S} \text{ is not satisfiable}\},$$

where $\bigwedge \mathcal{S}$ denotes the conjunction of all k–clauses in $\mathcal{S}$.

By applying the same techniques used for the colourability games, we show that $\frac{1}{n}\binom{n}{k}$ is the threshold bias for the Waiter–Client and Client–Waiter k–SAT games, up to a factor that is exponential and polynomial in k respectively. More precisely, we prove Theorems 1.5.9 and 1.5.10 in Sections 4.3.4 and 4.3.4 respectively. We restate these results for convenience.

Theorem 1.5.9 ([87]) *Let k, q and n be positive integers, with n sufficiently large and $k \geqslant 2$ fixed, and consider the $(1 : q)$ Waiter–Client k–SAT game $(\mathcal{C}_n^{(k)}, \mathcal{F}_{SAT})$. When $q \leqslant \binom{n}{k}/(2(n+1))$, Waiter can ensure that the conjunction of all k–clauses claimed by Client by the end of the game is not satisfiable. However, when $q \geqslant 2^{3k/2}e^{k/2+1}k\binom{n}{k}/n$, Client can ensure that the conjunction of all k–clauses he claims remains satisfiable throughout the game.*

Theorem 1.5.10 ([87]) *Let k, q and n be positive integers, with n sufficiently large and $k \geqslant 2$ fixed, and consider the $(1 : q)$ Client–Waiter k–SAT game $(\mathcal{C}_n^{(k)}, \mathcal{F}_{SAT})$. When $q < \binom{n}{k}/n$, Client can ensure that the conjunction of all k–clauses he claims by the end of the game is not satisfiable. However, when $q \geqslant 2^9k^3\binom{n}{k}/n$, Waiter can ensure that the conjunction of all k–clauses claimed by Client is satisfiable throughout the game.*

4.2 Useful Tools

Along with tools from Chapter 2, we will make use of the following results in our proofs. Our proofs for Theorems 1.5.5 and 1.5.6 require the following well known result by Kim [71].

Theorem 4.2.1 ([71]) *Let G be a graph with maximum degree Δ and girth at least 5. Then*

$$\chi(G) \leqslant (1 + \nu(\Delta))\Delta/\log\Delta\,,$$

where $\nu(\Delta)$ is a function which tends to zero as Δ tends to infinity.

We will also use the following lemmas when proving Theorems 1.5.8 and 1.5.10 which result from a standard application of the Lovász Local Lemma ([42], see *e.g.* [55, 64] or Chapter 5 in [9]). For completeness, we include their proofs here.

Lemma 4.2.2 (Lovász Local Lemma (Symmetric Case)) *Let* $A_1, A_2, \ldots, A_n$ *be events in an arbitrary probability space. Suppose that each event* A_i *is mutually independent of all but at most* d *other events* A_j*, and that* $\mathbb{P}[A_i] \leqslant p$ *for all* $1 \leqslant i \leqslant n$*. If* $ep(d+1) \leqslant 1$*, then* $\mathbb{P}\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] > 0$.

Lemma 4.2.3 (Corollary 1 in [42]) *Let* $\mathcal{H}$ *be a* k*–uniform hypergraph with maximum degree* $\Delta(\mathcal{H}) \leqslant r^{k-3}/k$*. If* $k \geqslant 5$ *and* $r \geqslant 2$ *then* $\mathcal{H}$ *is* r*–colourable.*

Proof. Suppose that we assign a colour from the set $\{1, \ldots, r\}$ to each vertex of $V(\mathcal{H})$ uniformly at random and let us label the edges of $E(\mathcal{H})$ with $e_1, e_2, \ldots, e_{|E(\mathcal{H})|}$. For each $1 \leqslant i \leqslant |E(\mathcal{H})|$, let A_i denote the event that edge e_i is monochromatic. Then $\mathbb{P}[A_i] = 1/r^{k-1} =: p$ for each i. Since $\mathcal{H}$ has maximum degree $\Delta(\mathcal{H}) \leqslant r^{k-3}/k$, every edge of $\mathcal{H}$ meets at most $d := r^{k-3}$ other edges of $\mathcal{H}$. Hence, for each i, A_i is mutually independent of all but at most d other events A_j. Additionally,

$$ep(d+1) = \frac{e(r^{k-3}+1)}{r^{k-1}} \leqslant 1.$$

Thus, by Lemma 4.2.2, the probability that no edge is monochromatic, under our random colouring, is positive. Hence, $\mathcal{H}$ is r–colourable. □

Lemma 4.2.4 (see Theorem 1 in [55]) *Let* $k \geqslant 4$ *be an integer. Any* k*–CNF boolean formula in which no variable appears in more than* $2^{k-4}/k$ k*–clauses is satisfiable.*

Proof. Suppose that we assign a value from $\{0, 1\}$ to each boolean variable uniformly at random and let $c_1, c_2, \ldots, c_m$ denote the k–clauses in our boolean formula. For each

$1 \leqslant i \leqslant m$, let A_i denote the event that k–clause c_i is not satisfied (*i.e.* the random assignment has caused every literal in c_i to have value 0). Then $\mathbb{P}[A_i] = 2^{-k} =: p$ for each i. Since each boolean variable appears in at most $2^{k-4}/k$ k–clauses, each k–clause shares the same variable with at most $d := 2^{k-4}$ others. Hence, for each i, A_i is mutually independent of all but at most d other events A_j. Additionally, $ep(d+1) \leqslant e/8 \leqslant 1$. Thus, by Lemma 4.2.2, the probability that every k–clause is satisfied, under our random assignment, is positive. Hence, our boolean formula is satisfiable. □

4.2.1 Core Lemmas

To reduce repetition, we begin by presenting some core lemmas that will be useful in the proofs following this section.

Lemma 4.2.5 *Let S denote some set of $\lceil n/r \rceil$–cliques in $K_n^{(k)}$. In a $(1:q)$ game on $E(K_n^{(k)})$, a strategy to ensure that $\mathcal{H}_C$ contains an edge in every member of S at the end of the game exists for Waiter in the Waiter–Client version if $q \leqslant \binom{\lceil n/r \rceil}{k} \frac{\log 2}{2\log(2|S|)}$ and for Client in the Client–Waiter version if $q < \binom{\lceil n/r \rceil}{k} \log 2/\log(|S|)$.*

Proof. Let $\mathcal{F} = \{E(\mathcal{H}) : \mathcal{H} \in S\}$ and let us first suppose that $q \leqslant \binom{\lceil n/r \rceil}{k} \frac{\log 2}{2\log(2|S|)}$. Observe that

$$\sum_{A \in \mathcal{F}} 2^{-|A|/(2q-1)} < |S| 2^{-\binom{\lceil n/r \rceil}{k}/(2q)} \leqslant \frac{1}{2},$$

where the final inequality follows from our choice of q. Thus, by Theorem 2.2.7, Waiter can force Client to claim an edge in every member of S as stated.

Now suppose that $q < \binom{\lceil n/r \rceil}{k} \log 2/\log(|S|)$ and observe that

$$\sum_{A \in \mathcal{F}} \left(\frac{q}{q+1}\right)^{|A|} \leqslant \sum_{A \in \mathcal{F}} 2^{-|A|/q} \leqslant |S| 2^{-\binom{\lceil n/r \rceil}{k}/q} < 1,$$

where the final inequality follows from our choice of q. Thus, by Theorem 2.2.6, Client can claim an edge in every member of S by the end of the game as stated. □

Corollary 4.2.6 *In a* $(1:q)$ *game on* $E(K_n^{(k)})$*, a strategy to ensure that* $\chi(\mathcal{H}_C) > r$ *at the end of the game exists for Waiter in the Waiter–Client version if* $q \leqslant \binom{\lceil n/r \rceil}{k} \frac{\log 2}{2\left(\log 2 + \log \binom{n}{\lceil n/r \rceil}\right)}$ *and for Client in the Client–Waiter version if* $q < \binom{\lceil n/r \rceil}{k} \frac{\log 2}{\log \binom{n}{\lceil n/r \rceil}}$.

Proof. By taking S to be the set of all $\lceil n/r \rceil$–cliques in $K_n^{(k)}$, Lemma 4.2.5 provides strategies for Waiter and Client to ensure that $\alpha(\mathcal{H}_C) < \lceil n/r \rceil$ at the end of the game. Since $\chi(\mathcal{H}_C)\alpha(\mathcal{H}_C) \geqslant n$, the result follows. □

Lemma 4.2.7 *For* $k \geqslant 2$*, let* $\mathcal{H}^{\mathcal{P}}$ *denote some* k*–uniform hypergraph with a partition* $\mathcal{P} = \{V_1, \ldots, V_n\}$ *of its vertex set such that each part has the same size* $\ell \in \{1, 2\}$ *and contains at most one vertex from any edge in* $E(\mathcal{H}^{\mathcal{P}})$*. Consider a* $(1:q)$ *Waiter–Client game on* $E(\mathcal{H}^{\mathcal{P}})$*. If*

$$q \geqslant (2\ell^r)^{k/r} e^{k/r+1} k \binom{n}{k} \frac{1}{n},$$

then Client has a strategy to ensure that, for every $S \subseteq \mathcal{P}$*, there exists some* $A \in S$ *such that* $d_{\mathcal{H}_C[\cup S]}(A) \leqslant r - 1$ *at the end of the game.*

Proof. Let $\mathcal{F} = \{F : \exists S \subseteq \mathcal{P} \text{ s.t. } S \neq \emptyset, F \subseteq E(\mathcal{H}^{\mathcal{P}}[\cup S]) \text{ and } |F| = \lceil r|S|/k \rceil\}$. Observe that

$$\begin{aligned}
\Phi(\mathcal{F}) &= \sum_{A \in \mathcal{F}} (q+1)^{-|A|} \leqslant \sum_{t=k+\ell-1}^{n} \binom{n}{t} \binom{\binom{\ell t}{k}}{\lceil rt/k \rceil} q^{-\lceil rt/k \rceil} \leqslant \sum_{t=k+\ell-1}^{n} \left(\frac{en}{t}\right)^t \left(\frac{e\binom{\ell t}{k}}{\lceil rt/k \rceil q}\right)^{\lceil rt/k \rceil} \\
&\leqslant \sum_{t=k+\ell-1}^{n} \left(\frac{en}{t}\right)^t \left(\frac{ek\binom{\ell t}{k}}{rtq}\right)^{\lceil rt/k \rceil} \leqslant \sum_{t=k+\ell-1}^{n} \left[\frac{en}{t} \left(\frac{ek\binom{\ell t}{k}}{rtq}\right)^{r/k}\right]^t \\
&\leqslant \sum_{t=1}^{n} \left[\frac{en}{t} \left(\frac{e\ell^k t^{k-1}}{rq(k-1)!}\right)^{r/k}\right]^t \leqslant \sum_{t=1}^{n} \left[\frac{en}{t} \left(\frac{1}{(2e)^{k/r}} \left(\frac{t}{n}\right)^{k-1}\right)^{r/k}\right]^t \\
&= \sum_{t=1}^{n} \left[\frac{1}{2} \left(\frac{t}{n}\right)^{\frac{r}{k}(k-1)-1}\right]^t < \sum_{t=1}^{\infty} \left[\frac{1}{2}\right]^t = 1,
\end{aligned}$$

where the fourth and sixth inequalities follow from our choice of q and for n sufficiently large. Thus, by Theorem 2.2.3, Client can avoid claiming any member of $\mathcal{F}$. In particular, this means that, for every $S \subseteq \mathcal{P}$, Client must have strictly less than $r|S|/k$ edges in $\mathcal{H}^{\mathcal{P}}[\cup S]$ at the end of the game. Hence, in every $S \subseteq \mathcal{P}$ there exists some $A \in S$ for which $d_{\mathcal{H}_C[\cup S]}(A) \leqslant r-1$, as stated. □

Lemma 4.2.8 *For $k \geqslant 2$, let $\mathcal{H}^{\mathcal{P}}$ denote some k–uniform hypergraph with a partition $\mathcal{P} = \{V_1, \ldots, V_n\}$ of its vertex set such that each part has the same size $\ell \in \{1, 2\}$ and contains at most one vertex from any edge in $E(\mathcal{H}^{\mathcal{P}})$. Consider a $(1 : q)$ Client–Waiter game on $E(\mathcal{H}^{\mathcal{P}})$ where $q \geqslant 2k-2$. Waiter has a strategy to ensure that*

$$d_{\mathcal{H}_C}(V_j) < \frac{2kd_{\mathcal{H}^{\mathcal{P}}}(V_j)}{q} + 2$$

for every $j \in [n]$ at the end of the game.

Proof. In the first round, Waiter offers $q+1$ arbitrary free edges. In each round i, let us denote the edge claimed by Client by e_i and the parts of $\mathcal{P}$ in which e_i has a vertex $V_{i_1}, \ldots, V_{i_k}$ ordered arbitrarily. In round $i+1$, Waiter responds to Client's claim of e_i in the following way. With

$$S_{i_j} = \{e \in E(\mathcal{H}_F) : V_{i_j} \cap e \neq \emptyset\}$$

for each $j \in [k]$, let $F_{i_1} \subseteq S_{i_1}$ with size

$$|F_{i_1}| = \min\{d_{\mathcal{H}_F}(V_{i_1}), \lfloor (q+1)/k \rfloor\}$$

and, for each $2 \leqslant j \leqslant k$, let $F_{i_j} \subseteq S_{i_j} \setminus \bigcup\{F_{i_\ell} : 1 \leqslant \ell < j\}$ with size

$$|F_{i_j}| = \min\left\{\left|S_{i_j} \setminus \bigcup\{F_{i_\ell} : 1 \leqslant \ell < j\}\right|, \lfloor (q+1)/k \rfloor\right\}.$$

Immediately after Client has claimed e_i, Waiter offers all edges in $\bigcup\{F_{i_j} : j \in [k]\}$. Recall that, in any round of a Client–Waiter game, Waiter may offer less than $q+1$ edges if he desires. If no free edge contains a vertex from $\bigcup_{j=1}^{k} V_{i_j}$, Waiter performs his response on an edge that Client claimed earlier on in the game. If this is not possible, Waiter simply offers $\min\{q+1, |E(\mathcal{H}_F)|\}$ arbitrary free edges. It is clear that, by responding to each of Client's moves in this way, Waiter offers every edge of $\mathcal{H}^{\mathcal{P}}$ in the game.

Consider an arbitrary part V_j from $\mathcal{P}$. Every time Client claims an edge containing a vertex from V_j, Waiter offers at least $\lfloor (q+1)/k \rfloor$ free edges containing a vertex from V_j in the next round, except for perhaps the last time he offers edges touching V_j when there may be less than $\lfloor (q+1)/k \rfloor$ such edges available. Every time Waiter offers edges touching V_j, Client may claim at most one of these. Hence, at the end of the game,

$$d_{\mathcal{H}_C}(V_j) \leqslant \left\lceil \frac{d_{\mathcal{H}^{\mathcal{P}}}(V_j) - 1}{\lfloor (q+1)/k \rfloor} \right\rceil + 1 < \frac{2k d_{\mathcal{H}^{\mathcal{P}}}(V_j)}{q} + 2,$$

for every $j \in [n]$, by our choice of q, as stated. □

Lemma 4.2.9 *For $k \geqslant 2$, let $\mathcal{H}^{\mathcal{P}}$ denote some k–uniform hypergraph with a partition $\mathcal{P} = \{V_1, \ldots, V_n\}$ of its vertex set such that each part has the same size $\ell \in \{1, 2\}$ and contains at most one vertex from any edge in $E(\mathcal{H}^{\mathcal{P}})$. Consider a $(1 : q)$ Client–Waiter game on $E(\mathcal{H}^{\mathcal{P}})$. If $q \geqslant k\Delta_{\mathcal{P}}(\mathcal{H}^{\mathcal{P}})$, then Waiter can ensure that $\mathcal{H}_C$ is a linear forest with respect to partition $\mathcal{P}$ that contains no pair of complementary edges at the end of the game.*

Proof. Let us denote by e_i the edge claimed by Client in round i and let $V_{i_1}, \ldots, V_{i_k}$ denote the members of $\mathcal{P}$ within which e_i has at least one vertex. Waiter's strategy is as follows.

In the first round, Waiter chooses an arbitrary part V_{0_1} with non–empty intersection with at least one free edge of $\mathcal{H}^{\mathcal{P}}$. He then offers all free edges that intersect with V_{0_1}. Note that this is possible since every part of $\mathcal{P}$ has non–empty intersection with at most

$\Delta_{\mathcal{P}}(\mathcal{H}^{\mathcal{P}}) \leqslant k\Delta_{\mathcal{P}}(\mathcal{H}^{\mathcal{P}})$ edges in $\mathcal{H}^{\mathcal{P}}$, which is at most $q+1$ by our hypothesis. By doing this, Waiter ensures that $d_{\mathcal{H}_F}(V_{0_1}) = 0$ at the end of this first round. Therefore, since the edge e_1 claimed by Client in this first round has non–empty intersection with V_{0_1}, every edge complementary to e_1 is no longer free, and so cannot be claimed by Client, once round 1 is over.

In round i, for every $i \geqslant 2$, Waiter offers all free edges with non–empty intersection with the parts in $\mathcal{P}(e_{i-1})$. Again, this is possible since every part of $\mathcal{P}$ has non–empty intersection with at most $\Delta_{\mathcal{P}}(\mathcal{H}^{\mathcal{P}})$ edges in $\mathcal{H}^{\mathcal{P}}$ and therefore, all k parts in $\mathcal{P}(e_{i-1})$ intersect at most $k\Delta_{\mathcal{P}}(\mathcal{H}^{\mathcal{P}}) \leqslant q < q+1$ edges of $\mathcal{H}^{\mathcal{P}}$ in total. Offering all such edges that are free therefore ensures that $d_{\mathcal{H}_F}(V_{(i-1)_j}) = 0$ for every part $V_{(i-1)_j} \in \mathcal{P}(e_{i-1})$ and hence, no edge complementary to the edge e_i claimed by Client in round i is free once round i is complete. Additionally, this means that e_i is the only edge claimed by Client after round $i-1$ that can intersect with a part in $\mathcal{P}(e_{i-1})$. If there are no free edges intersecting the parts of $\mathcal{P}(e_{i-1})$ at the beginning of round i, Waiter proceeds as dictated for round 1.

Thus, at the end of the game $\mathcal{H}_C$ is a linear forest with respect to partition $\mathcal{P}$ that does not contain any pair of complementary edges. □

4.3 Main Proofs

In this section, we present the proofs of our results in Section 4.1.

4.3.1 The Non–r–Colourability Game $(E(K_n), \mathcal{NC}_r^{(2)})$

The Waiter–Client Non–r–Colourability Game

Proof of Theorem 1.5.5. Let n be sufficiently large.

Client's Strategy: Let $q \geqslant (8e+\alpha)n/(r\log r)$ be an integer and let ν be the function appearing in the statement of Theorem 4.2.1. Fix an arbitrarily small constant $\varepsilon > 0$ and let r_0 be the smallest integer such that $\log\log r_0 \geqslant \log 3 - \log(1-\varepsilon)$ and

$\nu((1-\varepsilon)r\log r/3) \leqslant \varepsilon$ holds for every $r \geqslant r_0$. Assume first that $r \geqslant \max\{r_0, 1000\}$. Client's strategy is based on Theorem 2.2.3. In order to present it we first consider several sums.

