Fusion systems on p-groups of sectional rank 3

by

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A thesis submitted to

The University of Birmingham

for the degree of

DOCTOR OF PHILOSOPHY

School of Mathematics

College of Engineering and Physical Sciences

The University of Birmingham

March 2017

Ad Alessandro,

che continua a credere in me..

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ABSTRACT

In this thesis we study saturated fusion systems on p-groups having sectional rank 3, for p odd. We obtain a complete classification of simple fusion systems $\mathcal{F}$ on p-groups having sectional rank 3 for $p \geq 5$, exhibiting a new simple exotic fusion system on a 7-group of order 7^5. We introduce the notion of pearls, defined as essential subgroups isomorphic to the groups $\mathrm{C}_p \times \mathrm{C}_p$ and p_+^{1+2} (for p odd), and we illustrate some properties of fusion systems involving pearls. As for $p = 3$, we determine the isomorphism type of a certain section of the 3-groups considered.

ACKNOWLEDGEMENT

First and foremost, I would like to thank my supervisor, *Professor Chris Parker*, for his patient and enthusiastic guidance throughout the course of this work. His vast knowledge and constant smile motivated me to improve day by day and enabled me to complete this thesis. Through his caring attitude and encouragement, he has also helped me greatly in taking my first steps into an academic career. His love for Mathematics will always be a source of inspiration in my life.

Un grazie particolare va alla mia famiglia. A *mamma e papà*, che non hanno mai smesso di incoraggiarmi e mi hanno insegnato a dare sempre il meglio di me. A *Betty e Gio* che continuano a rendermi orgogliosa di loro e sono i miei fan più entusiasti. *A nonna Adriana*, che mi ha sempre fatta sentire speciale.

Grazie di cuore ad *Alessandro*, a cui dedico questa tesi. Grazie per essere sempre rimasto al mio fianco, anche quando chilometri di distanza ci separavano, per avermi fatta sentire amata e per avermi aiutata ad affrontare con il sorriso le sfide di ogni giorno.

My sincere thanks go out to all the people that I've met over these last four years.

To my office mates, in particular *Esen, David, Raul, Mate, Cristina, Alessandro and Wei En.* Thank you for the laughs, the coffee breaks, the lunches and all the everyday experiences that made this journey unforgettable.

To *Columba*, who has been a good housemate and a great friend. Thank you for our long chats and for all the moments we shared.

To *the Algebra group*, for the good times spent before, during and after the weekly seminar. Thanks particularly *to Ryan*, who was always able to make me smile, and to *Dr David Craven*, who convinced me to be more social and helped me when I needed it.

To *Janette*, who has been like a mum for the PhD students of the School of Mathematics and has had answers to all of my questions.

Thanks to *all my friends* around the world who keep caring about me, even when the distance makes it harder. Grazie in particolare a *Sara*, su cui so che posso sempre contare.

I'd like to thank the *University of Birmingham* for all the opportunities it has given me that have contributed to the person that I am today. Thank you in particular to the Universitas 21 network that gave me the chance to visit the amazing New Zealand and to collect memories that will last for a lifetime.

Finally, I thankfully acknowledge financial support from the EPSRC and the University of Birmingham.

CONTENTS

LIST OF NOTATION

Throughout this work p denotes a prime and we consider only finite groups. All our groups are denoted by upper case Roman letters. Also, we write mappings on the right hand side.

Let G be a finite group and let $P, Q \leq G$ be subgroups.

- $\mathrm{C}_P(Q) = \{x \in P \mid xy = yx \text{ for every } y \in Q\}$ is the centralizer in P of Q;
- $\mathrm{N}_P(Q) = \{x \in P \mid Q^x = Q\}$ is the normalizer in P of Q;
- $\mathrm{N}_G^1(P) = \mathrm{N}_G(P)$ and $\mathrm{N}_G^i(P) = \mathrm{N}_G(\mathrm{N}_G^{i-1}(P))$ for every $i > 1$
  (if there is no danger of confusion we shall simply write $\mathrm{N}^i(P)$);
- if x, y are elements of G then $[x, y] = x^{-1}y^{-1}xy$;
- $[P, Q] = \langle [x, y] \mid x \in P, y \in Q \rangle$ is the commutator of P and Q;
- $G_2 = [G, G]$ is the commutator subgroup of G and $G_{i+1} = [G_i, G]$ is the i-th term of the lower central series of G;
- $\mathrm{Z}_1(G) = \mathrm{Z}(G)$ is the center of G and $\mathrm{Z}_i = \mathrm{Z}_i(G) \leq G$ such that $\mathrm{Z}_i/\mathrm{Z}_{i-1} = \mathrm{Z}(G/\mathrm{Z}_{i-1})$ is the i-th term of the upper central series of G;
- $\mathrm{Syl}_p(G)$ is the set of Sylow p-subgroups of G;

- $O_p(G)$ is the largest normal p-subgroup of G,

$$O_p(G) = \bigcap_{S \in \mathrm{Syl}_p(G)} S;$$

- $O^{p'}(G) = \langle \mathrm{Syl}_p(G) \rangle$ ($G/O^{p'}(G)$ is the largest p'-group onto which G surjects);

- $\Phi(G)$ is the Frattini subgroup of G;

- $\mathrm{Aut}(G)$ is the group of automorphisms of G;

- $\mathrm{Inn}(G)$ is the group of inner automorphisms of G;

- $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is the group of outer automorphisms of G;

- $|G|_p$ is the highest power of p that divides the order of G;

- $\Omega_1(G) = \langle g \in G \mid g^p = 1 \rangle$;

- $\mho(G) = G^p = \langle g^p \mid g \in G \rangle$;

- $\mathbf{A}(G) = \{A \leq G \mid A \text{ is abelian and of maximal order}\}$;

- $\mathrm{J}(G) = \langle \mathbf{A}(G) \rangle$ is the Thompson subgroup of G;

- $P \times Q$ is the direct product of P and Q;

- $P \colon Q$ is a semidirect product of P and Q, where $Q \leq \mathrm{N}_G(P)$, with non-trivial action (not uniquely determined).

We use the following notation for specific groups:

- $\mathrm{Sym}(n)$ is the symmetric group of degree n;
- $\mathrm{Alt}(n)$ is the alternating group of degree n;
- C_n is the cyclic group of order n;
- D_n is the dihedral group of order n;
- $13\colon 3$ is the Frobenius group of order 39;
- if p is odd, then p_+^{1+2n} is the extraspecial group of order p^{1+2n} and exponent p;
- if p is odd, then p_-^{1+2n} is the extraspecial group of order p^{1+2n} and exponent p^2;
- $\mathrm{GF}(p^n)$ is the finite field of order p^n;
- $\mathrm{GL}_n(p)$ is the general linear group of degree n over $\mathrm{GF}(p)$;
- $\mathrm{SL}_n(p)$ is the special linear group of degree n over $\mathrm{GF}(p)$;
- $\mathrm{PGL}_n(p)$ is the projective general linear group of degree n over $\mathrm{GF}(p)$;
- $\mathrm{PSL}_n(p)$ is the projective special linear group of degree n over $\mathrm{GF}(p)$;
- $\mathrm{U}_n(p)$ is the unitary group of degree n over $\mathrm{GF}(p^2)$;
- $\mathrm{PGU}_n(p)$ is the projective unitary group of degree n over $\mathrm{GF}(p)$;
- $\mathrm{Sp}_{2n}(p)$ is the symplectic group of degree $2n$ over $\mathrm{GF}(p)$;
- $\mathrm{Sz}(2^n)$ is the Suzuki group over $\mathrm{GF}(2^n)$;
- $\mathrm{Ree}(3^n)$ is the Ree group over $\mathrm{GF}(3^n)$.
- $\mathrm{G}_2(p^n)$ is the automorphism group of the octonion algebra over $\mathrm{GF}(p^n)$.

INTRODUCTION

In finite group theory, the word *fusion* refers to the study of conjugacy maps between subgroups of a group. This concept has been investigated for over a century, probably starting with Burnside, and the modern way to solve problems involving fusion is via the theory of fusion systems. A *saturated fusion system* on a p-group S is a category whose objects are the subgroups of S and whose morphisms are the monomorphisms between subgroups which satisfy certain axioms, motivated by conjugacy relations among p-subgroups of a given finite group. The precise axioms were formulated in the nineties by the representation theorist Puig, who finally published his work in 2006 in [Pui06]. The methods were introduced into topology by Broto, Levi and Oliver in 2003 [BLO03].

Given any finite group of order divisible by the prime p, there is a natural construction of a saturated fusion system on its Sylow p-subgroups. There are however saturated fusion systems which do not arise in this way; these fusion systems are called *exotic*. For the prime $p = 2$, it is possible that there is just one infinite family of exotic fusion systems. In contrast, for odd primes p, the work by Craven, Oliver and Semeraro [COS16] reveals a plethora of examples. This leads to an interesting research direction which was suggested by Oliver [AKO11, III.7]:

Try to better understand how exotic fusion systems arise at odd primes.

Many researchers around the world are currently working on classifying all simple fusion systems at the prime 2, and on classifying important families of simple fusion systems at odd primes. Here is one of the main questions:

Given a class of p-groups, can we determine all simple fusion systems on them?

Taking inspiration from the Classification of Finite Simple Groups, an important class to examine is the class of p-groups of *small sectional rank.* The *rank* of a group is the smallest size of a generating set for it and the *sectional rank* of a p-group S is the largest rank of a section Q/R where $R \trianglelefteq Q \leq S$. In the elementary case in which S has sectional rank 1, the group S is cyclic and all saturated fusion systems on S are completely determined by the automorphism group of S (by adapting Burnside's result for groups with abelian Sylow p-subgroup). If p is an odd prime then all saturated fusion systems on p-groups of sectional rank 2 have been classified by Diaz, Ruiz and Viruel ([DRV07]). If $p = 2$ then all simple fusion systems on 2-groups of sectional rank at most 4 have been classified by Oliver ([Oli16], 2016).

This thesis aims to study saturated fusion systems on p-groups of sectional rank 3, when p is an odd prime.

Main Theorem. *Let $p \geq 5$ be a prime, let S be a p-group having sectional rank 3 and let $\mathcal{F}$ be a saturated fusion system on S such that $O_p(\mathcal{F}) = 1$. Then*

- *either S is isomorphic to a Sylow p-subgroup of the group $\mathrm{Sp}_4(p)$;*
- *or $p = 7$, S is isomorphic to a maximal subgroup of a Sylow 7-subgroup of the group $G_2(7)$ and $\mathcal{F}$ is the exotic fusion system completely determined by $\mathrm{Inn}(S)$, $\mathrm{Out}_{\mathcal{F}}(S) \cong \mathrm{C}_6$ and $\mathrm{Out}_{\mathcal{F}}(E) \cong \mathrm{SL}_2(7)$, where E is a subgroup of S isomorphic to the group $\mathrm{C}_7 \times \mathrm{C}_7$. Also, $\mathcal{F}$ is simple.*

If S is isomorphic to a Sylow p-subgroup of the group $\mathrm{Sp}_4(p)$ and $\mathcal{F}$ is simple then $\mathcal{F}$ is reduced (as defined in [AKO11, Definition III.6.2]) and it is among the fusion systems classified in [Oli14] and [COS16].

The situation is more complicated for $p = 3$. The fusion systems realized by the groups $\mathrm{SL}_4(q)$ and $\mathrm{P\Gamma L}_3(q^{3^a})$ (with $q \equiv 1 \mod 3$) on a Sylow 3-subgroup show that if $\mathcal{F}$ is a saturated fusion system on a 3-group S having sectional rank 3 then there is no bound for the order of S.

The first chapter of this thesis collects various results in finite group theory that constitute the background needed to prove our *Main Theorem*. In particular we introduce the p-Stability theorem ([Gor80, Theorem 3.8.3]) and Stellmacher's Pushing Up Theorem ([Ste86, Theorem 1]) that will play an important role in Chapter 4. We also present the new concept of normalizer tower of a subgroup E of the p-group S, defined as the sequence of distinct subgroups of S defined recursively by

$$\mathrm{N}^0(E) = E \quad \text{and} \quad \mathrm{N}^i(E) = \mathrm{N}_S(\mathrm{N}^{i-1}) \quad \text{for every } 1 \leq i \leq m,$$

where m is the smallest integer such that $\mathrm{N}^m(E) = S$. Such tower is maximal if $[S\colon E] = p^m$ (i.e. $[\mathrm{N}^i(E)\colon \mathrm{N}^{i-1}(E)] = p$ for every $1 \leq i \leq m$).

A class of subgroups of a p-group S having maximal normalizer tower is the class of soft subgroups of S, first introduced by Héthelyi in [Hét84]. These are self-centralizing subgroups of S having index p in their normalizer in S.

A finite group G has a strongly p-embedded subgroup H if p is a prime, H is proper in G, p divides the order of H and p does not divide the order of $H \cap H^g$ for every $g \in G \backslash H$. We prove that if G has a strongly p-embedded subgroup and acts faithfully on a 3-dimensional vector space, then the group $O^{p'}(G)$ is isomorphic to either $\mathrm{SL}_2(p)$ or $\mathrm{PSL}_2(p)$ or $13\colon 3$ (for $p = 3$).

The last part of Chapter 1 is dedicated to amalgams and weak BN-pairs of rank 2, whose classification given in the Delgado-Stellmacher's Theorem is used to prove the *Main Theorem*.

In Chapter 2 we introduce the notion of fusion system, recalling the definitions and notations used in [AKO11, Part I]. If $\mathcal{F}$ is a fusion system on the p-group S and $P \leq S$ then we denote by $\mathrm{Hom}_{\mathcal{F}}(P, S)$ the set of injective homomorphisms from P to S belonging to the category $\mathcal{F}$. Also, the $\mathcal{F}$-automorphism group of P, denoted $\mathrm{Aut}_{\mathcal{F}}(P)$, is the group of automorphisms of P belonging to the category $\mathcal{F}$ and the outer $\mathcal{F}$-automorphism group of P, denoted $\mathrm{Out}_{\mathcal{F}}(P)$, is the quotient of $\mathrm{Aut}_{\mathcal{F}}(P)$ by the inner automorphism group $\mathrm{Inn}(P)$. A subgroup Q of P is said to be $\mathcal{F}$-characteristic in P if it is normalized by every $\mathcal{F}$-automorphism of P.

The starting point for the classification of saturated fusion systems comes from the Alperin-Goldschmidt Fusion Theorem [AKO11, Theorem 3.5], which guarantees that every saturated fusion system on a p-group S is completely determined by the group of $\mathcal{F}$-automorphisms of S and by the group of $\mathcal{F}$-automorphisms of certain subgroups of S, called *$\mathcal{F}$-essential* subgroups.

Definition. Let p be a prime, let S be a p-group and let $\mathcal{F}$ be a saturated fusion system on S. A proper subgroup E of S is *$\mathcal{F}$-essential* if

1. E is $\mathcal{F}$-centric: $\mathrm{C}_S(E\alpha) \leq E\alpha$ for every $\alpha \in \mathrm{Hom}_{\mathcal{F}}(E, S)$;
2. E is fully normalized in $\mathcal{F}$: $|\mathrm{N}_S(E)| \geq |\mathrm{N}_S(E\alpha)|$ for every $\alpha \in \mathrm{Hom}_{\mathcal{F}}(E, S)$;
3. $\mathrm{Out}_{\mathcal{F}}(E)$ contains a strongly p-embedded subgroup.

Since we want to investigate fusion systems $\mathcal{F}$ on p-groups having sectional rank 3, we are interested in $\mathcal{F}$-essential subgroups having rank at most 3. Since $\mathcal{F}$-essential subgroups are self-centralizing in S and a subgroup of their outer $\mathcal{F}$-automorphism group is strongly p-embedded, the outer $\mathcal{F}$-automorphism groups of $\mathcal{F}$-essential subgroups have a very restricted structure, as described earlier. Applying the results on groups having a strongly p-embedded subgroup obtained in the previous chapter, we get the following theorem.

Theorem 1 (Structure Theorem for $\mathrm{Out}_{\mathcal{F}}(E)$)**.** *Let $\mathcal{F}$ be a saturated fusion system on the p-group S and let $E \leq S$ be an $\mathcal{F}$-essential subgroup of rank at most 3. Then*

1. *If $|E/\Phi(E)| = p^2$, then $\mathrm{SL}_2(p) \leq \mathrm{Out}_{\mathcal{F}}(E) \leq \mathrm{GL}_2(p)$;*

2. *If $|E/\Phi(E)| = p^3$ and the action of $\mathrm{Out}_{\mathcal{F}}(E)$ on $E/\Phi(E)$ is reducible then*

$$\mathrm{SL}_2(p) \leq \mathrm{Out}_{\mathcal{F}}(E) \leq \mathrm{GL}_2(p) \times \mathrm{GL}_1(p);$$

3. *If $|E/\Phi(E)| = p^3$ and the action of $\mathrm{Out}_{\mathcal{F}}(E)$ on $E/\Phi(E)$ is irreducible then*

   (a) either $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{PSL}_2(p) \cong \Omega_3(p)$;

   (b) or $p = 3$ and $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong 13\colon 3$.

A direct consequence of the previous theorem is that every $\mathcal{F}$-essential subgroup of rank at most 3 has index p in its normalizer in S. In particular abelian $\mathcal{F}$-essential subgroups of the p-group S of rank at most 3 are soft subgroups of S and so have maximal normalizer tower in S. We prove that this implies that every morphism in $\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(E)}(\mathrm{Aut}_S(E))$ is a restriction of a morphism in $\mathrm{Aut}_{\mathcal{F}}(S)$.

Note that if S is a p-group and $\mathcal{F}$ is a saturated fusion system on S such that none of the subgroups of S is $\mathcal{F}$-essential, then $\mathcal{F}$ is completely determined by $\mathrm{Aut}_{\mathcal{F}}(S)$. In particular the fusion subsystem $\mathcal{F}_S(S)$ is normal in $\mathcal{F}$ (Lemma 2.17). So if $\mathcal{F}$ is simple and no subgroups of S are $\mathcal{F}$-essential then $\mathcal{F} = \mathcal{F}_S(S)$ and S is cyclic of order p ([Cra11, Lemma 5.76]). Thus we may always assume that there exists at least one $\mathcal{F}$-essential subgroup of S.

Chapter 3 focuses on the study of saturated fusion systems on p-groups containing at least one $\mathcal{F}$-essential subgroup that is either elementary abelian of order p^2 or extraspecial of order p^3 and exponent p. Note that the first of these groups is the smallest candidate for an abelian $\mathcal{F}$-essential subgroup (since $\mathcal{F}$-essential subgroups are not cyclic) and the second is the smallest candidate for a non-abelian $\mathcal{F}$-essential subgroup. The presence of such very small $\mathcal{F}$-essential subgroups enriches the structure of the p-group considered as a jewel is made precious by a pearl.

Definition (Pearl)**.** Let p be an odd prime, let S be a p-group and let $\mathcal{F}$ be a saturated fusion system on S. A subgroup of S is a *pearl* if it is an $\mathcal{F}$-essential subgroup of S that is either elementary abelian of order p^2 or extraspecial of order p^3 and exponent p.

We denote by $\mathcal{P}(\mathcal{F})$ the set of pearls of S with respect to the fusion system $\mathcal{F}$ on S, by $\mathcal{P}(\mathcal{F})_a$ the set of abelian pearls in $\mathcal{P}(\mathcal{F})$ and by $\mathcal{P}(\mathcal{F})_e$ the set of extraspecial pearls in $\mathcal{P}(\mathcal{F})$. Note that $\mathcal{P}(\mathcal{F}) = \mathcal{P}(\mathcal{F})_a \cup \mathcal{P}(\mathcal{F})_e$.

In Lemma 3.3 we show that p-groups containing pearls have maximal nilpotency class.

Note that pearls have rank 2 and by Theorem 1 if $E \in \mathcal{P}(\mathcal{F})$ then $\mathrm{SL}_2(p) \leq \mathrm{Out}_{\mathcal{F}}(E) \leq \mathrm{GL}_2(p)$. In particular if $E \in \mathcal{P}(\mathcal{F})_a$ and G is a model for $\mathrm{N}_{\mathcal{F}}(E)$, then the quotient $\mathrm{N}_S(E)/E$ is isomorphic to a Sylow p-subgroup of the group $\mathrm{GL}_2(p)$ and since the Sylow p-subgroups of $\mathrm{GL}_2(p)$ generate $\mathrm{SL}_2(p)$ (see for example [Gor80, Theorem 2.8.4]), we get

$$\langle \mathrm{N}_S(E)^G \rangle \cong (\mathrm{C}_p \times \mathrm{C}_p).\mathrm{SL}_2(p).$$

Thus $\langle \mathrm{N}_S(E)^G \rangle$ is a so-called $Qd(p)$ group, as defined by Glauberman.

Fusion systems that do not involve $Qd(p)$ groups have been studied in [HSZ17]. In this paper the authors also describe all the finite simple groups that involve the groups $Qd(p)$ and $\widetilde{Qd}(p)$ (defined as the group $p_+^{1+2}\mathrm{SL}_2(p)$, that is related to our extraspecial pearls).

For $i \geq 2$, let S_i denote the i-th term of the lower central series of S. In the first part of Chapter 3 we recall the structure of a p-group S having maximal nilpotency class, studying in particular the properties of the group $S_1 = \mathrm{C}_S(S_2/S_4)$. For example we prove that if S_1 is neither abelian nor extraspecial then $\mathrm{Aut}(S)$ has a normal Sylow p-subgroup P and the quotient $\mathrm{Aut}(S)/P$ is isomorphic to a cyclic group of order dividing $p-1$.

We then determine a general bound for the order of groups having maximal nilpotency class depending on their sectional rank.

Theorem 2. *Let S be a p-group of maximal nilpotency class and sectional rank k. If $p \geq k+2$ then $|S| \leq p^{2k}$ (with strict inequality if $S_1 = \mathrm{C}_S(\mathrm{Z}_2(S))$). Also, if $p = 3$ and $k \geq 3$ then $|S| = 3^4$.*

As a consequence, if p is odd and S has sectional rank 3, then either $p = 3$ and $|S| = 3^4$ or $p \geq 5$ and $|S| \leq p^6$.

Next, we describe the candidates for $\mathcal{F}$-essential subgroups of p-groups having maximal nilpotency class.

Theorem 3. *Let $\mathcal{F}$ be a saturated fusion system on the p-group S, that has maximal nilpotency class. Let E be an $\mathcal{F}$-essential subgroup of S. Then one of the following holds:*

1. *E is a pearl;*

2. *$E \leq S_1$ (and if S_1 is extraspecial or abelian then $E = S_1$); or*

3. *$E \leq \mathrm{C}_S(\mathrm{Z}_2(S))$, $E \nleq S_1$, $[E\colon \mathrm{Z}_i(S)] = p$ for some $i \in \{2,3,4\}$ and either $E \cong \mathrm{C}_p \times \mathrm{C}_p \times \mathrm{C}_p$ or $E/\mathrm{Z}_2(S)$ is isomorphic to either $\mathrm{C}_p \times \mathrm{C}_p$ or p_+^{1+2}.*

Also, if $O_p(\mathcal{F}) = 1$, S_1 is extraspecial and $\mathrm{C}_S(\mathrm{Z}_2(S))$ is $\mathcal{F}$-essential then $p \geq 5$, S is isomorphic to a Sylow p-subgroup of the group $\mathrm{G}_2(p)$ (with $p = 7$ if there is a pearl) and $\mathcal{F}$ is one of the fusion systems classified by Parker and Semeraro in [PS16].

When the group S_1 is extraspecial and there is a pearl, we can indeed determine the size of S and the nature of the pearl.

Theorem 4. *Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on a p-group S containing a pearl E. Then the following are equivalent:*

1. *S_1 is extraspecial;*

2. *$S_1 \neq \mathrm{C}_S(\mathrm{Z}_2(S))$;*

3. *$E \cong \mathrm{C}_p \times \mathrm{C}_p$, $|S| = p^{p-1}$ and S_1 is not abelian.*

Also, if one (and then all) of the cases above occurs, then $p \geq 7$, $S_1 \cong p_+^{1+(p-3)}$ and S has exponent p.