First, let $\mathcal{F}_1 = \{E(C_1) \cup E(C_2) : C_1$ and C_2 are distinct cycles of K_n, $|C_1|, |C_2| \in \{3,4\}$ and $V(C_1) \cap V(C_2) \neq \emptyset\}$ and note that $\mathcal{F}_1 = \{E(G) : G \in \mathcal{F}_{1,t}$ for some $t \in \{4,5,6,7\}\}$, where $\mathcal{F}_{1,t} = \{C_1 \cup C_2 : C_1$ and C_2 are distinct vertex intersecting cycles of $K_n, |V(C_1)|, |V(C_2)| \in \{3,4\}$ and $|V(C_1 \cup C_2)| = t\}$ for each $t \in \{4,5,6,7\}$. Then,

$$\begin{aligned}\Phi(\mathcal{F}_1) &= \sum_{A \in \mathcal{F}_1} (q+1)^{-|A|} = \sum_{t=4}^{7} \sum_{G \in \mathcal{F}_{1,t}} (q+1)^{-|E(G)|} \\ &\leqslant n^4 \left(\frac{r\log r}{(8e+\alpha)n}\right)^5 + 3n^5 \left(\frac{r\log r}{(8e+\alpha)n}\right)^6 + 2n^6 \left(\frac{r\log r}{(8e+\alpha)n}\right)^7 \\ &+ n^7 \left(\frac{r\log r}{(8e+\alpha)n}\right)^8 = o(1)\,. \end{aligned} \tag{4.3.1}$$

For $\mathcal{F}_2 = \{F : \exists S \subseteq V(K_n)$ such that $S \neq \emptyset,\ F \subseteq E_{K_n}(S)$ and $|F| = |S| r\log r/16\}$,

$$\begin{aligned}\Phi(\mathcal{F}_2) &= \sum_{A \in \mathcal{F}_2} (q+1)^{-|A|} \leqslant \sum_{t=1}^{n} \binom{n}{t} \binom{\binom{t}{2}}{tr\log r/16} (q+1)^{-tr\log r/16} \\ &\leqslant \sum_{t=1}^{n} \left[\frac{en}{t} \left(\frac{8et}{r\log r} \cdot \frac{r\log r}{(8e+\alpha)n}\right)^{r\log r/16}\right]^t \leqslant \sum_{t=1}^{n} \left[e\left(\frac{8e}{8e+\alpha}\right)^{r\log r/16}\right]^t \\ &< 1/3\,, \end{aligned} \tag{4.3.2}$$

where the third inequality holds since r is assumed to be sufficiently large and the last inequality holds for an appropriately chosen $\alpha = \alpha(r)$; it is not hard to see that α can be chosen such that it tends to zero as r tends to infinity.

Finally, for $\mathcal{F}_3 = \{F \cup F' : \exists S \subseteq V(K_n)$ such that $S \neq \emptyset,\ F \subseteq E_{K_n}(S)$,

$F' \subseteq E_{K_n}(S, V(K_n) \setminus S),\ |F| = |S|r/6$ and $|F'| = |S|r\log r/8\}$,

$$\begin{aligned}\Phi(\mathcal{F}_3) &= \sum_{A \in \mathcal{F}_3} (q+1)^{-|A|} \leqslant \sum_{t=1}^{n} \binom{n}{t} \binom{\binom{t}{2}}{tr/6} \binom{t(n-t)}{tr\log r/8} (q+1)^{-(tr/6 + tr\log r/8)} \\ &\leqslant \sum_{t=1}^{n} \left[\frac{en}{t} \left(\frac{3et}{r} \cdot \frac{r\log r}{(8e+\alpha)n} \right)^{r/6} \left(\frac{8e(n-t)}{r\log r} \cdot \frac{r\log r}{(8e+\alpha)n} \right)^{r\log r/8} \right]^t \\ &\leqslant \sum_{t=1}^{n} \left[e \left(\frac{3e}{8e+\alpha} \right)^{r/6} \left(\frac{8e}{8e+\alpha} \right)^{r\log r/8} (\log r)^{r/6} \right]^t < 1/3\,, \qquad (4.3.3)\end{aligned}$$

where the third inequality holds since r is assumed to be sufficiently large and the last inequality holds for an appropriately chosen $\alpha = \alpha(r)$; it is not hard to see that α can be chosen such that it tends to zero as r tends to infinity.

Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$. Combining (4.3.1), (4.3.2) and (4.3.3), it follows from Theorem 2.2.3 that Client has a strategy to build a graph G_C which satisfies the following three properties:

(a) If C_1 and C_2 are cycles of length at most 4 in G_C, then $V(C_1) \cap V(C_2) = \emptyset$.

(b) $e_{G_C}(S) \leqslant |S|r\log r/16$ for every $S \subseteq V(K_n)$.

(c) For every $S \subseteq V(K_n)$, if $e_{G_C}(S) \geqslant |S|r/6$, then $e_{G_C}(S, V(K_n) \setminus S) < |S|r\log r/8$.

It remains to prove that a graph which satisfies Properties (a), (b) and (c), has chromatic number at most r. Let $X = \{u \in V(K_n) : d_{G_C}(u) \leqslant (1-\varepsilon)r\log r/3\}$ and let $Y = V(K_n) \setminus X$. Let $X_1 \cup X_2$ be a partition of X such that both $G_C[X_1]$ and $G_C[X_2]$ have girth at least 5; such a partition exists by Property (a). Clearly $\Delta(G_C[X_i]) \leqslant \Delta(G_C[X]) \leqslant (1-\varepsilon)r\log r/3$ holds for $i \in \{1,2\}$ by the definition of X.

Since $r \geqslant r_0$, using Theorem 4.2.1 we infer that

$$\begin{aligned}\chi(G_C[X_i]) &\leqslant (1+\nu((1-\varepsilon)r\log r/3))\cdot\frac{(1-\varepsilon)r\log r/3}{\log((1-\varepsilon)r\log k/3)}\\ &\leqslant (1+\varepsilon)\cdot\frac{(1-\varepsilon)r\log r/3}{\log((1-\varepsilon)r\log r/3)}\leqslant r/3\,,\end{aligned}$$

holds for $i \in \{1,2\}$. Hence, $\chi(G_C[X]) \leqslant \chi(G_C[X_1]) + \chi(G_C[X_2]) \leqslant 2r/3$.

Suppose for a contradiction that $\chi(G_C) \geqslant r+1$. Since

$$\chi(G_C) \leqslant \chi(G_C[X]) + \chi(G_C[Y]) \leqslant 2r/3 + \chi(G_C[Y]),$$

it follows that $\chi(G_C[Y]) \geqslant r/3+1$. Therefore, there exists a set $Z \subseteq Y$ such that $\delta(G_C[Z]) \geqslant r/3$, entailing $e_{G_C}(Z) \geqslant |Z|r/6$. It follows by Property (b) that $e_{G_C}(Z) \leqslant |Z|r\log r/16$. We then have $e_{G_C}(Z, V(K_n)\setminus Z) \geqslant |Z|r\log r/8$, by the definition of Y. However, this contradicts Property (c). We conclude that $\chi(G_C) \leqslant r$ as claimed.

Assume then that $2 \leqslant r < \max\{r_0, 1000\}$. For $\alpha = \alpha(r)$ sufficiently large on this range of r, $q \geqslant (8e+\alpha)/(r\log r) \geqslant (2e)^{2/r+1}\binom{n}{2}/n$. Thus, by Lemma 4.2.7, Client has a strategy to build a graph G_C such that every subgraph of G_C admits a vertex of degree at most $r-1$. Hence, $\chi(G_C) \leqslant 1 + \max_{G'\subseteq G}\delta(G') \leqslant r$.

Waiter's Strategy: Next, assume that $q \leqslant cn/(r\log r)$, where $c \leqslant \log 2/4 - \alpha$. Since

$$\frac{cn}{r\log r} \leqslant \binom{\lceil n/r\rceil}{k}\frac{\log 2}{2\left(\log 2 + \log\binom{n}{\lceil n/r\rceil}\right)},$$

for sufficiently large n and by our choice of c, Corollary 4.2.6 gives Waiter a strategy to ensure that $\chi(\mathcal{H}_C) > r$ at the end of the game. □

The Client–Waiter Non–r–Colourability Game

Proof of Theorem 1.5.6. Let n be sufficiently large.

Waiter's Strategy: Let $q \geqslant (4+\alpha)n/(r\log r)$ be an integer and let ν be the function appearing in the statement of Theorem 4.2.1. Fix an arbitrarily small constant $\varepsilon > 0$ and let r_0 be the smallest integer such that $\log\log r_0 \geqslant \log 2 - \log(1-\varepsilon)$ and $\nu((1-\varepsilon)r\log r/2) \leqslant \varepsilon$ holds for every $r \geqslant r_0$. Assume first that $r \geqslant r_0$. We present a strategy for Waiter; it is divided into the following two stages:

Stage I: Waiter forces Client to build a graph H_1 of maximum degree at most $(1-\varepsilon)r\log r/2$ and girth at least 5 such that $d_{G_F}(u) \leqslant (r\log r)^3$ holds for every $u \in V(K_n)$ at the end of this stage.

Stage II: Waiter forces Client to build a linear forest $H_2 := G_C \setminus H_1$.

We will prove that Waiter can indeed follow the proposed strategy. First, we introduce some notation and terminology. An edge $e \in E(G_F)$ is called *dangerous* if adding it to G_C creates a cycle of length 3 or 4. Note that once an edge becomes dangerous, it remains dangerous for as long as it is free. At any point during Stage I, we will denote the set of dangerous edges by D *i.e.* once an edge becomes dangerous in Stage I, it is immediately added to D.

With partition $\mathcal{P} = \{\{v\} : v \in V(K_n)\}$ and $k = 2$, Lemma 4.2.8 tells us that there exists a strategy $\mathcal{S}$ for Waiter to ensure that $d_{G_C}(u) < 4(n-1)/q + 2$ for every vertex $u \in V(K_n)$ at the end of the game. In Stage I, Waiter follows $\mathcal{S}$ except, whenever $\mathcal{S}$ instructs him to offer a dangerous edge, he only *imagines* that he does so. Therefore, if the edges that $\mathcal{S}$ tells Waiter to offer in some round are all dangerous, Waiter just imagines that it occurs and chooses an arbitrary edge within the instructed offering to represent what Client would have claimed in that round. Note that, since Waiter can offer less than $q+1$ edges per round in a Client–Waiter game, this is a valid strategy for Waiter. Once all free edges are dangerous, Stage I ends and Waiter proceeds to Stage II.

Since Waiter never offers Client any dangerous edges in Stage I, it is evident that Client's graph will have girth at least 5 at the end of the stage. Also, despite only carrying out the will of strategy $\mathcal{S}$ when instructed to offer non–dangerous edges, Waiter proceeds with $\mathcal{S}$ as if he followed its instructions exactly. Therefore,

$$d_{H_1}(u) < \frac{4(n-1)}{q} + 2 \leqslant (1-\varepsilon)\frac{r\log r}{2},$$

still holds for each $u \in V(K_n)$ at the end of Stage I, where the last inequality holds if α is chosen to be sufficiently large compared to ε. Thus, $\Delta(H_1) \leqslant (1-\varepsilon)r\log r/2$ holds at the end of Stage I.

At the end of Stage I, fix some vertex $u \in V(K_n)$ and let $v \in V(K_n)$ be such that $uv \in D$. It follows that there exists a vertex $z \in V(K_n)$ such that $uz, zv \in E(H_1)$ or vertices $x, y \in V(K_n)$ such that $ux, xy, yv \in E(H_1)$. That is, there is a path of length 2 or 3 between u and v in H_1. Since the number of paths of length t in H_1, starting at u, is at most $\Delta(H_1)^t$, we conclude that $|\{e \in D : u \in e\}| \leqslant (r\log r/2)^2 + (r\log r/2)^3 \leqslant (r\log r)^3$ holds for every $u \in V(K_n)$ as claimed.

In Stage II, Waiter follows the strategy whose existence is given by Lemma 4.2.9 on graph G_F with partition $\mathcal{P}$ and integer k as defined previously. Since

$$k\Delta_{\mathcal{P}}(G_F) = 2\Delta(G_F) \leqslant 2(r\log r)^3 \leqslant q$$

at the beginning of this stage, Waiter therefore ensures that the graph H_2 built by Client in Stage II is a linear forest with respect to $\mathcal{P}$.

It remains to prove that, by following the proposed strategy, Waiter forces Client's graph to be r–colourable. It follows from Theorem 4.2.1 that

$$\chi(H_1) \leqslant (1+\nu((1-\varepsilon)r\log r/2))\cdot\frac{(1-\varepsilon)r\log r/2}{\log((1-\varepsilon)r\log r/2)} \leqslant (1+\varepsilon)\cdot\frac{(1-\varepsilon)r\log r/2}{\log((1-\varepsilon)r\log r/2)} \leqslant r/2,$$

where the second and third inequalities follow by our choice of r_0. Moreover, it is evident that $\chi(H_2) \leqslant 2$. We conclude that $\chi(G_C) = \chi(H_1 \cup H_2) \leqslant \chi(H_1)\chi(H_2) \leqslant r$.

Assume then that $3 \leqslant r < r_0$. By following the strategy given by Lemma 4.2.8 and using the same partition $\mathcal{P}$ as above, Waiter can ensure that

$$d_{G_C}(u) < \frac{4(n-1)}{q} + 2 \leqslant \frac{4}{4+\alpha} r\log r + 2 < \frac{4}{4+\alpha} e^{2/(1-\varepsilon)}\frac{2}{1-\varepsilon} + 2 \leqslant 3 \leqslant r,$$

for sufficiently large $\alpha = \alpha(r)$. Thus, $\chi(G_C) \leqslant 1 + \max_{G' \subseteq G} \delta(G') \leqslant r$.

When $r = 2$, Waiter follows the strategy given by Lemma 4.2.9 to ensure that, at the end of the game, G_C is a linear forest with respect to partition $\mathcal{P}$ described previously. This is possible since, for sufficiently large α, $q \geqslant 2(n-1) = k\Delta_{\mathcal{P}}(K_n)$. Thus, $\chi(G_C) \leqslant 2$.

Client's Strategy: Next, assume that $q \leqslant cn/(r\log r)$, where $c \leqslant \log 2/2 - \alpha$. Since n is sufficiently large and by our choice of c,

$$\frac{cn}{r\log r} \leqslant \binom{\lceil n/r\rceil}{2}\frac{\log 2}{\log\binom{n}{\lceil n/r\rceil}}.$$

Therefore, Corollary 4.2.6 provides a strategy for Client to ensure that $\chi(G_C) > r$ at the end of the game. □

4.3.2 The Non–Bipartite Game $(E(K_n), \mathcal{NC}_2^{(2)})$

The Waiter–Client Non–Bipartite Game

Proof of Theorem 4.1.1. Let n be sufficiently large.

Waiter's Strategy: Let $\varepsilon \geqslant \sqrt{3/n}$ and $q \leqslant (1/2 - \varepsilon)n$. It suffices to show that Waiter can force Client to build an odd cycle.

He first equipartitions $V(K_n)$ into two parts A and B and forces Client to build a path on at least εn vertices by offering only edges from $E_{K_n}(A, B)$. His strategy for this is very similar to the proof of Claim 3.2.1. Waiter first picks an arbitrary vertex $v_1 \in A$, sets $P_1 = v_1$ and offers $q+1$ free edges of $E_{K_n}(A, B)$ with v_1 as an endpoint. By claiming one of these edges, say $v_1 v_2$, Client creates a path $P_2 = v_1 v_2$. Waiter continues in a similar way, responding to the creation of path $P_i = v_1 \dots v_i$ in Client's graph, for a positive integer i, by offering $q+1$ arbitrary free edges of $E_{K_n}(A, B) \setminus E(P_i)$ that all contain v_i as an endpoint. By claiming any one of these edges, Client extends P_i to a path $P_{i+1} = v_1 v_2 \dots v_i v_{i+1}$. Once Waiter can no longer offer in this way, we must have $(n-i)/2 - 1 \leqslant \min\{|A \setminus V(P_i)|, |B \setminus V(P_i)|\} < q+1$ which entails $i > n - 2(q+2) \geqslant \varepsilon n$. Let us call Client's final path P and let $V_1 = A \cap V(P)$ and $V_2 = B \cap V(P)$.

Note that, once Client's path is complete, all edges of $E_{K_n}(V_1) \cup E_{K_n}(V_2)$ are still free. Also note that, since P is a connected bipartite graph, every edge in $E_{K_n}(V_1) \cup E_{K_n}(V_2)$ closes an odd cycle. Since $E_{K_n}(V_1) \cup E_{K_n}(V_2)$ has size

$$\binom{|V_1|}{2} + \binom{|V_2|}{2} \geqslant \binom{\lfloor \varepsilon n/2 \rfloor}{2} + \binom{\lceil \varepsilon n/2 \rceil}{2} \geqslant \varepsilon^2 n^2/5 \geqslant q+1,$$

for sufficiently large n, where the final inequality follows from our choice of ε and q, Waiter can force Client to claim a member of $E_{K_n}(V_1) \cup E_{K_n}(V_2)$ in the round following the completion of P by simply offering $q+1$ edges within this set. This move forces Client to close an odd cycle in G_C and Waiter plays arbitrarily in subsequent rounds.

Client's Strategy: Let $q \geqslant n + \alpha$ where $\alpha \geqslant (1 - \tanh(2))n/\tanh(2)$. It suffices to show that Client can avoid building an odd cycle for the duration of the game. So let $X = E(K_n)$ and $\mathcal{F} = \{E(C) : C \text{ is an odd cycle in } K_n\}$. Observe that

$$\begin{aligned}\Phi(\mathcal{F}) &= \sum_{A \in \mathcal{F}} (q+1)^{-|A|} = \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} \frac{(2k)!}{2} (q+1)^{-(2k+1)} \leqslant \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{n^{2k+1}}{2(2k+1)(q+1)^{2k+1}} \\ &< \frac{1}{2} \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{2k+1} \left(\frac{n}{q}\right)^{2k+1} < \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{2k+1} \left(\frac{n}{n+\alpha}\right)^{2k+1} = \frac{1}{2} \tanh^{-1}\left(\frac{n}{n+\alpha}\right) \\ &\leqslant \frac{1}{2} \tanh^{-1}\left(\frac{n}{n/\tanh(2)}\right) = 1,\end{aligned}$$

where the penultimate equality follows from the Taylor expansion $\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1} = \tanh^{-1}(x)$ for $|x| < 1$. So by Theorem 2.2.3 from Chapter 1, Client has a strategy to avoid building an odd cycle as required. □

The Client–Waiter Non–Bipartite Game

Proof of Theorem 4.1.2. Let n be sufficiently large.

Waiter's Strategy: When $q \geqslant \lceil n/2 \rceil - 1$, Waiter simply follows the strategy described in the proof of Theorem 1.5.2 to ensure that Client's graph is acyclic and therefore bipartite.

Client's Strategy: Suppose that $q \leqslant (1/4 - o(1))n$ and let

$$\mathcal{F} = \{E(K_s) \cup E(K_{n-s}) : K_s, K_{n-s} \subseteq K_n,\ V(K_s) \cap V(K_{n-s}) = \emptyset\}.$$

If Client fails to build an odd cycle by the end of the game, he must end with a bipartite graph. Since the vertex classes of this graph must be independent sets in Client's graph, they must appear as disjoint cliques covering all vertices of K_n in Waiter's graph. Hence, if Client can prevent Waiter from fully claiming any member of $\mathcal{F}$, Client's graph cannot be bipartite at the end of the game. Observe that,

$$\sum_{A\in\mathcal{F}}\left(\frac{q}{q+1}\right)^{|A|} \leqslant \sum_{A\in\mathcal{F}} 2^{-|A|/q} = \sum_{i=1}^{\lfloor n/2\rfloor}\binom{n}{i}\cdot 2^{-\left(\binom{n}{2}-i(n-i)\right)/q}$$
$$\leqslant \frac{n}{2}\cdot\binom{n}{n/2}\cdot 2^{-\left(\binom{n}{2}-n^2/4\right)/((1/4-o(1))n)} \leqslant \frac{n}{2}\cdot 2^n\cdot 2^{-(1+o(1))n} < 1,$$

for sufficiently large n. Hence, by Theorem 2.2.6, Client can prevent Waiter from claiming a member of $\mathcal{F}$, thereby enabling Client to build an odd cycle. □

4.3.3 The Non–r–Colourability Game $(E(K_n^{(k)}), \mathcal{NC}_r^{(k)})$

The Waiter–Client Non–r–Colourability Game

Proof of Theorem 1.5.7. Fix $k, r \geqslant 2$ and let n be sufficiently large.

Waiter's Strategy: Since

$$q \leqslant \binom{\lceil n/2\rceil}{k}\frac{\log 2}{2((1+\log 2)n+\log 2)} \leqslant \binom{\lceil n/2\rceil}{k}\frac{\log 2}{2(\log 2+\log\binom{n}{\lceil n/2\rceil})},$$

Waiter's strategy follows immediately from Corollary 4.2.6.

Client's Strategy: Suppose that $q \geqslant 2^{k/r}e^{k/r+1}k\binom{n}{k}/n$. Then, by choosing partition $\mathcal{P} = \{\{v\} : v \in V(K_n^{(k)})\}$ and $\ell = 1$ in Lemma 4.2.7, Client can ensure that, for every $S \subseteq V(\mathcal{H}_C)$, there exists a vertex $v \in S$ with $d_{\mathcal{H}_C[S]}(v) \leqslant r-1$ at the end of the game. Thus, $\chi(\mathcal{H}_C) \leqslant 1 + \max_{\mathcal{H}'\subseteq\mathcal{H}_C}\delta(\mathcal{H}') \leqslant r$. □

The Client–Waiter Non–r–Colourability Game

Proof of Theorem 1.5.8. Fix $k, r \geqslant 2$ and let n be sufficiently large.

Client's Strategy: Client's strategy follows immediately from Corollary 4.2.6.

Waiter's Strategy: Let $k_0 = \min\{k : (r-1)r^{k-4}/k \geqslant 2\}$ and suppose that $q \geqslant k^3 r^{-k+5}\binom{n}{k}/n$. We first consider the case $k \geqslant k_0$. By Lemma 4.2.3, it suffices for Waiter to ensure that $\Delta(\mathcal{H}_C) \leqslant r^{k-3}/k$ at the end of the game. Indeed, with partition $\mathcal{P} = \{\{v\} : v \in V(K_n^{(k)})\}$, Lemma 4.2.8 tells us that Waiter can ensure that

$$d_{\mathcal{H}_C}(u) < \frac{2k\binom{n-1}{k-1}}{q} + 2 \leqslant \frac{r^{k-3}}{k},$$

for each $u \in V(K_n^{(k)})$, where the final inequality follows from our choice of k and q.

Now consider the case $2 \leqslant k < k_0$. When $r \geqslant 3$, Waiter plays as in the previous case, but this time his strategy from Lemma 4.2.8 ensures that

$$d_{\mathcal{H}_C}(u) < \frac{2k\binom{n-1}{k-1}}{q} + 2 \leqslant \frac{2r^{k-5}}{k} + 1 < \frac{4}{r(r-1)} + 2 \leqslant 3 \leqslant r,$$

for each $u \in V(K_n^{(k)})$, where the second inequality follows from our choice of q, the third inequality follows from our choice of k and the penultimate and final inequalities follow from $r \geqslant 3$. Thus, $\Delta(\mathcal{H}_C) \leqslant r-1$ and we readily obtain that $\chi(\mathcal{H}_C) \leqslant r$.

Now suppose that $r = 2$. Using the same partition $\mathcal{P}$ as above and noting that

$$q \geqslant k^3 2^{-k+5}\binom{n}{k}\frac{1}{n} \geqslant k\Delta_{\mathcal{P}}(K_n^{(k)})$$

for our choice of q and k in this case, Lemma 4.2.9 gives Waiter a strategy to ensure that $\mathcal{H}_C$ is a linear forest with respect to $\mathcal{P}$ at the end of the game. In particular, $\delta(\mathcal{H}') \leqslant 1$ for each $\mathcal{H}' \subseteq \mathcal{H}_C$ and thus, $\chi(\mathcal{H}_C) \leqslant 1 + \max_{\mathcal{H}' \subseteq \mathcal{H}} \delta(\mathcal{H}') \leqslant 2$. □

4.3.4 The k–SAT Game $(\mathcal{C}_n^{(k)}, \mathcal{F}_{SAT})$

For the sake of clarity, in both of the following proofs we analyse a game that is analogous to the k–SAT game; namely $(E(K_{2n}^{(k)}), \mathcal{F}'_{SAT})$. In this, each vertex of $V(K_{2n}^{(k)})$ is labeled with a unique literal in $\cup_{i\in[n]}\{x_i, \neg x_i\}$ over the set $\{x_i : i \in [n]\}$ of boolean variables and $\mathcal{P}_{SAT} = \{B_i : i \in [n]\}$ denotes a fixed partition of $V(K_{2n}^{(k)})$, where $B_i = \{x_i, \neg x_i\}$ for each $i \in [n]$. An edge containing a pair of vertices that lie within the same part B_i will be referred to as a *satisfied edge* and any edge that is not satisfied will be called *unsatisfied*. The winning sets are defined by

$$\mathcal{F}'_{SAT} = \{F \subseteq E(K_{2n}^{(k)}) : \wedge\{\vee e : e \in F\} \text{ is } \textit{not} \text{ satisfiable}\}.$$

We will use the following corollary of Lemma 4.2.5 in our proofs.

Corollary 4.3.1 *In a $(1:q)$ game on $E(K_{2n}^{(k)})$, a strategy to ensure that $E(\mathcal{H}_C) \notin \mathcal{F}'_{SAT}$ at the end of the game exists for Waiter in the Waiter–Client version if $q \leqslant \binom{n}{k}/(2(n+1))$ and for Client in the Client–Waiter version if $q < \binom{n}{k}/n$.*

Proof. In the statement of Lemma 4.2.5, replace n with $2n$, set $r = 2$ and let

$$S = \{\mathcal{H} \subseteq K_{2n}^{(k)} : \mathcal{H} \text{ is an } n\text{–clique without a satisfied edge}\}.$$

Note that, therefore, $|S| = 2^n$. Then, Lemma 4.2.5 tells us that there exist strategies $\mathcal{S}_W$ and $\mathcal{S}_C$, for Waiter in the $(1:q)$ Waiter–Client game when $q \leqslant \binom{n}{k}/(2(n+1))$ and for Client in the $(1:q)$ Client–Waiter game when $q < \binom{n}{k}/n$ respectively, to ensure that $\mathcal{H}_C$ contains an edge in every member of S at the end of the game. We will show this means that $E(\mathcal{H}_C) \notin \mathcal{F}'_{SAT}$.

We first claim that every $\{0,1\}$–assignment to the boolean variables $x_1, \ldots, x_n$ defines a 2–colouring of $V(K_{2n}^{(k)})$ where each colour class contains a member of S. Indeed, by definition of negation, if some boolean variable x_i is assigned a value $z \in \{0,1\}$, the value of its negation $\neg x_i$ must be the sole member of $\{0,1\} \setminus \{z\}$. Therefore, every $\{0,1\}$–assignment π to the boolean variables partitions the set $\cup_{i\in[n]}\{x_i, \neg x_i\}$ of literals into two parts, each of size n, where every pair of literals in the same part have the same value under the given assignment π. Additionally, no part contains a variable and its negation. Since each vertex of our board $V(K_{2n}^{(k)})$ is labeled with a unique literal from $\cup_{i\in[n]}\{x_i, \neg x_i\}$, we can translate any $\{0,1\}$–assignment π to a $\{0,1\}$–colouring of the vertices in $V(K_{2n}^{(k)})$ by giving each vertex v the colour that matches the value given by π to the literal that labels v. The partition of the set of literals that is produced by π therefore translates in this way to a 2–colouring in which each colour class has size n and no colour class contains two vertices with labels x_i and $\neg x_i$ for some $i \in [n]$. In particular, this means that no colour class contains a satisfied edge and therefore, all n vertices in a single colour class form an n–clique without a satisfied edge in $V(K_{2n}^{(k)})$ *i.e.* a member of S.

Hence, if strategies $\mathcal{S}_W$ and $\mathcal{S}_C$ ensure that Client has an edge in every member of S by the end of the game, the translation of any $\{0,1\}$–assignment π of the boolean variables to the vertices of Client's hypergraph $\mathcal{H}_C$ will produce a 2–colouring in which at least one edge of Client's is monochromatic in colour 0. This then translates to a 0–valued k–clause in the boolean formula $\phi := \wedge\{\vee e : e \in E(\mathcal{H}_C)\}$, which causes ϕ to evaluate to 0 under assignment π. Since this is true for all assignments π, the k–CNF boolean formula corresponding to Client's edges at the end of the game cannot be satisfiable. Hence, the game ends with $E(\mathcal{H}_C) \notin \mathcal{F}'_{SAT}$. □

The Waiter–Client k–SAT Game

Proof of Theorem 1.5.9. Fix $k \geqslant 2$ and let n be sufficiently large.

Waiter's Strategy: Waiter's strategy follows immediately from Corollary 4.3.1.

Client's Strategy: Let $q \geqslant 2^{3k/2}e^{k/2+1}k\binom{n}{k}/n$. Also, in the statement of Lemma 4.2.7, let $r = \ell = 2$ and $\mathcal{P} = \mathcal{P}_{SAT}$ with $\mathcal{H}^{\mathcal{P}}$ denoting the subhypergraph of $K_{2n}^{(k)}$ with vertex set $V(K_{2n}^{(k)})$ and edge–set consisting of all edges in $E(K_{2n}^{(k)})$ that are unsatisfied. Note that, since all edges of $\mathcal{H}^{\mathcal{P}}$ are unsatisfied, each part $B \in \mathcal{P}_{SAT}$ contains at most one vertex from any edge in $E(\mathcal{H}^{\mathcal{P}})$ as required. Thus, due to our lower bound on q, Lemma 4.2.7 tells us that there exists a strategy $\mathcal{S}_C$ for Client to ensure that, for every $S \subseteq \mathcal{P}_{SAT}$, there exists some $B \in S$ such that $d_{\mathcal{H}_C[\cup S]}(B) \leqslant 1$ when playing a $(1 : q)$ Waiter–Client game on $E(\mathcal{H}^{\mathcal{P}})$. In particular, since an edge, once claimed, is no longer free for the remaining duration of the game, this is true at any point in the game. We claim that Client can use strategy $\mathcal{S}_C$ to ensure that this result still holds when playing on $E(K_{2n}^{(k)})$.

Indeed, Client achieves this by playing as follows. In every round where Waiter offers entirely within $E(\mathcal{H}^{\mathcal{P}})$, Client follows $\mathcal{S}_C$. In any other round, Client claims an arbitrary satisfied edge. Recall that $E(\mathcal{H}^{\mathcal{P}})$ consists entirely of the unsatisfied edges of $K_{2n}^{(k)}$ and thus, any offering that does not lie completely within $E(\mathcal{H}^{\mathcal{P}})$ must contain a satisfied edge.

Since Client follows $\mathcal{S}_C$ whenever Waiter plays entirely within $E(\mathcal{H}^{\mathcal{P}})$, and the edges of $E(\mathcal{H}^{\mathcal{P}})$ that Waiter claims in rounds where he offers some satisfied edges are still deemed free from the perspective of $\mathcal{S}_C$, the goal of $\mathcal{S}_C$ is still realised at the end of the game *i.e.* for every $S \subseteq \mathcal{P}_{SAT}$, there exists some $B \in S$ such that $d_{\mathcal{H}_C[\cup S]}(B) \leqslant 1$. As far as $\mathcal{S}_C$ is concerned, Waiter and Client simply stop playing the game on $E(\mathcal{H}^{\mathcal{P}})$ sooner than expected. Consequently, at the end of the game, there exists an ordering $B_{i_1}, \ldots, B_{i_n}$ of the elements of $\mathcal{P}_{SAT}$ such that, for each $j \in [n]$, Client has at most one unsatisfied edge contained in $\cup\{B_{i_t} : t \leqslant j\}$ with a vertex $v_{i_j} \in B_{i_j}$. Assigning the value, 0 or 1, to the variable x_{i_j} such that the literal labelling v_{i_j} has value 1 for every $j \in [n]$, and assigning arbitrary values to any remaining variables, provides a satisfying assignment for the formula corresponding to the edges in $E(\mathcal{H}_C) \cap E(\mathcal{H}^{\mathcal{P}})$ at the end of the game.

Since a k–clause corresponding to a satisfied edge of $K_{2n}^{(k)}$ is satisfiable under every $\{0,1\}$–assignment to the boolean variables x_i, its conjunction with any k–CNF boolean formula ϕ does not affect the satisfiability of ϕ. Hence, the boolean formula corresponding to all edges in $E(\mathcal{H}_C)$ at the end of the game is also satisfiable. □

The Client–Waiter k–SAT Game

Proof of Theorem 1.5.10. Fix $k \geqslant 2$ and let n be sufficiently large.

Client's Strategy: Client's strategy follows immediately from Corollary 4.3.1.

Waiter's Strategy: Let $q \geqslant 2^9 k^3 \binom{n}{k}/n$. Waiter's strategy consists of two stages. In Stage 1, Waiter only offers satisfied edges until no more are left, at which point Stage 2 begins. We denote Client's hypergraph built during Stage 1 and Stage 2 by $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Stage 2 depends on the following two cases.

We first consider the case $k \geqslant 10$. In the statement of Lemma 4.2.8, let $\ell = 2$ and $\mathcal{P} = \mathcal{P}_{SAT}$ with $\mathcal{H}^{\mathcal{P}}$ denoting the subhypergraph of $K_{2n}^{(k)}$ with vertex set $V(K_{2n}^{(k)})$ and edge–set consisting of all edges in $E(K_{2n}^{(k)})$ that are unsatisfied. Note that, since all edges of $\mathcal{H}^{\mathcal{P}}$ are unsatisfied, each part $B \in \mathcal{P}_{SAT}$ contains at most one vertex from any edge in $E(\mathcal{H}^{\mathcal{P}})$ as required. Additionally, since Waiter only offers satisfied edges in Stage 1, all edges in $E(\mathcal{H}^{\mathcal{P}})$ are free at the beginning of Stage 2. Lemma 4.2.8 tells us that Waiter can ensure that

$$d_{\mathcal{H}_2}(B_i) < \frac{2^{k+1} k \binom{n-1}{k-1}}{q} + 2 \leqslant \frac{2^{k-4}}{k},$$

for each $i \in [n]$, at the end of the game, where the final inequality follows from our choice of k and q. Hence, Waiter can ensure that every $B_i \in \mathcal{P}_{SAT}$ has non–empty intersection with at most $2^{k-4}/k$ edges in $\mathcal{H}_2$ at the end of the game. By Lemma 4.2.4, this means that $E(\mathcal{H}_2) \in \mathcal{F}'_{SAT}$. Thus, in Stage 2, Waiter follows the strategy given by Lemma 4.2.8.

Now consider the case $2 \leqslant k \leqslant 9$. By using partition $\mathcal{P}$, ℓ and $\mathcal{H}^{\mathcal{P}}$ as before, and noting that $q \geqslant 2^9 k^3 \binom{n}{k}/n \geqslant k\Delta_{\mathcal{P}_{SAT}}(K_{2n}^{(k)})$ for our choice of q and k, Lemma 4.2.9 gives Waiter a strategy to ensure that $\mathcal{H}_2$ is a linear forest with respect to $\mathcal{P}_{SAT}$ that does not contain any pair of complementary edges at the end of the game. Hence, there exists an ordering $e_1, \ldots, e_m$ of the edges in $E(\mathcal{H}_2)$ such that $\mathcal{P}_{SAT}(e_i) \cap \mathcal{P}_{SAT}(e_j) \neq \emptyset$ only when $j \in \{i-1, i, i+1\}$.

A $\{0,1\}$–assignment to the boolean variables $x_1, \ldots, x_n$ that satisfies $\wedge\{\vee e : e \in E(\mathcal{H}_2)\}$ is as follows. First, assign every literal labelling a vertex in e_1 value 1. Since $\mathcal{H}_2$ contains no complementary edges, $\mathcal{P}_{SAT}(e_{i+1}) \setminus \mathcal{P}_{SAT}(e_i) \neq \emptyset$ for every $1 \leqslant i \leqslant m-1$. For every part $V_{(i+1)_j} \in \mathcal{P}_{SAT}(e_{i+1}) \setminus \mathcal{P}_{SAT}(e_i)$, assign the literal labelling the vertex in $e_{i+1} \cap V_{(i+1)_j}$ value 1. Assign values from $\{0,1\}$ arbitrarily to those variables x_i for which B_i is untouched by $\mathcal{H}_2$. The only way that this assignment can produce an edge of $\mathcal{H}_2$ whose vertices are only labelled by literals of value 0 is if this edge is complementary to another edge of $\mathcal{H}_2$. Since this is not possible, we conclude that our assignment satisfies $\wedge\{\vee e : e \in E(\mathcal{H}_2)\}$ as claimed.

Since any k–clause corresponding to a satisfied edge of $K_{2n}^{(k)}$ is satisfiable under every $\{0,1\}$–assignment to the boolean variables x_i, the conjunction of the formula corresponding to $\mathcal{H}_1$ with that corresponding to $\mathcal{H}_2$ is also satisfiable in both of our considered cases. Hence, Waiter can ensure that $E(\mathcal{H}_C) = E(\mathcal{H}_1) \cup E(\mathcal{H}_2) \notin \mathcal{F}_{SAT}$ at the end of the game.

□

Chapter 5

Hamiltonicity Games on a Random Graph

5.1 Results

As mentioned in Chapter 1, the bias of a positional game is not the only parameter we can vary. We may also vary the board itself in a process known as *thinning the board*. This leads us to consider games played on the edge–set of the binomial random graph $\mathcal{G}(n, p)$. In particular, this chapter focuses on the $(1 : q)$ Waiter–Client and Client–Waiter Hamiltonicity games $(E(\mathcal{G}(n, p)), \mathcal{HAM})$, where

$$\mathcal{HAM} = \{E(G) : G \subseteq K_n \text{ is Hamiltonian}\}.$$

We determine the minimum density that a graph typically needs to ensure that *a.a.s.* its edge–set is a board on which Waiter/Client wins the Waiter–Client/Client–Waiter version of this game. More precisely, for every fixed positive integer q, we find *sharp* thresholds for the graph properties $\mathcal{W}^q_{\mathcal{HAM}}$ and $\mathcal{C}^q_{\mathcal{HAM}}$ (see Section 1.5.5 in Chapter 1). Theorem 1.5.11, restated below for convenience, shows that, for every fixed positive integer q, a sharp threshold for $\mathcal{W}^q_{\mathcal{HAM}}$ coincides with a sharp threshold for the appearance of a

Hamilton cycle in $\mathcal{G}(n,p)$ [73, 24].

Theorem 1.5.11 ([68]) *Let q be a positive integer. Then $\log n/n$ is a sharp threshold for the property $\mathcal{W}^q_{\mathcal{HAM}}$.*

Proof Overview.

Client's Strategy: If $p \leqslant (1-o(1))\log n/n$, then $\mathcal{G}(n,p)$ *a.a.s.* has minimum degree at most 1 (this is a standard result in random graphs, see *e.g.* [25]) and therefore does not contain a Hamilton cycle. Hence, no matter what strategy Client follows, he will not have a Hamilton cycle at the end of the game.

Waiter's Strategy: When $p \geqslant (1+o(1))\log n/n$, Waiter can *a.a.s.* force Client to build a graph with minimum degree large enough so that G_C is a connected *expander* (see Section 5.2.1). Then, by offering only *boosters* (see Section 5.2.1), Waiter forces Client to build a Hamilton cycle. □

In contrast to the Waiter–Client game, the sharp threshold for the property $\mathcal{C}^q_{\mathcal{HAM}}$ grows with q, and even for $q = 1$, is already larger than the threshold for the Hamiltonicity of $\mathcal{G}(n,p)$. This is shown by Theorem 1.5.12, restated here for convenience.

Theorem 1.5.12 ([68]) *Let q be a positive integer. Then $(q+1)\log n/n$ is a sharp threshold for the property $\mathcal{C}^q_{\mathcal{HAM}}$.*

Proof Overview.

Client's Strategy: When $p \geqslant (q+1+o(1))\log n/n$, Client builds a graph that satisfies the conditions of Theorem 5.2.4 (see Section 5.2.1), using tools from Chapter 2, to obtain a Hamiltonian graph.

Waiter's Strategy: When $p \leqslant (q + 1 - o(1)) \log n/n$, Waiter identifies an independent set I_k consisting of vertices of degree k in $\mathcal{G}(n, p)$, for some positive integer k. He then plays the box game (see Section 5.2.3) on the edges of $\mathcal{G}(n, p)$ that touch I_k to isolate a vertex in Client's graph. This therefore prevents Client from building a Hamilton cycle. □

We present our proofs of Theorems 1.5.11 and 1.5.12 in Section 5.3.

5.2 Useful Tools

5.2.1 Building a Hamilton Cycle

Given a connected, non–Hamiltonian graph, how do we add edges to create a Hamilton cycle? A popular method of choice hinges on the so–called *rotation–extension technique* which was first developed by Pósa in 1976 to aid his investigation into the Hamiltonicity of the random graph [82]. This involves repeatedly deforming a given path in a graph, with the use of extra edges, to either increase its length by appending a new vertex, or close it to make a Hamilton cycle.