Note that the fusion systems described in the previous theorem are close to the ones determined by Parker and Stroth in [PS15]. Also, Theorem 4 suggests a distinct avenue for research: the study of fusion systems on p-groups having a maximal subgroup that is extraspecial. This is the current research project of Moragues Moncho.

A combination of the previous results enables us to determine the candidates for $\mathcal{F}$-essential subgroups of p-groups containing pearls (**Theorem 5**).

In the last part of this chapter we consider p-groups having sectional rank 3 and containing pearls. By Theorem 2 these groups have order at most p^6. After studying the structure of p-groups containing pearls and having order at most p^6 (**Theorem 6**), we classify saturated fusion systems on p-groups having sectional rank 3 and containing pearls.

Theorem 7. *Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on a p-group S of sectional rank 3 containing a pearl E. Then S has maximal nilpotency class and either S is isomorphic to a Sylow p-subgroup of $\mathrm{Sp}_4(p)$ or the following hold:*

1. *$p = 7$ and S is isomorphic to the group indexed in* Magma *as SmallGroup($7^5, 37$);*

2. *$E \cong \mathrm{C}_7 \times \mathrm{C}_7$ and $\mathrm{Aut}_{\mathcal{F}}(E) \cong \mathrm{SL}_2(7)$;*

3. *$\mathcal{F}$ is completely determined by $\mathrm{Inn}(S)$, $\mathrm{Aut}_{\mathcal{F}}(E)$ and $\mathrm{Out}_{\mathcal{F}}(S) \cong \mathrm{C}_6$; and*

4. *$\mathcal{F}$ is simple and exotic. Also, such an $\mathcal{F}$ exists and is unique.*

To prove that the fusion system presented in parts *1.–4.* of Theorem 7 is exotic we use the Classification of Finite Simple Groups.

Note that the fusion systems described in Theorem 7 are the same appearing in the *Main Theorem*. Indeed in Chapter 5 we show that if $p \geq 5$, S is a p-group having sectional rank 3 and $\mathcal{F}$ is a saturated fusion system on S such that $O_p(\mathcal{F}) = 1$, then S contains a pearl (Theorem 20).

Chapter 4 aims to describe the automorphism group and the structure of the $\mathcal{F}$-essential subgroups of p-groups having sectional rank 3. In this chapter, p is an odd prime, S is a p-group having sectional rank 3 and $\mathcal{F}$ is a saturated fusion system on S.

We start strengthening the result given in Theorem 1.

Theorem 8. *Let $E \le S$ be an $\mathcal{F}$-essential subgroup.*

- *If E is $\mathcal{F}$-characteristic in S and there exists an $\mathcal{F}$-essential subgroup $P \neq E$ such that every morphism in $\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_S(P))$ is a restriction of an $\mathcal{F}$-automorphism of S, then*
  - *either $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(p)$*
  - *or $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{PSL}_2(p)$, $O^{p'}(\mathrm{Out}_{\mathcal{F}}(P)) \cong \mathrm{SL}_2(p)$ and S has rank 2.*
- *If E is not $\mathcal{F}$-characteristic in S then $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(p)$ and*
  - *either $[E\colon \Phi(E)] = p^2$ and $\mathrm{SL}_2(p) \le \mathrm{Out}_{\mathcal{F}}(E) \le \mathrm{GL}_2(p)$;*
  - *or $[E\colon \Phi(E)] = p^3$ and $\mathrm{SL}_2(p) \le \mathrm{Out}_{\mathcal{F}}(E) \le \mathrm{GL}_2(p) \times \mathrm{GL}_1(p)$.*

Note that a subgroup P of S is $\mathcal{F}$-characteristic in S if and only if it is $\mathcal{F}$-characteristic in $\mathrm{N}_S(P)$. Indeed if P is $\mathcal{F}$-characteristic in S then P is normal in S and so $\mathrm{N}_S(P) = S$. On the other hand, if P is $\mathcal{F}$-characteristic in $\mathrm{N}_S(P)$ then P is normal in $\mathrm{N}_S(\mathrm{N}_S(P))$ and so $S = \mathrm{N}_S(P)$.

Theorem 8 is proved by considering the interplay of two distinct $\mathcal{F}$-essential subgroups E_1 and E_2 of S having the same normalizer N in S. An important role is played by the largest subgroup of $E_1 \cap E_2$ that is normalized by $\mathrm{Aut}_{\mathcal{F}}(E_1)$, $\mathrm{Aut}_{\mathcal{F}}(E_2)$ and $\mathrm{Aut}_{\mathcal{F}}(N)$.

Definition ($\mathcal{F}$-core of E_1 and E_2)**.** Let $E_1 \le S$ and $E_2 \le S$ be $\mathcal{F}$-essential subgroups of S such that $\mathrm{N}_S(E_1) = \mathrm{N}_S(E_2)$. We define the *$\mathcal{F}$-core of E_1 and E_2*, denoted $\mathrm{core}_{\mathcal{F}}(E_1, E_2)$, as the largest subgroup of $E_1 \cap E_2$ that is $\mathcal{F}$-characteristic in E_1, E_2 and $\mathrm{N}_S(E_1)$.

We set $\mathrm{core}_{\mathcal{F}}(E_1) = \mathrm{core}_{\mathcal{F}}(E_1, E_1)$ and we call it the *$\mathcal{F}$-core of* E_1.

If $\langle \mathcal{E}_1, \mathcal{E}_2 \rangle_P$ denotes the smallest fusion subsystem of $\mathcal{F}$ on $P \le S$ containing the fusion subsystems $\mathcal{E}_1$ and $\mathcal{E}_2$ (both defined on P), then $\mathrm{core}_{\mathcal{F}}(E_1, E_2) = O_p(\langle \mathrm{N}_{\mathcal{F}}(E_1), \mathrm{N}_{\mathcal{F}}(E_2) \rangle_{\mathrm{N}_S(E_1)})$.

If E is an $\mathcal{F}$-essential subgroup of S and $\alpha \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}_S(E))$ then $\mathrm{N}_S(E\alpha) = \mathrm{N}_S(E)$, $E\alpha$ is an $\mathcal{F}$-essential subgroup of S (Lemma 2.26(6)) and we show that $\mathrm{core}_{\mathcal{F}}(E) = \mathrm{core}_{\mathcal{F}}(E, E\alpha) = \mathrm{core}_{\mathcal{F}}(E\alpha)$. Thus, if E is an $\mathcal{F}$-essential subgroup of S not $\mathcal{F}$-characteristic in $\mathrm{N}_S(E)$, then the $\mathcal{F}$-core of E can always be described as the $\mathcal{F}$-core of two distinct $\mathcal{F}$-essential subgroups of S.

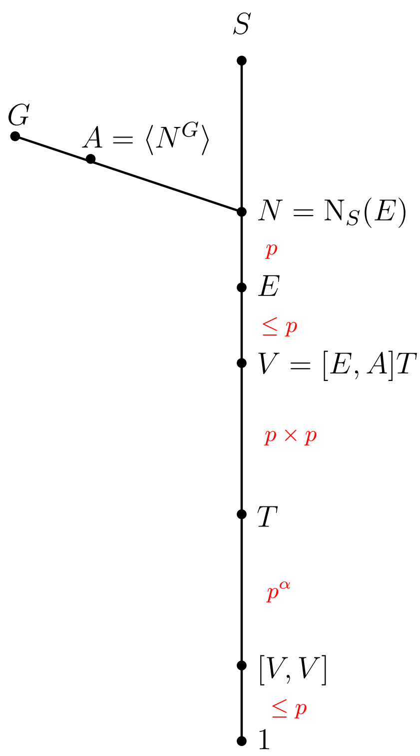
The properties of the $\mathcal{F}$-core of two distinct $\mathcal{F}$-essential subgroups E_1 and E_2 of S are described by **Theorem 9** and **Theorem 10** (stated in the introduction of Chapter 4).

We now focus our attention on $\mathcal{F}$-essential subgroups of S that are not $\mathcal{F}$-characteristic in S.

Theorem 11. *Let E be an $\mathcal{F}$-essential subgroup of S not $\mathcal{F}$-characteristic in S and let $T = \mathrm{core}_{\mathcal{F}}(E)$. Set $V = [E, O^{p'}(\mathrm{Out}_{\mathcal{F}}(E))]T$. Then*

1. *V/T is a natural $\mathrm{SL}_2(p)$-module for $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E))$;*
2. *$\mathrm{N}_S(E)/T$ has exponent p;*
3. *E/T is elementary abelian and $p^2 \leq [E\colon T] \leq p^3$;*
4. *$[E/T\colon \mathrm{Z}(\mathrm{N}_S(E)/T)] = p$;*
5. *T is abelian, $T \leq \mathrm{Z}(V)$, $|[V,V]| \leq p$ and $T/[V,V]$ is a cyclic group.*

Moreover, if $[E\colon T] = p^2$, then $T \leq \mathrm{Z}(\mathrm{N}_S(E))$.



The previous results are enough to show that $\mathcal{F}$-essential subgroups of rank 2 that are not $\mathcal{F}$-characteristic in S are pearls.

Theorem 12. *Let $E \leq S$ be an $\mathcal{F}$-essential subgroup of S not $\mathcal{F}$-characteristic in S. If E has rank 2 then E is a pearl and $\mathcal{F}$ is one of the fusion systems described in Theorem 7.*

In general, when the $\mathcal{F}$-core of E has index p^2 in E we have the following:

Theorem 13. *Suppose that $E \leq S$ is an $\mathcal{F}$-essential subgroup of S and let $T = \mathrm{core}_{\mathcal{F}}(E)$. If $[E\colon T] = p^2$ and $|T| = p^a$ then*

- *either $E \cong \mathrm{C}_p \times \mathrm{C}_p \times \mathrm{C}_{p^a}$;*
- *or $E \cong \frac{\Omega_1(E) \times T}{(\mathrm{Z}(\Omega_1(E)) = \Omega_1(T))} \cong p_+^{1+2} \circ \mathrm{C}_{p^a}$;*
- *or $E \cong p_+^{1+2} \times \mathrm{C}_{p^{a-1}}$.*

The next step is to consider $\mathcal{F}$-essential subgroups of S that have rank 3.

Theorem 14. *Suppose that $E \leq S$ is an $\mathcal{F}$-essential subgroup of S having rank 3. If $p \geq 5$ then $E \trianglelefteq S$.*

If $p = 3$, E has rank 3 and $\mathrm{N}_S(E) < S$, then in Lemmas 4.2 and 4.4 we determine the isomorphism type of the quotient $\mathrm{N}_S(\mathrm{N}_S(E))/\Phi(E)$. More precisely, using the notation of the normalizer tower introduced in Chapter 1, if $\mathrm{N}^2(S) < S$ then either the quotient $\mathrm{N}^2(S)/\Phi(E)$ is isomorphic to a section of a Sylow 3-subgroup of the group $\mathrm{SL}_4(19)$ or $\mathrm{N}^3(S)/\Phi(E)$ is isomorphic to a Sylow 3-subgroup of the group $\mathrm{P\Gamma L}_3(64)$. Note that to prove such results we do not require $\mathcal{F}$ to satisfy the condition $O_p(\mathcal{F}) = 1$.

This fact suggests to look at the Sylow 3-subgroups of the groups $\mathrm{SL}_4(q)$ and $\mathrm{P\Gamma L}_3(q^{3^a})$, where $q \equiv 1 \mod 3$. These examples show that if $\mathcal{F}$ is a saturated fusion system on a 3-group S having sectional rank 3 and at least one $\mathcal{F}$-essential subgroup $E \leq S$, then in general we cannot bound the index of E in S (more details about these examples are given in the introduction of Chapter 4).

In the final section of Chapter 4 we consider the interplay of two $\mathcal{F}$-essential subgroups of S that are $\mathcal{F}$-characteristic in S.

Theorem 15. *Let $E_1 \leq S$ and $E_2 \leq S$ be distinct $\mathcal{F}$-essential subgroups $\mathcal{F}$-characteristic in S and let $T = \mathrm{core}_{\mathcal{F}}(E_1, E_2)$. Then*

- *either $S/T \cong p_+^{1+2}$ and for every $1 \leq i \leq 2$ the group E_i is abelian and $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_i)) \cong \mathrm{SL}_2(p)$;*
- *or S/T is isomorphic to a Sylow p-subgroup of $\mathrm{Sp}_4(p)$ and there exist $1 \leq i, j \leq 2$ such that $i \neq j$, $\mathrm{Z}(S) = \mathrm{Z}(E_i)$ is the preimage in S of $\mathrm{Z}(S/T)$ and the following holds:*
    1. *$E_i/T \cong p_+^{1+2}$ and $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_i)) \cong \mathrm{SL}_2(p)$;*
    2. *E_j is abelian, $T = \Phi(E_j)$ and $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_j)) \cong \mathrm{PSL}_2(p)$.*

Finally, we show that if $\mathcal{F}$ is simple and there are two $\mathcal{F}$-essential subgroups of S that are $\mathcal{F}$-characteristic in S then S is isomorphic to a Sylow p-subgroup of the group $\mathrm{Sp}_4(p)$.

Theorem 16. *Let $E_1 \leq S$ and $E_2 \leq S$ be distinct $\mathcal{F}$-essential subgroups $\mathcal{F}$-characteristic in S. Then the group $\mathrm{core}_{\mathcal{F}}(E_1, E_2)$ is normal in $\mathcal{F}$. In particular, if $O_p(\mathcal{F}) = 1$ then S is isomorphic to a Sylow p-subgroup of the group $\mathrm{Sp}_4(p)$, $E_i \cong p_+^{1+2}$ and $E_j \cong \mathrm{C}_p \times \mathrm{C}_p \times \mathrm{C}_p$ for some $i, j \in \{1, 2\}$, $i \neq j$, and $\mathcal{F}$ is one of the fusion systems classified in [COS16].*

In Chapter 5 we put together all the results presented in the previous chapters to prove our *Main Theorem*. From now on, p is an odd prime, S is a p-group having sectional rank 3 and $\mathcal{F}$ is a saturated fusion system on S satisfying the condition $O_p(\mathcal{F}) = 1$.

For every subgroup $P \leq S$ such that $\mathrm{Z}(S) \leq P$ we define

$$Z_P = \langle\ \Omega_1(\mathrm{Z}(S))^{\mathrm{Aut}_{\mathcal{F}}(P)}\ \rangle.$$

Note that $Z_S = \Omega_1(\mathrm{Z}(S))$ and $Z_S \leq Z_P \leq \Omega_1(\mathrm{Z}(P))$. In particular Z_P is elementary abelian and since S has sectional rank 3 we deduce $|Z_P| \leq p^3$.

By assumption, there exists at least one $\mathcal{F}$-essential subgroup E of S such that $Z_S < Z_E$.

Theorem 17. *Let $E \leq S$ be an $\mathcal{F}$-essential subgroup of S not $\mathcal{F}$-characteristic in S. Then*

$$Z_S = Z_E \text{ if and only if } \mathrm{Z}_S \leq \mathrm{core}_{\mathcal{F}}(E).$$

Theorem 18. *Let $E \leq S$ be an $\mathcal{F}$-essential subgroup of S not $\mathcal{F}$-characteristic in S. If $Z_S < Z_E$ then E is abelian and if $O_p(\mathcal{F}) = 1$ then E is not normal in S.*

We now have all the ingredients to prove our final results.

Theorem 19. *Let p be an odd prime, let S be a p-group having sectional rank 3 and let $\mathcal{F}$ be a saturated fusion system on S satisfying the condition $O_p(\mathcal{F}) = 1$. Then one of the following holds:*

1. *S is isomorphic to a Sylow p-subgroup of the group $\mathrm{Sp}_4(p)$;*
2. *there exists an $\mathcal{F}$-essential subgroup of S that is not normal in S.*

Theorem 20. *Let $p \geq 5$ be a prime, let S be a p-group having sectional rank 3 and let $\mathcal{F}$ be a saturated fusion system on S such that $O_p(\mathcal{F}) = 1$. Then $\mathcal{F}$ contains a pearl.*

Proof of the Main Theorem. By Theorem 20 there exists an $\mathcal{F}$-essential subgroup of S that is a pearl and we conclude by Theorem 7. □

The Appendix contains some results for $p = 3$, that might be used for future projects.

To summarize:

- in Chapter 1 we present the group theoretical background;
- in Chapter 2 we introduce fusion systems and $\mathcal{F}$-essential subgroups, proving some properties of $\mathcal{F}$-essential subgroups of rank at most 3;
- in Chapter 3 we study saturated fusion systems containing pearls (for p odd) and we determine such saturated fusion systems when the p-group considered has sectional rank 3 (discovering a new exotic fusion system);
- in Chapter 4 we study the structure of $\mathcal{F}$-essential subgroups of p-groups having sectional rank 3 (for p odd), proving that if $p \geq 5$ then all $\mathcal{F}$-essential subgroups of rank 3 are normal in the p-group considered. When $p = 3$, we determine the isomorphism type of a specific section of the 3-group studied.

  Note that in Chapters 3 and 4 we do not impose $O_p(\mathcal{F}) = 1$ and so the results can be used modulo certain subgroups.
- in Chapter 5 we use the results of Chapters 3 and 4 and we give the last ingredients required to classify saturated fusion systems $\mathcal{F}$ on p-groups having sectional rank 3 such that $O_p(\mathcal{F}) = 1$ and $p \geq 5$ (*Main Theorem*).

A guide for the proofs of Theorems 1–20 presented in this introduction is given on page xix.

CHAPTER 1

GROUP METHODS

‘A piece of music can be accompanied by words, movement, or dance, or can simply be appreciated on its own. It is the same with groups. They can be seen as groups of symmetries, permutations, or motions, or can simply be studied and admired in their own right.’

[Mark Ronan]

In this chapter we collect definitions and results about finite groups that form the background needed to work with fusion systems. We start with the definition of conjugation map, setting the notation and recalling some fundamental properties. We then focus our attention on p-groups and on their automorphism groups. For example we see how knowing the action of a morphism on elements x, y of a p-group can give information on its action on $[x, y]$. This remark will be important in the study of the structure of p-groups in Chapter 4. Of particular relevance are then the p-stability Theorem, which proves that under specific conditions the automorphism group of an elementary abelian p-group involves the group $\mathrm{SL}_2(p)$, and Stellmacher's Pushing Up Theorem, which will be used to determine the structure of certain p-groups.

The goal of Section 1.3 is to introduce the concepts of normalizer tower and soft subgroups and to illustrate the structure of a p-group containing a soft subgroup.

The main focus of our study are the so-called *essential subgroups* of a fusion system and their automorphism groups. Because of that, we need to examine strongly p-embedded subgroups. In Section 1.4 we prove that if G is a finite group having a strongly p-embedded subgroup and acting faithfully on a 3-dimensional vector space, then $O^{p'}(G)$ is isomorphic to either $\mathrm{SL}_2(p)$, or $\mathrm{PSL}_2(p)$, or $13\colon 3$ (for $p = 3$).

The final section's aim is to introduce the concepts of amalgams, weak BN-pairs of rank 2 and symplectic amalgams. We will need these objects when we consider the case of *essential subgroups* characteristic in the p-group studied.

1.1 Commutators and automorphisms of finite groups

Let G be a finite group and let $g \in G$ be an element. The *conjugation* map by g is the automorphism of G defined as

$$\begin{aligned} c_g \colon G &\to G, \\ h &\mapsto h^g = g^{-1}hg. \end{aligned}$$

If $P \leq G$ is a subgroup of G, we set $P^g = \{x^g \mid x \in P\}$ and we define the *normalizer* and the *centralizer* in G of P as follows:

$$\mathrm{N}_G(P) = \{g \in G \mid P^g = P\},$$

$$\mathrm{C}_G(P) = \{g \in G \mid x^g = x \text{ for every } x \in P\} \leq \mathrm{N}_G(P).$$

If x and y are elements of G, then the commutator of x and y is given by

$$[x, y] = x^{-1}y^{-1}xy.$$

If $P, Q \leq G$ are subgroups, then the commutator of P and Q is defined as

$$[P, Q] = \langle [x, y] \mid x \in P, y \in Q \rangle.$$

Note that $[P, \mathrm{N}_G(P)] \leq P$ and $[P, \mathrm{C}_G(P)] = 1$.

We can extend this definition to automorphisms $\alpha \in \mathrm{Aut}(P)$, defining

$$[x, \alpha] = x^{-1} \cdot (x)\alpha \quad \in P,$$

for every $x \in P$. We set

$$[P, \alpha] = \langle [x, \alpha] \mid x \in P \rangle.$$

Note that for all $x, g \in G$ we have $[x, c_g] = [x, g]$.

Recall that a group G is *nilpotent* if $\mathrm{Z}_c(G) = G$ for some $c \in \mathbb{N}$ (or equivalently if $G_{c+1} = 1$). The smallest integer c satisfying $\mathrm{Z}_c(G) = G$ is the *nilpotency class* of G. Every p-group P is a nilpotent group and if $|P| = p^n$ then we say that P has *maximal nilpotency class* if it has class $n-1$.

Lemma 1.1. *Let G be a nilpotent group and let $P, Q \leq G$ be subgroups. If $[P, Q] = P$ then $P = 1$.*

Proof. Suppose $[P, Q] = P$. Then $[P, Q, Q] = [P, Q] = P$. In particular $P \leq G_i$ for every $i \geq 2$. Let c be the nilpotency class of G. Then $P \leq G_c = 1$, implying $P = 1$. □

Lemma 1.2. *[Gor80, Theorem 2.3.3] Let G be a nilpotent group and let $P < G$ be a proper subgroup. Then $P < \mathrm{N}_G(P)$.*

Lemma 1.3. *Given elements x, y, z of a finite group G, we have the following equalities.*

1. $[xy, z] = [x, z]^y [y, z]$;
2. $[x, yz] = [x, z][x, y]^z$;
3. *if $[x, y]$ commutes with both x and y then for all i, j we have*
   - $[x^i, y^j] = [x, y]^{ij}$; *and*
   - $(yx)^i = [x, y]^{\frac{i(i-1)}{2}} y^i x^i$.

A proof of these elementary statements can be found for example in [Gor80, Theorem 2.2.1 and Lemma 2.2.2]. These equalities will be taken for granted and used many times throughout the paper (mostly without citing the previous lemma). The next lemma illustrates one of the main applications.

When we write $u = v \mod Z$ we mean $u = vz$ for some $z \in Z$. In particular, the statement $u = 1 \mod Z$ is equivalent to $u \in Z$.

Lemma 1.4. *Let $x, y \in G$ be elements and let $\alpha \in \mathrm{Aut}(G)$ be an automorphism of G acting on x and y as follows:*

$$x\alpha = x^i g, \quad y\alpha = y^j h,$$

for some $i, j \geq 1$ and $g, h \in G$. If there exists a subgroup $Z \trianglelefteq G$ such that

$$[g, y^j], [x^i, h], [g, h] \in Z \text{ and } [x, y]Z \in \mathrm{Z}(G/Z),$$

then

$$[x, y]\alpha = [x, y]^{ij} \mod Z.$$

Proof. From Lemma 1.3 we have

$$[x^i g, y^j h] = [x^i, y^j h]^g [g, y^j h] = ([x^i, h][x^i, y^j]^h)^g [g, h][g, y^j]^h.$$

By assumptions we then get $[x^i g, y^j h] = [x^i, y^j]^{hg} \mod Z$. Also, $[x, y]Z$ is in the center of G/Z. Thus $[x^i, y^j] = [x, y]^{ij} \mod Z$ and $[x^i, y^j]$ commutes with g and h modulo Z. Hence

$$[x, y]\alpha = [x\alpha, y\alpha] = [x^i g, y^j h] = [x^i, y^j]^{hg} = [x, y]^{ij} \mod Z.$$

□

Thus, knowing the action of α on x and y we can deduce its action on $[x, y] \mod Z$. This technique will be fundamental in the characterization of $\mathcal{F}$-essential subgroups.

Lemma 1.5. *Let p be an odd prime, let P be a p-group and let $x, y \in P$ be elements. If $[x, y] = [x^{-1}, y]$ then $[x, y] = 1$.*

Proof. Suppose $[x, y] = [x^{-1}, y]$. Then $x^{-1}y^{-1}xy = xy^{-1}x^{-1}y$, so $x^{-2}y^{-1}x^2 = y^{-1}$. Therefore $x^2 \in \mathrm{C}_P(y^{-1})$ and since p is odd and P is a p-group we deduce that $x \in \mathrm{C}_P(y)$. Hence $[x, y] = 1$. □

We see another application of Lemma 1.3.