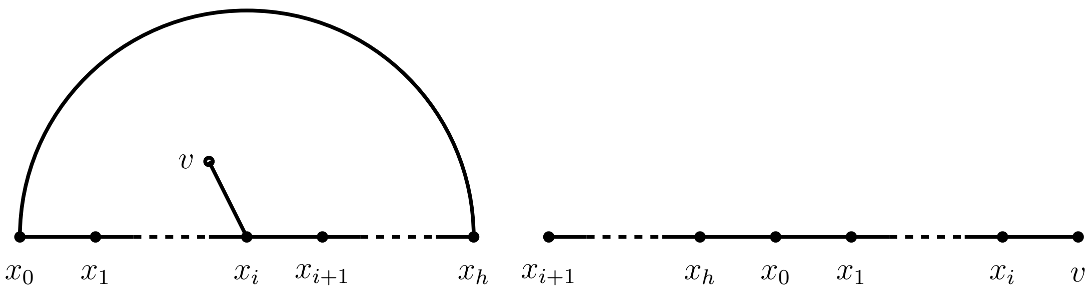

Figure 5.1: Before and after the extension of path $x_0x_1 \ldots x_h$.

We proceed as follows. Suppose we have a connected, non–Hamiltonian n–vertex graph G. We want to add edges from $E(K_n) \setminus E(G)$ to $E(G)$ to make a Hamiltonian graph. First, we take a longest path $P = x_0x_1 \ldots x_h$ in G. Let us assume that $x_0x_h \notin E(G)$; we will soon see that our choice of P and G guarantees this. Then we add the edge x_0x_h to G to create a new graph G' containing the cycle $x_0x_1\dots x_hx_0$. If this cycle is Hamiltonian, we are done. If not, then P was not a Hamiltonian path in G. Hence, there exists some vertex $v \in V(G)\setminus V(P)$ for which $x_iv \in E(G)$ for some i in the range $1 \leqslant i \leqslant h-1$, due to the connectedness of G. Note that $x_0v, x_hv \notin E(G)$, otherwise P would not be a longest path in G. Thus, $P' = x_{i+1}\dots x_hx_0\dots x_iv$ is a longer path in G' than P, which uses the new edge x_0x_h that we added to G (see Figure 5.1). Observe then that $x_0x_h \notin E(G)$, otherwise we could have created a longer path than P using this method without needing to add any new edges to G, and this would contradict our choice of P. Continuing in this way, we gradually absorb each vertex that lay outside our initial path P into the path we're working with. Eventually, we must end up with a Hamiltonian path, at which point we add the edge connecting its endpoints and create a Hamilton cycle.

In each stage of this process, the edges we add to our graph are known as *boosters*. Formally, these are defined as follows.

Definition 5.2.1 (Booster) *A non–edge uv of a graph G, where $u, v \in V(G)$, is called a* booster *with respect to G if $G \cup \{uv\}$ is Hamiltonian or its longest path is strictly longer than that of G. We denote the set of boosters with respect to G by $\mathcal{B}_G$.*

What if we are trying to build a Hamiltonian graph inside a random graph $\mathcal{G}(n,p)$? In this case, we cannot add boosters unless they lie in the random graph. Or suppose that we are playing the Hamiltonicity game against an adversary. Then we can only add those boosters that have not already been taken by our opponent. What can we do if we try to perform the above process and find that the booster we need to add to our graph is not available for us to take? We try to *rotate* our path. Indeed, if we have a path $P = x_0x_1\dots x_h$ in our graph G such that $x_ix_h \in E(G)$ for some i in the range $0 \leqslant i \leqslant h-2$, we can delete the edge x_ix_{i+1} from P and add the edge x_ix_h instead to create a new path $P' = x_0\dots x_ix_hx_{h-1}\dots x_{i+1}$ with the same length and vertex set as P, but with a different endpoint x_{i+1}. This is called an *elementary rotation* and it provides us with a different booster than before. If this booster is also unavailable to us, we try to rotate again until we find a booster that we can add to our graph. Note, however, that in a sequence of rotations, the same booster may appear more than once. We would like to know how many *different* boosters we can obtain through subsequent rotations of our path.

This is where *Pósa's lemma* can help us. It says that if our graph has a certain expansion property, then rotating our path many times provides us with many *different* boosters that we could potentially add to our graph if they are available to us. More precisely, this is true if our graph is a $(t, 2)$–expander.

Definition 5.2.2 (Expander) *Let $G = (V, E)$ be a graph on n vertices and let $t = t(n)$ and $k = k(n)$. The graph G is called a (t, k)–*expander *if $|N_G(U)| \geqslant k|U|$ for every set $U \subseteq V$ of size at most t.*

Note that adding edges to an expander preserves its expansion properties. In other words, the graph property of being an expander is monotone increasing. The following lemma (see e.g. [51]), which is essentially due to Pósa [82], asserts that expanders have many boosters.

Lemma 5.2.3 (Pósa's Lemma) *If G is a connected non–Hamiltonian $(t, 2)$–expander, then $|\mathcal{B}_G| \geqslant (t + 1)^2/2$.*

Therefore, if we have a connected $(k, 2)$–expander, we need only check that the set of boosters, whose size is guaranteed by Pósa's lemma, and the set of edges available for us to add to our graph have non–empty intersection. If they do, we know that we can add a booster to our graph and continue with the rotation–extension technique. In fact, Pósa's lemma allows us to forgo the manual implementation of this technique altogether. If we can show that a booster is available for us to add to our graph up to n times if necessary, then by the definition of a booster, we must end this process with a Hamiltonian graph. This is the method that we will employ in our treatment of the $(1 : q)$ Waiter–Client Hamiltonicity game on $E(\mathcal{G}(n,p))$ in Theorem 1.5.11.

For the Client–Waiter version in Theorem 1.5.12, we will use the following sufficient condition for Hamiltonicity from [66]; this is based on expansion and high connectivity.

Theorem 5.2.4 ([66]) *Let $12 \leqslant d \leqslant e^{\sqrt[3]{\log n}}$ and let G be a graph on n vertices which satisfies the following two properties.*

P1 *For every $S \subseteq V(G)$, if $|S| \leqslant \frac{n \log\log n \log d}{d \log n \log\log\log n}$, then $|N_G(S)| \geqslant d|S|$.*

P2 *There exists an edge in G between any two disjoint subsets $A, B \subseteq V(G)$ of size $|A|, |B| \geqslant \frac{n \log\log n \log d}{4130 \log n \log\log\log n}$.*

Then G is Hamiltonian for sufficiently large n.

5.2.2 Forcing Large Minimum Degree

A simple step in Waiter's strategy to force Client to build a Hamilton cycle, is to force him to quickly build a graph with large minimum degree. Our next result shows that this is indeed possible.

Lemma 5.2.5 *Let G be a graph on n vertices with minimum degree $\delta(G) \geqslant \delta$ and let q and $\gamma \leqslant \left\lfloor \frac{\delta}{2(q+1)} \right\rfloor$ be positive integers. When playing a $(1 : q)$ Waiter–Client game on $E(G)$, Waiter has a strategy to force Client to build a spanning subgraph of G with minimum degree at least γ, by offering at most $(q+1)\gamma n$ edges of G.*

Proof. Let $u_1, \ldots, u_n$ denote the vertices of G. We define a new graph G^*, where $G^* = G$ if $d_G(u_i)$ is even for every $1 \leqslant i \leqslant n$, and otherwise G^* is the graph obtained from G by adding a new vertex v^* and connecting it to every vertex of odd degree in G. Since all degrees of G^* are even, it admits an Eulerian orientation $\overrightarrow{G^*}$. For every $1 \leqslant i \leqslant n$, let $E(u_i) = \{u_iu_j \in E(G) : u_iu_j$ is directed from u_i to u_j in $\overrightarrow{G^*}\}$. It is evident that $E(u_i) \geqslant \lfloor \delta/2 \rfloor \geqslant (q+1)\gamma$ for every $1 \leqslant i \leqslant n$ and that the sets $E(u_1), \ldots, E(u_n)$ are pairwise disjoint.

For every $1 \leqslant i \leqslant n$ and every $1 \leqslant j \leqslant \gamma$, in the $((i-1)\gamma + j)$th round of the game, Waiter offers Client $q+1$ arbitrary free edges of $E(u_i)$. It is evident that, after offering at most $(q+1)\gamma$ edges of $E(u_i)$ for every $1 \leqslant i \leqslant n$, the minimum degree of Client's graph is at least γ. □

5.2.3 Box Games

Waiter's strategy to prevent Client from building a Hamilton cycle in the Client–Waiter game hinges on isolating one of Client's vertices. A very useful tool that one can use to aid Waiter in this is the analysis of a *box game*.

Box games describe positional games whose winning sets belong to a family $\mathcal{F} = \{A_1, \ldots, A_n\}$ of *pairwise disjoint* subsets (boxes) of the board. In the special case where $t-1 \leqslant |A_1| \leqslant \ldots \leqslant |A_n| = t$ for some positive integer t, we say that the box game is *canonical* of type t. We may also describe the family $\mathcal{F}$ whose members satisfy this criteria as being canonical. Box games were first introduced by Chvátal and Erdős in their seminal paper [29] where the canonical Maker–Breaker box game was studied. The non–canonical Maker–Breaker box game was subsequently fully analysed by Hamidoune and Las Vergnas in [58]. Generally speaking, these papers show that Breaker should always claim in the smallest boxes and, to counteract this, Maker should try and balance the number of free elements amongst the boxes not yet touched by Breaker. Although little in the literature is devoted explicitly to box games between other players, strategies similar to those of Maker and Breaker in the box game do feature. For example, in [60], Hefetz, Krivelevich, Stojaković and Szabó used a strategy similar to Maker's box game strategy to enable Avoider to avoid having positive minimum degree in his graph. We will also use a similar approach for the following box game between Waiter and Client.

For Waiter's strategy in the Client–Waiter Hamiltonicity game, we are interested in a box game where Waiter aims to fully claim some box in $\mathcal{F}$. Since both Waiter–Client and Client–Waiter games are only concerned with Client's claiming of a winning set, we must express our desired box game as the $(1 : q)$ Client–Waiter *transversal* game $(\bigcup_{i=1}^{n} A_i, \mathcal{F}^*)$. In particular, since Waiter's strategy in the Hamiltonicity game will involve Waiter isolating a vertex from a set of independent vertices of the *same* degree in $\mathcal{G}(n,p)$, we need only consider the case where $\mathcal{F}$ is canonical (see Remark 5.2.8 for comments on the non-canonical version).

It will be helpful to have the following perspective as our box game progresses. Suppose that, at some point during the box game on $\mathcal{F}$, Client claims an element of A_i for some $1 \leqslant i \leqslant n$. Since Waiter can no longer claim all elements of A_i, neither player has any incentive to claim more elements from this set. Therefore, we can pretend that A_i was removed from $\mathcal{F}$. If on the other hand, Waiter claims an element $a \in A_i$, then we can pretend that instead of trying to fully claim A_i, his goal is now to fully claim $A_i \setminus \{a\}$. Hence, we can view the family $\mathcal{F}$, on which the game is played, as changing throughout the game as follows. Assume that $\mathcal{F}_i$ denotes the (multi) family representing the game immediately before the ith round; in particular $\mathcal{F}_1 = \mathcal{F}$. Let W_i denote the set of elements Waiter offers Client in the ith round, let $c_i \in W_i$ denote the element claimed by Client and let j denote the unique integer for which $c_i \in A_j$. Then we define $\mathcal{F}_{i+1} = \{A \setminus W_i : A \in \mathcal{F}_i \text{ and } A \neq A_j\}$. Using this point of view, we see that Waiter wins the $(1 : q)$ Client–Waiter transversal game $(\bigcup_{i=1}^{n} A_i, \mathcal{F}^*)$ if and only if $\emptyset \in \mathcal{F}_i$ for some positive integer i.

Proposition 5.2.6 *Let q and t be positive integers and let $\mathcal{F}$ be a canonical family of type t. If $|\mathcal{F}| \geqslant 2(q+1)^{t+1}/q^t$, then Waiter has a winning strategy for the $(1 : q)$ Client–Waiter box game on $\mathcal{F}$.*

Remark 5.2.7 *In light of Theorem 2.2.6, Proposition 5.2.6 is not far from being best possible.*

Proof of Proposition 5.2.6. Waiter plays so as to keep the families $\mathcal{F}_i$ canonical; this is achieved as follows. For every positive integer i, let $t_i = \max\{|A| : A \in \mathcal{F}_i\}$, let $\mathcal{L}_i = \{A \in \mathcal{F}_i : |A| = t_i\}$ and let $\ell_i = |\mathcal{L}_i|$. In the ith round, Waiter offers Client an arbitrary set $W_i \subseteq \bigcup_{A \in \mathcal{L}_i} A$ of size $\min\{q+1, \ell_i\}$ such that $|A \cap W_i| \leqslant 1$ for every $A \in \mathcal{L}_i$. We claim that this is a winning strategy for Waiter.

For every $0 \leqslant j \leqslant t$, let i_j denote the smallest integer such that $\mathcal{F}_{i_j}$ is canonical of type j (to make this well–defined, we view the empty family as being canonical of every type). In particular, $i_t = 1$ and, in order to prove our claim, it suffices to show that $|\mathcal{F}_{i_0}| \geqslant 1$. We will in fact prove a more general claim, namely, that

$$|\mathcal{F}_{i_j}| \geqslant \left(\frac{q}{q+1}\right)^{t-j} |\mathcal{F}| - (q+1)\left(1 - \left(\frac{q}{q+1}\right)^{t-j}\right), \tag{5.2.1}$$

holds for every $0 \leqslant j \leqslant t$. This is indeed a more general result as, in particular, it follows from (5.2.1) that

$$|\mathcal{F}_{i_0}| \geqslant \left(\frac{q}{q+1}\right)^t \cdot \frac{2(q+1)^{t+1}}{q^t} - (q+1) + \frac{q^t}{(q+1)^{t-1}} = (q+1)\left(\left(\frac{q}{q+1}\right)^t + 1\right) \geqslant 1,$$

where the first inequality follows from our assumption that $|\mathcal{F}| \geqslant 2(q+1)^{t+1}/q^t$.

We prove (5.2.1) by reverse induction on j. The base case $j = t$ holds trivially. Assume that (5.2.1) holds for some $1 \leqslant j \leqslant t$; we prove it holds for $j - 1$ as well. It follows by Waiter's strategy that $i_{j-1} \leqslant i_j + \lceil |\mathcal{F}_{i_j}|/(q+1) \rceil$. Since, moreover, Client claims exactly one offered element per round, we conclude that

$$\begin{aligned}
|\mathcal{F}_{i_{j-1}}| &\geqslant |\mathcal{F}_{i_j}| - \left\lceil \frac{|\mathcal{F}_{i_j}|}{q+1} \right\rceil \geqslant \frac{q}{q+1}|\mathcal{F}_{i_j}| - 1 \\
&\geqslant \frac{q}{q+1}\left[\left(\frac{q}{q+1}\right)^{t-j}|\mathcal{F}| - (q+1)\left(1 - \left(\frac{q}{q+1}\right)^{t-j}\right)\right] - 1 \\
&= \left(\frac{q}{q+1}\right)^{t-j+1}|\mathcal{F}| - (q+1)\left(1 - \left(\frac{q}{q+1}\right)^{t-j+1}\right).
\end{aligned}$$

□

Remark 5.2.8 *Waiter's strategy for fully claiming a box in a canonical family can also be used to help Waiter do the same in a non–canonical family $\mathcal{F}$. Indeed, he does this by playing a sequence of* mini *canonical box games in the following way. He first identifies a maximal canonical subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of type $t = \max\{|A| : A \in \mathcal{F}\}$. For each $A \in \mathcal{F}'$, Waiter then chooses an arbitrary subset $D(A) \subseteq A$ of size $|A| - t'$, where $t' = \max\{|A| : A \in \mathcal{F} \setminus \mathcal{F}'\}$, and plays the canonical box game on $\{D(A) : A \in \mathcal{F}'\}$, using the strategy given in the proof of Proposition 5.2.6 to fully claim as many boxes in $\{D(A) : A \in \mathcal{F}'\}$ as he can. After this, Waiter removes any box with an element belonging to Client from $\mathcal{F}$ and, from each box that remains, he removes any elements that are no longer free. A new mini canonical box game is then created and played in the same way with this updated family $\mathcal{F}$. Performing this process repeatedly eventually enables Waiter to fully claim a box from the original non–canonical family, provided that this family is sufficiently large. An analysis similar to that in the proof of Proposition 5.2.6 can be performed to quantify how large this family must be for Waiter's strategy to succeed. However, since this is quite technical and unnecessary for our purposes in the Client–Waiter Hamiltonicity game, we do not present the analysis here.*

5.2.4 Concentration Inequalities

Throughout this chapter, we will use the following well–known concentration inequalities (see e.g. [9]).

Theorem 5.2.9 (Markov) *If X is a non–negative random variable and $a > 0$, then*

$$\mathbb{P}[X \geqslant a] \leqslant \frac{\mathbb{E}[X]}{a}.$$

Theorem 5.2.10 (Chernoff) *If $X \sim Bin(n, p)$, then*

(i) $\mathbb{P}[X < (1-a)np] < \exp\left(-\frac{a^2np}{2}\right)$ *for every $a > 0$.*

(ii) $\mathbb{P}[X > (1+a)np] < \exp\left(-\frac{a^2np}{3}\right)$ *for every $0 < a < 1$.*

Theorem 5.2.11 (Chebyshev) *If X is a random variable with $\mathbb{E}[X] < \infty$ and $Var[X] < \infty$, then for any $k > 0$,*

$$\mathbb{P}[|X - \mathbb{E}[X]| \geqslant k] \leqslant \frac{Var[X]}{k^2}.$$

5.2.5 Properties of Random Graphs

In this section we will prove several technical results about the binomial random graph $\mathcal{G}(n, p)$ for various edge probabilities p. These results will be useful in the proofs of Theorems 1.5.11 and 1.5.12.

Lemma 5.2.12 *Let $G \sim \mathcal{G}(n, p)$, where $p = c \log n/n$ for some constant $c > 0$ and let $t = t(n)$ be such that $\lim_{n\to\infty} t \log n = \infty$. Then* a.a.s. *we have $e_G(A) \leqslant 2ct|A| \log n$ for every $A \subseteq V(G)$ of size $1 \leqslant |A| \leqslant tn$.*

Proof.

$$\begin{aligned}
&\mathbb{P}\left[\exists A \subseteq V(G) \text{ such that } 1 \leqslant |A| \leqslant tn \text{ and } e_G(A) \geqslant 2ct|A|\log n\right] \\
&\leqslant \sum_{a=1}^{tn} \binom{n}{a} \binom{\binom{a}{2}}{2cta\log n} p^{2cta\log n} \leqslant \sum_{a=1}^{tn} \left(\frac{en}{a}\right)^a \left(\frac{e\binom{a}{2}p}{2cta\log n}\right)^{2cta\log n} \\
&\leqslant \sum_{a=1}^{tn} \left[\frac{en}{a} \cdot \left(\frac{ea}{4tn}\right)^{2ct\log n}\right]^a \\
&= \sum_{a=1}^{tn} \left[\exp\left\{1 + \log\left(\frac{n}{a}\right) + 2ct\log n\left(1 - \log\left(\frac{n}{a}\right) - \log(4t)\right)\right\}\right]^a = o(1). \qquad \square
\end{aligned}$$

Lemma 5.2.13 *Let $G \sim \mathcal{G}(n,p)$ and let $k = k(n)$ be an integer satisfying $kp \geqslant 100\log(n/k)$. Then* a.a.s. *$e_G(X,Y) \geqslant k^2p/2$ holds for any pair of disjoint sets $X, Y \subseteq V(G)$ of size $|X| = |Y| = k$.*

Proof. Let $X, Y \subseteq V(G)$ be arbitrary disjoint sets of size $|X| = |Y| = k$. Then $e_G(X,Y) \sim \mathrm{Bin}(k^2, p)$ and thus

$$\mathbb{P}\left[e_G(X,Y) < k^2p/2\right] = \mathbb{P}\left[e_G(X,Y) < \mathbb{E}[e_G(X,Y)]/2\right] < e^{-k^2p/8},$$

where the last inequality holds by Theorem 5.2.10(i).

A union bound over all choices of X and Y of size k then gives

$$\begin{aligned}
&\mathbb{P}\left[\exists X, Y \subseteq V(G) \text{ such that } |X| = |Y| = k,\ X \cap Y = \emptyset, \text{ and } e_G(X,Y) < k^2p/2\right] \\
&\leqslant \binom{n}{k}^2 \cdot e^{-k^2p/8} \leqslant \left[\left(\frac{en}{k}\right)^2 \cdot e^{-kp/8}\right]^k = \left[\exp\{2 + 2\log(n/k) - kp/8\}\right]^k = o(1),
\end{aligned}$$

where the last equality holds by our assumption on k. $\square$

Lemma 5.2.14 *Let $c > 0$ be a constant and let $G \sim \mathcal{G}(n,p)$, where $p = c/n$. Then,* a.a.s. *$e(G) \leqslant cn$.*

Proof. Clearly $e(G) \sim \text{Bin}(\binom{n}{2}, p)$ and therefore $\mathbb{E}[e(G)] = \binom{n}{2}p = c(n-1)/2$. Hence,

$$\mathbb{P}[e(G) > cn] \leqslant \mathbb{P}\left[e(G) > 1.5\binom{n}{2}p\right] < \exp\left\{-\frac{\binom{n}{2}p}{12}\right\} \leqslant \exp\left\{-cn/25\right\} = o(1),$$

where the second inequality holds by Theorem 5.2.10(ii). $\square$

An important part of proving Client's side in Theorem 1.5.12, is to show that *a.a.s.* the sum $\sum_{v \in V(G)} \left(\frac{q}{q+1}\right)^{d_G(v)}$ is very small, where $G \sim \mathcal{G}(n, p)$. The following lemma will play a key role in this endeavour.

Lemma 5.2.15 *Let q be a positive integer and let $G \sim \mathcal{G}(n, p)$. For every $0 \leqslant i \leqslant n-1$, let $X_i = |\{u \in V(G) : d_G(u) = i\}|$ and let $\mu_i = \mathbb{E}[X_i]$. Then,*

$$\sum_{i=0}^{n-1} \left(\frac{q}{q+1}\right)^i \mu_i = n\left(1 - \frac{p}{q+1}\right)^{n-1}.$$

Proof. Let $\tilde{G} \sim \mathcal{G}\left(n, \frac{p}{q+1}\right)$ and let Y denote the number of isolated vertices in $\tilde{G}$. Then,

$$\mathbb{E}[Y] = n\left(1 - \frac{p}{q+1}\right)^{n-1}. \tag{5.2.2}$$

An alternative way of generating $\tilde{G}$ is by first generating $G \sim \mathcal{G}(n, p)$ and then deleting each edge of G with probability $\frac{q}{q+1}$, independently of all other edges. It is then apparent that, for any $v \in V(G)$ with $d_G(v) = i$, we have

$$\mathbb{P}[d_{\tilde{G}}(v) = 0] = \left(\frac{q}{q+1}\right)^i.$$

Hence,

$$\mathbb{E}[Y] = \sum_{i=0}^{n-1} \left(\frac{q}{q+1}\right)^i \mu_i. \tag{5.2.3}$$

Combining (5.2.2) and (5.2.3) we conclude that

$$\sum_{i=0}^{n-1} \left(\frac{q}{q+1}\right)^i \mu_i = n\left(1 - \frac{p}{q+1}\right)^{n-1},$$

as stated. □

Lemma 5.2.16 *Let $\varepsilon > 0$ be a constant, let q be a positive integer and let $G \sim \mathcal{G}(n,p)$, where $p = (q+1-\varepsilon)\log n/n$. For every $0 \leqslant i \leqslant n-1$, let $X_i = |\{u \in V(G) : d_G(u) = i\}|$ and let $\mu_i = \mathbb{E}[X_i]$. If $0 \leqslant k \leqslant 9(q+1-\varepsilon)\log n$ is an integer such that $\mu_k \to \infty$, then* a.a.s. *$X_k \geqslant \mu_k/2$.*

Proof. Since

$$\mathbb{P}[X_k < \mu_k/2] \leqslant \mathbb{P}[|X_k - \mu_k| \geqslant \mu_k/2] \leqslant \frac{4\mathrm{Var}[X_k]}{\mu_k^2},$$

where the last inequality holds by Chebyshev's inequality (Theorem 5.2.11), it suffices to show that $\mathrm{Var}[X_k]/\mu_k^2 = o(1)$.

Let $v_1, \ldots, v_n$ denote the vertices of G. For every $1 \leqslant i \leqslant n$, let Y_i be the indicator random variable taking the value 1 if $d_G(v_i) = k$ and 0 otherwise. Then

$$\mathbb{E}[Y_i] = \mathbb{P}[Y_i = 1] = \binom{n-1}{k} p^k (1-p)^{n-1-k}.$$

Moreover, $X_k = \sum_{i=1}^n Y_i$ and thus

$$\mu_k = \sum_{i=1}^n \mathbb{E}[Y_i] = n\binom{n-1}{k} p^k (1-p)^{n-1-k}.$$

For every $1 \leqslant i \leqslant n$ we have $\mathrm{Var}[Y_i] = \mathbb{E}[Y_i^2] - (\mathbb{E}[Y_i])^2 = \mathbb{E}[Y_i] - (\mathbb{E}[Y_i])^2 \leqslant \mathbb{E}[Y_i]$, where the second equality holds since $Y_i^2 = Y_i$. Hence,

$$\sum_{i=1}^n \mathrm{Var}[Y_i] \leqslant \sum_{i=1}^n \mathbb{E}[Y_i] = \mu_k.$$

Fix some $1 \leqslant i \neq j \leqslant n$. Then

$$\begin{aligned}\mathbb{E}[Y_iY_j] &= \mathbb{P}[Y_iY_j = 1] = \mathbb{P}[(Y_i = 1) \wedge (Y_j = 1)] \\ &= p\left[\binom{n-2}{k-1}p^{k-1}(1-p)^{n-1-k}\right]^2 + (1-p)\left[\binom{n-2}{k}p^k(1-p)^{n-2-k}\right]^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Cov}[Y_i, Y_j] &= \mathbb{E}[Y_iY_j] - \mathbb{E}[Y_i]\mathbb{E}[Y_j] \\ &= p\left[\binom{n-2}{k-1}p^{k-1}(1-p)^{n-1-k}\right]^2 + (1-p)\left[\binom{n-2}{k}p^k(1-p)^{n-2-k}\right]^2 \\ &\qquad - \left[\binom{n-1}{k}p^k(1-p)^{n-1-k}\right]^2 \\ &= \left[\binom{n-1}{k}p^k(1-p)^{n-1-k}\right]^2 \left[\left(\frac{k}{n-1}\right)^2 \frac{1}{p} + \left(1 - \frac{k}{n-1}\right)^2 \frac{1}{1-p} - 1\right].\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{\mu_k^2} \sum_{1 \leqslant i \neq j \leqslant n} \mathrm{Cov}[Y_i, Y_j] &= \frac{n(n-1)}{\mu_k^2}\left[\binom{n-1}{k}p^k(1-p)^{n-1-k}\right]^2 \\ &\qquad \cdot \left[\left(\frac{k}{n-1}\right)^2 \frac{1}{p} + \left(1 - \frac{k}{n-1}\right)^2 \frac{1}{1-p} - 1\right] \\ &= \frac{n-1}{n}\left[\left(\frac{k}{n-1}\right)^2 \frac{1}{p} + \left(1 - \frac{k}{n-1}\right)^2 \frac{1}{1-p} - 1\right] \\ &\leqslant \left(\frac{k}{n-1}\right)^2 \frac{1}{p} + \frac{p}{1-p} \\ &\leqslant \left(\frac{9(q+1-\varepsilon)\log n}{n-1}\right)^2 \frac{n}{(q+1-\varepsilon)\log n} + \frac{(q+1-\varepsilon)\log n}{n-(q+1-\varepsilon)\log n} \\ &\leqslant \frac{82(q+1)\log n}{n-1} + \frac{2(q+1)\log n}{n} = o(1). \qquad (5.2.4)\end{aligned}$$

Moreover, by our assumption on k we have

$$\frac{1}{\mu_k} = o(1). \tag{5.2.5}$$

We conclude that

$$\frac{\mathrm{Var}[X_k]}{\mu_k^2} = \frac{1}{\mu_k^2}\left(\sum_{i=1}^{n}\mathrm{Var}[Y_i] + \sum_{1\leqslant i\neq j\leqslant n}\mathrm{Cov}[Y_i,Y_j]\right) \leqslant \frac{1}{\mu_k} + \frac{1}{\mu_k^2}\sum_{1\leqslant i\neq j\leqslant n}\mathrm{Cov}[Y_i,Y_j] = o(1),$$

where the last equality holds by (5.2.4) and (5.2.5). □

Lemma 5.2.17 *Let $G \sim \mathcal{G}(n,p)$, where $p = c\log n/n$ for some constant $c > \frac{2}{9\log 3}$. For every $0 \leqslant i \leqslant n-1$, let $X_i = |\{u \in V(G) : d_G(u) = i\}|$ and let $\mu_i = \mathbb{E}[X_i]$. Then*

$$\sum_{i=9c\log n}^{n-1} \mu_i = o(1).$$

Proof. We first observe that the function $f(i) = (enp/i)^i$ is decreasing for $i \geqslant 9c\log n$. Indeed,

$$\frac{f(i)}{f(i+1)} = \left(1+\frac{1}{i}\right)^i \cdot \frac{i+1}{enp} > \frac{i+1}{enp} \geqslant \frac{9c\log n}{ec\log n} = \frac{9}{e} > 1\,. \tag{5.2.6}$$

Then

$$\begin{aligned}\sum_{i=9c\log n}^{n-1} \mu_i = n\sum_{i=9c\log n}^{n-1}\binom{n-1}{i}p^i(1-p)^{n-1-i} &\leqslant n\sum_{i=9c\log n}^{n-1}\left(\frac{enp}{i}\right)^i \\ &\leqslant n^2\left(\frac{e}{9}\right)^{9c\log n} \leqslant \exp\{2\log n - 9c\log n\cdot\log 3\} = o(1),\end{aligned}$$

where the second inequality holds by (5.2.6) and the last equality follows from our choice of c. □

Corollary 5.2.18 *Let $G \sim \mathcal{G}(n,p)$, where $p = c\log n/n$ for some constant $c > \frac{2}{9\log 3}$. Then,* a.a.s. $\Delta(G) \leqslant 9c\log n$.

Proof. For every $0 \leqslant i \leqslant n-1$, let $X_i = |\{u \in V(G) : d_G(u) = i\}|$ and let $\mu_i = \mathbb{E}[X_i]$. Then

$$\mathbb{P}[\Delta(G) \geqslant 9c \log n] = \mathbb{P}[\exists i \text{ such that } 9c\log n \leqslant i \leqslant n-1 \text{ and } X_i > 0] \leqslant \sum_{i=9c\log n}^{n-1} \mu_i = o(1),$$

where the first inequality follows from Theorem 5.2.9 and a union bound, and the last equality follows from Lemma 5.2.17. □

Lemma 5.2.19 *Let $\varepsilon > 0$ be a constant and let $G \sim \mathcal{G}(n,p)$, where $p \geqslant (1+\varepsilon)\log n/n$. Then there exists a constant $\gamma = \gamma(\varepsilon) > 0$ such that* a.a.s. $\delta(G) \geqslant \gamma \log n$.

Proof. By monotonicity, we can assume that $p = (1+\varepsilon)\log n/n$. Let $0 < \gamma < 1$ be a constant satisfying $\gamma \log(e(1+\varepsilon)/\gamma) < \varepsilon/3$; such a constant exists since $\lim_{\gamma\to 0} \gamma \log(1/\gamma) = 0$. We first observe that the function $f(i) = (enp/i)^i$ is increasing for $1 \leqslant i \leqslant \gamma \log n$. Indeed,

$$\frac{f(i)}{f(i+1)} = \left(1 + \frac{1}{i}\right)^i \cdot \frac{i+1}{enp} \leqslant \frac{i+1}{np} \leqslant \frac{\gamma \log n + 1}{(1+\varepsilon)\log n} < \gamma < 1\,, \tag{5.2.7}$$

where the last inequality holds by our choice of γ.

Let X be the random variable that counts the number of vertices of degree at most $\gamma \log n$ in G. Then,

$$\begin{aligned}
\mathbb{E}[X] &= n \sum_{i=0}^{\gamma\log n} \binom{n-1}{i} p^i (1-p)^{n-1-i} \leqslant n \sum_{i=0}^{\gamma \log n} \binom{n}{i} p^i \exp\{-p(n-1-i)\} \\
&\leqslant n\exp\{-p(n-1)\} + n\exp\{-p(n-2\gamma\log n)\} \sum_{i=1}^{\gamma\log n} \left(\frac{enp}{i}\right)^i \\
&\leqslant n\exp\{-(1+\varepsilon/2)\log n\} + n\exp\{-(1+\varepsilon/2)\log n\} \cdot \gamma \log n \left(\frac{e(1+\varepsilon)\log n}{\gamma \log n}\right)^{\gamma\log n} \\
&\leqslant n^{-\varepsilon/2}\left(1 + \exp\left\{\log\gamma + \log\log n + \gamma\log n \log\left(\frac{e(1+\varepsilon)}{\gamma}\right)\right\}\right) = o(1),
\end{aligned}$$

where the third inequality holds by (5.2.7) and the last equality follows from our choice of γ. Using Theorem 5.2.9 we conclude that

$$\mathbb{P}[\delta(G) \leqslant \gamma \log n] = \mathbb{P}[X > 0] \leqslant \mathbb{E}[X] = o(1).$$

Hence, *a.a.s.* $\delta(G) \geqslant \gamma \log n$. $\square$

Lemma 5.2.20 *Let $r > 0$ be a constant and let $G \sim \mathcal{G}(n,p)$, where $p = c \log n/n$ for some constant $c > \frac{2}{9\log 3}$. Then* a.a.s.

$$\binom{d_G(v)}{r \log n} \leqslant n^{r(1+\log(9c)+\log(1/r))}$$

holds for any vertex $v \in V(G)$.

Proof. Since, by Corollary 5.2.18, *a.a.s.* $\Delta(G) \leqslant 9c \log n$, it follows that *a.a.s.*

$$\binom{d_G(v)}{r \log n} \leqslant \left(\frac{e \cdot d_G(v)}{r \log n}\right)^{r \log n} \leqslant \left(\frac{e \cdot 9c}{r}\right)^{r \log n} = \exp\left\{r \log n \left(1 + \log(9c) + \log(1/r)\right)\right\}$$
$$= n^{r(1+\log(9c)+\log(1/r))}.$$

$\square$

Lemma 5.2.21 *Let $\varepsilon > 0$ be a constant, let q be a positive integer and let $G \sim \mathcal{G}(n,p)$, where $p = (q+1+\varepsilon)\log n/n$. Then* a.a.s.

$$\sum_{v \in V(G)} \left(\frac{q}{q+1}\right)^{d_G(v)} \leqslant n^{-\varepsilon/(4(q+1))}.$$

Proof. For every $0 \leqslant i \leqslant n-1$, let $X_i = |\{u \in V(G) : d_G(u) = i\}|$ and let $\mu_i = \mathbb{E}[X_i]$. Setting

$$X = \sum_{i=0}^{n-1} \left(\frac{q}{q+1}\right)^{i} X_i = \sum_{v \in V(G)} \left(\frac{q}{q+1}\right)^{d_G(v)},$$

it suffices to prove that *a.a.s.* $X \leqslant n^{-\varepsilon/(4(q+1))}$. Indeed, we have

$$\mathbb{E}[X] = \sum_{i=0}^{n-1} \left(\frac{q}{q+1}\right)^i \mu_i = n\left(1 - \frac{p}{q+1}\right)^{n-1} \leqslant n \exp\left\{-\frac{(q+1+\varepsilon)\log n}{(q+1)n} \cdot (n-1)\right\}$$
$$\leqslant n \exp\left\{-\left(1 + \frac{\varepsilon}{2(q+1)}\right)\log n\right\} = n^{-\varepsilon/(2(q+1))},$$

where the second equality follows from Lemma 5.2.15. Therefore,

$$\mathbb{P}\left[X \geqslant n^{-\varepsilon/(4(q+1))}\right] \leqslant n^{\varepsilon/(4(q+1))} \cdot \mathbb{E}[X] \leqslant n^{\varepsilon/(4(q+1))-\varepsilon/(2(q+1))} = n^{-\varepsilon/(4(q+1))},$$

where the first inequality follows from Theorem 5.2.9. □

Lemma 5.2.22 *Let $\varepsilon > 0$ be a constant, let q be a positive integer and let $G \sim \mathcal{G}(n,p)$, where $p = (q+1+\varepsilon)\log n/n$. Then there exists a constant $r > 0$ such that the following holds. For every $v \in V(G)$, let $E(v) = \{e \in E(G) : v \in e\}$ and let $\mathcal{F}_1 = \bigcup_{v \in V(G)} \mathcal{A}(v)$, where $\mathcal{A}(v) = \{A(v) \subseteq E(v) : |A(v)| = d_G(v) - r\log n\}$. Then*

$$\sum_{A \in \mathcal{F}_1} \left(\frac{q}{q+1}\right)^{|A|} = o(1).$$

Proof. By Lemma 5.2.19 there exists a constant $\gamma > 0$ such that $\delta(G) \geqslant \gamma \log n$. Let $0 < r < \gamma$ be a constant satisfying

$$r\left(1 + \log(9(q+1+\varepsilon)) + \log(1/r) + \log\left(\frac{q+1}{q}\right)\right) < \frac{\varepsilon}{4(q+1)}.$$

Such a constant r exists since $\lim_{r \to 0} r\log(1/r) = 0$.

Using this r in the definition of $\mathcal{F}_1$, we obtain

$$
\begin{aligned}
\sum_{A\in\mathcal{F}_1}\left(\frac{q}{q+1}\right)^{|A|} &= \sum_{v\in V(G)}\binom{d_G(v)}{r\log n}\left(\frac{q}{q+1}\right)^{d_G(v)-r\log n}\\
&\leqslant \left(\frac{q+1}{q}\right)^{r\log n}\cdot n^{r(1+\log(9(q+1+\varepsilon))+\log(1/r))}\cdot\sum_{v\in V(G)}\left(\frac{q}{q+1}\right)^{d_G(v)}\\
&\leqslant \exp\left\{r\log n\cdot\log\left(\frac{q+1}{q}\right)\right\}\cdot n^{r(1+\log(9(q+1+\varepsilon))+\log(1/r))-\varepsilon/(4(q+1))}\\
&= n^{r\left(1+\log(9(q+1+\varepsilon))+\log(1/r)+\log\left(\frac{q+1}{q}\right)\right)-\varepsilon/(4(q+1))} = o(1),
\end{aligned}
$$

where the first inequality follows from Lemma 5.2.20, the second inequality follows from Lemma 5.2.21 and the last equality follows from our choice of r. □

Lemma 5.2.23 *Let $\varepsilon>0$ be a constant, let q be a positive integer and let $G\sim\mathcal{G}(n,p)$, where $p=(q+1+\varepsilon)\log n/n$. Then there exists a constant $\lambda>0$ for which*

$$
\sum_{A\in\mathcal{F}_2}\left(\frac{q}{q+1}\right)^{|A|}=o(1),
$$

where $\mathcal{F}_2=\left\{E_G(X,Y): X,Y\subseteq V(G),\ |X|=|Y|=\frac{\lambda n\log\log n}{\log n} \text{ and } X\cap Y=\emptyset\right\}$.