Lemma 1.6. *Let p be a prime and let P be a p-group.*

- *If $P/\mathrm{Z}(P)$ is cyclic then P is abelian.*
- *If $[P\colon \mathrm{Z}(P)] = p^2$ then $[P, P] = \langle [x, y] \rangle$, where $x, y \in P$ are such that $P = \langle x, y \rangle \mathrm{Z}(P)$ and $|[P, P]| = p$.*

Proof. Suppose that $P/\mathrm{Z}(P)$ is cyclic and let $g \in P$ be such that $P = \langle g \rangle \mathrm{Z}(P)$. Take $x, y \in P$. Then $x = g^i z_1$ and $y = g^j z_2$ for some $i, j \in \mathbb{N}$ and $z_1, z_2 \in \mathrm{Z}(P)$. Thus

$$xy = g^i z_1 g^j z_2 = g^{i+j} z_1 z_2 = g^j z_2 g^i z_1 = yx.$$

So x and y commute and we deduce that P is abelian.

Suppose that $[P\colon \mathrm{Z}(P)] = p^2$. By what we proved above, $P/\mathrm{Z}(P)$ is not cyclic, so we have $P/\mathrm{Z}(P) \cong \mathrm{C}_p \times \mathrm{C}_p$. Let $x, y \in P$ be such that $P = \langle x, y \rangle \mathrm{Z}(P)$. Since $[x, y] \in \mathrm{Z}(P)$, by Lemma 1.3 we deduce that

$$[P, P] = [\langle x, y \rangle, \langle x, y \rangle] = \langle [x, y] \rangle.$$

Also, since $x^p \in \mathrm{Z}(P)$, we get $[x, y]^p = [x^p, y] = 1$. Therefore $|[P, P]| = p$. □

Another important subgroup of a finite group G is the *Frattini subgroup*, denoted $\Phi(G)$ and defined as the intersection of all maximal subgroups of G. It is a characteristic subgroup of G and the quotient $G/\Phi(G)$ is called *Frattini quotient*.

In the case of a p-group P, we have

$$\Phi(P) = [P, P]P^p,$$

where $P^p = \langle x^p \mid x \in P \rangle \leq P$.

Recall that an *elementary abelian* p-group V is an abelian group that has exponent p, i.e. $x^p = 1$ for every $x \in V$. Every elementary abelian p-group of order p^n can be considered as a vector space of dimension n over the field $\mathrm{GF}(p)$. In particular, if V is elementary abelian of order p^n then $\mathrm{Aut}(V) \cong \mathrm{GL}_n(p)$.

From the equation $\Phi(P) = [P, P]P^p$ we deduce that the Frattini quotient of a p-group is an elementary abelian group.

We call *rank* the smallest size of a generating set for a group. It is a known result that a p-group P has rank r if and only $P/\Phi(P)$ has order p^r ([Ber08, Theorem 1.12]).

Definition 1.7. If $R, Q \leq G$ are subgroups of G with $R \trianglelefteq Q$ then the quotient Q/R is a *section* of G. We say that G involves the group H if H is isomorphic to a section of G.

Definition 1.8. We say that a p-group P has *sectional rank* k if every elementary abelian section of P has order at most p^k, and k is the smallest integer with this property.

Thus P has sectional rank k if and only if every subgroup of P has rank at most k.

To calculate the sectional rank of a p-group it is therefore important to be able to establish the exponent of its subgroups.

Given a finite p-group P, we denote by $\Omega_1(P)$ the group generated by the elements of P of order p:

$$\Omega_1(P) = \langle x \in P \mid x^p = 1 \rangle.$$

Note that in general the group $\Omega_1(P)$ can have exponent larger than p. As an example if $P \cong D_8$, the dihedral group of order 8, then $P = \Omega_1(P)$ and P has exponent 4. The next lemma gives some sufficient conditions for the group $\Omega_1(P)$ to have exponent p.

Lemma 1.9. *[LGM02, Lemmas 1.2.11, 1.2.13] Let P be a p-group. If $\Omega_1(P)$ is abelian or $|\Omega_1(P)| \leq p^p$ then $\Omega_1(P)$ has exponent p.*

The action of a p'-automorphism of a p-group P on the Frattini quotient $P/\Phi(P)$, gives us information about the action on P, as Burnside's Theorem states.

Theorem 1.10 (Burnside)**.** *[Gor80, Theorem 5.1.4] Let p be a prime and let α be a p'-automorphism of the p-group P. If α induces the identity on $P/\Phi(P)$, then α is the identity automorphism of P.*

In general, the action of a p'-automorphism on a p-group P leads to a decomposition of the group P. We refer to it as decomposition by *coprime action*.

Theorem 1.11 (Coprime Action)**.** *[Gor80, Theorems 5.2.3 and 5.3.5] Let p be a prime and let A be a p'-group of automorphisms of the p-group P. Then*

$$P = \mathrm{C}_P(A)[P, A],$$

where $\mathrm{C}_P(A) = \{x \in P \mid x\alpha = x \text{ for every } \alpha \in A\}$ is the centralizer in P of A.

If P is abelian then $P \cong \mathrm{C}_P(A) \times [P, A]$.

The next two lemmas are applications of Theorem 1.11.

Lemma 1.12. *Let p be a prime and let A be a p'-group of automorphisms of the p-group P. Then*

$$[P, A] = [P, A, A],$$

where $[P, A, A] = [[P, A], A]$.

Proof. By Theorem 1.11 we have $P = \mathrm{C}_P(A)[P, A]$. Hence

$$[P, A] = [\mathrm{C}_P(A)[P, A], A] = [\mathrm{C}_P(A), A][[P, A], A] = [P, A, A].$$

$\square$

Lemma 1.13. *Let p be a prime and let A be a p'-group of automorphisms of the p-group P. If $[P \colon \mathrm{C}_P(A)] = p$ then $|[P, A]| = p$ and*

$$P \cong \mathrm{C}_P(A) \times [P, A].$$

Proof. We prove the statement by induction on the order of P. If P is cyclic then it is abelian and the statement is true by Theorem 1.11. Suppose P is not cyclic. Then $[P \colon \Phi(P))] \geq p^2$. Note that $\Phi(P) \leq \mathrm{C}_P(A)$ and $\mathrm{C}_P(A)/\Phi(P) \leq \mathrm{C}_{P/\Phi(P)}(A) < P/\Phi(P)$ (by Theorem 1.10). Hence we have $[P/\Phi(P) \colon \mathrm{C}_{P/\Phi(P)}(A)] = p$ and by inductive hypothesis we get

$$P/\Phi(P) \cong \mathrm{C}_{P/\Phi(P)}(A) \times [P/\Phi(P), A],$$

with $|[P/\Phi(P), A]| = p$. Thus there exists a maximal subgroup M of P such that $[P, A] \leq M$. Since $P = M\mathrm{C}_P(A)$ and $[P \colon \mathrm{C}_P(A)] = p$ by assumption, we deduce that

$$[M \colon \mathrm{C}_M(A)] = [M \colon M \cap \mathrm{C}_P(A)] = [M\mathrm{C}_P(A) \colon \mathrm{C}_P(A)] = p.$$

Also, A is a group of automorphisms of M, since $[P, A] \leq M$. Hence by inductive hypothesis we have $|[M, A]| = p$. Also by Lemma 1.12 we get

$$[M, A] \leq [P, A] = [P, A, A] \leq [M, A].$$

So $[M, A] = [P, A]$ and $|[P, A]| = p$. By Theorem 1.11 we know $P = \mathrm{C}_P(A)[P, A]$.

Since $|[P, A]| = p$ we get $\mathrm{C}_P(A) \cap [P, A] = 1$. Also, since $\mathrm{C}_P(A)$ and $[P, A]$ are both normal in P, we get $[\mathrm{C}_P(A), [P, A]] \leq \mathrm{C}_P(A) \cap [P, A] = 1$. Hence $\mathrm{C}_P(A)$ commutes with $[P, A]$ and we conclude that $P \cong \mathrm{C}_P(A) \times [P, A]$. □

Other information about the automorphism group of a p-group can be obtained using Thompson's $A \times B$-Lemma.

Lemma 1.14 (Thompson's $A \times B$-Lemma). *[Gor80, Theorem 5.3.4] Let p be a prime and let $A \times B$ be a group of automorphisms of the p-group P, with A a p'-group and B a p-group. If $[\mathrm{C}_P(B), A] = 1$, then $A = 1$.*

We now present a consequence of Maschke's Theorem.

Theorem 1.15. *[Gor80, Theorem 3.3.2] Let A be a p'-group of automorphisms of an abelian p-group V and suppose that V_1 is a non-trivial direct factor of V normalized by A. Then there exists a subgroup $V_2 \leq V$ normalized by A and such that $V = V_1 \times V_2$.*

If there is an automorphism φ of order prime to p acting on a p-group $P \cong \mathrm{C}_p \times \mathrm{C}_p$ and normalizing a maximal subgroup $V_1 \leq P$, then the previous theorem implies that there exists a maximal subgroup $V_2 \leq P$ distinct from V_1 and normalized by φ. This situation will appear many times in the next chapters.

To better understand the structure of a p-group P, we consider the set $\mathbf{A}(P)$ of subgroups of P that are abelian and of maximal order. More precisely, define the integer $a = \max\{|A| \mid A \leq P \text{ is abelian}\}$. Then $\mathbf{A}(P) = \langle A \mid A \leq P \text{ is abelian and } |A| = a\rangle$. The *Thompson subgroup of* P is the subgroup generated by $\mathbf{A}(P)$:

$$\mathrm{J}(P) = \langle A \mid A \in \mathbf{A}(P)\rangle.$$

The Thompson subgroup of P is characteristic in P and satisfies the following property

$$\text{if } \mathrm{J}(P) \leq Q \leq P \text{ then } \mathrm{J}(P) = \mathrm{J}(Q).$$

See [Gor80, Lemma 8.2.2] for a proof. The following lemmas characterize the natural action of P on its abelian subgroups of maximal order.

Lemma 1.16. *If $A \in \mathbf{A}(P)$, then $A = \mathrm{C}_P(A)$.*

Proof. Since A is abelian, we have $A \leq \mathrm{C}_P(A)$. Let $x \in \mathrm{C}_P(A)$. Then $A\langle x\rangle$ is an abelian subgroup of P and from the maximality of A we conclude $|A| = |A\langle x\rangle|$. So $x \in A$ and $\mathrm{C}_P(A) \leq A$. Therefore $A = \mathrm{C}_P(A)$. □

Lemma 1.17. *Let $A \in \mathbf{A}(P)$ and let B be a subgroup of P. Then B normalizes A if and only if $[B, A, A] = 1$.*

Proof. Assume B normalizes A. Then $[B, A] \leq A$ and so $[B, A, A] \leq [A, A] = 1$, since A is abelian. Conversely, assume $[B, A, A] = 1$. Then $[B, A] \leq \mathrm{C}_P(A)$ and by Lemma 1.16 we have $[B, A] \leq A$. Thus B normalizes A. □

We can now state the Thompson Replacement Theorem, that guarantees that given an abelian subgroup A of maximal order normalizing an abelian subgroup B, if B does not normalize A then we can replace B by an abelian subgroup A^* of maximal order normalizing A and satisfying $A \cap B < A^* \cap B$.

Theorem 1.18 (Thompson Replacement Theorem)**.** *[Gor80, Theorem 8.2.5] Let $A \in \mathbf{A}(P)$ and let B be an abelian subgroup of the p-group P. Assume A normalizes B, but B does not normalize A. Then there exists a group A^* in $\mathbf{A}(P)$ with the following properties:*

1. *$A \cap B < A^* \cap B$, and*

2. *A^* normalizes A.*

Lemma 1.19. *[KS04, Lemma 5.1.8] Let P be a p-group and let $Q, R \leq P$ be such that*

$$[Q, R] \leq Q \cap R \quad \textit{and} \quad |[Q, R]| \leq p.$$

Then $[Q \colon \mathrm{C}_Q(R)] = [R \colon \mathrm{C}_R(Q)]$.

Recall that a p-group P is *extraspecial* if $\Phi(P) = \mathrm{Z}(P)$ and $|\mathrm{Z}(P)| = p$. It is well known that if p is an odd prime then there are exactly two extraspecial groups of order p^{1+2n} for every $n \geq 1$, one of exponent p, denoted p_+^{1+2}, and one of exponent p^2, denoted p_-^{1+2}. As a consequence of Lemma 1.19, we can compute the order of a maximal abelian subgroup of an extraspecial group.

Lemma 1.20. *Let P be an extraspecial group of order p^{1+2n} and let Q be an abelian subgroup of P of maximal order. Then $|Q| = p^{n+1}$.*

Proof. By Lemma 1.16 we have $\mathrm{C}_P(Q) = Q$. In particular $\Phi(P) = \mathrm{Z}(P) \leq Q$ and so $[Q, P] \leq Q$. Since P is extraspecial we also have $|[Q, P]| \leq |[P, P]| = p$. Therefore by Lemma 1.19 we get $[P \colon Q] = [Q \colon \mathrm{C}_Q(P)] = [Q \colon \mathrm{Z}(P)]$.

Suppose $|Q| = p^a$. Then $[Q \colon \mathrm{Z}(P)] = p^{a-1}$ and the following holds

$$p^{1+2n} = |P| = [P \colon Q][Q \colon \mathrm{Z}(P)]|\mathrm{Z}(P)| = (p^{a-1})^2 p.$$

Hence $a = n + 1$ and $|Q| = p^{n+1}$. $\square$

Definition 1.21. Let G be a finite group acting on a set X. We say that G *acts faithfully* on X if no non-trivial element of G acts trivially on X.

Definition 1.22. Let G be a group that acts on the elementary abelian p-group V. We say that G *acts quadratically* on V if $[V, G, G] = 1$ and $[V, G] \neq 1$.

Recall that the p-core of a finite group G, denoted $O_p(G)$, is the largest normal p-subgroup of G.

Theorem 1.23 (p-stability)**.** *[Gor80, Theorem 3.8.3] Let p be an odd prime and let G be a group that acts faithfully on the elementary abelian p-group V. If $O_p(G) = 1$ and there is a non-trivial p-subgroup of G that acts quadratically on V, then G involves* $\mathrm{SL}_2(p)$.

This p-Stability theorem will be used in Chapter 4 to prove that the automorphism group of the $\mathcal{F}$*-essential subgroups* (defined on page vii) involves the group $\mathrm{SL}_2(p)$. As we will see later, this is a crucial step in the classification of fusion systems on p-groups of small sectional rank.

Definition 1.24. Let G be a finite group acting on a vector space V. We say that the action of G is *reducible* if there exists a proper non-trivial subspace U of V that is normalized by G. The action of G is *irreducible* if it is not reducible.

The presence of a subgroup isomorphic to $\mathrm{SL}_2(p)$ allows us to apply Stellmacher's Pushing Up Theorem, that gives information regarding the structure of the p-group studied. Before stating the theorem, we recall the definition of a natural $\mathrm{SL}_2(p^n)$-module.

Definition 1.25. Let $G = \mathrm{SL}_2(p^n)$, $S \in \mathrm{Syl}_p(G)$ and V be a $\mathrm{GF}(p)G$-module. Then V is a *natural* $\mathrm{SL}_2(p^n)$*-module* if $|V| = p^{2n}$ and S acts quadratically on V.

Note that there is a unique natural $\mathrm{SL}_2(p)$-module, that is the faithful $\mathrm{SL}_2(p)$-module of dimension 2.

Theorem 1.26 (Stellmacher)**.** *[Ste86, Theorem* 1*][Nil79, Theorem 3.2] Let G be a finite group, p a prime and P a Sylow p-subgroup of G such that*

1. *No non-trivial characteristic subgroup of P is normal in G, and*

2. *$\overline{G}/\Phi(\overline{G}) \cong \mathrm{PSL}_2(p^n)$ for $\overline{G} = G/O_p(G)$.*

Let $Q = O_p(G)$ and $V = [Q, O^p(G)]$. Then either P is elementary abelian or there exists $\alpha \in \mathrm{Aut}(P)$ such that

$$L/V_0O_{p'}(L) \cong \mathrm{SL}_2(p^n)$$

where $L = V^\alpha O^p(G)$ and $V_0 = V(L \cap \mathrm{Z}(G))$, and one of the following holds:

1. *$P/\Omega_1(\mathrm{Z}(P))$ is elementary abelian, $V \leq \mathrm{Z}(Q)$ and V is a natural $\mathrm{SL}_2(p^n)$-module for $L/V_0O_{p'}(L)$;*

2. *$p = 2$, $P/\Omega_1(\mathrm{Z}(P))$ is elementary abelian, $V \leq \mathrm{Z}(Q)$, $n > 1$ and $V/(V \cap \mathrm{Z}(G))$ is a natural $\mathrm{SL}_2(2^n)$-module for $L/V_0O_{2'}(L)$;*

3. *$p \neq 2$, $\mathrm{Z}(V) \leq \mathrm{Z}(Q)$, $\Phi(V) = V \cap \mathrm{Z}(G)$ has order p^n, and $V/\mathrm{Z}(V)$ and $\mathrm{Z}(V)/\Phi(V)$ are natural $\mathrm{SL}_2(p^n)$-modules for $L/V_0O_{p'}(L)$.*

In addition, in case (3) *the group P has nilpotency class* 3*, $\Phi(\Phi(P)) = 1$, P does not act quadratically on $V/\Phi(V)$ and $p = 3$.*

Note that in case 3 the group $V/\Phi(V)$ is elementary abelian of order p^{4n}, thus Q has sectional rank at least $4n$. Since we will be interested in p-groups of sectional rank 3 with p odd, Stellmacher's result implies that the p-groups satisfying the hypothesis have to be as in point 1 of the theorem with $n = 1$.

1.2 Normalizer tower and soft subgroups

Let P be a p-group and let E be a subgroup of P. The *normalizer tower* of E in P is the sequence of distinct subgroups of P defined recursively as

$$\mathrm{N}^0(E) = E \quad \text{and} \quad \mathrm{N}^i(E) = \mathrm{N}_P(\mathrm{N}^{i-1}(E)) \quad \text{for every } 1 \leq i \leq m,$$

where $m \in \mathbb{N}$ be the smallest integer satisfying $\mathrm{N}^m(E) = P$ (such m is called the *subnormal depth* of E in S). We say that E has *maximal normalizer tower* in P if $[P\colon E] = p^m$ (i.e. $[\mathrm{N}^i\colon \mathrm{N}^{i-1}] = p$ for every $1 \leq i \leq m$).

In Chapter 4 we show that every essential subgroup E of a p-group P of sectional rank 3 for p odd, contains its centralizer and has index p in its normalizer:

$$\mathrm{C}_P(E) \leq E \quad \text{and} \quad [\mathrm{N}_P(E)\colon E] = p. \tag{1.1}$$

An abelian group satisfying property 1.1 is called a *soft subgroup*, as defined and studied by Héthelyi. In particular, combining [Hét84, Lemma 2, Corollary 3], [Hét90, Theorem 1, Lemma 1 and Corollary 6] and [BH97, Theorem 2.1] we get the following theorem.

Theorem 1.27. *Let P be a p-group and let E be a soft subgroup of P with $[P\colon E] = p^m \geq p^2$. Set*

$$H_i = \begin{cases} \mathrm{Z}_i(\mathrm{N}^i) & \text{if} \quad 1 \leq i \leq m-1 \\ \mathrm{Z}(\mathrm{N}^1)[P,P] & \text{if} \quad i = m \end{cases}$$

Then

1. *E has maximal normalizer tower in P and the members of such a tower are the only subgroups of P containing E;*

2. *the group* N^i *has nilpotency class* $i+1$ *for every* $i \leq m-1$*;*

3. $H_i \leq \mathrm{N}^{i-1}$ *and* H_i *is characteristic in* N^i*;*

4. $[H_{i+1} \colon H_i] = [\mathrm{N}^{i-1} \colon H_i] = p$*;*

5. $\mathrm{N}^i / H_i \cong \mathrm{C}_p \times \mathrm{C}_p$*;*

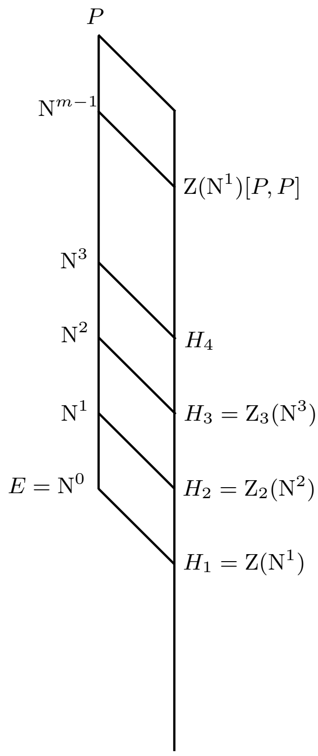
6. *the members of the sequence*

   $\mathrm{Z}(\mathrm{N}^1) = H_1 < H_2 < \cdots < H_{m-1} < H_m$

   are the only subgroups of H_m *normalized by* E *that contain* $\mathrm{Z}(\mathrm{N}^1)$*;*

7. *if* Q *is a soft subgroup of* P *with* $[P \colon Q] \geq p^2$*, then* $H_m = \mathrm{Z}(\mathrm{N}_P(Q))[P, P]$*;*

8. *if* Q *is a soft subgroup of* P *and* $Q \leq \mathrm{N}^{m-1}(E)$ *then there exists* $g \in P$ *such that* $Q^g = E$*.*



Parts 7 and 8 imply that the group H_m does not depend on the soft subgroup considered and that there are at most $p+1$ conjugacy classes of soft subgroups of a p-group.

1.3 Groups with a strongly p-embedded subgroup

Let p be a prime and let G be a finite group. We denote by $|G|_p$ the order of a Sylow p-subgroup of G. If p does not divide $|G|$, we write $|G|_p = 1$. A proper subgroup H of G is *strongly p-embedded* in G if $|H|_p > 1$ and for each $x \in G \backslash H$ we have $|H \cap H^x|_p = 1$.

Lemma 1.28. *A subgroup $H < G$ is strongly p-embedded in G if and only if $|H|_p > 1$ and $\mathrm{N}_G(Q) \leq H$ for every non-trivial p-subgroup Q of H.*

In particular if H is strongly p-embedded in G then $\mathrm{Syl}_p(H) \subseteq \mathrm{Syl}_p(G)$.

Proof. Suppose H is strongly p-embedded in G and let $1 \neq Q \leq H$ be a p-subgroup of H. Let $x \in \mathrm{N}_G(Q)$. Then $Q \leq H \cap H^x$ and so $|H \cap H^x|_p \neq 1$. Thus $\mathrm{N}_G(Q) \leq H$.

Suppose $|H|_p > 1$ and $\mathrm{N}_G(Q) \leq H$ for every p-subgroup $Q \leq H$. Let $x \in G$ be such that $|H \cap H^x|_p \neq 1$. We want to prove that $x \in H$. Let $Q \in \mathrm{Syl}_p(H \cap H^x)$. Then $\mathrm{N}_G(Q) \leq H$. In particular $Q \leq \mathrm{N}_{H^x}(Q) \leq H \cap H^x$. Hence $Q \in \mathrm{Syl}_p(\mathrm{N}_{H^x}(Q))$. Let $P \in \mathrm{Syl}_p(H^x)$ be such that $Q \leq P$. Then $\mathrm{N}_P(Q)$ is a p-subgroup of $\mathrm{N}_{H^x}(Q)$. Therefore $\mathrm{N}_P(Q) = Q$ and we conclude $P = Q \in \mathrm{Syl}_p(H^x)$. Hence $Q^{x^{-1}} \in \mathrm{Syl}_p(H)$ and since $|Q| = |Q^{x^{-1}}|$ we deduce that $Q \in \mathrm{Syl}_p(H)$ as well. So there exists $h \in H$ such that $Q^{x^{-1}h} = Q$. Hence $x^{-1}h \in \mathrm{N}_G(Q) \leq H$ and $x \in H$ as wanted.

Let $P \in \mathrm{Syl}_p(H)$ and let $S \in \mathrm{Syl}_p(G)$ be such that $P \leq S$. Then $\mathrm{N}_S(P) \leq H$ and since P is a Sylow p-subgroup of H we get $P = \mathrm{N}_S(P)$. Since S is a p-group, we conclude $P = S \in \mathrm{Syl}_p(G)$. □

An example of group with strongly p-embedded subgroup is a group G having a cyclic Sylow p-subgroup S with $\Omega_1(S)$ not normal in G.

Lemma 1.29. *If G has a cyclic Sylow p-subgroup S then either $G = \mathrm{N}_G(\Omega_1(S))$ or $\mathrm{N}_G(\Omega_1(S))$ is strongly p-embedded in G.*

Proof. Let $H = \mathrm{N}_G(\Omega_1(S))$ and assume $H < G$. Notice that $S \in \mathrm{Syl}_p(H)$ so $|H|_p > 1$. Let $1 \neq Q \leq S$. Since S is cyclic we have $\Omega_1(Q) = \Omega_1(S)$. Hence $\mathrm{N}_G(Q) \leq \mathrm{N}_G(\Omega_1(Q)) = \mathrm{N}_G(\Omega_1(S)) = H$. By Lemma 1.28 we conclude that the group H is strongly p-embedded in G. □

Corollary 1.30. *Let G be a group isomorphic to one of the following:*

$$\mathrm{SL}_2(p), \mathrm{GL}_2(p), \mathrm{PSL}_2(p) \textit{ and } \mathrm{PGL}_2(p).$$

If S is a Sylow p-subgroup of G then $\mathrm{N}_G(S)$ is a strongly p-embedded subgroup of G.

Lemma 1.31. *Let p be a prime, let G be a finite group whose order is divisible by p and let $N \trianglelefteq G$ be a normal subgroup of G such that $|N|_p = 1$. Let H be a subgroup of G such that $N \leq H$. Then H is a strongly p-embedded subgroup of G if and only if H/N is a strongly p-embedded subgroup of G/N.*

Proof. Note that $|H/N| = |H|/|N|$ and since $|N|_p = 1$ we get $|H|_p = |H/N|_p$. Since N is normal in G and is contained in H, we have $N \leq H \cap H^g$ for every $g \in G$. Also, $(H \cap H^g)/N = H/N \cap H^g/N$. So $|H \cap H^g|_p = |H/N \cap H^g/N|_p$. Finally notice that $g \in H$ if and only if $gN \in H/N$. Hence H is strongly p-embedded in G if and only if H/N is strongly p-embedded in G/N. □

The next lemma gives a necessary condition for a group to have a strongly p-embedded subgroup.

Lemma 1.32. *If G contains a strongly p-embedded subgroup, then $O_p(G) = 1$.*

Proof. Assume $H < G$ is strongly p-embedded. By Lemma 1.28 we have $\mathrm{Syl}_p(H) \subseteq \mathrm{Syl}_p(G)$, so $O_p(G) \leq H$, and if $O_p(G)$ is non-trivial then $\mathrm{N}_G(O_p(G)) \leq H$. Since $G = \mathrm{N}_G(O_p(G))$ and $G \neq H$, we conclude that $O_p(G) = 1$. □

Note that the condition $O_p(G) = 1$ is not sufficient for G to have a strongly p-embedded subgroup, as the next lemma shows.

Lemma 1.33. *The groups* $\mathrm{SL}_3(p)$ *and* $\mathrm{PSL}_3(p)$ *do not have strongly* p*-embedded subgroups.*

Proof. Let $G = \mathrm{SL}_3(p)$. Aiming for a contradiction, let $H \leq G$ be a strongly p-embedded subgroup and let $S \in \mathrm{Syl}_p(H)$. Then we may assume

$$S = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix} \mid x, y, z \in \mathrm{GF}(p) \right\}$$

Consider the following subgroups of S of order p:

$$H_z = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\rangle, \quad H_x = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle, \quad H_y = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \right\rangle.$$

Then

$$N_z = \mathrm{N}_G(H_z) = \left\{ \begin{pmatrix} a & 0 & 0 \\ c & b & 0 \\ d & e & (ab)^{-1} \end{pmatrix} \mid a, b \in \mathrm{GF}(p)^*, c, d, e \in \mathrm{GF}(p) \right\};$$

$$N_x = \mathrm{N}_G(H_x) = \left\{ \begin{pmatrix} a & 0 & 0 \\ c & b & f \\ d & 0 & (ab)^{-1} \end{pmatrix} \mid a, b \in \mathrm{GF}(p)^*, c, d, f \in \mathrm{GF}(p) \right\};$$

$$N_y = \mathrm{N}_G(H_y) = \left\{ \begin{pmatrix} a & f & 0 \\ 0 & b & 0 \\ d & e & (ab)^{-1} \end{pmatrix} \mid a, b \in \mathrm{GF}(p)^*, d, e, f \in \mathrm{GF}(p) \right\}.$$

By Lemma 1.28 we get $\langle N_z, N_x, N_y \rangle \leq H$.

In particular $\left\langle \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \leq N_y \leq H, \quad \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \leq N_x \leq H,$ and

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in H.$$

Let E be the set of matrices of the form $\mathbb{I}_3 + E_{i,j}(\lambda)$ for $i \neq j$ and $\lambda \in \mathrm{GF}(p)^*$, where $\mathbb{I}_3$ is the identity (3×3)-matrix and $E_{i,j}(\lambda)$ is a (3×3)-matrix with the (i, j)-entry equal to λ and every other entry equal to 0. We showed that $\langle E \rangle \leq H$. However it is well known that $\langle E \rangle = \mathrm{SL}_3(p)$ and so $H = G$, which is a contradiction.

Thus the group $\mathrm{SL}_3(p)$ does not have a strongly p-embedded subgroup.

Recall that $\mathrm{PSL}_3(p) = \mathrm{SL}_3(p)/\mathrm{Z}(\mathrm{SL}_3(p))$ and $|\mathrm{Z}(\mathrm{SL}_3(p))| = \gcd(3, p-1)$. Suppose by contradiction that there exists a subgroup $\overline{H} \leq \mathrm{PSL}_3(p)$ that is strongly p-embedded in $\mathrm{PSL}_3(p)$ and let H be the preimage of $\overline{H}$ in $\mathrm{SL}_3(p)$. Thus by Lemma 1.31 we conclude that H is a strongly p-embedded subgroup of $\mathrm{SL}_3(p)$, contradicting what we proved above. So $\mathrm{PSL}_3(p)$ does not have a strongly p-embedded subgroup. □

In order to prove that a group G has a strongly p-embedded subgroup, from now on we will first check that $O_p(G) = 1$. For this reason, the following result is decisive.

Lemma 1.34. *[Gor80, Corollary 5.3.3] Let p be a prime and let A be a group of automorphisms of the p-group P. Consider a sequence of subgroups*

$$P_0 \leq P_1 \leq \cdots \leq P_n = P$$

all normalized by A and satisfying $P_0 \leq \Phi(P)$ and $P_i \trianglelefteq P_{i+1}$ for every $0 \leq i \leq n-1$. Let $H \leq A$ be the subgroup generated by the elements of A that centralize every quotient P_{i+1}/P_i. Then $H \leq O_p(A)$.

As a direct consequence, if a p-group P has a group of automorphisms A that has a strongly p-embedded subgroup, and there is a sequence of subgroups $P_0 \leq P_1 \leq \cdots \leq P_n = P$ of the form described by the previous lemma, then every automorphism in A acts non-trivially on at least one quotient P_{i+1}/P_i. This idea will be used many times.

We now consider a group G with a strongly p-embedded subgroup acting faithfully on a 3-dimensional vector space over $\mathrm{GF}(p)$. We prove that if the action is reducible then G has a subgroup isomorphic to $\mathrm{SL}_2(p)$, otherwise it has a subgroup isomorphic to either $\mathrm{SL}_2(p)$, or $\mathrm{PSL}_2(p)$, or $13\colon 3$ (and in the last case $p = 3$). The determination of the automorphism group of $\mathcal{F}$-essential subgroups, discussed in Chapter 4, is a corollary of this result.

Lemma 1.35. *Let p be an odd prime, let V be a 3-dimensional vector space over the field $\mathrm{GF}(p)$ and let $G \leq \mathrm{Aut}(V)$. Suppose that G has a strongly p-embedded subgroup and acts reducibly on V. Then there exist unique subspaces $U, W \leq V$ normalized by G such that $\dim(U) = 1$, $\dim(W) = 2$ and*

$$\mathrm{SL}_2(p) \leq G \leq \mathrm{Aut}(U) \times \mathrm{Aut}(W) \cong \mathrm{GL}_2(p) \times \mathrm{GL}_1(p).$$

Proof. Let S be a Sylow p-subgroup of G and set $H = \langle S^G \rangle$. Then $\mathrm{Syl}_p(H) = \mathrm{Syl}_p(G)$ and so $O_p(H) = O_p(G) = 1$ by Lemma 1.32.

Let U be a proper subspace of V normalized by G. Then U is normalized by H and $1 \leq \dim(U) \leq 2$.

1. Suppose $\dim(U) = 1$. Then $[S, U] = 1$ for every Sylow p-subgroup S of G. Thus $[H, U] = 1$. So the subgroup $\mathrm{C}_H(V/U)$ centralizes each quotient of two consecutive subspaces in the sequence $1 < U < V$ and by Lemma 1.34 and the fact that $O_p(H) = 1$ we deduce that $\mathrm{C}_H(V/U) = 1$. Therefore $H \hookrightarrow \mathrm{Aut}(V/U) \cong \mathrm{GL}_2(p)$ and so $H \cong \mathrm{SL}_2(p)$.

Let $t \in \mathrm{Z}(H)$ be an involution. Then by coprime action (Theorem 1.11) we get

$$V = [V, t] \oplus \mathrm{C}_V(t).$$

Note that from $t \in \mathrm{Z}(H)$ we deduce that the subspaces $[V, t]$ and $\mathrm{C}_V(t)$ are normalized by G. Also, $U \leq \mathrm{C}_V(t)$. If $U \neq \mathrm{C}_V(t)$, then the quotients $V/\mathrm{C}_V(t)$ and $\mathrm{C}_V(t)/U$ have dimension 1. In particular H centralizes every quotient of two consecutive subgroups in the sequence $1 < U < \mathrm{C}_V(t) < V$ and we get a contradiction by Lemma 1.34. Hence $U = \mathrm{C}_V(t)$ and $W := [V, t]$ is a 2-dimensional space such that $V = W \oplus U$ and W is normalized by G.

2. Suppose $\dim(U) = 2$. The group G acts on the dual space $V^* = \mathrm{Hom}(V, \mathrm{GF}(p))$, that is a 3-dimensional vector space over $\mathrm{GF}(p)$. Also, since it normalizes U, it normalizes the subspace

$$U^{\perp} = \{\varphi \in V^* \mid u\varphi = 0 \text{ for every } u \in U\} \subseteq V^*.$$

Note that $U^{\perp}$ has dimension $1 = \dim V - \dim U$. Thus G normalizes a 1-dimensional subspace of a vector space of dimension 3. Hence, with an argument similar to the one used in part 1, we can show that there exists a 2-dimensional space W^* of V^* normalized by G. In particular the corresponding subspace $W := (W^*)^{\perp}$ of V is a 1-dimensional subspace normalized by G.

Suppose by contradiction there exist distinct 1-dimensional subspaces U_1 and U_2 of V that are normalized by G. Then H centralizes U_1 and the quotients V/U_1U_2 and U_1U_2/U_1. So $H \leq O_p(G)$ by Lemma 1.34, contradicting the fact that $O_p(G) = 1$. Thus there exists a unique 1-dimensional space normalized by G, and so a unique decomposition of V as $U \oplus W$, for some 2-dimensional subspace W of V. This proves the result. $\square$

To study the irreducible case, we first need to show that the property of having a strongly p-embedded subgroup is inherited by the subgroup $O^{p'}(G)$ of G, that is the intersection of all normal subgroups N of G such that G/N has order prime to p. Equivalently, $O^{p'}(G) = \langle \mathrm{Syl}_p(G) \rangle$.

Lemma 1.36. *Suppose that the group G has a strongly p-embedded subgroup H and let $K \trianglelefteq G$ be such that $O^{p'}(G) \leq K$. Then $H \cap K$ is a strongly p-embedded subgroup of K.*

Proof. By Lemma 1.28 there exists a Sylow p-subgroup S of G that is contained in H. Hence $S \leq H \cap O^{p'}(G) \leq H \cap K$ and so $|H \cap K|_p \neq 1$. Lemma 1.28 also tells us that $\mathrm{N}_G(S) \leq H$. By the Frattini argument we have $G = K\mathrm{N}_G(S) \leq KH$. Since $H < G$, we deduce that $K \not\leq H$ and so $H \cap K < K$. Take $g \in K \backslash (H \cap K)$. Then $g \in G \backslash H$ and since K is normal in G we have $(H \cap K)^g = H^g \cap K$. Using the fact that H is strongly p-embedded in G we get

$$|(H \cap K) \cap (H \cap K)^g|_p = |H \cap H^g \cap K|_p \leq |H \cap H^g|_p = 1.$$

So $H \cap K$ is a strongly p-embedded subgroup of K. $\square$

Another required ingredient is the list of subgroups of the group $\mathrm{PSL}_3(p)$, illustrated by the next results.

Lemma 1.37. *Let $p \in \{3, 5\}$ and let H be a subgroup of $\mathrm{SL}_3(p)$ having a strongly p-embedded subgroup. Then H is isomorphic to one of the following groups:*

1. $\mathrm{PSL}_2(p)$;
2. $\mathrm{PGL}_2(p)$;
3. $\mathrm{SL}_2(p)$;
4. $\mathrm{GL}_2(p)$;

5. $13\colon 3$ *with* $p = 3$;

6. $\mathrm{SL}_2(5)\colon \mathrm{C}_2$ *(isomorphic to a non-split extension of* C_4 *by* $\mathrm{Alt}(5)$*) with* $p = 5$.

Proof. By definition and Lemma 1.32 we are looking for subgroups of $\mathrm{SL}_3(p)$ whose order is divisible by p and with trivial p-core. We use *Magma* to identify such groups.

```
G1:=SL(3,3);
X1:=[H : H in Subgroups(G1) |♯H‘subgroup mod 3 eq 0
    and ♯pCore(H‘subgroup,3) eq 1];
X1;

G2:=SL(3,5);
X2:=[H : H in Subgroups(G2) |♯H‘subgroup mod 5 eq 0
    and ♯pCore(H‘subgroup,5) eq 1];
X2;
```

We can then check with the command *IsIsomorphic* that the groups in $X1$ and $X2$ are either isomorphic to the groups in the list or to the group $\mathrm{SL}_3(p)$ (with $p = 3$ or 5, respectively). Note that all groups in the list have cyclic (non normal) Sylow p-subgroup, hence they have a strongly p-embedded subgroup by Lemma 1.29. Finally notice that $\mathrm{SL}_3(p)$ does not have a strongly p-embedded subgroup by Lemma 1.33. □

Theorem 1.38. *[GLS98, Theorem 6.5.3] Let G be a subgroup of* $\mathrm{PSL}_3(p)$ *that acts irreducibly on the natural module for* $\mathrm{SL}_3(p)$*. If $p \geq 7$ then G is isomorphic to a subgroup of one of the following groups:*

1. *a Frobenius group with kernel of order* $(p^2 + p + 1)/3$ *and complement of order* 3;

2. *the group* $\mathrm{U}_3(2)$*, and if* 27 *divides* $|G|$ *the group* $\mathrm{PGU}_3(2)$*, for* $p \equiv 1 \mod 3$;

3. *the group* $\mathrm{PGL}_2(p)$;

4. *the group* $\mathrm{PSL}_3(2)$*, for* $p \equiv 1, 2, 4 \mod 7$;

5. *the group* $\mathrm{Alt}(6)$*, for* $p \equiv 1, 4 \mod 15$*.*

We can now describe the subgroup $O^{p'}(G)$ of a group G having a strongly p-embedded subgroup and acting on a 3-dimensional vector space over $\mathrm{GF}(p)$.

Theorem 1.39. *Let p be an odd prime, let V be a 3-dimensional vector space over the field $\mathrm{GF}(p)$ and let $G \leq \mathrm{Aut}(V)$. Suppose that G has a strongly p-embedded subgroup. Then one of the following holds:*

1. $O^{p'}(G) \cong \mathrm{SL}_2(p)$ *and* $G \leq \mathrm{Aut}(U) \times \mathrm{Aut}(W) \cong \mathrm{GL}_2(p) \times \mathrm{GL}_1(p)$*, for unique subspaces* $U, W \subset V$*;*

2. $O^{p'}(G) \cong \mathrm{PSL}_2(p)$ *and G acts irreducibly on V;*

3. $p = 3$ *and* $O^{p'}(G) \cong 13\colon 3$ *and G acts irreducibly on V.*

Proof. Since $\mathrm{Aut}(V) \cong \mathrm{GL}_3(p)$, the group G is isomorphic to a subgroup of $\mathrm{GL}_3(p)$. To simplify the notation we assume $G \leq \mathrm{GL}_3(p)$.

Let H be a strongly p-embedded subgroup of G and set $K = O^{p'}(G)$. Then $K \leq \mathrm{SL}_3(p)$ and $H \cap K$ is a strongly p-embedded subgroup of K by Lemma 1.36.

If $p = 3$ or $p = 5$ then K is isomorphic to one of the groups listed in Lemma 1.37. Since $K = O^{p'}(G)$ we conclude that either $K \cong \mathrm{SL}_2(p)$ or $K \cong \mathrm{PSL}_2(p)$, or $K \cong 13\colon 3$ (and $p = 3$). In particular G acts reducibly on V if and only if $K \cong \mathrm{SL}_2(p)$ and in this case we conclude by Lemma 1.35.

Suppose $p \geq 7$. If the action of K on V is reducible, then $K \cong \mathrm{SL}_2(p)$ by Lemma 1.35 and there exist unique subspaces $U, W \subset V$ such that $G \leq \mathrm{Aut}(U) \times \mathrm{Aut}(W) \cong \mathrm{GL}_2(p) \times \mathrm{GL}_1(p)$.

Suppose the action of K is irreducible and set $Z = \mathrm{Z}(\mathrm{SL}_3(p))$. By Lemma 1.36 and the fact that $K \leq KZ \leq \mathrm{SL}_3(p)$ we deduce that $H \cap \mathrm{SL}_3(p)$ is a strongly p-embedded subgroup of $G \cap \mathrm{SL}_3(p)$ and $H \cap KZ$ is a strongly p-embedded subgroup of KZ. In particular $Z \leq H \cap KZ$ and by Lemma 1.31 we conclude that the group $\overline{H} = (H \cap KZ)/Z$ is a strongly p-embedded subgroup of the group $\overline{K} = KZ/Z$.

Note that $\overline{K} \leq \mathrm{PSL}_3(p)$. Also, $\overline{K} \neq \mathrm{PSL}_3(p)$ by Lemma 1.33 and $O_p(\overline{K}) = 1$ by Lemma 1.32. We consider the classification of maximal subgroups of $\mathrm{PSL}_3(p)$ given by Theorem 1.38. Using the fact that p divides the order of $\overline{K}$ and $O_p(\overline{K}) = 1$ we get that either $\overline{K}$ is isomorphic to a subgroup of $\mathrm{PGL}_2(p)$ (case 3) or $p = 7$ and $\overline{K}$ is isomorphic to a subgroup of $\mathrm{PSL}_3(2) \cong \mathrm{PSL}_2(7)$ (case 4).

Since $K = O^{p'}(G)$, we conclude that $\overline{K} \cong \mathrm{PSL}_2(p)$ for every $p \geq 7$.

The group $\mathrm{PSL}_2(p)$ has a Schur multiplier of order 2 ([Hup67, Satz V.25.7]) and $|\mathrm{Z}(\mathrm{SL}_3(p))|$ is either 1 or 3. Hence $\overline{K} \cong K$. □

1.4 Amalgams and weak BN-pairs of rank 2

A *rank* 2 *amalgam* $\mathcal{A} = \mathcal{A}(P_1, P_2, P_{12})$ consists of three finite groups P_1, P_2 and P_{12} and two monomorphisms $\phi_1 \colon P_{12} \hookrightarrow P_1$ and $\phi_2 \colon P_{12} \hookrightarrow P_2$. A group G is called a *faithful completion* of $\mathcal{A}(P_1, P_2, P_{12})$ if there exist two monomorphisms $\psi_1 \colon P_1 \hookrightarrow G$ and $\psi_2 \colon P_2 \hookrightarrow G$ such that $G = \langle P_1\psi_1, P_2\psi_2 \rangle$ and $\phi_1\psi_1 = \phi_2\psi_2$.

$$\begin{tikzcd}
 & G & \\
P_1 \arrow[ur, hook, "\psi_1"] & & P_2 \arrow[ul, hook', "\psi_2"'] \\
 & P_{12} \arrow[ul, hook', "\phi_1"] \arrow[ur, hook, "\phi_2"'] &
\end{tikzcd}$$

We identify P_1, P_2 and P_{12} by their images under ψ_1 and ψ_2. Note that the free amalgamated product of P_1 and P_2 over P_{12} is a faithful completion of $\mathcal{A}$.

Definition 1.40. Let G be a faithful completion of the amalgam $\mathcal{A}(P_1, P_2, P_{12})$. Let X be the free amalgamated product of P_1 and P_2 over P_{12}.

- A group H is *locally isomorphic* to G if there exists a free normal subgroup Y of X such that $X/Y \cong H$.
- A group H is *parabolic isomorphic* to G if H is a faithful completion of the amalgam $\mathcal{A}(Q_1, Q_2, Q_{12})$ and $P_1 \cong Q_1$, $P_2 \cong Q_2$ and $P_{12} \cong Q_{12}$.

Definition 1.41. We say that the rank 2 amalgam $\mathcal{A}$ is a *weak BN-pair of rank* 2 if no non-trivial subgroup of P_{12} is normalized by both P_1 and P_2 and there exist $P_1^* \trianglelefteq P_1$ and $P_2^* \trianglelefteq P_2$ such that for every $i \in \{1, 2\}$ we have

1. $O_p(P_i) \leq P_i^*$ and $P_i = P_i^* P_{12}$;
2. $C_{P_i}(O_p(P_i)) \leq O_p(P_i)$;
3. $P_i^* \cap P_{12}$ is the normalizer of a Sylow p-subgroup of P_i^*; and
4. $P_i^*/O_p(P_i)$ is isomorphic to one of the following groups:
   (a) $\mathrm{PSL}_2(p^{n_i}), \mathrm{SL}_2(p^{n_i}), \mathrm{U}_3(p^{n_i}), \mathrm{SU}_3(p^{n_i}), \mathrm{Sz}(2^{n_i})$, or
   (b) D_{10} and $p = 2$, or
   (c) $\mathrm{Ree}(3^{n_i})$ or $\mathrm{Ree}(3)'$ and $p = 3$.

We write $\mathcal{A} = \mathcal{A}(P_1, P_2, P_{12}, P_1^*, P_2^*)$.

The following result is due to Delgado and Stellmacher and describes the faithful completions of weak BN-pairs of rank 2.

Theorem 1.42. *[DGS85, Theorem II.4.A] Let p be a prime and let G be a faithful completion of a weak BN-pair of rank 2. Then one of the following holds*

1. *G is locally isomorphic to a group X such that $H \leq X \leq \mathrm{Aut}(H)$ and H is one of the following:*

$$\mathrm{PSL}_3(p^n),\ \mathrm{PSp}_4(p^n),\ \mathrm{U}_4(p^n),\ \mathrm{U}_5(p^n),\ \mathrm{G}_2(p^n),\ {}^3\mathrm{D}_4(p^n),$$
$${}^2\mathrm{F}_4(2^n),\ \mathrm{G}_2(2)',\ {}^2\mathrm{F}_4(2)',\ \mathrm{M}_{12},\ \mathrm{J}_2,\ \textit{or}\ \mathrm{F}_3.$$

2. *G is parabolic isomorphic to $\mathrm{G}_2(2)'$, J_2, $\mathrm{Aut}(\mathrm{J}_2)$, M_{12} or $\mathrm{Aut}(\mathrm{M}_{12})$.*

3. *G is of type ${}^2\mathrm{F}_4(2)'$, ${}^2\mathrm{F}_4(2)$ or F_3.*

In particular if $\mathcal{A} = \mathcal{A}(P_1, P_2, P_{12}, P_1^*, P_2^*)$ is a weak BN-pair and $\mathrm{Syl}_p(P_{12}) \subseteq \mathrm{Syl}_p(P_1^*) \cap \mathrm{Syl}_p(P_2^*)$ then every $S \in \mathrm{Syl}_p(P_{12})$ is isomorphic to a Sylow p-subgroup of one of the groups listed in Theorem 1.42.