Proof. Let $\lambda\geqslant 100$ be a constant satisfying $\lambda\log\left(\frac{q+1}{q}\right)>2$. Then

$$
\begin{aligned}
\sum_{A\in\mathcal{F}_2}\left(\frac{q}{q+1}\right)^{|A|} &\leqslant \binom{n}{\frac{\lambda n\log\log n}{\log n}}^2\left(\frac{q}{q+1}\right)^{\frac{\lambda^2 n(\log\log n)^2}{\log n}}\\
&\leqslant \left[\left(\frac{e\log n}{\lambda\log\log n}\right)^2\left(\frac{q}{q+1}\right)^{\lambda\log\log n}\right]^{\frac{\lambda n\log\log n}{\log n}}\\
&\leqslant \left[\exp\left\{2\log\log n-\lambda\log\log n\log\left(\frac{q+1}{q}\right)\right\}\right]^{\frac{\lambda n\log\log n}{\log n}} = o(1),
\end{aligned}
$$

where the first inequality follows since $q\geqslant 1$ and by Lemma 5.2.13 which is applicable since $\lambda\geqslant 100$, and the last equality follows since $\lambda\log\left(\frac{q+1}{q}\right)>2$ by assumption. □

5.2.6 Expanders in Random Graphs

Here we present two results concerning expanders in $\mathcal{G}(n,p)$. The first result asserts that typically, for subgraphs of a random graph, large minimum degree is enough to ensure expansion.

Lemma 5.2.24 *Let $G \sim \mathcal{G}(n,p)$, where $p = c\log n/n$ for some constant $c > 0$, and let $\alpha = \alpha(n)$ and $k = k(n)$ be such that $\lim_{n\to\infty} \alpha k \log n = \infty$. Then* a.a.s. *every spanning subgraph $G' \subseteq G$ with minimum degree $\delta(G') \geqslant r\log n$ for some constant $r \geqslant 4c\alpha(k+1)^2 > 0$ is an $(\alpha n, k)$–expander.*

Proof. Suppose for a contradiction that there exists a set $A \subseteq V(G)$ of size $1 \leqslant |A| \leqslant \alpha n$ and a spanning subgraph $G' \subseteq G$, with minimum degree $\delta(G') \geqslant r\log n$ as in the statement of the lemma, such that $|N_{G'}(A)| < k|A|$. Then,

$$|A \cup N_{G'}(A)| < (k+1)|A| \leqslant (k+1)\alpha n.$$

It thus follows by Lemma 5.2.12 that *a.a.s.*

$$e_{G'}(A \cup N_{G'}(A)) \leqslant 2c(k+1)\alpha|A \cup N_{G'}(A)|\log n < 2c(k+1)^2\alpha|A|\log n \leqslant r|A|\log n/2. \tag{5.2.8}$$

On the other hand, since $\delta(G') \geqslant r\log n$, we have

$$e_{G'}(A \cup N_{G'}(A)) \geqslant r|A|\log n/2$$

which clearly contradicts (5.2.8). We conclude that G' is indeed an $(\alpha n, k)$–expander. □

Using Pósa's lemma (Lemma 5.2.3), the following result shows that a sparse graph within $\mathcal{G}(n,p)$, with good expansion, has many boosters in the random graph.

Lemma 5.2.25 *Let $\varepsilon, s_1 < 1$ and $s_2 < 1/100$ be positive constants and let $G \sim \mathcal{G}(n,p)$, where $p = (1+\varepsilon)\log n/n$. If $s_1(1-\log s_1) < 1/400$, then* a.a.s. *every connected non–Hamiltonian $(n/5,2)$–expander $\Gamma \subseteq G$ with at most $s_1 n\log n$ edges has at least $s_2 n \log n$ boosters in G.*

Proof. For a connected non–Hamiltonian $(n/5,2)$–expander $\Gamma \subseteq G$ with at most $s_1 n \log n$ edges, let $X_\Gamma = |\mathcal{B}_\Gamma \cap E(G)|$. Then $X_\Gamma \sim \mathrm{Bin}(|\mathcal{B}_\Gamma|, p)$ and, by Lemma 5.2.3, $|\mathcal{B}_\Gamma| \geqslant n^2/50$. Therefore,

$$\mathbb{P}[X_\Gamma < s_2 n \log n] < \exp\left\{-\left(1-\frac{50s_2}{1+\varepsilon}\right)^2 n^2p/100\right\} \leqslant \exp\{-n\log n/400\},$$

where the first inequality follows from Theorem 5.2.10(i) with $a = 1-50s_2/(1+\varepsilon)$ and the last inequality holds since $s_2 \leqslant 1/100$ and $\varepsilon > 0$.

Taking a union bound over all spanning subgraphs of G which are connected non–Hamiltonian $(n/5,2)$–expanders with at most $s_1 n\log n$ edges, we conclude that the probability that there exists such a subgraph with less than $s_2 n \log n$ boosters in G is at most

$$\begin{aligned}
&\sum_{m=1}^{s_1 n\log n}\binom{\binom{n}{2}}{m}p^m \cdot \exp\{-n\log n/400\} \leqslant \exp\{-n\log n/400\}\cdot \sum_{m=1}^{s_1 n \log n}\left(\frac{en\log n}{m}\right)^m \\
&\leqslant \exp\{-n\log n/400\}\cdot s_1 n\log n \cdot \left(\frac{e}{s_1}\right)^{s_1 n\log n} \\
&\leqslant \exp\{2\log n + s_1 n\log n(1-\log s_1) - n\log n/400\} = o(1),
\end{aligned}$$

where the first inequality holds since $\varepsilon < 1$, the second inequality holds since $f(m) = (en\log n/m)^m$ is increasing for $1 \leqslant m \leqslant s_1 n \log n$ as can be shown by a calculation similar to (5.2.7), and the last equality holds since $s_1(1-\log s_1) < 1/400$ by assumption. □

5.3 Main Proofs

Here we present the proofs of our results in Section 5.1.

The Waiter–Client Hamiltonicity Game

Proof of Theorem 1.5.11. Fix some constant $\varepsilon > 0$ and let n be sufficiently large.

Client's Strategy: For $p = (1-\varepsilon)\log n/n$, it is well–known (see, e.g., [25, 69]) that *a.a.s.* $G \sim \mathcal{G}(n,p)$ has an isolated vertex and therefore is not Hamiltonian. Hence, *a.a.s.* Client wins the $(1:q)$ Waiter–Client Hamiltonicity game on $E(G)$ regardless of his strategy.

Waiter's Strategy: Assume then that $G \sim \mathcal{G}(n,p)$, where $p = (1+\varepsilon)\log n/n$ for some constant $\varepsilon > 0$. We present a strategy for Waiter to win the $(1:q)$ Waiter–Client Hamiltonicity game on $E(G)$ and then prove that *a.a.s.* he can play according to this strategy. Waiter's strategy consists of the following four stages.

Preparation Stage: Waiter splits G into two spanning subgraphs, the main graph G_M and a reservoir graph R, by placing each edge of G in R independently with probability $\overline{p} = \overline{c}/\log n$, for some positive constant $\overline{c}$ (defined later), and then setting $E(G_M) = E(G) \setminus E(R)$.

Stage I: By only offering edges from G_M and using Lemmas 5.2.5 and 5.2.24, Waiter forces Client to build a $(c_1 n, 2)$–expander G_1 with at most $c_2 n \log n$ edges for some positive constants c_1 and c_2 (defined later).

Stage II: By only offering the edges of R and following the strategy given by Theorem 2.2.7, Waiter forces Client to build a graph G_2 such that $G_1 \cup G_2$ is an $(n/5, 2)$–expander.

Stage III: For as long as G_C is not Hamiltonian, in each round Waiter offers Client $q+1$ free boosters with respect to G_C. Once G_C becomes Hamiltonian, Waiter plays arbitrarily for the remainder of the game.

It is evident from the description of Stage III of the proposed strategy that, if Waiter is able to play according to this strategy, then he wins the game. Moreover, it is clear that Waiter can follow the Preparation Stage of the strategy. It thus remains to prove that he can follow Stages I–III as well. We consider each stage in turn.

Stage I: We first observe that

$$p(1-\overline{p}) = (1+\varepsilon)(\log n - \overline{c})/n \geqslant (1+\varepsilon/2)\log n/n,$$

and that $G_M \sim \mathcal{G}(n, p(1-\overline{p}))$. It then follows from Lemma 5.2.19 that *a.a.s.* $\delta(G_M) \geqslant \gamma \log n$ for some constant $\gamma > 0$. Let $0 < c_2 < 1/(600(q+1))$ be a constant satisfying $\lfloor \gamma \log n/(2(q+1)) \rfloor \geqslant c_2 \log n$ and $3c_2(1-\log(3c_2)) < 1/400$. By Lemma 5.2.5, Waiter has a strategy to force Client to build a spanning subgraph G_1 of G_M with minimum degree $\delta(G_1) \geqslant c_2 \log n$, by offering at most $(q+1)c_2 n \log n$ edges of G_M; in particular, $e(G_1) \leqslant c_2 n \log n$. Finally, it follows by Lemma 5.2.24 that G_1 is a $(c_1 n, 2)$–expander, for a sufficiently small constant $c_1 > 0$.

Stage II: Let $\mathcal{F} = \{E_R(X,Y) : X, Y \subseteq V(G),\ |X| = |Y| = c_1 n \text{ and } X \cap Y = \emptyset\}$. Since $R \sim \mathcal{G}(n, p\overline{p})$ and $p\overline{p} = (1+\varepsilon)\overline{c}/n$, we have

$$\begin{aligned}\sum_{A \in \mathcal{F}} 2^{-|A|/(2q-1)} &\leqslant \binom{n}{c_1 n}^2 2^{-0.5c_1^2\overline{c}(1+\varepsilon)n/(2q-1)} \leqslant \left(\frac{e}{c_1}\right)^{2c_1 n} 2^{-c_1^2\overline{c}n/(4q)} \\ &= \exp\left\{2c_1 n\,(1-\log c_1) - \frac{c_1^2\overline{c}n\log 2}{4q}\right\} = o(1),\end{aligned}$$

where the first inequality follows from Lemma 5.2.13 which is applicable for a sufficiently large constant $\overline{c}$, and the last equality holds for sufficiently large $\overline{c}$. Hence, by Theorem 2.2.7, and since all edges of R are free at the beginning of Stage II, Waiter has a strategy to force Client to claim an edge of R between every pair of disjoint sets of vertices of G, each of size $c_1 n$.

Let G_2 denote the graph built by Client in Stage II. We claim that $G_1 \cup G_2$ is an $(n/5, 2)$–expander. Since G_1 is a $(c_1 n, 2)$–expander and expansion is a monotone increasing property, it suffices to demonstrate expansion for sets $A \subseteq V(G)$ of size $c_1 n \leqslant |A| \leqslant n/5$. Suppose for a contradiction that $A \subseteq V(G)$ is a set of size $c_1 n \leqslant |A| \leqslant n/5$ and yet $|N_{G_1 \cup G_2}(A)| < 2|A|$. Then $|V(G) \setminus (A \cup N_{G_1 \cup G_2}(A))| > n - 3|A| \geqslant 2n/5 \geqslant c_1 n$ and there are no edges of $G_1 \cup G_2$ between A and $V(G) \setminus (A \cup N_{G_1 \cup G_2}(A))$. This contradicts the way G_2 was constructed. We conclude that $G_1 \cup G_2$ is indeed an $(n/5, 2)$–expander at the end of Stage II.

Stage III: Observe that, at the end of Stage II, Client's graph G_C is connected. Indeed, since $G_1 \cup G_2$ is an $(n/5, 2)$–expander, each of its connected components must have size at least $3n/5$ and thus there can be only one such component. It follows that, at the beginning of Stage III, Client's graph is a connected $(n/5, 2)$–expander. Since connectivity and expansion are monotone increasing properties, this remains true for the remainder of the game. We will show that this allows Waiter to offer Client $q+1$ free boosters in every round of Stage III until G_C becomes Hamiltonian.

It is evident from Definition 5.2.1 that one needs to sequentially add at most n boosters to an n–vertex graph to make it Hamiltonian. Hence, in order to prove that Waiter can follow Stage III of the proposed strategy, it suffices to show that, for every $1 \leqslant i \leqslant n$, if G_C is not Hamiltonian at the beginning of the ith round of Stage III, then $|\mathcal{B}_{G_C} \cap E(G_F)| \geqslant q + 1$ holds at this point. By the description of Stage I we have $e(G_1) \leqslant c_2 n \log n$ and by the description of Stage II we have

$$e(G_2) \leqslant e(R) \leqslant (1 + \varepsilon)\bar{c}n,$$

where the last inequality holds *a.a.s.* by Lemma 5.2.14. Hence, *a.a.s.* $e(G_1 \cup G_2) \leqslant 2c_2 n \log n$.

Fix an integer $1 \leqslant i \leqslant n$ and suppose that G_C is not Hamiltonian at the beginning of the ith round of Stage III. Then G_C is a connected, non–Hamiltonian $(n/5, 2)$–expander with at most $2c_2 n \log n + (i - 1) \leqslant 3c_2 n \log n$ edges. Since, moreover, c_2 was chosen such that $3c_2(1 - \log(3c_2)) < 1/400$, it follows by Lemma 5.2.25 that $|\mathcal{B}_{G_C} \cap E(G)| \geqslant n \log n/200$. We conclude that

$$|\mathcal{B}_{G_C} \cap E(G_F)| \geqslant |\mathcal{B}_{G_C} \cap E(G)| - (e(G_C) + e(G_W)) \geqslant n \log n/200 - 3c_2(q+1)n \log n \geqslant q+1,$$

where the last inequality holds since $c_2 < 1/(600(q + 1))$ by assumption. $\square$

The Client–Waiter Hamiltonicity Game

Proof of Theorem 1.5.12. Fix some constant $\varepsilon > 0$ and let n be sufficiently large.

Client's Strategy: Assume first that $G \sim \mathcal{G}(n, p)$, where $p = (q + 1 + \varepsilon) \log n/n$. We will present a strategy for Client for the $(1 : q)$ Client–Waiter Hamiltonicity game on $E(G)$; it is based on the sufficient condition for Hamiltonicity from Theorem 5.2.4. Let r and $\mathcal{F}_1$ be as in Lemma 5.2.22 and let λ and $\mathcal{F}_2$ be as in Lemma 5.2.23. Note that $\sum_{A \in \mathcal{F}_1} \left(\frac{q}{q+1}\right)^{|A|} = o(1)$ holds by Lemma 5.2.22 and that $\sum_{A \in \mathcal{F}_2} \left(\frac{q}{q+1}\right)^{|A|} = o(1)$ holds by Lemma 5.2.23. Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. Then

$$\sum_{A \in \mathcal{F}} \left(\frac{q}{q+1}\right)^{|A|} = \sum_{A \in \mathcal{F}_1} \left(\frac{q}{q+1}\right)^{|A|} + \sum_{A \in \mathcal{F}_2} \left(\frac{q}{q+1}\right)^{|A|} = o(1).$$

It thus follows by Theorem 2.2.6 that Client has a winning strategy for the $(1 : q)$ Client–Waiter game $(E(G), \mathcal{F}^*)$.

We claim that if Client follows this strategy, then his graph at the end of the game satisfies properties **P1** and **P2** from Theorem 5.2.4, with $d = (\log n)^{1/3}$, and is therefore Hamiltonian. Indeed, it follows from the definition of $\mathcal{F}_1$ that, at the end of the game, the minimum degree in Client's graph will be at least $r \log n$. Using Lemma 5.2.24, it is then easy to verify that Client's graph is an $(n/\log n, (\log n)^{1/3})$–expander and thus satisfies property **P1**. Moreover, a straightforward calculation shows that, by the definition of $\mathcal{F}_2$, at the end of the game, Client's graph will satisfy property **P2** as well.

Waiter's Strategy: Next, assume that $G \sim \mathcal{G}(n,p)$, where $p = (q+1-\varepsilon)\log n/n$. We will present a strategy for Waiter to isolate a vertex in Client's graph.

Let k be a positive integer and let I_k be an independent set in G such that $|I_k| \geqslant 2(q+1)^{k+1}/q^k$ and $d_G(u) = k$ for every $u \in I_k$. For every $u \in I_k$, let $E(u) = \{e \in E(G) : u \in e\}$ and let $X = \bigcup_{u \in I_k} E(u)$. Waiter isolates a vertex of I_k in Client's graph by following the strategy for the $(1:q)$ box game on $\{E(u) : u \in I_k\}$ which is described in the proof of Proposition 5.2.6.

Since $|I_k| \geqslant 2(q+1)^{k+1}/q^k$, it follows by Proposition 5.2.6 that Waiter can indeed isolate a vertex in Client's graph. Hence, it remains to prove that Waiter can play according to the proposed strategy. In order to do so, it suffices to show that *a.a.s.* a positive integer k and an independent set I_k as above exist.

For every $0 \leqslant i \leqslant n-1$, let $X_i = |\{u \in V(G) : d_G(u) = i\}|$ and let $\mu_i = \mathbb{E}[X_i]$. Then

$$\sum_{i=0}^{n-1} \left(\frac{q}{q+1}\right)^i \mu_i = n\left(1 - \frac{p}{q+1}\right)^{n-1} \tag{5.3.1}$$

$$\geqslant n \exp\left\{-\frac{n-1}{n}\left(\frac{(q+1-\varepsilon)\log n}{q+1} + \frac{(q+1-\varepsilon)^2\log^2 n}{n(q+1)^2}\right)\right\}$$

$$\geqslant n \exp\left\{-\left(1 - \frac{\varepsilon}{2(q+1)}\right)\log n\right\} \geqslant n^{\delta}, \tag{5.3.2}$$

where the first equality holds by Lemma 5.2.15, the first inequality follows from the fact that $e^{-(x+x^2)} \leqslant 1-x$ holds for sufficiently small $x > 0$ by the Taylor expansion of e^{-y}, and the last inequality holds for a sufficiently small constant $\delta > 0$. Since, moreover,

$$\sum_{i=9(q+1-\varepsilon)\log n}^{n-1} \left(\frac{q}{q+1}\right)^i \mu_i = o(1),$$

holds by Lemma 5.2.17, it follows from (5.3.1) that

$$\sum_{i=0}^{9(q+1-\varepsilon)\log n} \left(\frac{q}{q+1}\right)^i \mu_i \geqslant n^\delta/2.$$

Hence, there exists an integer $0 \leqslant k \leqslant 9(q+1-\varepsilon)\log n$ such that

$$\left(\frac{q}{q+1}\right)^k \mu_k \geqslant \frac{n^\delta}{18(q+1)\log n}.$$

In particular, $\mu_k \to \infty$ as $n \to \infty$ holds for this value of k and thus, by Lemma 5.2.16, a.a.s $X_k \geqslant \mu_k/2$. It follows that *a.a.s.*

$$\frac{q^k}{2(q+1)^{k+1}} \cdot X_k \geqslant \frac{q^k}{2(q+1)^{k+1}} \cdot \frac{\mu_k}{2} \geqslant \frac{n^\delta}{72(q+1)^2\log n}. \tag{5.3.3}$$

Let $S_k = \{u \in V(G) : d_G(u) = k\}$ and let $I_k \subseteq S_k$ be an independent set of maximum size. It is easy to see that

$$|I_k| \geqslant \frac{|S_k|}{k+1} = \frac{X_k}{k+1} \geqslant \frac{n^\delta}{72(k+1)(q+1)^2\log n} \cdot \frac{2(q+1)^{k+1}}{q^k} \geqslant \frac{2(q+1)^{k+1}}{q^k},$$

where the second inequality holds by (5.3.3) and the last inequality holds for sufficiently large n since $k \leqslant 9(q+1-\varepsilon)\log n$. $\square$

Chapter 6

Conclusion and Open Problems

6.1 Conclusion

This thesis focuses on two types of biased positional games: Waiter–Client and Client–Waiter games played on graphs, hypergraphs and clauses of boolean variables, with various properties such as graph colourability and satisfiability of a boolean formula defining the winning sets. By developing winning strategies for Waiter and Client, we determine the winner of each game we study across almost all values of Waiter's bias when Client's bias is fixed at 1. More precisely, we give an approximate value for the threshold bias of these games, whose close proximity to the threshold bias predicted by the typical outcome of a game played by random players adds the games we study to those that exhibit the probabilistic intuition. We also characterise those probabilities for which Waiter and Client can *a.a.s.* build a Hamilton cycle in the binomial random graph when playing a fixed bias Waiter–Client and Client–Waiter game respectively. In short, we concern ourselves with discovering which player wins when Waiter and Client play a selection of positional games optimally.