Another family of amalgams that we are going to use is the family of symplectic amalgams.

Definition 1.43. Let $\mathcal{A} = \mathcal{A}(P_1, P_2, P_{12})$ be a rank 2 amalgam, let p be a prime and let $S \in \mathrm{Syl}_p(P_{12})$. Set

$$L_i = \langle S^{P_i} \rangle;$$
$$Q_i = O_p(P_i);$$
$$W_1 = (Q_1 \cap Q_2)^{L_1}.$$

Then $\mathcal{A}$ is a *symplectic amalgam* over $\mathrm{GF}(p)$ if the following holds:

1. no non-trivial subgroup of P_{12} is normal in both P_1 and P_2;

2. $S \in \mathrm{Syl}_p(P_1) \cap \mathrm{Syl}_p(P_2)$;

3. $C_{P_i}(O_p(P_i)) \leq O_p(P_i)$ for every i;

4. $L_1/Q_1 \cong SL_2(p)$;

5. $P_{12} = N_{P_1}(S)$;

6. $P_2 = P_{12}\langle W_1^{L_2}\rangle$ and $O^p(L_2) \leq \langle W_1^{L_2}\rangle$;

7. $\langle \Omega_1(Z(S))^{P_2}\rangle = \Omega_1(Z(S)) = \Omega_1(Z(L_2))$;

8. $\langle \Omega_1(Z(S))^{P_1}\rangle \leq Q_2$ and there exists $x \in P_2$ such that $\langle \Omega_1(Z(S))^{P_1}\rangle \nleq Q_1^x$.

As an example, let $G = G_2(p)$ and $S \in \mathrm{Syl}_p(G)$. Set $Q_1 = C_S(Z_2(S))$ and let Q_2 be the unique maximal subgroup of S isomorphic to p_+^{1+4}. Then the amalgam

$$\mathcal{A}(N_G(Q_1), N_G(Q_2), N_G(Q_1) \cap N_G(Q_2))$$

is a symplectic amalgam.

Recall that a finite group is called a $\mathcal{K}$-group if all its non-abelian simple sections are isomorphic to one of the simple groups listed in the classification theorem of finite simple groups.

Theorem 1.44. *[PR02, Theorem 1.10] Let $\mathcal{A}(P_1, P_2, P_{12})$ be a symplectic amalgam over* $GF(p)$ *(with the notation introduced in Definition 1.43). Suppose that P_2 is a $\mathcal{K}$-group, $p \geq 11$ and there are two non-central chief factors of L_1 inside Q_1. Then*

$$Q_2\langle W_1^{L_2}\rangle \text{ is of shape } p_+^{1+4}.SL_2(p).$$

We will use Theorem 1.44 in Chapter 3 to prove that a certain p-group is isomorphic to a Sylow p-subgroup of the group $G_2(p)$.

CHAPTER 2

INTRODUCTION TO FUSION SYSTEMS

'Not all those who wander are lost.'

[J.R.R. Tolkien]

In this chapter we begin our journey in the world of Fusion Systems.

Starting from the definition of fusion category of a group, we generalize it to the notion of (abstract) fusion system. We then introduce the concept of saturation, motivated by properties of Sylow subgroups of a group.

In Section 2.2 we study fusion subsystems, giving the definition of normal fusion subsystem, of simple fusion system, of normalizer fusion subsystem and of subgroup normal in the fusion system. We conclude this section stating the Model Theorem for constrained fusion system that guarantees that a constrained fusion system is realizable by a finite group.

Section 2.3 is dedicated to *$\mathcal{F}$-essential subgroups.* We define this class of subgroups, underlying their importance in the classification of saturated fusion system, and we analyse some of their properties, that will be used in the next chapters. We prove that the outer $\mathcal{F}$-automorphism group of an $\mathcal{F}$-essential subgroup of rank r is isomorphic to a subgroup of the general linear group $\mathrm{GL}_r(p)$ and that the normalizer fusion system $\mathrm{N}_{\mathcal{F}}(E)$ of an $\mathcal{F}$-essential subgroup E of S is realizable by a finite group (weak model theorem). We then show that under a certain assumption, the presence of two $\mathcal{F}$-essential subgroups of S characteristic in S allows us to construct a weak BN-pair associated to the fusion system considered and to describe the isomorphism type of a quotient of S.

Recalling that the final goal of this work is the classification of fusion systems on p-groups of sectional rank 3, in Section 2.4 we focus our attention on properties of $\mathcal{F}$-essential subgroups related to their rank. We start determining the outer automorphism group of $\mathcal{F}$-essential subgroups of rank at most 3.

Theorem 1 (Structure Theorem for $\mathrm{Out}_{\mathcal{F}}(E)$)**.** *Let $\mathcal{F}$ be a saturated fusion system on the p-group S and let $E \leq S$ be an $\mathcal{F}$-essential subgroup of rank at most 3. Then*

1. *If $|E/\Phi(E)| = p^2$, then $\mathrm{SL}_2(p) \leq \mathrm{Out}_{\mathcal{F}}(E) \leq \mathrm{GL}_2(p)$;*
2. *If $|E/\Phi(E)| = p^3$ and the action of $\mathrm{Out}_{\mathcal{F}}(E)$ on $E/\Phi(E)$ is reducible then $\mathrm{SL}_2(p) \leq \mathrm{Out}_{\mathcal{F}}(E) \leq \mathrm{GL}_2(p) \times \mathrm{GL}_1(p)$;*
3. *If $|E/\Phi(E)| = p^3$ and the action of $\mathrm{Out}_{\mathcal{F}}(E)$ on $E/\Phi(E)$ is irreducible then*
   (a) either $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{PSL}_2(p)$;
   (b) or $p = 3$ and $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong 13\colon 3$.

Notice that when E has rank at most 3 then as a consequence of Theorem 1 the quotient $\mathrm{N}_S(E)/E$ has order p. We then prove an extension property for such $\mathcal{F}$-essential subgroups. In particular we show that if E is an abelian $\mathcal{F}$-essential subgroup of rank at most 3, then every $\mathcal{F}$-automorphism of E that normalizes the quotient $\mathrm{N}_S(E)/E$ is the restriction of an $\mathcal{F}$-automorphism of the group S.

2.1 Fusion categories of groups and fusion systems

Let G be a finite group. The subgroup of the automorphism group of G generated by the conjugation maps c_g, for $g \in G$, is called *inner automorphism group* of G and is denoted by $\mathrm{Inn}(G)$. We have

$$\mathrm{Inn}(G) \cong G/\mathrm{Z}(G).$$

Given subgroups P and Q of G we define the set of conjugation maps that conjugate P into Q as

$$\mathrm{Hom}_G(P, Q) = \{c_g \colon P \to Q \mid g \in G, P^g \leq Q\}.$$

For every subgroup $P \leq G$, set $\mathrm{Aut}_G(P) = \mathrm{Hom}_G(P, P)$. Then $\mathrm{Aut}_G(P)$ is a subgroup of $\mathrm{Aut}(P)$ containing $\mathrm{Inn}(P)$ and

$$\mathrm{Aut}_G(P) \cong \mathrm{N}_G(P)/\mathrm{C}_G(P).$$

Note that $\mathrm{Aut}_G(G) = \mathrm{Inn}(G)$.

Definition 2.1. Let G be a finite group and let $S \leq G$ be a p-subgroup. The *fusion category of G on S*, denoted $\mathcal{F}_S(G)$, is the category given by

$$\mathrm{Obj}(\mathcal{F}_S(G)) = \{P \mid P \leq S\} \quad \text{and}$$
$$\mathrm{Mor}_{\mathcal{F}_S(G)}(P, Q) = \mathrm{Hom}_G(P, Q) \quad \text{for all } P, Q \leq S.$$

If $H \leq G$ is a subgroup containing S such that $\mathcal{F}_S(H) = \mathcal{F}_S(G)$ then we say that H *controls fusion* in S.

Lemma 2.2. *If $H < G$ is a strongly p-embedded subgroup of G and $S \leq H$ then H controls fusion in S.*

Proof. Note that $\mathcal{F}_S(G)$ and $\mathcal{F}_S(H)$ have the same set of objects, namely the subgroups of S. Since H is a subgroup of G, we deduce that $\mathrm{Hom}_H(P,Q) \subseteq \mathrm{Hom}_G(P,Q)$ for every $P,Q \leq S$. Let $\alpha \in \mathrm{Hom}_G(P,Q)$ for some $P,Q \leq S$. Then $\alpha = c_g$, for some $g \in G$. Hence $P \leq S \leq H$ and $P^g \leq Q \leq S \leq H$, so $P^g \leq H \cap H^g$. If $P = 1$ then $\alpha = \mathrm{id_P}$ so $\alpha \in \mathrm{Hom}_H(P,Q)$. If $P \neq 1$ then $|H \cap H^g|_p \neq 1$ and since H is strongly p-embedded in G, we deduce that $g \in H$. Therefore $\alpha = c_g \in \mathrm{Hom}_H(P,Q)$. Hence we conclude that $\mathrm{Hom}_G(P,Q) = \mathrm{Hom}_H(P,Q)$ for every $P,Q \leq S$ and so $\mathcal{F}_S(G) = \mathcal{F}_S(H)$. □

In general it is not true that a subgroup that controls fusion is strongly p-embedded. As an example, let $G = \mathrm{Sym}(3) \times \mathrm{Sym}(3)$ and let S be a Sylow 2-subgroup of G. Notice that S has a normal 2-complement and the Frobenius Normal 2-Complement Theorem implies that $\mathcal{F}_S(G) = \mathcal{F}_S(S)$. However S is not strongly 2-embedded in G.

We now want to generalize the concept of fusion category 'forgetting' about the group G and considering collections of monomorphisms between subgroups of the p-group S that behave as conjugation maps. What we obtain is an abstract fusion system on the p-group S.

Definition 2.3. Let S be a finite p-group. A *fusion system* $\mathcal{F}$ on S is a category with set of objects $\mathrm{Obj}(\mathcal{F}) = \{P \mid P \leq S\}$ and set of morphisms $\mathrm{Mor}(\mathcal{F}) = \bigcup_{P,Q \leq S} \mathrm{Hom}_{\mathcal{F}}(P,Q)$, where

$$\mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \{\alpha \colon P \hookrightarrow Q \mid \alpha \text{ is an injective homomorphism of groups } \}$$

and for every $P,Q \leq S$ the following holds:

1. $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q)$;

2. each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ is the composition of an isomorphism $\alpha \in \mathrm{Mor}(\mathcal{F})$ and an inclusion $\beta \in \mathrm{Mor}(\mathcal{F})$.

The definition of fusion system given above is the one presented in [AKO11, Definition I.2.1]. Indeed, the terminology and the notation introduced in this chapter are the same used in [AKO11, Part I].

The easiest way to construct a fusion system is to consider all the injective morphisms between subgroups of S. The fusion system $\mathcal{U}_S$ that we obtain in this way is called *universal fusion system* on S. The fusion category $\mathcal{F}_S(G)$ of a group G containing S is another example of fusion system.

From now on $\mathcal{F}$ will always denote a fusion system on a p-group S. If $P, Q \leq S$ are subgroups then we define the following sets:

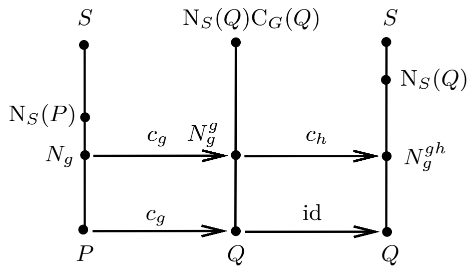
1. $\mathrm{Iso}_{\mathcal{F}}(P,Q) = \{\alpha \in \mathrm{Hom}_{\mathcal{F}}(P,Q) \mid P\alpha = Q\}$;
2. $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Iso}_{\mathcal{F}}(P,P)$;
3. $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$.

We refer to any $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ as an *$\mathcal{F}$-automorphism* of P and we say that $R \leq P$ is *$\mathcal{F}$-characteristic* in P, denoted $R \operatorname{char}_{\mathcal{F}} P$, if R is normalized by every $\mathcal{F}$-automorphism of P.

The fusion category of a group on one of its Sylow p-subgroups satisfies the Sylow and extension properties illustrated by the next lemma, which we will refer to as *saturation properties.*

Lemma 2.4 (Saturation)**.** *Let G be a finite group and let $S \leq G$ be a Sylow p-subgroup. Then for each subgroup $P \leq S$ there exists an element $g \in G$ such that $P^g \leq S$ and if $Q = P^g$ then*

1. *(Sylow Property)* $\mathrm{N}_S(Q) \in \mathrm{Syl}_p(\mathrm{N}_G(Q))$ *and* $\mathrm{Aut}_S(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_G(Q))$*;*
2. *(Extension Property) if $N_g = \{x \in \mathrm{N}_S(P) \mid x^g \in \mathrm{N}_S(Q)\mathrm{C}_G(Q)\}$, then there exists $h \in \mathrm{C}_G(Q)$ such that $N_g^{gh} \leq S$.*



Proof. Let $T \in \mathrm{Syl}_p(\mathrm{N}_G(P))$ be such that $P \leq T$. Since $S \in \mathrm{Syl}_p(G)$, there exists $g \in G$ such that $T^g \leq S$. Set $Q = P^g \leq S$. Note $T^g \in \mathrm{Syl}_p(\mathrm{N}_G(P^g))$ and $T^g \leq S$, which is a Sylow p-subgroup of G. Thus $T^g = \mathrm{N}_S(Q)$. Moreover $\mathrm{N}_S(Q)/\mathrm{C}_S(Q) \cong \mathrm{N}_S(Q)\mathrm{C}_G(Q)/\mathrm{C}_G(Q) \in \mathrm{Syl}_p(\mathrm{N}_G(Q)/\mathrm{C}_G(Q))$ so $\mathrm{Aut}_S(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_G(Q))$.

By assumption $N_g^g \leq \mathrm{N}_S(Q)\mathrm{C}_G(Q)$. Note that $\mathrm{N}_S(Q) \in \mathrm{Syl}_p(\mathrm{N}_S(Q)\mathrm{C}_G(Q))$. Thus there exists $h \in \mathrm{C}_G(Q)$ such that $N_g^{gh} \leq \mathrm{N}_S(Q) \leq S$. □

We are interested in the class of fusion systems satisfying similar *saturation properties*.

Definition 2.5 (Saturation)**.** Let $P, Q \leq S$ be subgroups. For every $\alpha \in \mathrm{Iso}_{\mathcal{F}}(P, Q)$, we set

$$\mathrm{N}_\alpha = \{g \in \mathrm{N}_S(P) \mid (c_g)^\alpha \in \mathrm{Aut}_S(Q)\}.$$

1. We say that Q is *$\mathcal{F}$-conjugate* to P, denoted $Q \in P^{\mathcal{F}}$, if $\mathrm{Iso}_{\mathcal{F}}(P, Q) \neq \emptyset$;

2. we say that Q is *fully automized* in $\mathcal{F}$ if $\mathrm{Aut}_S(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(Q))$;

3. we say that Q is *receptive* in $\mathcal{F}$ if for every $P \in Q^{\mathcal{F}}$ and every $\alpha \in \mathrm{Iso}_{\mathcal{F}}(P, Q)$ there exists $\overline{\alpha} \in \mathrm{Hom}_{\mathcal{F}}(N_\alpha, S)$ such that $\overline{\alpha}|_P = \alpha$ (where $\overline{\alpha}|_P$ denotes the restriction of the morphism $\overline{\alpha}$ to the group P).

A fusion system $\mathcal{F}$ on a p-group S is *saturated* if each subgroup of S is $\mathcal{F}$-conjugate to a subgroup which is fully automized and receptive.

By Lemma 2.4, the fusion category of a group on one of its Sylow p-subgroups is a saturated fusion system.

Definition 2.6. Let $\mathcal{F}$ be a saturated fusion system on S. If $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G with $S \in \mathrm{Syl}_p(G)$, then we say that $\mathcal{F}$ is *realized* by G. If $\mathcal{F}$ cannot be realized by a finite group, then we say that $\mathcal{F}$ is *exotic*.

Exotic fusion systems are not rare for p odd. For example, Ruiz and Viruel proved in [RV04] that there are 3 distinct exotic fusion systems on the extraspecial group 7_+^{1+2}.

As remarked in [RS09, Section 2], $\mathcal{F}$-conjugate subgroups of S have isomorphic $\mathcal{F}$-automorphism group.

Lemma 2.7. *Let $P, Q \leq S$ be such that $Q \in P^{\mathcal{F}}$. Then every morphism $\alpha \in \mathrm{Iso}_{\mathcal{F}}(P, Q)$ induces a group isomorphism between $\mathrm{Aut}_{\mathcal{F}}(P)$ and $\mathrm{Aut}_{\mathcal{F}}(Q)$ that sends $\mathrm{Inn}(P)$ to $\mathrm{Inn}(Q)$:*

$$\hat{\alpha}\colon \mathrm{Aut}_{\mathcal{F}}(P) \to \mathrm{Aut}_{\mathcal{F}}(Q), \quad \beta \mapsto \alpha^{-1}\beta\alpha.$$

A direct consequence of the extension property of a receptive subgroup Q is that every $\mathcal{F}$-automorphism of Q normalizing the group $\mathrm{Aut}_S(Q) \cong \mathrm{N}_S(Q)/\mathrm{C}_S(Q)$ is the restriction of an $\mathcal{F}$-automorphism of $\mathrm{N}_S(Q)$.

Lemma 2.8. *Let $Q \leq S$ be a receptive subgroup. Then for every $\alpha \in \mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(Q)}(\mathrm{Aut}_S(Q))$ there exists $\overline{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}_S(Q))$ such that $\overline{\alpha}|_Q = \alpha$.*

Proof. Let $\alpha \in \mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(Q)}(\mathrm{Aut}_S(Q))$. Then $\mathrm{N}_\alpha = \mathrm{N}_S(Q)$ by definition. Therefore there exists $\overline{\alpha} \in \mathrm{Hom}_{\mathcal{F}}(\mathrm{N}_S(Q), S)$ such that $\overline{\alpha}|_Q = \alpha$. It remains to prove that $\overline{\alpha} \in \mathrm{Aut}(\mathrm{N}_S(Q))$. Let $g \in \mathrm{N}_S(Q)$. Then

$$Q^{g\overline{\alpha}} = (Q^g)\alpha = Q\alpha = Q.$$

Thus $g\overline{\alpha} \in \mathrm{N}_S(Q)$ and $\mathrm{N}_S(Q)\overline{\alpha} \leq \mathrm{N}_S(Q)$. Since $\overline{\alpha}$ is injective we conclude $\overline{\alpha} \in \mathrm{Aut}(\mathrm{N}_S(Q))$. $\square$

There are several equivalent definitions of saturated fusion systems in the literature (see for example [AKO11, Section I.9]). We now present a characterization of saturation that was given as a definition in [BLO03, Definition 1.2]. We first need the notions of fully centralized and fully normalized subgroup.

Definition 2.9. Let $\mathcal{F}$ be a fusion system on the p-group S.

1. $P \leq S$ is *fully centralized* in $\mathcal{F}$ if $|\mathrm{C}_S(P)| \geq |\mathrm{C}_S(Q)|$ for all $Q \in P^{\mathcal{F}}$.

2. $P \leq S$ is *fully normalized* in $\mathcal{F}$ if $|\mathrm{N}_S(P)| \geq |\mathrm{N}_S(Q)|$ for all $Q \in P^{\mathcal{F}}$.

Lemma 2.10. *[RS09, Propositions 3.7 and 4.4] Let $\mathcal{F}$ be a fusion system on the p-group S. Then*

1. *every receptive subgroup of S is fully centralized;*

2. *every subgroup of S which is fully automized and receptive is fully normalized.*

Thus if $\mathcal{F}$ is saturated, then every subgroup of S is $\mathcal{F}$-conjugate to a fully normalized subgroup of S.

Theorem 2.11. *[RS09, Theorem 5.2] Let $\mathcal{F}$ be a saturated fusion system on the p-group S. Then*

1. *every fully normalized subgroup of S is fully centralized and fully automized (Sylow axiom), and*

2. *every fully centralized subgroup of S is receptive (Extension axiom).*

2.2 Fusion subsystems and model theorem

A *(fusion) subsystem* of $\mathcal{F}$ is a subcategory $\mathcal{E}$ of $\mathcal{F}$ that is a fusion system on some $P \leq S$. We write $\mathcal{E} \subseteq \mathcal{F}$. Note that $\mathcal{F}_P(P) \subseteq \mathcal{F}$ for every $P \leq S$. In particular, $\mathcal{F}_1(1)$ is the smallest fusion subsystem on S with respect to the partial order $\subseteq$ and is called the *trivial* fusion system. The universal fusion system $\mathcal{U}_S$ is the largest among the fusion systems on S with respect to $\subseteq$. In particular if $P \leq S$ then

$$1 \subseteq \mathcal{F}_P(P) \subseteq \mathcal{U}_P \subseteq \mathcal{U}_S.$$

Also, it can be shown that if $\mathcal{F}$ and $\mathcal{E}$ are fusion subsystems of $\mathcal{U}_S$ then the category $\mathcal{F} \cap \mathcal{E}$, called *the intersection* of $\mathcal{F}$ and $\mathcal{E}$, is a fusion subsystem of $\mathcal{F}$ and $\mathcal{E}$ (see [AKO11, Section I.3]).

Definition 2.12. Let S be a p-group and let $K \subseteq \mathrm{Mor}(\mathcal{U}_S)$. We define the *fusion system on S generated by K*, denoted $\langle K \rangle_S$, as

$$\langle K \rangle_S = \bigcap \{\mathcal{E} \subseteq \mathcal{U}_S \mid \mathcal{E} \text{ is a fusion system on } S \text{ with } K \subseteq \mathrm{Mor}(\mathcal{E})\}.$$

In other words, $\langle K \rangle_S$ is the smallest fusion system on S containing K.

In analogy with the definition of simple group, a saturated fusion system is *simple* if it does not contain proper nontrivial normal fusion subsystems. The following is the definition of normal fusion subsystems given by Aschbacher ([Asc08]).

Definition 2.13. A fusion subsystem $\mathcal{E} \subseteq \mathcal{F}$ on $P \leq S$ is *weakly normal* in $\mathcal{F}$ if $\mathcal{E}$ and $\mathcal{F}$ are both saturated and the following holds:

1. P is *strongly $\mathcal{F}$-closed*: for every $x \in P$ and every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle x \rangle, S)$ we have $x\varphi \in P$;

2. *Invariance condition*: for each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ and each $\varphi \in \mathrm{Hom}_{\mathcal{E}}(R, Q)$ we have $\varphi^{\alpha} = \alpha^{-1}|_{(R\alpha)} \circ \varphi \circ \alpha|_{Q} \in \mathrm{Hom}_{\mathcal{E}}(R\alpha, Q\alpha)$;

3. *Frattini condition*: for each $R \leq P$ we have $\mathrm{Hom}_{\mathcal{F}}(R, P) = \mathrm{Hom}_{\mathcal{E}}(R, P)\mathrm{Aut}_{\mathcal{F}}(P)$.

We say that $\mathcal{E}$ is *normal* in $\mathcal{F}$ (denoted $\mathcal{E} \trianglelefteq \mathcal{F}$) if $\mathcal{E}$ is weakly normal and for each $\varphi \in \mathrm{Aut}_{\mathcal{E}}(P)$ there exists $\overline{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(P\mathrm{C}_S(P))$ such that $\overline{\varphi}|_P = \varphi$ and $[\mathrm{C}_S(P), \overline{\varphi}] \leq \mathrm{Z}(P)$.

If $\mathcal{F}$ is a simple fusion system realized by a finite group, then there exists a simple group realizing it. The converse is not true (for example the fusion category of the group $\mathrm{Alt}(5)$ on one of its 2-Sylow subgroups is not simple, as a consequence of [Cra11, Theorem 5.71 and Lemma 5.77]).

For the rest of this section let $\mathcal{F}$ be a saturated fusion system on the p-group S.

We now introduce another family of fusion subsystems: the class of normalizer fusion systems of a subgroup of S in $\mathcal{F}$.

Definition 2.14. Let $P \leq S$ be a subgroup. The *normalizer fusion system* of P in $\mathcal{F}$ is the fusion subsystem $\mathrm{N}_{\mathcal{F}}(P) \subseteq \mathcal{F}$ on $\mathrm{N}_S(P)$ with set of morphisms

$$\begin{aligned}\mathrm{Hom}_{\mathrm{N}_{\mathcal{F}}(P)}(Q, R) = \{\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, R) \mid {} & \text{there exists } \overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(QP, RP) \\ & \text{with } \overline{\varphi}|_Q = \varphi \text{ and } \overline{\varphi}|_P \in \mathrm{Aut}_{\mathcal{F}}(P)\}.\end{aligned}$$

for every $Q, R \leq \mathrm{N}_S(P)$.

Note that $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathrm{N}_{\mathcal{F}}(P)}(P)$ and $\mathrm{N}_{\mathcal{F}}(S) = \langle \mathrm{Aut}_{\mathcal{F}}(S) \rangle_S$.

The following result goes back to Puig.

Lemma 2.15. *[Pui06, Proposition 2.15][BLO03, Proposition A.6] If $P \leq S$ is fully normalized in $\mathcal{F}$ then the fusion system $\mathrm{N}_{\mathcal{F}}(P)$ is saturated.*

Definition 2.16. A subgroup $P \leq S$ is *normal* in $\mathcal{F}$, denoted $P \trianglelefteq \mathcal{F}$, if for every $Q, R \leq S$ and every $\alpha \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$ there exists $\overline{\alpha} \in \mathrm{Hom}_{\mathcal{F}}(QP, RP)$ such that $\overline{\alpha}|_Q = \alpha$ and $\overline{\alpha}|_P \in \mathrm{Aut}_{\mathcal{F}}(P)$. We write $O_p(\mathcal{F})$ for the largest subgroup of S that is normal in $\mathcal{F}$.

Note that we can talk about the largest subgroup of S that is normal in S because if $P, Q \leq S$ are normal in $\mathcal{F}$ then it follows from the definition of normal subgroup that the subgroup PQ of S is normal in $\mathcal{F}$.

A subgroup P of S is normal in $\mathcal{F}$ if and only if $\mathcal{F} = \mathrm{N}_{\mathcal{F}}(P)$. Also, we have the following.

Lemma 2.17. *If $P \leq S$ is normal in $\mathcal{F}$ then $\mathcal{F}_P(P) \trianglelefteq \mathcal{F}$.*

Proof. Suppose P is normal in $\mathcal{F}$. First notice that the fusion system $\mathcal{F}_P(P)$ is saturated, since it is the fusion category of a group and $P \in \mathrm{Syl}_p(P)$ (Lemma 2.4).

Let $x \in P$. Then for every morphism $\alpha \in \mathrm{Hom}_{\mathcal{F}}(\langle x \rangle, S)$ there exists a morphism $\overline{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(S)$ such that $\overline{\alpha}|_{\langle x \rangle} = \alpha$. In particular $x\alpha \in P$, so P is strongly $\mathcal{F}$-closed.

For every $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ and each $\varphi \in \mathrm{Mor}(\mathcal{F}_P(P))$ we have $\varphi = c_g$ for some $g \in P$ and $(c_g)^\alpha = c_{g\alpha} \in \mathrm{Mor}(\mathcal{F}_P(P))$, thus the invariance condition of Definition 2.13 is satisfied.

If $R \leq P$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R, P)$ then, since P is normal, there exists $\overline{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(P)$ such that $\varphi = \overline{\varphi}|_R$, so the Frattini condition of Definition 2.13 is satisfied.

Finally, every morphism $\varphi \in \mathrm{Mor}(\mathcal{F}_P(P))$ is a conjugation map by an element g of P. Thus $\varphi = c_g$ can be seen as a conjugation map by an element of S and $[\mathrm{C}_S(P), c_g] = 1$.

Therefore the fusion system $\mathcal{F}_P(P)$ is normal in $\mathcal{F}$. $\square$

If P is a strongly $\mathcal{F}$-closed subgroup of S, then P is normal in $\mathcal{F}$ if and only if $\mathcal{F}_P(P) \trianglelefteq \mathcal{F}$ (see [R.06, Proposition 6.2]).

Therefore if $\mathcal{F}$ is a simple fusion system on the p-group S, then either $\mathcal{F} = \mathcal{F}_S(S)$ (and S is cyclic of order p by [Cra11, Lemma 5.76]) or $O_p(\mathcal{F}) = 1$.

We end this section giving sufficient conditions for a saturated fusion system to be realized by a finite group. We first need to define $\mathcal{F}$-centric subgroups.

Definition 2.18. A subgroup $P \leq S$ is *$\mathcal{F}$-centric* if $\mathrm{C}_S(Q) \leq Q$ for every $Q \in P^{\mathcal{F}}$.

In particular if P is $\mathcal{F}$-centric then every subgroup of S that is $\mathcal{F}$-conjugate to P is $\mathcal{F}$-centric. Note that the group S is always $\mathcal{F}$-centric. Also, if $P \leq R \leq S$ and P is $\mathcal{F}$-centric then R is $\mathcal{F}$-centric. Indeed, if $\alpha \in \mathrm{Hom}_{\mathcal{F}}(R, S)$, then $\mathrm{C}_S(R\alpha) \leq \mathrm{C}_S(P\alpha) \leq P\alpha \leq R\alpha$.

Definition 2.19. If there exists a subgroup of S that is $\mathcal{F}$-centric and normal in $\mathcal{F}$, then we say that the fusion system $\mathcal{F}$ is *constrained*. If $\mathcal{F}$ is constrained, a *model* for $\mathcal{F}$ is a finite group G such that $S \in \mathrm{Syl}_p(G)$, $\mathcal{F} = \mathcal{F}_S(G)$, and $C_G(O_p(G)) \leq O_p(G)$.

Theorem 2.20 (Model Theorem for Constrained Fusion Systems, [AKO11], Theorem I.4.9)**.** *Let $\mathcal{F}$ be a constrained fusion system on a p-group S. Let $P \trianglelefteq S$ be $\mathcal{F}$-centric and normal in $\mathcal{F}$. Then the following holds.*

(a) *There are models for $\mathcal{F}$.*

(b) *If G_1 and G_2 are two models for $\mathcal{F}$, then there exists an isomorphism $\phi : G_1 \to G_2$ such that $\phi|_S = \mathrm{Id_S}$.*

(c) *For any finite group G containing S as a Sylow p-subgroup such that $P \trianglelefteq G$, $\mathrm{C}_G(P) \leq P$, and $\mathrm{Aut}_G(P) = \mathrm{Aut}_{\mathcal{F}}(P)$, there is a model of $\mathcal{F}$ that is isomorphic to G.*

A p-group P is called *resistant* if for any saturated fusion system $\mathcal{E}$ on P we have $\mathcal{E} = \mathrm{N}_{\mathcal{E}}(P)$. In particular, if S is resistant then there exists a model for $\mathcal{F}$.

2.3 $\mathcal{F}$-Essential subgroups

The study of $\mathcal{F}$-essential subgroups is a crucial step of the classification of saturated fusion systems, due to the Alperin-Goldschmidt Fusion Theorem.

Theorem 2.21 (Alperin-Goldschmidt Fusion Theorem)**.** *Let $\mathcal{F}$ be a saturated fusion system on a p-group S. Then $\mathcal{F}$ is completely determined by the group $\mathrm{Aut}_{\mathcal{F}}(S)$ and by the $\mathcal{F}$-automorphism groups of the $\mathcal{F}$-essential subgroups of S:*

$$\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(E) \mid E = S \textit{ or } E \textit{ is an } \mathcal{F}\textit{-essential subgroup of } S \; \rangle_S.$$

This result is due to Puig ([Pui06], Corollary 5.10) and is inspired by the original fusion theorems of Alperin ([Alp67]) and Goldschmidt ([Gol70]). In this section we define $\mathcal{F}$-essential subgroups and we explore their properties.

Let S be a p-group and let $\mathcal{F}$ be a saturated fusion system on S.

Definition 2.22. A proper subgroup E of S is $\mathcal{F}$*-essential* if

1. E is $\mathcal{F}$*-centric*,
2. E is fully normalized in $\mathcal{F}$, and
3. $\mathrm{Out}_{\mathcal{F}}(E)$ has a strongly p-embedded subgroup.

The condition of being fully normalized guarantees that every $\mathcal{F}$-essential subgroup is fully automized and receptive. Also, since $\mathrm{Out}_{\mathcal{F}}(E)$ has a strongly p-embedded subgroup, we get $O_p(\mathrm{Out}_{\mathcal{F}}(E)) = 1$ by Lemma 1.32. We say that a subgroup $P \leq S$ is $\mathcal{F}$*-radical* if $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$. Thus every $\mathcal{F}$-essential subgroup is $\mathcal{F}$-radical.

Lemma 2.23. *The p-group S is $\mathcal{F}$-centric, fully normalized in $\mathcal{F}$ and $\mathcal{F}$-radical.*

Proof. Clearly $\mathrm{C}_S(S) = \mathrm{Z}(S) \leq S$ and $\mathrm{N}_S(S) = S$. So S is $\mathcal{F}$-centric and fully normalized in $\mathcal{F}$. In particular it is fully automized, so $\mathrm{Inn}(S) = \mathrm{Aut}_S(S) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(S))$. Hence $\mathrm{Out}_{\mathcal{F}}(S)$ has order prime to p and $O_p(\mathrm{Out}_{\mathcal{F}}(S)) = 1$. Thus S is $\mathcal{F}$-radical. □

Lemma 2.24. *Let $E \leq S$ be an $\mathcal{F}$-radical subgroup of S. Consider the sequence of subgroups:*

$$E_0 \leq E_1 \leq \cdots \leq E_n = E$$

such that $E_0 \leq \Phi(E)$ and for every $0 \leq i \leq n$ the group E_i is normalized by $\mathrm{Aut}_{\mathcal{F}}(E)$. Let $\varphi \in \mathrm{Aut}_{\mathcal{F}}(E)$ be a morphism that centralizes every quotient E_i/E_{i-1} for $1 \leq i \leq n$. Then $\varphi \in \mathrm{Inn}(E)$.

Proof. Since E is $\mathcal{F}$-radical we have $O_p(\mathrm{Aut}_{\mathcal{F}}(E)) = \mathrm{Inn}(E)$. For every i the group E_i is normal in E, since it is normalized by $\mathrm{Aut}_{\mathcal{F}}(E)$. The statement is then a direct consequence of Lemma 1.34. □

Lemma 2.25. *Let $P < S$. If $\mathrm{Aut}_S(P) \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(P)$ then P is not $\mathcal{F}$-essential. In particular $\mathcal{F}$-essential subgroups are not cyclic.*

Proof. Aiming for a contradiction assume that P is $\mathcal{F}$-essential. Then it is $\mathcal{F}$-centric, so $\mathrm{C}_S(P) = \mathrm{Z}(P)$. By assumption $\mathrm{Aut}_S(P) \leq O_p(\mathrm{Aut}_{\mathcal{F}}(P))$. Note that

$$\mathrm{Inn}(P) \cong P/\mathrm{Z}(P) < \mathrm{N}_S(P)/\mathrm{Z}(P) \cong \mathrm{Aut}_S(P).$$

So $O_p(\mathrm{Out}_{\mathcal{F}}(P)) \neq 1$, contradicting the fact that $O_p(\mathrm{Out}_{\mathcal{F}}(P))$ has a strongly p-embedded subgroup. Thus P is not $\mathcal{F}$-essential. In particular, if P is cyclic then $\mathrm{Aut}_{\mathcal{F}}(P)$ is abelian so $\mathrm{Aut}_S(P) \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(P)$. Hence $\mathcal{F}$-essential subgroups are not cyclic. □

Lemma 2.26. *Let $E \leq S$ be an $\mathcal{F}$-essential subgroup. Then*

1. *if $E < P \leq S$ then $\mathrm{Z}(P) < E$ (in particular P is not abelian);*

2. *$\mathrm{Out}_S(E) \cong \mathrm{N}_S(E)/E$ and $\mathrm{Out}_S(E) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(E))$;*

3. *$\mathrm{C}_{\mathrm{Aut}_{\mathcal{F}}(E)}(E/\Phi(E)) = \mathrm{Inn}(E)$ and $\mathrm{Out}_{\mathcal{F}}(E)$ acts faithfully on $E/\Phi(E)$;*

4. *if E has rank r then $\mathrm{Out}_{\mathcal{F}}(E)$ is isomorphic to a subgroup of $\mathrm{GL}_r(p)$;*

5. *if $[\mathrm{N}_S(E) \colon E] = p$ then every subgroup $P \in E^{\mathcal{F}}$ is $\mathcal{F}$-essential;*

6. *if $\alpha \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}_S(E))$ then $\mathrm{N}_S(E\alpha) = \mathrm{N}_S(E)$ and $E\alpha$ is $\mathcal{F}$-essential.*

Proof.

1. Since E is $\mathcal{F}$-centric we have $\mathrm{Z}(P) \leq \mathrm{C}_S(E) \leq E$. If $\mathrm{Z}(P) = E$, then $P \leq \mathrm{C}_S(E) \leq E$, which is a contradiction. Therefore $\mathrm{Z}(P) < E$.

2. As $E < S$ we have $\mathrm{N}_S(E) \neq E$. Since E is $\mathcal{F}$-essential, it is fully automized. So $\mathrm{Aut}_S(E) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(E))$. Therefore $\mathrm{Aut}_S(E)/\mathrm{Inn}(E) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(E))$. Notice that $\mathrm{Aut}_S(E) \cong \mathrm{N}_S(E)/\mathrm{C}_S(E) = \mathrm{N}_S(E)/\mathrm{Z}(E)$ because E is $\mathcal{F}$-centric, and $\mathrm{Inn}(E) \cong E/\mathrm{Z}(E)$. So $\mathrm{N}_S(E)/E \cong \mathrm{Out}_S(E)$.

3. Recall that $O_p(\mathrm{Out}_{\mathcal{F}}(E)) = 1$ and so $O_p(\mathrm{Aut}_{\mathcal{F}}(E)) = \mathrm{Inn}(E)$. Lemma 2.24 implies $\mathrm{C}_{\mathrm{Aut}_{\mathcal{F}}(E)}(E/\Phi(E)) = \mathrm{Inn}(E)$. Thus $\mathrm{Out}_{\mathcal{F}}(E)$ acts faithfully on $E/\Phi(E)$.

4. By 3. the group $\mathrm{Out}_{\mathcal{F}}(E)$ acts faithfully on $E/\Phi(E)$, which is an elementary abelian p-group of order p^r. Thus $\mathrm{Out}_{\mathcal{F}}(E)$ is isomorphic to a subgroup of $\mathrm{Aut}(E/\Phi(E)) \cong \mathrm{GL}_r(p)$.

5. Let $P \in E^{\mathcal{F}}$. Since E is fully normalized and S is a p-group, we have

$$|E| = |P| < |\mathrm{N}_S(P)| \leq |\mathrm{N}_S(E)| = |E| \cdot p.$$

Thus $|\mathrm{N}_S(P)| = |\mathrm{N}_S(E)|$ and P is fully normalized. Since $P \in E^{\mathcal{F}}$ it is $\mathcal{F}$-centric and by Lemma 2.7 the group $\mathrm{Out}_{\mathcal{F}}(P) \cong \mathrm{Out}_{\mathcal{F}}(E)$ has a strongly p-embedded subgroup. So P is $\mathcal{F}$-essential.

6. As in the previous point, it is enough to prove that the group $E\alpha$ is fully normalized. Note that for every $g \in \mathrm{N}_S(E)$ we have

$$(E\alpha)c_g = E\alpha c_g \alpha^{-1}\alpha = Ec_{g\alpha^{-1}}\alpha = E\alpha.$$

Therefore $\mathrm{N}_S(E) \leq \mathrm{N}_S(E\alpha)$ and since E is fully normalized we deduce $\mathrm{N}_S(E) = \mathrm{N}_S(E\alpha)$. Thus $E\alpha$ is fully normalized and so it is $\mathcal{F}$-essential.

□

Lemma 2.27. *Let $E \leq S$ be an $\mathcal{F}$-essential subgroup. Then $\mathrm{N}_S(E)$ is not a resistant group.*

Proof. Assume by contradiction that $\mathrm{N}_S(E)$ is resistant. Since E is fully normalized, by Lemma 2.15 the fusion system $\mathrm{N}_{\mathcal{F}}(E)$ is a saturated fusion system on $\mathrm{N}_S(E)$. Therefore, as $\mathrm{N}_S(E)$ is resistant, $\mathrm{N}_{\mathcal{F}}(E) = \mathrm{N}_{\mathrm{N}_{\mathcal{F}}(E)}(\mathrm{N}_S(E))$. Thus for every $P, R \leq \mathrm{N}_S(E)$ we have

$$\mathrm{Hom}_{\mathrm{N}_{\mathcal{F}}(E)}(P,R) = \{\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,R) \mid \text{there exists } \overline{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}_S(E)), \overline{\varphi}|_P = \varphi\}.$$

Recall that $\mathrm{Aut}_{\mathcal{F}}(E) = \mathrm{Aut}_{\mathrm{N}_{\mathcal{F}}(E)}(E)$. Therefore for every $\varphi \in \mathrm{Aut}_{\mathcal{F}}(E)$ there exists $\overline{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}_S(E))$ such that $\overline{\varphi}|_E = \varphi$. In particular $\mathrm{Out}_S(E) \trianglelefteq \mathrm{Out}_{\mathcal{F}}(E)$ and so E is not $\mathcal{F}$-radical, contradicting the assumption that E is $\mathcal{F}$-essential. □

The next lemma gives a characterization of subgroups of S that are normal in the saturated fusion system $\mathcal{F}$.

Lemma 2.28. *[AKO11, Proposition I.4.5] For any $P \leq S$, the following conditions are equivalent:*

1. *P is normal in $\mathcal{F}$;*

2. *P is strongly $\mathcal{F}$-closed and P is contained in every subgroup of S that is $\mathcal{F}$-centric and $\mathcal{F}$-radical;*

3. *if $E \leq S$ is $\mathcal{F}$-essential or $E = S$ then $P \leq E$ and P is $\mathcal{F}$-characteristic in E.*

The last statement is the one we are going to use the most. Indeed, if $O_p(\mathcal{F}) = 1$ then for every subgroup P of S that is contained in all the $\mathcal{F}$-essential subgroups of S, there exists an $\mathcal{F}$-automorphism $\varphi \in \mathrm{Aut}_{\mathcal{F}}(E)$, where E is either an $\mathcal{F}$-essential subgroup or $E = S$, such that $P\varphi \neq P$.

Lemma 2.29. *If $E \leq S$ is $\mathcal{F}$-centric, $\mathcal{F}$-radical and fully normalized in $\mathcal{F}$ then*

$$E = O_p(\mathrm{N}_{\mathcal{F}}(E)).$$

Proof. Let $P = O_p(\mathrm{N}_{\mathcal{F}}(E)) \leq \mathrm{N}_S(E)$. Then for every $\alpha \in \mathrm{Aut}_{\mathrm{N}_{\mathcal{F}}(E)}(E)$ there exists $\overline{\alpha} \in \mathrm{Aut}_{\mathrm{N}_{\mathcal{F}}(E)}(P)$ such that $\overline{\alpha}|_E = \alpha$. Since $\mathrm{Aut}_{\mathcal{F}}(E) = \mathrm{Aut}_{\mathrm{N}_{\mathcal{F}}(E)}(E)$ and $\mathrm{N}_{\mathcal{F}}(E)$ is a fusion subsystem of $\mathcal{F}$, we have that for every $\alpha \in \mathrm{Aut}_{\mathcal{F}}(E)$ there exists $\overline{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(P)$ such that $\overline{\alpha}|_E = \alpha$. In particular $\mathrm{Aut}_P(E) \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(E)$. Since E is $\mathcal{F}$-radical, this implies $\mathrm{Aut}_P(E) = \mathrm{Inn}(E)$. Hence $P = E$ because E is $\mathcal{F}$-centric. $\square$

Theorem 2.30 (Weak Model Theorem)**.** *Let $\mathcal{F}$ be a saturated fusion system on a p-group S and let $E \leq S$ be $\mathcal{F}$-centric, $\mathcal{F}$-radical and fully normalized in $\mathcal{F}$. Then there exists a finite group G that is a model for $\mathrm{N}_{\mathcal{F}}(E)$. In particular*

1. $\mathrm{N}_S(E) \in \mathrm{Syl}_p(G)$;
2. $E = O_p(G)$;
3. $\mathrm{C}_G(E) \leq E$;
4. $G/E \cong \mathrm{Out}_{\mathcal{F}}(E)$.

Proof. Since E is fully normalized, by Lemma 2.15 the fusion system $\mathrm{N}_{\mathcal{F}}(E)$ is saturated. Moreover we have $E \trianglelefteq \mathrm{N}_{\mathcal{F}}(E)$ and E is $\mathrm{N}_{\mathcal{F}}(E)$-centric, since it is $\mathcal{F}$-centric and $\mathrm{N}_{\mathcal{F}}(E) \subseteq \mathcal{F}$. Therefore $\mathrm{N}_{\mathcal{F}}(E)$ is a constrained fusion system and by the model theorem there exists a finite group G that is a model for $\mathrm{N}_{\mathcal{F}}(E)$. So $\mathrm{N}_S(E) \in \mathrm{Syl}_p(G)$, $C_G(O_p(G)) \leq O_p(G)$, and $\mathrm{N}_{\mathcal{F}}(E) = \mathcal{F}_{\mathrm{N}_S(E)}(G)$. In particular by Lemma 2.29 we have $E = O_p(\mathrm{N}_{\mathcal{F}}(E)) = O_p(\mathcal{F}_{\mathrm{N}_S(E)}(G))$ so $E = O_p(G)$. Therefore $C_G(E) = \mathrm{Z}(E)$ and $\mathrm{Aut}_{\mathcal{F}}(E) = \mathrm{Aut}_{\mathrm{N}_{\mathcal{F}}(E)}(E) \cong G/Z(E)$, so $\mathrm{Out}_{\mathcal{F}}(E) \cong G/E$. □

Note that Theorem 2.30 applies to $\mathcal{F}$-essential subgroups and to the group S (by Lemma 2.23).