Given that each game we consider is played on a finite board, with each board element allowed to be claimed at most once by any player throughout the game, one may wonder why we invest our efforts developing explicit winning strategies for Waiter and Client when a computer could simply perform an exhaustive search of all possible sequences of moves to determine the optimal winner. In fact, classical game theory deems positional games like those we study here trivially solved for precisely this reason. However, although theoretically a computer could find a solution for us, in reality the amount of time required to achieve this is often too great. A good example of this is the generalised 3–dimensional version of Tic–Tac–Toe played on an $n \times n \times n$ board. Even when n is as small as 5, we require around 3^{125} steps to explore all possible game routes. In comparison, the estimated age of the universe is less than 3^{54} seconds old. In light of this, an exhaustive search method is particularly impractical for the games addressed in this thesis since, although finite, each board considered can be as large as you like. In fact, the larger the board, the more accurate our results are.

It is tempting to believe that a smarter search process, one that exploits patterns in the game say, can be utilised to make this method a more feasible option (existing techniques for exhaustive search are discussed in [79]). However, despite the simplistic setting of a positional game, patterns in the game play are not obviously present and this unpredictability makes their analysis difficult. This is evident from the lack of knowledge we have regarding the outcome of generalised d–dimensional Tic–Tac–Toe on an n^d board. We only have solutions for the case $d = 3$, $n = 3, 4$; $n = 3$ is trivially a win for the first player and the proof that this is also true for the case $n = 4$, developed by Patashnik [80] in 1980, is much more difficult and computer aided.

How do we overcome this *combinatorial chaos*? One may suggest that an analysis of the random game could help. However, since we are interested in explicit winning strategies that guarantee a win for the player that obeys them, simulating the random game alone is not enough. Positional game theory bridges the gap through potential–type arguments to prove results like the Erdős–Selfridge theorem discussed in Chapter 2. There, we saw how these are used to convert a probabilistic analysis of the random game into deterministic optimal winning strategies. In fact, this approach can also be used to de–randomise randomised algorithms in a similar way. These potential arguments enable us to develop explicit winning strategies in Maker–Breaker, Avoider–Enforcer, Waiter–Client and Client–Waiter games without the use of a computer or exhaustive search through the game tree. For the latter two games, we have witnessed the power of such arguments throughout this thesis. Despite strong games like Tic–Tac–Toe being notoriously more difficult to analyse than the aforementioned weak games, strategies for the latter can sometimes be adapted to work for strong games too. For example, this is true for the Maker–Breaker perfect matching [46], Hamiltonicity [46], and k–connectivity games [47].

The probabilistic intuition also has the potential to use what happens in the random game to determine who wins the optimal game, provided we arrive at some characterisation of the positional games that exhibit it. Unlike the potential–type arguments, this heuristic cannot provide explicit winning strategies along with the knowledge of which player wins. However, one may be encouraged to use Monte–Carlo methods to investigate possible strategies if one knows the game exhibits the probabilistic intuition. These methods involve players choosing their next elements to claim based on the percentage of times such a choice leads to a win amongst many simulated random games that start from the current point of play. It makes sense for us to have more confidence in these Monte–Carlo methods if we know the player with a winning strategy is the player who wins a random game most of the time. These methods have certainly found success in finding strategies for the game Go. Initially, the computer fared poorly when playing against a human opponent. However, when computers started using variations of Monte–Carlo methods, they were able to challenge much higher level Go players. A huge breakthrough in this area arose in 2016 when the Google DeepMind program AlphaGo beat the number one world Go champion Lee Sodel without handicap, winning four out of the five matches played, by combining Monte–Carlo tree search with deep neural networks. A greater understanding of which games exhibit the probabilistic intuition and why may enable us to develop more of an intuition about the success of these Monte–Carlo methods. For more in depth conversation regarding the probabilistic intuition, its connection to Monte–Carlo methods, and combinatorial chaos, the interested reader is invited to read [13] and [64], whose material inspired much of this discussion.

The most common approach towards gaining more understanding about this heuristic is to continue locating the threshold bias of unexplored games, using the aforementioned potential methods to create winning strategies, and compare their values with what is predicted by the probabilistic intuition. The many examples of Waiter–Client games that exhibit strong probabilistic intuition, together with the fact that games with Waiter and Client are currently less well studied than Maker–Breaker and Avoider–Enforcer games, motivated us to do this for the games in this thesis.

For any given game, the relationship between the inverse of the threshold bias, the probability threshold of a winning set appearing in the random graph and the probability threshold for the appearance of a graph on which a specific player has a winning strategy is also of interest. We know of Maker–Breaker games for which these three parameters are equal. For example, this is true for the Maker–Breaker games mentioned previously, whose strategies can be adapted to work for strong games. Our study of the Hamiltonicity game on the random graph may help to understand this relationship further in the Waiter–Client and Client–Waiter settings.

6.2 Open Problems

6.2.1 Complete–Minor and Planarity Games

In Chapter 3, we found that the asymptotic threshold bias of the Waiter–Client K_t–minor game $(E(K_n), \mathcal{M}_t)$, for every t in the range $4 \leqslant t = \mathcal{O}(\sqrt{n})$, is $(1 + o(1))n$. By devoting further attention to the case where t is large, we additionally found that the threshold bias can be pushed below n for such t. We also found that the Client–Waiter version has asymptotic threshold bias $(1/2 + o(1))n$. As discussed in Chapter 1, these results evidence an exhibition of the probabilistic intuition by both games. Most notably, the threshold bias of the Waiter–Client version is asymptotically equivalent to that predicted by the heuristic and therefore exhibits *strong* probabilistic intuition. Due to Kuratowski's Theorem, the Waiter–Client and Client–Waiter non–planarity games $(E(K_n), \mathcal{NP})$ inherit the same asymptotic threshold biases as their complete–minor counterparts and therefore also exhibit the probabilistic intuition. These findings give rise to the following two open problems.

The Contraction Clique Number

For a graph G, $ccl(G)$ denotes its *contraction clique number*; the largest t such that G contains a K_t–minor. Much is known about this graph invariant in the context of the binomial random graph $\mathcal{G}(n,p)$ for a wide range of probabilities p. In the *sub–critical regime* (when $p \leqslant (1-\varepsilon)/n$ for any $\varepsilon > 0$) $\mathcal{G}(n,p)$ *a.a.s.* contains at most one cycle (see *e.g.* [25, 69]) and hence $ccl(\mathcal{G}(n,p)) \leqslant 3 = \mathcal{O}(1)$. In the *super–critical regime* (when $p \geqslant (1+\varepsilon)/n$ for any $\varepsilon > 0$) Fountoulakis, Kühn and Osthus [48] found that $ccl(\mathcal{G}(n,p)) = \Theta(\sqrt{n})$. Additionally, in the same paper, they also studied $ccl(\mathcal{G}(n,p))$ in the *critical window i.e.* when $p = (1+\lambda n^{-1/3})/n$ for some $\lambda \in \mathbb{R}$. Indeed, by building on research by Łuczak in [88] and [89], they found that *a.a.s.* $ccl(\mathcal{G}(n,p)) = \Theta(\lambda^{3/2})$ for such p with $1 \ll \lambda \ll n^{1/3}$.

Since Client's graph G_C at the end of a $(1:q)$ game on $E(K_n)$ has the same number of edges as those expected to appear in the random graph $\mathcal{G}(n, 1/(q+1))$, we can transfer these notions of the sub–critical regime, the super–critical regime and the critical window to the game setting. More precisely, we can say that the sub–critical regime describes the range $(1+\varepsilon)n \leqslant q \leqslant \binom{n}{2} - 1$ and the super–critical regime describes $1 \leqslant q \leqslant (1-\varepsilon)n$, for some $\varepsilon > 0$. In the same way, the critical window consists of those bias values q for which $q = (1 - \lambda n^{-1/3})n$ for some $\lambda \in \mathbb{R}$. We showed in Theorem 1.5.1 of Chapter 3 that Client's graph G_C at the end of the $(1 : q)$ Waiter–Client K_t–minor game, in which both players play optimally, *a.a.s.* satisfies $ccl(G_C) = \Theta(ccl(\mathcal{G}(n, 1/(q+1))))$ in both the sub–critical and super–critical regimes. It would be interesting to see if this is also true in the critical window. For this range of q, Theorem 1.5.1 provides the non–trivial lower bound $ccl(G_C) \geqslant c\lambda^2 n^{-1/6}$, for some constant $c > 0$ when $1 \ll \lambda \ll n^{1/3}$ (note that, since $c\lambda^2 n^{-1/6} \ll 1$ when $1 \ll \lambda \ll n^{1/12}$, the aforementioned lower bound is trivially true for this range of λ). However, as $\lambda^2 n^{-1/6} \ll \lambda^{3/2}$ when $1 \ll \lambda \ll n^{1/3}$, we have some way to go before we can show that $ccl(G_C) = \Theta(ccl(\mathcal{G}(n, 1/(q+1))))$ in the critical window.

Building a Larger Complete–Minor

Since we've seen that Waiter can force a complete–minor of order $\Theta(\sqrt{n})$ to be built in G_C when playing the Waiter–Client K_t–minor game, we would also like to see if Client can do the same in the Client–Waiter version. Even when $q = (1/2 - \varepsilon)n$, for some arbitrarily small but fixed $\varepsilon > 0$, Theorem 1.5.2 only provides a strategy for Client to build a complete–minor of order $\Theta(n^\gamma)$, where $\gamma = \gamma(\varepsilon) > 0$ is a small constant.

6.2.2 Colourability and k–SAT Games

In Chapter 4, we showed that the threshold bias for both the Waiter–Client and Client–Waiter versions of the non–r–colourability game $(E(K_n), \mathcal{NC}_r^{(2)})$ has order $\Theta(n/(r\log r))$. We also gave tighter bounds on the threshold bias for the case $r = 2$. Following this, we generalised our proofs to the hypergraph setting to show that the Waiter–Client and Client–Waiter non–r–colourability games $(E(K_n^{(k)}), \mathcal{NC}_r^{(k)})$ have threshold bias $\frac{1}{n}\binom{n}{k}r^{\mathcal{O}_k(k)}$ and $\frac{1}{n}\binom{n}{k}r^{-k(1+o_k(1))}$ respectively. The case $r = 2$ in these hypergraph games additionally gave rise to proving that the threshold bias of the Waiter–Client and Client–Waiter k–SAT games $(\mathcal{C}_n^{(k)}, \mathcal{F}_{SAT})$ is $\frac{1}{n}\binom{n}{k}$ up to a factor that is exponential and polynomial in k

respectively. As discussed in Chapter 1, our findings show that all of these games exhibit the probabilistic intuition.

Tighter Bounds on the Threshold Bias

Although our bounds on the threshold bias for the Waiter–Client and Client–Waiter colourability and k–SAT games are accurate enough to show that these games exhibit the probabilistic intuition, they are not as tight in comparison to the asymptotic threshold biases found for the corresponding K_t–minor and non–planarity games. Therefore, we would like to see these bounds improved, especially in the Waiter–Client versions of $(E(K_n^{(k)}), \mathcal{NC}_r^{(k)})$ and $(\mathcal{C}_n^k, \mathcal{F}_{SAT})$ where the multiplicative gap is exponential in k. In particular, we conjecture that all of the games discussed in Chapter 4 exhibit *strong* probabilistic intuition, as witnessed to be true for the Waiter–Client games discussed in Chapter 3.

Conjecture 6.2.1 *Let the threshold bias for the* $(1:q)$ *Waiter–Client and Client–Waiter non–r–colourability games* $(E(K_n), \mathcal{NC}_r^{(2)})$ *be denoted by* $b^{WC}_{\mathcal{NC}_r^{(2)}}$ *and* $b^{CW}_{\mathcal{NC}_r^{(2)}}$ *respectively and let*

$$h(x) := \lim_{n\to\infty} \frac{n}{x \cdot 2r \log r}.$$

Then $h\left(b^{WC}_{\mathcal{NC}_r^{(2)}}\right) = h\left(b^{CW}_{\mathcal{NC}_r^{(2)}}\right) = 1.$

Conjecture 6.2.2 *Let the threshold bias for the* $(1:q)$ *Waiter–Client and Client–Waiter non–r–colourability games* $(E(K_n^{(k)}), \mathcal{NC}_r^{(k)})$ *be denoted by* $b^{WC}_{\mathcal{NC}_r^{(k)}}$ *and* $b^{CW}_{\mathcal{NC}_r^{(k)}}$ *respectively and let*

$$f(x) := \lim_{k\to\infty} \left\{ \lim_{n\to\infty} \frac{1}{n} \binom{n}{k} \frac{1}{x \cdot r^{k-1} \log r} \right\}.$$

Then $f\left(b^{WC}_{\mathcal{NC}_r^{(k)}}\right) = f\left(b^{CW}_{\mathcal{NC}_r^{(k)}}\right) = 1.$

Conjecture 6.2.3 *Let the threshold bias for the* $(1:q)$ *Waiter–Client and Client–Waiter*

k–SAT games $(\mathcal{C}_n^{(k)}, \mathcal{F}_{SAT})$ be denoted by $b^{WC}_{\mathcal{F}_{SAT}}$ and $b^{CW}_{\mathcal{F}_{SAT}}$ respectively and let

$$g(x) := \lim_{k\to\infty}\left\{\lim_{n\to\infty}\frac{1}{n}\binom{n}{k}\frac{1}{x\log 2}\right\}.$$

Then $g\left(b^{WC}_{\mathcal{F}_{SAT}}\right) = g\left(b^{CW}_{\mathcal{F}_{SAT}}\right) = 1.$

Despite Theorems 4.1.1 and 4.1.2 improving our bounds on the threshold bias for the Waiter–Client and Client–Waiter non–r–colourability games $(E(K_n), \mathcal{NC}_r^{(2)})$ in the case $r = 2$, there is still room for further improvement here. It was conjectured in [18] that Client can avoid building any cycle if $q \geqslant (1 + o(1))n$. Thus, we believe our upper bound on the threshold bias given in Theorem 4.1.1 should match this. For the Client–Waiter version, we believe that the lower bound on the threshold bias given in Theorem 4.1.2 can be improved to match the upper bound, and therefore believe that its asymptotic threshold bias should be $(1/2 + o(1))n$.

6.2.3 Hamiltonicity Games on the Random Graph

In Chapter 5, we determined sharp thresholds for the $(1 : q)$ Waiter–Client and Client–Waiter Hamiltonicity games $(E(\mathcal{G}(n, p)), \mathcal{HAM})$, for every fixed positive integer q. For the Waiter–Client version, it is $\log n/n$; in particular it does not depend on q and is asymptotically the same as the sharp threshold for the appearance of a Hamilton cycle in $\mathcal{G}(n, p)$. On the other hand, the sharp threshold for the Client–Waiter Hamiltonicity game on $\mathcal{G}(n, p)$ is $(q + 1)\log n/n$ and thus does grow with q.

Non–Constant q

It is natural to study the behaviour of these thresholds for non–constant values of q as well. As discussed in Chapter 1, this was done by Ferber, Glebov, Krivelevich and Naor in [45] for $(1 : q)$ Maker–Breaker and Avoider–Enforcer Hamiltonicity games on $\mathcal{G}(n, p)$. However, since a player with a strategy to avoid a winning set, when playing on

$E(K_n)$, may use this same strategy to avoid when playing on a sparser board, we are only interested in finding probability thresholds for $\mathcal{W}^q_{\mathcal{HAM}}$ and $\mathcal{C}^q_{\mathcal{HAM}}$ when the winner of the $(1:q)$ game on K_n is the player wishing to build a Hamilton cycle in G_C *i.e.* Waiter in the Waiter–Client game and Client in the Client–Waiter game. It was proved in [18] by Bednarska–Bzdęga, Hefetz, Krivelevich and Łuczak that the largest q for which Waiter has a winning strategy in the $(1:q)$ Waiter–Client Hamiltonicity game $(E(K_n), \mathcal{HAM})$ is of linear order. Moreover, using a similar argument to the one employed in [77], Bednarska–Bzdęga, Hefetz and Łuczak [19] showed that the largest q for which Client has a winning strategy in the Client–Waiter version is $(1-o(1))n/\log n$. Thus, it would be interesting to determine threshold probabilities for $\mathcal{W}^q_{\mathcal{H}}$ for every $q = \mathcal{O}(n)$ and for $\mathcal{C}^q_{\mathcal{H}}$ for every $q \leqslant (1-o(1))\log n/n$.

The Threshold Bias as a Function of Probability p

Another open problem related to the Hamiltonicity game on $\mathcal{G}(n,p)$ is to understand how the threshold biases $b^{WC}_{\mathcal{HAM}}$ and $b^{CW}_{\mathcal{HAM}}$, for the Waiter–Client and Client–Waiter games respectively, vary as a function of probability p. From the aforementioned results concerning play on K_n, we know that $b^{WC}_{\mathcal{HAM}} = \Theta(n)$ and $b^{CW}_{\mathcal{HAM}} = (1-o(1))n/\log n$ when $p = 1$. Additionally, since we found that $\log n/n$ and $(q+1)\log n/n$ are sharp thresholds for $\mathcal{W}^q_{\mathcal{HAM}}$ and $\mathcal{C}^q_{\mathcal{HAM}}$ respectively when q is constant, $b^{WC}_{\mathcal{HAM}}(p) = 1$ for every $p \leqslant (1-\varepsilon)\log n/n$, where $\varepsilon > 0$, and $b^{CW}_{\mathcal{HAM}}(p) = 1$ for every $p = \mathcal{O}(\log n/n)$. However, the behaviour of these threshold biases remains unknown for other values of p.

Appendix A
Notation at a Glance

$V(G)$	vertex set of graph/hypergraph G
$E(G)$	edge set of graph/hypergraph G
$v(G)$	number of vertices in $V(G)$
$e(G)$	number of edges in $E(G)$
$E_G(A)$	set of edges of graph G with both endpoints in set A
$e_G(A)$	number of edges in $E_G(A)$
$E_G(A,B)$	set of edges of graph G with one endpoint in A and one endpoint in B
$e_G(A,B)$	number of edges in $E_G(A,B)$
$G[S]$	subgraph/subhypergraph of graph/hypergraph G induced on set S
$N_G(A)$	outer neighbourhood of set A in graph G
$d_G(u)$	degree of vertex u in graph G
$\Delta(G)$	maximum degree of graph/hypergraph G
$\delta(G)$	minimum degree of graph/hypergraph G
$\alpha(G)$	independence number of graph/hypergraph G
K_n	complete graph on n vertices
$\omega(G)$	clique number of graph/hypergraph G
$\chi(G)$	chromatic number of graph G
E_W	set of all edges currently owned by Waiter
E_C	set of all edges currently owned by Client
E_F	set of all edges currently free
G_W	graph with edge set E_W and vertex set equal to the board
G_C	graph with edge set E_C and vertex set equal to the board

G_F	graph with edge set E_F and vertex set equal to the board
$E_{\mathcal{H}}(S)$	set of edges of hypergraph $\mathcal{H}$ with exactly one endpoint in set S
$d_{\mathcal{H}}(S)$	number of edges in $E_{\mathcal{H}}(S)$ (degree of set S in $\mathcal{H}$)
$d_{\mathcal{H}}(v)$	$d_{\mathcal{H}}(\{v\})$ for vertex $v \in V(\mathcal{H})$
$\chi(\mathcal{H})$	weak chromatic number of hypergraph $\mathcal{H}$
$K_n^{(k)}$	complete k–uniform hypergraph on n vertices
$\mathcal{H}_C$	hypergraph with edge set E_C and vertex set equal to the board
$\mathcal{H}_F$	hypergraph with edge set E_F and vertex set equal to the board
$\mathcal{G}(n, p)$	binomial random graph on n vertices with edge probability p
$\mathcal{M}_t$	set of edge–sets of all K_t–minors contained in K_n
$\mathcal{NP}$	set of edge–sets of all non–planar subgraphs of K_n
$\mathcal{NC}_r^{(k)}$	set of edge–sets of all non–r–colourable subgraphs of $K_n^{(k)}$
$\mathcal{F}_{SAT}$	set of sets of clauses whose conjunction is a non–satisfiable k–CNF boolean formulae on n boolean variables
$\mathcal{C}_n^{(k)}$	set of all k–clauses on n boolean variables
$\bigwedge \mathcal{S}$	conjunction of all k–clauses in set S
$\mathcal{HAM}$	set of edge–sets of all Hamiltonian subgraphs of K_n
$\mathcal{W}_{\mathcal{HAM}}^q$	set of all subgraphs of K_n on which Waiter wins the $(1 : q)$ Waiter–Client Hamiltonicity game
$\mathcal{C}_{\mathcal{HAM}}^q$	set of all subgraphs of K_n on which Client wins the $(1 : q)$ Client–Waiter Hamiltonicity game
$\mathcal{P}(e)$	set of all parts of partition $\mathcal{P}$ that contain a vertex of edge e
$\Delta_{\mathcal{P}}(\mathcal{H})$	maximum number of edges in hypergraph $\mathcal{H}$ that contain a vertex of a single part in partition $\mathcal{P}$ of $V(\mathcal{H})$
a.a.s.	asymptotically almost surely; with probability tending to 1 as n tends to infinity
log	$\log_e$ or ln

Appendix B
Counterexamples for Bias Monotonicity

The following example shows that Waiter–Client games are not bias monotone in Client's bias.