Lemma 2.31. *Let $E \leq S$ be $\mathcal{F}$-centric, $\mathcal{F}$-radical and fully normalized in $\mathcal{F}$ and suppose E is $\mathcal{F}$-characteristic in S. Let G be a model for $\mathrm{N}_{\mathcal{F}}(E)$ (whose existence is guaranteed by Lemma 2.30). Then the group $\mathrm{N}_G(S)$ is a model for $\mathrm{N}_{\mathcal{F}}(S)$.*

Proof. Note that as E is $\mathcal{F}$-characteristic in S we have $S = \mathrm{N}_S(E)$. So $S \in \mathrm{Syl}_p(G)$ by definition of G, which implies $S \in \mathrm{Syl}_p(\mathrm{N}_G(S))$ and $S = O_p(\mathrm{N}_G(S))$. Also, $E = O_p(\mathrm{N}_{\mathcal{F}}(E))$ by Lemma 2.29 and $\mathrm{N}_{\mathcal{F}}(E) = \mathcal{F}_S(G)$. So $E = O_p(G)$ and

$$\mathrm{C}_{\mathrm{N}_G(S)}(S) \leq \mathrm{C}_G(S) \leq \mathrm{C}_G(E) \leq E \leq S.$$

It remains to prove that $\mathrm{N}_{\mathcal{F}}(S) = \mathcal{F}_S(\mathrm{N}_G(S))$. Note that $\mathrm{N}_{\mathcal{F}}(S) = \langle \mathrm{Aut}_{\mathcal{F}}(S) \rangle_S$ by definition. Since E is $\mathcal{F}$-characteristic in S, for every $\varphi \in \mathrm{Aut}_{\mathcal{F}}(S)$ we have $\varphi|_E \in \mathrm{Aut}_{\mathcal{F}}(E)$. Hence $\mathrm{Aut}_{\mathcal{F}}(S) \subseteq \mathrm{Mor}(\mathrm{N}_{\mathcal{F}}(E))$, by definition of normalizer fusion system. Thus for every morphism $\varphi \in \mathrm{Aut}_{\mathcal{F}}(S)$ there exists $g \in G$ such that $\varphi = c_g$. In particular $g \in \mathrm{N}_G(S)$ and we conclude $\mathrm{N}_{\mathcal{F}}(S) \subseteq \mathcal{F}_S(\mathrm{N}_G(S))$. Now notice that $\mathcal{E} = \mathrm{N}_{\mathcal{F}}(E) = \mathcal{F}_S(G)$ is a subsystem of $\mathcal{F}$, so $\mathrm{Aut}_{\mathcal{E}}(S) \subseteq \mathrm{Aut}_{\mathcal{F}}(S)$. We have

$$\mathrm{Aut}_{\mathcal{E}}(S) = \{c_g \colon S \to S | g \in G, S^g = S\} = \mathrm{Mor}(\mathcal{F}_S(\mathrm{N}_S(G))).$$

Therefore $\mathcal{F}_S(\mathrm{N}_S(G)) \subseteq \langle \mathrm{Aut}_{\mathcal{F}}(S) \rangle_S = \mathrm{N}_{\mathcal{F}}(S)$. Therefore we conclude that $\mathrm{N}_{\mathcal{F}}(S) = \mathcal{F}_S(\mathrm{N}_G(S))$ and so $\mathrm{N}_G(S)$ is a model for $\mathrm{N}_{\mathcal{F}}(S)$. $\square$

As a direct consequence of Lemma 2.31, we prove that whenever we have two $\mathcal{F}$-essential subgroups of S that are $\mathcal{F}$-characteristic in S, we can build an amalgam of rank 2 associated to the fusion system $\mathcal{F}$.

Lemma 2.32. *Let E_1 and E_2 be $\mathcal{F}$-characteristic subgroups of S that are $\mathcal{F}$-centric and $\mathcal{F}$-radical. Let G_1, G_2 and G_{12} be models for $\mathrm{N}_{\mathcal{F}}(E_1)$, $\mathrm{N}_{\mathcal{F}}(E_2)$ and $\mathrm{N}_{\mathcal{F}}(S)$ respectively. Then there exist two monomorphisms $\phi_1 \colon G_{12} \to G_1$ and $\phi_2 \colon G_{12} \to G_2$ such that $\phi_1|_S = \phi_2|_S = \mathrm{id_S}$ and $\mathcal{A} = \mathcal{A}(G_1, G_2, G_{12})$ is an amalgam of rank 2.*

Note that E_1 and E_2 are normal in S (because they are $\mathcal{F}$-characteristic in S) and so they are fully normalized in $\mathcal{F}$. The existsence of G_1, G_2 and G_{12} is guaranteed by Theorem 2.30.

Proof. By Lemma 2.31 the groups $\mathrm{N}_{G_1}(S)$ and $\mathrm{N}_{G_2}(S)$ are models for $\mathrm{N}_{\mathcal{F}}(S)$. Therefore by the model theorem (Theorem 2.20(b)) there exists two isomorphisms $\overline{\phi_1} \colon G_{12} \to \mathrm{N}_{G_1}(S)$ and $\overline{\phi_2} \colon G_{12} \to \mathrm{N}_{G_2}(S)$ such that $\overline{\phi_1}|_S = \overline{\phi_2}|_S = \mathrm{id_S}$. We now define $\phi_i = \overline{\phi_i}\iota_i$, where ι_i is the natural inclusion of $\mathrm{N}_{G_i}(S)$ in G_i. $\square$

Remark 2.33. Let $G = G_1 *_{G_{12}} G_2$ be the free amalgamated product of G_1 and G_2 over G_{12}. Then G is an infinite group. We say that a finite p-subgroup P of G is a Sylow p-subgroup of G if every finite p-subgroup of G is conjugate to a subgroup of P. According to this definition the group S is a Sylow p-subgroup of G. Also, we can define the fusion category $\mathcal{F}_S(G)$ in the same way used for finite groups (note that since G is infinite the fusion category $\mathcal{F}_S(G)$ might not be saturated). In this setup, by Robinson's Theorem ([CP10, Theorem 3.1]) we get the following equalities:

$$\mathcal{F}_S(G) = \langle \mathrm{Mor}(\mathcal{F}_S(G_1)), \mathrm{Mor}(\mathcal{F}_S(G_2)) \rangle_S = \langle \mathrm{Aut}_{\mathcal{F}}(E_1), \mathrm{Aut}_{\mathcal{F}}(E_2) \rangle_S.$$

We now prove that under certain conditions the rank 2 amalgam constructed in Lemma 2.32 is a weak BN-pair of rank 2.

Theorem 2.34. *Let $\mathcal{F}$ be a saturated fusion system on a p-group S and let $E_1, E_2 \leq S$ be two $\mathcal{F}$-characteristic subgroups of S that are $\mathcal{F}$-centric and $\mathcal{F}$-radical. Let G_1, G_2 and G_{12} be models for $\mathrm{N}_{\mathcal{F}}(E_1), \mathrm{N}_{\mathcal{F}}(E_2)$ and $\mathrm{N}_{\mathcal{F}}(S)$, respectively, and let T be the largest subgroup of $E_1 \cap E_2$ that is normalized by G_1 and G_2. Suppose that for every $i \in \{1, 2\}$ we have $\mathrm{C}_{G_i/T}(E_i/T) \leq E_i/T$ and that the quotient $\langle S^{G_i} \rangle / E_i$ is isomorphic to one of the following groups:*

$$\mathrm{PSL}_2(p^{n_i}), \mathrm{SL}_2(p^{n_i}), \mathrm{U}_3(p^{n_i}), \mathrm{SU}_3(p^{n_i}), \mathrm{Sz}(2^{n_i}).$$

Then $\mathcal{A}(G_1/T, G_2/T, G_{12}/T, \langle S^{G_1} \rangle /T, \langle S^{G_2} \rangle /T)$ is a weak BN-pair of rank 2 and S/T is isomorphic to a Sylow p-subgroup of one of the groups listed in Theorem 1.42.

Note that when $T = 1$ the inclusion $\mathrm{C}_{G_i}(E_i) \leq E_i$ follows directly from Theorem 2.30.

Proof. By the weak model theorem (Theorem 2.30) the group G_i exists for every $i \in \{1,2\}$, $S \in \mathrm{Syl}_p(G_i)$, $E_i = O_p(G_i)$ and $\mathrm{C}_{G_i}(E_i) \leq E_i$. By assumption we also have $\mathrm{C}_{G_i/T}(E_i/T) \leq E_i/T$.

By Lemma 2.32 there exist monomorphisms $\phi_1 \colon G_{12} \to G_1$ and $\phi_2 \colon G_{12} \to G_2$ such that $G_{12}\phi_i = \mathrm{N}_{G_i}(S)$ and $\phi_1|_S = \phi_2|_S = \mathrm{id}_S$. Set $A_i = \langle S^{G_i} \rangle$. Then $O_p(G_i) = E_i \leq A_i \trianglelefteq G_i$ and $S \in \mathrm{Syl}_p(A_i)$. Also $\mathrm{N}_{A_i}(S) = A_i \cap \mathrm{N}_{G_i}(S) = A_i \cap G_{12}\phi_i$ and by the Frattini argument we have $G_i = A_i \mathrm{N}_{G_i}(S) = A_i G_{12}\phi_i$.

Let H be a subgroup of G_{12} such that $H\phi_i$ is normalized by G_i for every i. We prove that $H \leq T$.

Note that $H \cap S \in \mathrm{Syl}_p(H)$ and since $S = O_p(G_{12})$, we deduce that $H \cap S \trianglelefteq H$. Thus $H \cap S$ is the unique Sylow p-subgroup of H and is therefore characteristic in H. Hence $S \cap H \trianglelefteq G_i$ for every i, that implies $S \cap H \leq O_p(G_1) \cap O_p(G_2) = E_1 \cap E_2$. By the maximality of T we deduce $S \cap H \leq T$. In particular we have

$$[E_i, H] \leq H \cap E_i \leq H \cap S \leq T.$$

So $H\phi_i/T$ is a subgroup of G_i/T centralizing E_i/T for every i. By assumption we deduce $H \leq E_1 \cap E_2$, so $H = H \cap S$ and $H \leq T$.

In other words, no proper non-trivial subgroups H/T of G_{12}/T is such that $H\phi_i/T$ is normalized by G_i/T for every i. Therefore $\mathcal{A}(G_1/T, G_2/T, G_{12}/T, A_1/T, A_2/T)$ is a weak BN-pair of rank 2 and the last statement follows from Theorem 1.42. $\square$

We end this section with properties of the normalizer in S of the $\mathcal{F}$-essential subgroups of S.

Lemma 2.35. *Let $E \leq S$ be an $\mathcal{F}$-essential subgroup. Then*

$$\Phi(E) < [\mathrm{N}_S(E), E]\Phi(E) < E.$$

Proof. If $[\mathrm{N}_S(E), E] \leq \Phi(E)$ then the automorphism group $\mathrm{Aut}_S(E)$ centralizes the quotient $E/\Phi(E)$. Hence $\mathrm{Aut}_S(E) = \mathrm{Inn}(E)$ by Lemma 2.24 and $\mathrm{N}_S(E)/\mathrm{C}_S(E) \cong E/\mathrm{Z}(E)$, contradicting the fact that E is $\mathcal{F}$-centric and proper in S. So $\Phi(E) < [\mathrm{N}_S(E), E]\Phi(E)$. If $[\mathrm{N}_S(E), E]\Phi(E) = E$ then $[\mathrm{N}_S(E), E] = E$, contradicting the fact that S is nilpotent. Thus $[\mathrm{N}_S(E), E]\Phi(E) < E$. □

Lemma 2.36. *Let $E \leq S$ be an $\mathcal{F}$-essential subgroup. If E has maximal normalizer tower in S and $[S\colon E] = p^m$ then for every $1 \leq i \leq m$ we have*

$$\Phi(\mathrm{N}^{i-1}) < \Phi(\mathrm{N}^i) \quad \textit{and} \quad \mathrm{rank}(\mathrm{N^i}) \leq \mathrm{rank}(\mathrm{N^{i-1}}).$$

Proof. Note that E having maximal normalizer tower implies $\Phi(\mathrm{N}^i) \leq \mathrm{N}^{i-1}$ for every $i \geq 1$. By Lemma 2.35 we have $\Phi(E) < \Phi(\mathrm{N}^1)$. Suppose $2 \leq i \leq m$. Then $\Phi(\mathrm{N}^{i-1}) \leq \Phi(\mathrm{N}^i) \leq \mathrm{N}^{i-1}$ and $\Phi(\mathrm{N}^{i-1}) \leq \mathrm{N}^{i-2}$. If $\Phi(\mathrm{N}^{i-1}) = \Phi(\mathrm{N}^i)$ then $\mathrm{N}^{i-2} \trianglelefteq \mathrm{N}^i$ and by definition of the normalizer tower we get $\mathrm{N}^i = \mathrm{N}^{i-1} = S$, which is a contradiction.

Therefore for every $1 \leq i \leq m$ we have $\Phi(\mathrm{N}^{i-1}) < \Phi(\mathrm{N}^i)$ and

$$p^{\mathrm{rank}(\mathrm{N^i})} = [\mathrm{N}^i\colon \Phi(\mathrm{N}^i)] = [\mathrm{N}^i\colon \mathrm{N}^{i-1}][\mathrm{N}^{i-1}\colon \Phi(\mathrm{N}^i)] \quad < \quad p[\mathrm{N}^{i-1}\colon \Phi(\mathrm{N}^{i-1})] = p^{\mathrm{rank}(\mathrm{N^{i-1}})+1}.$$

Hence $\mathrm{rank}(\mathrm{N^i}) \leq \mathrm{rank}(\mathrm{N^{i-1}})$. □

2.4 $\mathcal{F}$-Essential subgroups of rank at most 3

We end this chapter by considering properties of $\mathcal{F}$-essential subgroups depending on their rank. Due to the fact that we want to classify fusion systems on p-groups of sectional rank 3, we are particularly interested in $\mathcal{F}$-essential subgroups of rank at most 3.

Theorem 2.37 (Structure Theorem for $\mathrm{Out}_{\mathcal{F}}(E)$)**.** *Let $\mathcal{F}$ be a saturated fusion system on the p-group S and let $E \leq S$ be an $\mathcal{F}$-essential subgroup of rank at most 3. Then*

1. *If $|E/\Phi(E)| = p^2$, then $\mathrm{SL}_2(p) \leq \mathrm{Out}_{\mathcal{F}}(E) \leq \mathrm{GL}_2(p)$;*
2. *If $|E/\Phi(E)| = p^3$ and the action of $\mathrm{Out}_{\mathcal{F}}(E)$ on $E/\Phi(E)$ is reducible then*
$$\mathrm{SL}_2(p) \leq \mathrm{Out}_{\mathcal{F}}(E) \leq \mathrm{GL}_2(p) \times \mathrm{GL}_1(p);$$
3. *If $|E/\Phi(E)| = p^3$ and the action of $\mathrm{Out}_{\mathcal{F}}(E)$ on $E/\Phi(E)$ is irreducible then*
   - *(a) either $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{PSL}_2(p) \cong \Omega_3(p)$;*
   - *(b) or $p = 3$ and $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong 13\colon 3$.*

Remark 2.38. With abuse of notation, when we write $X \leq \mathrm{Out}_{\mathcal{F}}(E) \leq Y$ we mean that $\mathrm{Out}_{\mathcal{F}}(E)$ contains a subgroup isomorphic to X and is contained in a subgroup isomorphic to Y. This notation will be used throughout this thesis.

Proof. By assumption E has rank at most 3 and is not cyclic, so $p^2 \leq |E/\Phi(E)| \leq p^3$. In particular $\mathrm{Out}_{\mathcal{F}}(E) \leq \mathrm{GL}_3(p)$ by Lemma 2.26.

If $|E/\Phi(E)| = p^2$, then $\mathrm{Out}_{\mathcal{F}}(E) \leq \mathrm{GL}_2(p)$. Since $O_p(\mathrm{Out}_{\mathcal{F}}(E)) = 1$, the group $\mathrm{Out}_{\mathcal{F}}(E)$ contains at least two Sylow p-subgroups of $\mathrm{GL}_2(p)$ (and so all of them). By [Gor80, Theorem 2.8.4] the Sylow p-subgroups of $\mathrm{GL}_2(p)$ generate $\mathrm{SL}_2(p)$. Hence $\mathrm{SL}_2(p) \leq \mathrm{Out}_{\mathcal{F}}(E)$.

If $|E/\Phi(E)| = p^3$, then the result follows from Theorem 1.39. □

Remark 2.39. Recall that if E is $\mathcal{F}$-essential then $\mathrm{N}_S(E)/E \cong \mathrm{Out}_S(E) \in \mathrm{Syl}_p(\mathrm{Out}_\mathcal{F}(E))$. Thus Theorem 2.37 implies that if E has rank at most 3 then $[\mathrm{N}_S(E)\colon E] = p$. In particular by Lemma 2.26(5) every subgroup of S belonging to $E^\mathcal{F}$ is $\mathcal{F}$-essential.

What we proved is in accordance with the following result of Sambale that describes the order and the nature of the group $\mathrm{N}_S(E)/E$.

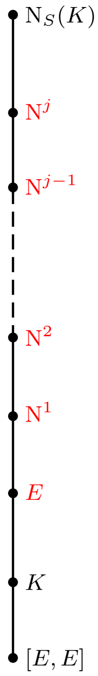
Theorem 2.40. *[Sam14, Theorem 6.9, Proposition 6.12] Let $E \leq S$ be an $\mathcal{F}$-essential subgroup and suppose that E has rank r. Set $N = \mathrm{N}_S(E)/E$.*

1. *If $r \leq 3$ then $|N| = p$.*
2. *If $p \geq 5$ then either N is cyclic of order $|N| \leq p^{\lceil \log_p(r) \rceil}$ or N is elementary abelian of order $|N| \leq p^{\lfloor r/2 \rfloor}$.*

Let E be an $\mathcal{F}$-essential subgroup of rank at most 3. Since E is fully normalized, it is receptive. Hence by Lemma 2.8 every $\mathcal{F}$-automorphism of E that normalizes the group $\mathrm{Aut}_S(E) \cong \mathrm{N}_S(E)/\mathrm{Z}(E)$ is the restriction of an $\mathcal{F}$-automorphism of the group $\mathrm{N}_S(E)$. Recall that $\mathrm{N}^i = \mathrm{N}^i(E)$ is the i-th term of the normalizer tower of E. We are interested in finding conditions that guarantee that a morphism $\varphi \in \mathrm{N}_{\mathrm{Aut}_\mathcal{F}(E)}(\mathrm{Aut}_S(E))$ is the restriction of an $\mathcal{F}$-automorphism of N^j, for some $j \geq 1$. In the best case, when $\mathrm{N}^j = S$, the existence of an 'extension' of φ enables us to describe the isomorphism type of S (as we will see in Chapter 3).

In the next lemma we select a specific section of S, namely N^j/K, that contains the group E/K as a soft subgroup. We can then apply the properties of soft subgroups stated in Theorem 1.27 and determine extension properties of the $\mathcal{F}$-automorphisms of E.

Lemma 2.41. *Let $E \leq S$ be an $\mathcal{F}$-essential subgroup of rank at most 3. Let $K \leq E$ be a subgroup of E containing $[E, E]$ but not $[\mathrm{N}^1, \mathrm{N}^1]$. Let $j \in \mathbb{N}$ be such that $\mathrm{N}^j \leq \mathrm{N}_S(K)$. Then E has maximal normalizer tower in N^j and the members of such tower are the only subgroups of N^j containing E. Also, N^i is fully normalized for every $i \leq j-1$.*



If moreover K $\mathrm{char}_{\mathcal{F}}$ N^i for some $i \leq j-1$, then

$$\mathrm{Aut}_S(\mathrm{N}^i) \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^i) \text{ and } \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^i) = \mathrm{Aut}_S(\mathrm{N}^i)\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^i)}(\mathrm{N}^{i-1}),$$

the group N^i is not $\mathcal{F}$-essential and for every morphism $\varphi \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^i)$ there exists $\overline{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^{i+1})$ such that $\overline{\varphi}|_{\mathrm{N}^i} = \varphi$.

In particular if K $\mathrm{char}_{\mathcal{F}}$ N^i for every $i \leq j-1$, then for every $\varphi \in \mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(E)}(\mathrm{Aut}_S(E))$ there exists $\hat{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^j)$ such that $\hat{\varphi}|_E = \varphi$.

Note that $[\mathrm{N}^1, \mathrm{N}^1] \leq E$ because E has rank at most 3 and so $[\mathrm{N}^1 \colon E] = p$ as a consequence of Theorem 2.37 (see Remark 2.39). Thus $K < E$.

Proof. Consider the group N^j/K. Notice that the subgroup E/K is abelian and for every $i \leq j$ we have $\mathrm{N}^i(E/K) = \mathrm{N}^i/K$. Since $[\mathrm{N}^1, \mathrm{N}^1] \nleq K$, we deduce that E/K is self-centralizing in N^j/K. Also by Lemma 2.40 we have $[\mathrm{N}^1 \colon E] = p$, so $[\mathrm{N}^1(E/K) \colon E/K] = p$. Therefore E/K is a soft subgroup of N^j/K. So by Theorem 1.27 E has maximal normalizer tower in N^j and the members of such tower are the only subgroups of N^j containing E.

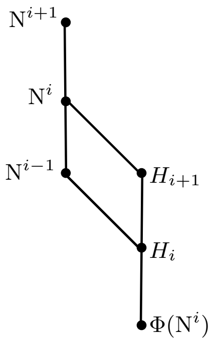
Since $\mathcal{F}$ is saturated, for every $i \leq j-1$ there exists $\alpha \in \mathrm{Hom}_{\mathcal{F}}(\mathrm{N}^i, S)$ such that $\mathrm{N}^i\alpha$ is fully normalized. Note that $E\alpha$ is an $\mathcal{F}$-essential subgroup of S by Lemma 2.26 and $K\alpha$ contains $[E, E]\alpha = [E\alpha, E\alpha]$ and is normalized by $\mathrm{Aut}_{\mathcal{F}}(E\alpha)$ and $\mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^1\alpha)$. Hence

$E\alpha$ has maximal normalizer tower in $\mathrm{N}^i\alpha$. Thus

$$[\mathrm{N}^{i+1}\alpha\colon \mathrm{N}^i\alpha] = p = [\mathrm{N}^{i+1}\colon \mathrm{N}^i].$$

Since $\mathrm{N}^i\alpha$ is fully normalized, we conclude that N^i is fully normalized.

Consider the sequence of subgroups $\overline{H_1} < \overline{H_2} < \ldots \overline{H_{j-1}} < \overline{H_j}$ of $\overline{\mathrm{N}^j} = \mathrm{N}^j/K$ as defined in Theorem 1.27 and let H_i be the preimage of $\overline{H_i}$ in N^j.



Note that $\mathrm{N}^i/H_i \cong \mathrm{C}_p \times \mathrm{C}_p$ and the group H_i/K is characteristic in N^i/K.

Suppose K $\mathrm{char}_{\mathcal{F}}$ N^i. Then H_i $\mathrm{char}_{\mathcal{F}}$ N^i. Also, $\mathrm{N}^i/H_i \cong \mathrm{C}_p \times \mathrm{C}_p$, so $\Phi(\mathrm{N}^i) \leq H_i$. The group $\mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^i)$ acts on N^i/H_i and since $\mathrm{N}^i \neq S$, the group $\mathrm{Aut}_S(\mathrm{N}^i) \cong \mathrm{N}^{i+1}/\mathrm{C}_S(\mathrm{N}^i)$ acts on the set of conjugates of N^{i-1} contained in N^i. Note that N^i/H_i has $p+1$ maximal subgroups and p of those are conjugates of N^{i-1}.

Thus the action of $\mathrm{Aut}_S(\mathrm{N}^i)$ is transitive and by the Frattini Argument we have

$$\mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^i) = \mathrm{Aut}_S(\mathrm{N}^i)\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^i)}(\mathrm{N}^{i-1})$$

Also, $H_{i+1} \notin {\mathrm{N}^{i-1}}^{\mathcal{F}}$ because $H_{i+1} \trianglelefteq \mathrm{N}^{i+1}$ and N^{i-1} is fully normalized. Therefore H_{i+1} $\mathrm{char}_{\mathcal{F}}$ N^i. Consider the sequence of subgroups

$$\Phi(\mathrm{N}^i) \leq H_i < H_{i+1} < \mathrm{N}^i.$$

They are all normalized by $\mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^i)$ and $[\mathrm{N}^1\colon H_{i+1}] = [H_{i+1}\colon H_i] = p$. By assumption E has rank at most 3. Hence by Lemma 2.36 the group N^i has rank at most 3 for every i. Thus $[\mathrm{N}^i\colon \Phi(\mathrm{N}^i)] \leq p^3$ and so $[H_i\colon \Phi(\mathrm{N}^i)] \leq p$. Hence every quotient of two consecutive subgroups in the sequence is centralized by $\mathrm{Aut}_S(\mathrm{N}^i)$ and by Lemma 1.34 we deduce that $\mathrm{Aut}_S(\mathrm{N}^i) \leq O_p(\mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^i))$. Thus $\mathrm{Aut}_S(\mathrm{N}^i) \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^i)$. Moreover, since N^i is receptive, we conclude that for every morphism $\varphi \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^i)$ there exists a morphism $\overline{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^{i+1})$ such that $\overline{\varphi}|_{\mathrm{N}^i} = \varphi$.