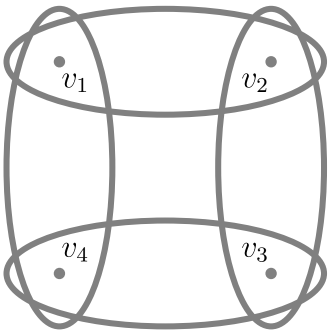

Figure B.1: Setting of the Waiter–Client game $(X, \mathcal{F})$ in Example B.0.4.

Example B.0.4 *Consider a Waiter–Client game* $(X, \mathcal{F})$ *with board* $X = \{v_1, v_2, v_3, v_4\}$ *and set* $\mathcal{F} = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_1, v_4\}\}$ *of winning sets (see Fig. B.1). A winning strategy for Waiter in the* $(1 : 1)$ *game is as follows. Waiter offers elements* v_1, v_3 *in the first round. He then offers the remaining elements* v_2, v_4 *in the second, and final, round. It is clear that Client cannot avoid fully claiming a winning set when Waiter plays in this way.*

However, Client wins the $(2 : 1)$ *game* $(X, \mathcal{F})$*. Indeed, no matter which three free elements Waiter offers in the first round, there must exist two elements that do not form a winning set. Client claims these two elements. Then, in the second, and final, round, only one free element remains for Waiter to offer which Client rejects. Thus, Client avoids fully claiming a winning set by playing in this way.*

Since Waiter wins the $(1 : 1)$ *game whilst Client wins the* $(2 : 1)$ *game, increasing Client's bias does not help Waiter. Its increase does not help Client either. This is illustrated by the fact that Waiter wins the* $(3 : 1)$ *game. Indeed, since every set of three* *elements of X contain a winning set, Waiter need only offer all elements of X in the first round of this game to win.*

The following example shows that Client–Waiter games are not bias monotone in Waiter's bias.

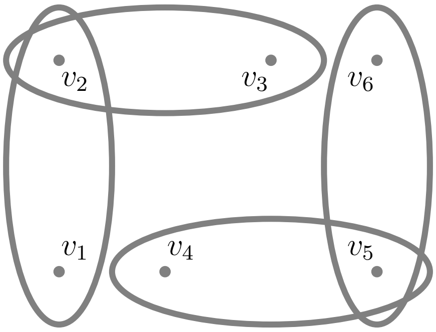

Figure B.2: Setting of the Client–Waiter game $(X, \mathcal{F})$ in Example B.0.5.

Example B.0.5 *Consider a Client–Waiter game $(X, \mathcal{F})$ with board $X = \{v_i : i \in [6]\}$ and set $\mathcal{F} = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_4, v_5\}, \{v_5, v_6\}\}$ of wining sets (see Fig. B.2). For any $i \geqslant 1$, in round i of this game, we denote the set of elements offered by Waiter by Z_i.*

The following is a winning strategy for Client in the $(1 : 1)$ game.

<u>If $\{v_2, v_5\} \cap Z_1 \neq \emptyset$</u>: Client claims an arbitrary element $x \in \{v_2, v_5\} \cap Z_1$. Since $|Z_1| = 2$, there exists a winning set $A \in \mathcal{F}$ such that $A \cap Z_1 = \{x\}$. As all winning sets consist of two elements, there exists $y \in A \setminus Z_1$ that is free immediately after round 1. Client then plays arbitrarily until Waiter offers y, at which point Client claims y to fully claim $A \in \mathcal{F}$.

<u>If $\{v_2, v_5\} \cap Z_1 = \emptyset$</u>: Then there exists $A \in \mathcal{F}$ such that $A \cap Z_1 = \{x\}$ for some $x \in X$ which Client claims in round 1, leaving some free element $y \in A \setminus Z_1$. Client plays arbitrarily until Waiter offers y, at which point Client claims y to fully claim $A \in \mathcal{F}$.

However, Waiter wins the $(1 : 2)$ if he plays such that $Z_1 = \{v_1, v_2, v_3\}$ and $Z_2 = \{v_4, v_5, v_6\}$. It is clear that, if Waiter does so, Client cannot fully claim a winning set. Thus, increasing Waiter's bias harms Client, since he wins the $(1 : 1)$ game but loses the $(1 : 2)$ game. In fact, increasing Waiter's bias does not help Waiter either since Client wins the $(1 : 3)$ game. His strategy is as follows.

Since $|Z_1| = 4$ in the $(1 : 3)$ game, there exists a winning set $A \in \mathcal{F}$ such that $A \cap Z_1 = \{x\}$ for some $x \in X$. Client claims x in the first round, leaving a free element $y \in A \setminus Z_1$. Client then plays arbitrarily until Waiter offers y, at which point Client claims y to fully claim $A \in \mathcal{F}$.

List of References

[1] D. Achlioptas, J. H. Kim, M. Krivelevich, and P. Tetali. Two–coloring random hypergraphs. *Random Structures and Algorithms*, 20(2):249–259, 2002.

[2] D. Achlioptas and C. Moore. The asymptotic order of the random k–SAT threshold. In *Proc. 43rd Annual Symposium on Foundations of Computer Science*, pages 126–127, 2002.

[3] D. Achlioptas and C. Moore. On the 2–colorability of random hypergraphs. In *Proc. 6th International Workshop on Randomization and Approximation Techniques*, pages 78–90, 2002.

[4] D. Achlioptas and C. Moore. Random k–SAT: Two moments suffice to cross a sharp threshold. *SIAM Journal of Computing*, 36:740–762, 2006.

[5] D. Achlioptas and A. Naor. The two possible values of the chromatic number of a random graph. *Annals of Mathematics*, 162:1335–1351, 2005.

[6] D. Achlioptas and Y. Peres. The threshold for random k–SAT is $2^k \log 2 - \mathcal{O}(k)$. *Journal of the American Mathematical Society*, 17(4):947–973, 2004.

[7] N. Alon and M. Krivelevich. The concentration of the chromatic number of random graphs. *Combinatorica*, 17:303–313, 1997.

[8] N. Alon and J. Spencer. A note on coloring random k–sets. Unpublished manuscript.

[9] N. Alon and J. H. Spencer. *The Probabilistic Method.* Wiley–Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, 4th edition, 2015.

[10] A. Ayre, A. Coja-Oghlan, and C. Greenhill. Hypergraph coloring up to condensation. arXiv:1508.01841.

[11] J. Beck. Remarks on positional games. *Acta Math. Acad. Sci. Hungar.*, 40:65–71, 1982.

[12] J. Beck. Positional games and the second moment method. *Combinatorica*, 22:169–216, 2002.

[13] J. Beck. *Combinatorial games: Tic–tac–toe theory.* Encyclopedia of Mathematics and its Applications 114. Cambridge University Press, Cambridge, 2008.

[14] M. Bednarska and O. Pikhurko. Biased positional games on matroids. *European Journal of Combinatorics*, 26:271–285, 2005.

[15] M. Bednarska and O. Pikhurko. Odd and even cycles in Maker–Breaker games. *European Journal of Combinatorics*, 29:742–745, 2008.

[16] M. Bednarska and T. Łuczak. Biased positional games for which random strategies are nearly optimal. *Combinatorica*, 20:477–488, 2000.

[17] M. Bednarska-Bzdęga. On weight function methods in Chooser–Picker games. *Theoretical Computer Science*, 475:21–33, 2013.

[18] M. Bednarska-Bzdęga, D. Hefetz, M. Krivelevich, and T. Łuczak. Manipulative Waiters with probabilistic intuition. *Combinatorics, Probability and Computing*, 25:823–849, 2016.

[19] M. Bednarska-Bzdęga, D. Hefetz, and T. Łuczak. On biased Chooser–Picker and Picker–Chooser games. Unpublished.

[20] M. Bednarska-Bzdęga, D. Hefetz, and T. Łuczak. Picker–Chooser fixed graph games. *Journal of Combinatorial Theory Series B*, 119:122–154, 2016.

[21] S. Ben-Shimon, A. Ferber, D. Hefetz, and M. Krivelevich. Hitting time results for Maker–Breaker games. *Random Structures and Algorithms*, 41:23–46, 2012.

[22] T. Bohman and A. Frieze. Avoiding a giant component. *Random Structures and Algorithms*, 19:75–85, 2001.

[23] T. Bohman and D. Kravitz. Creating a giant component. *Combinatorics, Probability and Computing*, 15:489–511, 2006.

[24] B. Bollobás. The evolution of sparse graphs. In *Graph Theory and Combinatorics*, pages 35–57, London, 1984. Academic Press.

[25] B. Bollobás. *Random Graphs.* Academic Press, London, 1985.

[26] B. Bollobás and P. Erdős. Cliques in random graphs. *Math. Proc. Cambridge Philos. Soc.*, 80:419–427, 1976.

[27] B. Bollobás and A. Thomason. Threshold functions. *Combinatorica*, 7(1):35–38, 1986.

[28] B. Bollobás and A. Thomason. The chromatic number of random graphs. *Combinatorica*, 8:49–55, 1988.

[29] V. Chvátal and P. Erdős. Biased positional games. *Annals of Discrete Math.*, 2:221–228, 1978.

[30] V. Chvátal and B. Reed. Mick gets some (the odds are on his side). In *Proc. 33rd Annual Symposium on Foundations of Computer Science*, pages 620–627, 1992.

[31] D. Clemens, J. Ehrenmüller, Y. Person, and T. Tran. Keeping Avoider's graph almost acyclic. *The Electronic Journal of Combinatorics*, 22(1):P1.60, 2015.

[32] A. Coja-Oghlan. Upper-bounding the k-colorability threshold by counting covers. *Electronic Journal of Combinatorics*, 20:P32, 2013.

[33] A. Coja-Oghlan and K. Panagiotou. The asymptotic k–SAT threshold. *Advances in Mathematics*, 288:985 –1068, 2016.

[34] A. Coja-Oghlan, K. Panagiotou, and A. Steger. On the chromatic number of random graphs. *Journal of Combinatorial Theory, Series B*, 98(5):980–993, 2008.

[35] A. Coja-Oghlan and D. Vilenchik. Chasing the k-colorability threshold. In *Proc. 54th FOCS*, pages 380–389, 2013.

[36] A. Coja-Oghlan and L. Zdeborová. The condensation transition in random hypergraph 2–coloring. In *Proceedings of the Twenty-third Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '12, pages 241–250, Philadelphia, PA, USA, 2012. Society for Industrial and Applied Mathematics.

[37] S. Cook. The complexity of theorem proving procedures. In *Proceedings of the Third Annual ACM Symposium on Theory of Computing*, pages 151–158, 1971.

[38] A. Csernenszky, C. I. Mándity, and A. Pluhár. On Chooser–Picker positional games. *Discrete Mathematics*, 309:5141–5146, 2009.

[39] V. Dani, J. Diaz, T. Hayes, and C. Moore. *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques: 16th International Workshop, APPROX 2013, and 17th International Workshop, RANDOM 2013, Berkeley, CA, USA, August 21-23, 2013. Proceedings*, chapter The Power of Choice for Random Satisfiability, pages 484–496. Springer Berlin Heidelberg, Berlin, Heidelberg, 2013.

[40] O. Dean and M. Krivelevich. Client–Waiter games on complete and random graphs. *Electronic Journal of Combinatorics*, 23(4):P4.38, 2016.

[41] M. Dyer, A. Frieze, and C. Greenhill. On the chromatic number of a random hypergraph. *Journal of Combinatorial Theory, Series B*, 113:68–122, 2015.

[42] P. Erdős and L. Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. In A. Hajnal et al., editor, *Infinite and Finite Sets*, pages 609–628. North–Holland, Amsterdam, 1975.

[43] P. Erdős and A. Rényi. On the evolution of random graphs. *Magyar Tud. Akad. Mat. Kutató Int. Kőzl.*, 5:1761, 1960.

[44] P. Erdős and J. L. Selfridge. On a combinatorial game. *Journal of Combinatorial Theory Series A*, 14:298–301, 1973.

[45] A. Ferber, R. Glebov, M. Krivelevich, and A. Naor. Biased games on random graphs. *Random Structures and Algorithms*, 46:651–676, 2015.

[46] A. Ferber and D. Hefetz. Winning strong games through fast strategies for weak games. *Electronic Journal of Combinatorics*, 18(1):P144, 2011.

[47] A. Ferber and D. Hefetz. Weak and strong k–connectivity games. *European Journal of Combinatorics*, 35:169–183, 2014.

[48] N. Fountoulakis, D. Kühn, and D. Osthus. Minors in random regular graphs. *Random Structures and Algorithms*, 35:444–463, 2009.

[49] J. Franco and M. Paull. Probabilistic analysis of the Davis-Putnam procedure for solving the satisfiability problem. *Discrete Applied Mathematics*, 5(1):77–87, 1983.

[50] E. Friedgut. Sharp thresholds of graph properties and the k–sat problem. *J. Amer. Math. Soc.*, 12:1017–1054, 1999.

[51] A. Frieze and M. Krivelevich. On two Hamilton cycle problems in random graphs. *Israel Journal of Mathematics*, 166:221–234, 2008.

[52] A. Frieze and N. C. Wormald. Random k–SAT: a tight threshold for moderately growing k. In *Proc. 5th International Symposium on Theory and Applications of Satisfiability Testing*, pages 1–6, 2002.

[53] A. M. Frieze and S. Suen. Analysis of two simple heuristics on a random instance of k–SAT. *Journal of Algorithms*, 20(2):312–355, 1996.

[54] D. Gale. The game of Hex and the Brouwer Fixed–Point Theorem. *The American Mathematical Monthly*, 86(10):818–827, 1979.

[55] H. Gebauer, R. A. Moser, D. Scheder, and E. Welzl. The Lovász Local Lemma and satisfiability. In *Efficient Algorithms*, pages 30–54. Springer, 2009.

[56] G. R. Grimmett and C. J. H. McDiarmid. On colouring random graphs. *Math. Proc. Cambridge Philos. Soc.*, 77:313–324, 1975.

[57] A. W. Hales and R. I. Jewett. Regularity and positional games. *Transactions of the American Mathematical Society*, 106:222–229, 1963.

[58] Y. O. Hamidoune and M. Las Vergnas. A solution to the box game. *Discrete Mathematics*, 65:157–171, 1987.

[59] D. Hefetz, M. Krivelevich, M. Stojaković, and T. Szabó. Planarity, colorability and minor games. *SIAM Journal on Discrete Mathematics*, 22(1):194–212, 2008.

[60] D. Hefetz, M. Krivelevich, M. Stojaković, and T. Szabó. Avoider–Enforcer: The rules of the game. *Electronic Notes in Discrete Mathematics*, 34:261–265, 2009.

[61] D. Hefetz, M. Krivelevich, M. Stojaković, and T. Szabó. Fast winning strategies in Maker-Breaker games. *Journal of Combinatorial Theory, Series B*, 99:39–47, 2009.

[62] D. Hefetz, M. Krivelevich, M. Stojaković, and T. Szabó. A sharp threshold for the Hamilton cycle Maker–Breaker game. *Random Structures and Algorithms*, 34(1):112–122, 2009.

[63] D. Hefetz, M. Krivelevich, M. Stojaković, and T. Szabó. Avoider–Enforcer games. In *Positional Games*, volume 44 of *Oberwolfach Seminars*, pages 43–60. Springer Basel, 2014.

[64] D. Hefetz, M. Krivelevich, M. Stojaković, and T. Szabó. *Positional Games*. Birkhäuser, 2014.

[65] D. Hefetz, M. Krivelevich, and T. Szabó. Avoider–Enforcer games. *Journal of Combinatorial Theory, Series A*, 114:840–853, 2007.

[66] D. Hefetz, M. Krivelevich, and T. Szabó. Hamilton cycles in highly connected and expanding graphs. *Combinatorica*, 29(5):547–568, 2009.

[67] D. Hefetz, M. Krivelevich, and W. E. Tan. Waiter–Client and Client–Waiter planarity, colorability and minor games. *Discrete Mathematics*, 339(5):1525–1536, 2016.

[68] D. Hefetz, M. Krivelevich, and W. E. Tan. Waiter–Client and Client–Waiter Hamiltonicity games on random graphs. *European Journal of Combinatorics*, 63:26–43, 2017.

[69] S. Janson, T. Łuczak, and A. Ruciński. *Random Graphs*. Wiley, 2000.

[70] J. R. Johnson, I. Leader, and M. Walters. Transitive avoidance games. Submitted, arXiv:1607.02945.

[71] J. H. Kim. On Brook's Theorem for sparse graphs. *Combinatorics, Probability and Computing*, 4:97–132, 1995.

[72] F. Knox. Two constructions relating to conjectures of Beck on positional games. Manuscript, arXiv:1212.3345.

[73] J. Komlós and E. Szemerédi. Limit distributions for the existence of Hamilton circuits in a random graph. *Discrete Mathematics*, 43:55–63, 1983.

[74] M. Krivelevich and G. Kronenberg. Random–player Maker–Breaker games. *The Electronic Journal of Combinatorics*, 22(4):P4.9, 2015.

[75] M. Krivelevich, P.-S. Loh, and B. Sudakov. Avoiding small subgraphs in Achlioptas processes. *Random Structures and Algorithms*, 34:165–195, 2009.

[76] M. Krivelevich, E. Lubetzky, and B. Sudakov. Hamiltonicity thresholds in Achlioptas processes. *Random Structures and Algorithms*, 37:1–24, 2010.

[77] M. Krivelevich and T. Szabó. Biased positional games and small hypergraphs with large covers. *Electronic Journal of Combinatorics*, 15:R70, 2008.

[78] A. Lehman. A solution of the Shannon switching game. *J. Soc. Indust. Appl. Math.*, 12:687–725, 1964.

[79] J. Nievergelt, R. Gasser, F. Mäumlser, and C. Wirth. All the needles in a haystack: Can exhaustive search overcome combinatorial chaos? In J. van Leeuwen, editor, *Computer Science Today: Lecture Notes in Computer Science*, volume 1000, pages 254–274. Berlin: Springer, 1995.

[80] O. Patashnik. Qubic: $4 \times 4 \times 4$ Tic-Tac-Toe. *Mathematics Magazine*, 53(4):202–216, 1980.

[81] W. Perkins. Random k–SAT and the power of two choices. *Random Structures and Algorithms*, 47(1):163–173, 2015.

[82] L. Pósa. Hamiltonian circuits in random graphs. *Discrete Mathematics*, 14:359–364, 1976.

[83] E. Shamir and J. Spencer. Sharp concentration of the chromatic number on random graphs $g_{n,p}$. *Combinatorica*, 7:121–129, 1987.

[84] A. Sinclair and D. Vilenchik. Delaying satisfiability for random 2SAT. *Random Structures and Algorithms*, 43(2):251–263, 2013.

[85] M. Stojaković. *Games on Graphs.* PhD thesis, ETH Zürich, 2005.

[86] M. Stojaković and T. Szabó. Positional games on random graphs. *Random Structures and Algorithms*, 26(1–2), 2005.

[87] W. E. Tan. Waiter–Client and Client–Waiter colourability and k–SAT games. *Electronic Journal of Combinatorics*, 24(2):P2.46, 2017.

[88] T. Łuczak. Component behaviour near the critical point of the random graph process. *Random Structures and Algorithms*, 1:287–310, 1990.

[89] T. Łuczak. Cycles in a random graph near the critical point. *Random Structures and Algorithms*, 2:421–440, 1991.

[90] T. Łuczak. A note on the sharp concentration of the chromatic number of random graphs. *Combinatorica*, 11:295–297, 1991.

[91] D. B. West. *Introduction to Graph Theory*. Prentice Hall, 2001.