Since E is receptive, by Lemma 2.8 for every $\varphi \in \mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(E)}(\mathrm{Aut}_S(E))$ there exists $\overline{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^1)$ such that $\overline{\varphi}|_E = \varphi$. Using what we proved above, we conclude that for every $\varphi \in \mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(E)}(\mathrm{Aut}_S(E))$ there exists $\hat{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^j)$ such that $\hat{\varphi}|_E = \varphi$. $\square$

Note that by definition the group H_i/K is $\mathcal{F}$-characteristic in the group N^i/K for every $i \leq j$. The assumption $K \operatorname{char}_{\mathcal{F}} \mathrm{N}^i$ is only used to deduce that $H_i \operatorname{char}_{\mathcal{F}} \mathrm{N}^i$. Hence the assumption $K \operatorname{char}_{\mathcal{F}} \mathrm{N}^i$ can be replaced by $H_i \operatorname{char}_{\mathcal{F}} \mathrm{N}^i$.

Corollary 2.42. *Let $E \leq S$ be an abelian $\mathcal{F}$-essential subgroup of rank at most 3. Then E has maximal normalizer tower and is not properly contained in any $\mathcal{F}$-essential subgroup of S. Moreover every morphism in $\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(E)}(\mathrm{Aut}_S(E))$ is a restriction of an $\mathcal{F}$-automorphism of S.*

Proof. The result is a direct consequence of Lemma 2.41 applied with $K = 1$ and $\mathrm{N}^j = S$, recalling that by Theorem 1.27 the members of the normalizer tower of E in S (that is maximal) are the only subgroups of S containing E. $\square$

CHAPTER 3

FUSION SYSTEMS CONTAINING PEARLS

'Great things are done by a series of small things brought together.'

[Vincent van Gogh]

Let p be an *odd* prime and let $\mathcal{F}$ be a saturated fusion system on a p-group S.

In this chapter we see how the presence of small $\mathcal{F}$-essential subgroups can lead to the determination of the isomorphism type of S. Note that the smallest candidate for being an abelian $\mathcal{F}$-essential subgroup is a group isomorphic to the direct product $\mathrm{C}_p \times \mathrm{C}_p$, since an $\mathcal{F}$-essential subgroup cannot be cyclic (see Lemma 2.25). When p is an odd prime, the smallest candidate for a non-abelian $\mathcal{F}$-essential subgroup is a group isomorphic to the extraspecial group p^{1+2}_+. Indeed, the group p^{1+2}_- has a characteristic subgroup of order p^2 (that is the unique subgroup of order p^2 and exponent p) and so cannot be $\mathcal{F}$-essential by Lemma 2.24. The structure of a p-group containing $\mathcal{F}$-essential subgroups isomorphic to either $\mathrm{C}_p \times \mathrm{C}_p$ or p^{1+2}_+ is enriched with nice properties as a jewel is made precious by a pearl.

Definition 3.1. A subgroup of S is a *pearl* if it is an $\mathcal{F}$-essential subgroup of S that is either elementary abelian of order p^2 or extraspecial of order p^3 and exponent p.

We denote by $\mathcal{P}(\mathcal{F})$ the set of pearls of $\mathcal{F}$, by $\mathcal{P}(\mathcal{F})_a$ the set of abelian pearls of $\mathcal{F}$ and by $\mathcal{P}(\mathcal{F})_e$ the set of extraspecial pearls of $\mathcal{F}$. Note that $\mathcal{P}(\mathcal{F}) = \mathcal{P}(\mathcal{F})_a \cup \mathcal{P}(\mathcal{F})_e$.

It is not hard to see that if a satruated fusion system $\mathcal{F}$ on a p-group S contains a pearl, then S has maximal nilpotency class due to the following fact:

Lemma 3.2. *[Ber08, Proposition 1.8] Let S be a p-group and let $E \cong \mathrm{C}_p \times \mathrm{C}_p$ be a subgroup of S such that $\mathrm{C}_S(E) = E$. Then S has maximal nilpotency class.*

Lemma 3.3. *Let $E \leq S$ be a pearl. Then S has maximal nilpotency class.*

Proof. Note that E is $\mathcal{F}$-centric, so $\mathrm{C}_S(E) \leq E$. If $E \cong \mathrm{C}_p \times \mathrm{C}_p$ then S has maximal nilpotency class by Lemma 3.2. Suppose $E \cong p_+^{1+2}$. Then $\mathrm{Z}(S) = \mathrm{Z}(E) = \Phi(E)$ and $|\mathrm{Z}(S)| = p$. Let $\overline{C} = \mathrm{C}_{S/\mathrm{Z}(S)}(E/\mathrm{Z}(S))$. Then $E/\mathrm{Z}(S) \leq \overline{C} \leq \mathrm{N}_S(E)/\mathrm{Z}(S)$. By Theorem 1 and the fact that E has rank 2 we deduce $[\mathrm{N}_S(E) \colon E] = p$. Suppose by contradiction that $E/\mathrm{Z}(S) < \overline{C}$. Then $\overline{C} = \mathrm{N}_S(E)/\mathrm{Z}(S) = \mathrm{N}_S(E)/\Phi(E)$, contradicting the fact that $\Phi(E) < [\mathrm{N}_S(E), E]\Phi(E)$ by Lemma 2.35. Therefore $E/\mathrm{Z}(S) = \overline{C} = \mathrm{C}_{S/\mathrm{Z}(S)}(E/\mathrm{Z}(S))$. Since $E/\mathrm{Z}(S) \cong \mathrm{C}_p \times \mathrm{C}_p$, the group $S/\mathrm{Z}(S)$ has maximal nilpotency class by Lemma 3.2. Since $|\mathrm{Z}(S)| = p$ we conclude that S has maximal nilpotency class. □

The study of fusion systems containing pearls is particularly relevant for us because, as we will see in Chapter 4, when p is odd, the $\mathcal{F}$-essential subgroups of rank 2 of a p-group of sectional rank 3 that are not $\mathcal{F}$-characteristic in S are pearls (Theorem 12).

The first section of this chapter is dedicated to properties of p-groups of maximal nilpotency class. Suppose that the p-group S has maximal nilpotency class and order p^n. For $i \geq 2$ set $S_2 = [S, S]$ and $S_{i+1} = [S_i, S]$. Then by definition S has nilpotency class $n-1$, $S_n = 1$ and $S_{n-1} = \mathrm{Z}(S) \neq 1$. Also, since the lower central series is of maximal length, we have $[S_i \colon S_{i+1}] = p$ for every $i \geq 2$.

An important role is played by the subgroup S_1 of S, defined as the centralizer in S of the quotient S_2/S_4. In Theorem 3.11 we see that whenever S_1 is neither abelian nor extraspecial, $\mathrm{Aut}(S) \cong P\colon H$, where $P \in \mathrm{Syl}_p(\mathrm{Aut}(S))$ and $H \leq \mathrm{C}_{p-1}$.

We also bound the order of a p-group of maximal nilpotency class by a function of its sectional rank.

Theorem 2. *Let S be a p-group of maximal nilpotency class and sectional rank k. If $p \geq k+2$ then $|S| \leq p^{2k}$ (with strict inequality if $S_1 = \mathrm{C}_S(\mathrm{Z}_2(S))$). Also, if $p = 3$ and $k \geq 3$ then $|S| = 3^4$.*

Therefore if p is odd and S has sectional rank 3, then either $p = 3$ and $|S| = 3^4$ or $p \geq 5$ and $|S| \leq p^6$. At the end of this chapter, we will use this bound to classify the saturated fusion systems on p-groups containing a pearl and having sectional rank 3.

In Section 3.2, after showing that any $\mathcal{F}$-essential subgroup having maximal nilpotency class is a pearl, we describe the $\mathcal{F}$-essential subgroups of p-groups of maximal nilpotency class.

Theorem 3. *Let $\mathcal{F}$ be a saturated fusion system on a p-group S, that has maximal nilpotency class. Let E be an $\mathcal{F}$-essential subgroup of S. Then one of the following holds:*

1. *E is a pearl;*
2. *$E \leq S_1$ (and if S_1 is extraspecial or abelian then $E = S_1$); or*
3. *$E \leq \mathrm{C}_S(\mathrm{Z}_2(S))$, $E \not\leq S_1$, $[E\colon \mathrm{Z}_i(S)] = p$ for some $i \in \{2,3,4\}$ and either $E \cong \mathrm{C}_p \times \mathrm{C}_p \times \mathrm{C}_p$ or $E/\mathrm{Z}_2(S)$ is isomorphic to either $\mathrm{C}_p \times \mathrm{C}_p$ or p_+^{1+2}.*

Also, if $O_p(\mathcal{F}) = 1$, S_1 is extraspecial and $\mathrm{C}_S(\mathrm{Z}_2(S))$ is $\mathcal{F}$-essential then $p \geq 5$, S is isomorphic to a Sylow p-subgroup of the group $\mathrm{G}_2(p)$ (with $p = 7$ if there is a pearl) and $\mathcal{F}$ is one of the fusion systems classified by Parker and Semeraro in [PS16].

When the group S_1 is extraspecial and S contains a pearl, we can determine the order and the exponent of S.

Theorem 4. *Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on a p-group S containing a pearl E. Then the following are equivalent:*

1. *S_1 is extraspecial;*
2. *$S_1 \neq \mathrm{C}_S(\mathrm{Z}_2(S))$;*
3. *$E \cong \mathrm{C}_p \times \mathrm{C}_p$, $|S| = p^{p-1}$ and S_1 is not abelian.*

Also, if one (and then each) of the cases above occurs then $p \geq 7$, $S_1 \cong p_+^{1+(p-3)}$ and S has exponent p.

As a corollary of the previous results, we classify the $\mathcal{F}$-essential subgroups of p-groups containing a pearl, depending on the nature of the group S_1.

Theorem 5. *Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on a p-group S of order p^n and suppose $\mathcal{P}(\mathcal{F}) \neq \emptyset$. Set $S_1 = \mathrm{C}_S(S_2/S_4)$ and let $\mathcal{E}$ be the set of $\mathcal{F}$-essential subgroups of S. Then S has maximal nilpotency class and the following hold:*

1. *If S_1 is extraspecial then $p \geq 7$, S has order p^{p-1} and exponent p and*

$$\mathcal{E} \subseteq \{S_1\} \cup \mathcal{P}(\mathcal{F})_a \cup \{E \leq \mathrm{C}_S(\mathrm{Z}_2(S)) \mid E \not\leq S_1\}$$

   where if $E \leq \mathrm{C}_S(\mathrm{Z}_2(S))$ and $E \not\leq S_1$ then $[E\colon \mathrm{Z}_i(S)] = p$ for $2 \leq i \leq 4$ and either $E \cong \mathrm{C}_p \times \mathrm{C}_p \times \mathrm{C}_p$ or $E/\mathrm{Z}_2(S)$ is isomorphic to either $\mathrm{C}_p \times \mathrm{C}_p$ or p_+^{1+2}.

   If moreover $p = 7$ then S is isomorphic to a Sylow 7-subgroup of the group $\mathrm{G}_2(7)$ (and this is always the case when $\mathrm{C}_S(\mathrm{Z}_2(S))$ is $\mathcal{F}$-essential).

2. *If S_1 is abelian then $\mathcal{E} \subseteq \{S_1\} \cup \mathcal{P}(\mathcal{F})$.*

   In particular the reduced fusion systems on the p-group S (as defined in [AKO11, Definition III.6.2]) have been classified by Craven, Oliver and Semeraro in [Oli14] and [COS16].

3. *If S_1 is neither abelian nor extraspecial then $|\mathrm{Aut}_{\mathcal{F}}(S)| = p^{n-1}(p-1)$, $S_1 = \mathrm{C}_S(\mathrm{Z}_2(S))$ and*

$$\mathcal{E} \subseteq \{E \leq S_1\} \cup \mathcal{P}(\mathcal{F}).$$

When S_1 is extraspecial, the saturated fusion systems on S are being investigated by Moragues Moncho. The case of S_1 neither abelian nor extraspecial is an open problem.

In Section 3.3 we focus on p-groups of maximal nilpotency class and sectional rank 3, for p odd. Recalling that a p-group of maximal nilpotency class and sectional rank 3 has order at most p^6, we first study the structure of small groups containing pearls.

Theorem 6. *Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on a p-group S. Suppose that S contains a pearl E, has sectional rank greater than 2 and $p^4 \leq |S| \leq p^6$. Then S has maximal nilpotency class and one of the following holds:*

1. *$|S| = p^4$ and S is isomorphic to a Sylow p-subgroup of the group $\mathrm{Sp}_4(p)$;*

2. *$|S| = p^5$ and either S_1 is elementary abelian or $p = 7, E \cong \mathrm{C}_7 \times \mathrm{C}_7$ and S is isomorphic to the group indexed in* Magma *as SmallGroup($7^5, 37$) (that is isomorphic to a maximal subgroup of the group $\mathrm{G}_2(7)$);*

3. *$|S| = 7^6$, $S_1 \cong 7_+^{1+4}$, $E \cong \mathrm{C}_7 \times \mathrm{C}_7$ and S isomorphic to a Sylow 7-subgroup of $\mathrm{G}_2(7)$;*

4. *$|S| = p^6$, S_2 is elementary abelian of order p^4 and one of the following holds:*

(a) S_1 *is abelian (and if* $\mathcal{F}$ *is reduced then it is among the fusion systems studied in [Oli14] and [COS16]);*

(b) $p = 5$*,* $E \cong \mathrm{C}_5 \times \mathrm{C}_5$*,* $S_3 = \mathrm{Z}(S_1)$ *and* $S \cong$ *SmallGroup(*$5^6, 636$*),*

(c) $p = 5$*,* $E \cong \mathrm{C}_5 \times \mathrm{C}_5$*,* $S_3 = \mathrm{Z}(S_1)$*,* $S \cong$ *SmallGroup(*$5^6, i$*), for* $i \in \{639, 640, 641, 642\}$ *and if* $P \in \mathcal{P}(\mathcal{F})$ *then* $P \in E^{\mathcal{F}}$ *(also* S *has exponent* 25 *and* S_1 *has exponent* 5 *if and only if* $i = 639$*);*

(d) $p = 7$*,* $E \cong 7^{1+2}_+$*,* $\mathrm{Z}_2(S) = \mathrm{Z}(S_1)$*,* S_1 *has exponent* 7 *and* $S \cong$ *SmallGroup(*$7^6, 813$*);*

(e) $p = 7$*,* $E \cong 7^{1+2}_+$*,* $\mathrm{Z}_3(S) = \mathrm{Z}(S_1)$*,* S_1 *has exponent* 7 *and* $S \cong$ *SmallGroup(*$7^6, 798$*).*

In particular we see that if S has order p^6 then it has sectional rank greater than 3.

We end this chapter with the classification of saturated fusion systems on p-groups of sectional rank 3 containing pearls. In particular, we discover a new exotic fusion system.

Theorem 7. *Let* p *be an odd prime and let* $\mathcal{F}$ *be a saturated fusion system on a* p*-group* S *of sectional rank* 3 *containing a pearl* E*. Then* S *has maximal nilpotency class and either* S *is isomorphic to a Sylow* p*-subgroup of* $\mathrm{Sp}_4(p)$ *or the following hold:*

1. $p = 7$ *and* S *is isomorphic to the group indexed in* Magma *as SmallGroup(*$7^5, 37$*);*
2. $E \cong \mathrm{C}_7 \times \mathrm{C}_7$ *and* $\mathrm{Aut}_{\mathcal{F}}(E) \cong \mathrm{SL}_2(7)$*;*
3. $\mathcal{F}$ *is completely determined by* $\mathrm{Inn}(S)$*,* $\mathrm{Aut}_{\mathcal{F}}(E)$ *and* $\mathrm{Out}_{\mathcal{F}}(S) \cong \mathrm{C}_6$*; and*
4. $\mathcal{F}$ *is simple and exotic. Also, such an* $\mathcal{F}$ *exists and is unique.*

Note that the group SmallGroup$(7^5, 37)$ is the unique 7-group of order 7^5 that has maximal nilpotency class, exponent 7 and no abelian maximal subgroups.

The following is a presentation of this group:

$$\begin{aligned} S := \langle x, s_1, s_2, s_3, s_4 \mid & [x, s_1] = s_2,\ [x, s_2] = s_3,\ [x, s_3] = s_4,\ [s_1, s_2] = s_4, \\ & [x, s_4] = [s_1, s_4] = [s_2, s_4] = [s_3, s_4] = 1, \\ & x^7 = s_1^7 = s_2^7 = s_3^7 = s_4^7 = 1\rangle. \end{aligned}$$

Such an S is isomorphic to a maximal subgroup of a Sylow 7-subgroup P of the group $G_2(7)$, distinct from $\mathrm{C}_P(\mathrm{Z}_2(P))$ and P_1.

In Chapter 5 we prove that if $\mathcal{F}$ is a saturated fusion system on a p-group of sectional rank 3 with $p \geq 5$ such that $O_p(\mathcal{F}) = 1$, then S contains a pearl (Theorem 20). Therefore Theorem 7 applies.

3.1 On p-groups of maximal nilpotency class

Let p be an odd prime and let S be a p-group of maximal nilpotency class.

Note that if $|S| = p^2$ then S has class 1 and so it is an abelian group of order p^2. Therefore either $S \cong \mathrm{C}_p \times \mathrm{C}_p$ or $S \cong C_{p^2}$. If $|S| = p^3$ then S is a non-abelian group of order p^3, and so either $S \cong p_+^{1+2}$ or $S \cong p_-^{1+2}$.

From now on we assume that $|S| = p^n > p^3$.

Recall that S_i denotes the i-th term of the lower central series for S for every $i \geq 2$ and $\mathrm{Z}_i(S)$ denotes the i-th term of the upper central series for S for every $i \geq 1$.

The next lemma introduces some basic properties of p-groups of maximal nilpotency class.

Lemma 3.4. *[Hup67, Lemma III.14.2]*

1. $[S \colon S_2] = p^2$ *and* $[S_i \colon S_{i+1}] = p$ *for every* $2 \leq i \leq n-1$*;*

2. *for* $2 \leq i \leq n$*, the group* S_i *is the unique normal subgroup of* S *of index* p^i*;*

3. $\mathrm{Z}_{n-1}(S) = S$ *and* $\mathrm{Z}_i(S) = S_{n-i}$ *for* $0 \leq i \leq n-2$.

Moreover, if P *is a* p*-group of nilpotency class* c *and there exists* $i \leq c$ *such that* $P/\mathrm{Z}_i(P)$ *has maximal nilpotency class and* $|\mathrm{Z}_i(P)| = p^i$ *then* P *has maximal nilpotency class.*

We remark that the normal subgroups of S of order at most p^{n-2} are all characteristic in S and form a chain:

$$1 = S_n < S_{n-1} < S_{n-2} < \cdots < S_2.$$

Every quotient of two consecutive subgroups in the chain has order p and $S/S_2 \cong \mathrm{C}_p \times \mathrm{C}_p$. We now complete the chain introducing an important maximal subgroup of S that contains S_2 and whose structure is closely related to the structure of S.

Definition 3.5. We set

$$S_1 = \mathrm{C}_S(S_2/S_4) = \{x \in S \mid [x, S_2] \le S_4\}.$$

The nature of the characteristic subgroup S_1 gives information about the group S.

Theorem 3.6. *[Hup67, Chapter* III.14]

1. *S_1 is a maximal subgroup of S;*

2. *$S_1 = \mathrm{C}_S(S_i/S_{i+2})$ for every $1 \le i \le n-3$;*

3. *S_1 and $\mathrm{C}_S(\mathrm{Z}_2(S))$ are the only maximal subgroups of S that do not have maximal nilpotency class.*

Note that in Theorem 3.6 the group S_1 can equal $\mathrm{C}_S(\mathrm{Z}_2(S))$. The next theorem tells us when this can happen.

Theorem 3.7. *[LGM02, Corollary 3.2.7, Theorem 3.2.11, Theorem 3.3.5]. Assume one of the following holds:*

1. *$n = 4$, or*

2. *$n > p + 1$, or*

3. *$5 \le n \le p + 1$ and n is odd.*

Then $S_1 = \mathrm{C}_S(\mathrm{Z}_2(S))$.

We also have a precise characterization of the exponent of the subgroups S_i.

Lemma 3.8. *[LGM02, Proposition 3.3.2, Corollary 3.3.6].*

1. *If $4 \le n \le p + 1$ then both $S/\mathrm{Z}(S)$ and $S_2 = [S, S]$ have exponent p.*

2. *If $n > p+1$ then $S_i^p = S_{i+p-1}$ for every $1 \leq i \leq n-p+1$.*

Note that if $|S| = p^n > p^{p+1}$ and $i \geq n-p+1$ then $S_i^p \leq S_{n-p+1}^p = S_n = 1$.

We now consider elements and subgroups of S not contained in S_1.

Lemma 3.9. *Suppose $x \in S$ is not contained in S_1. Then*

1. *$x^p \in \mathrm{Z}_2(S)$ and if $x \notin \mathrm{C}_S(\mathrm{Z}_2(S))$ then $x^p \in \mathrm{Z}(S)$;*

2. *if $z \in S_i \backslash S_{i+1}$ and either $x \notin \mathrm{C}_S(\mathrm{Z}_2(S))$ or $\mathrm{Z}(S) \leq \langle x, z\rangle$, then $S_i \leq \langle x, z\rangle$.*

Proof.

1. Suppose $x^p \neq 1$ and let $1 \leq i \leq n-1$ be such that $x^p \in S_i \backslash S_{i+1}$. We want to prove that $i \geq n-2$. Note that

$$[S_i, x] = [S_{i+1}\langle x^p\rangle, x] = [S_{i+1}, x].$$

If $i < n-2$, then $[S_i, x] \leq S_{i+2}$ and so $x \in \mathrm{C}_S(S_i/S_{i+2}) = S_1$, contradicting the assumption. Thus $i \geq n-2$. If $x \notin \mathrm{C}_S(\mathrm{Z}_2(S))$ the same argument implies $i = n-1$.

2. Let $s_i = z$ and $s_j = [x, s_{j-1}]$ for $i+1 \leq j \leq n-1$. Since x is not contained in S_1, we have $s_j \in S_j \backslash S_{j+1}$, and so $S_j = S_{j+1}\langle s_j\rangle$ for every $j \leq n-2$. Also, if $x \notin \mathrm{C}_S(\mathrm{Z}_2(S))$ then $s_{n-1} = [x, s_{n-2}] \neq 1$ and $S_{n-1} = \mathrm{Z}(S) \leq \langle x, s_i\rangle$. Thus in any case we have $\mathrm{Z}(S) \leq \langle x, s_i\rangle$. In conclusion $S_i = \langle s_i, s_{i+1}, \dots, s_{n-2}\rangle \mathrm{Z}(S) \leq \langle x, s_i\rangle$.

□

Lemma 3.10. *Let P be a proper subgroup of S not contained in S_1. Suppose that either $P \not\leq \mathrm{C}_S(\mathrm{Z}_2(S))$ or $\mathrm{Z}(S) \leq P$. If $|P| = p^m$ then $\mathrm{Z}_{m-1} \leq P$ and $[P\colon \mathrm{Z}_{m-1}] = p$. Moreover*

- *if $P \not\leq \mathrm{C}_S(\mathrm{Z}_2(S))$ and $|P| > p$ then P has maximal nilpotency class;*

- *if $P \leq \mathrm{C}_S(\mathrm{Z}_2(S))$ and $|P| > p^2$ then $P/\mathrm{Z}(S)$ has maximal nilpotency class;*

In particular if $P \nleq \mathrm{C}_S(\mathrm{Z}_2(S))$ and $|P| > p^2$ then $\mathrm{Z}(P) = \mathrm{Z}(S)$ and if $P \leq \mathrm{C}_S(\mathrm{Z}_2(S))$ and $|P| > p^3$ then $\mathrm{Z}(P) = \mathrm{Z}_2(S)$.

Proof. Note that $PS_1 = S$ and $[S\colon S_1] = p$, so $[P\colon P \cap S_1] = p$.

Suppose $P \nleq \mathrm{C}_S(\mathrm{Z}_2(S))$. If $P \cap S_1 = 1$, then $|P| = p$ and $P \cong \mathrm{C}_p$. Assume there exists $1 \neq z \in P \cap S_1$. Then there exists i such that $z \in S_i \backslash S_{i+1}$. Let $x \in P$ be such that x is not contained in S_1 nor $\mathrm{C}_S(\mathrm{Z}_2(S))$. Then by Lemma 3.9 we have $S_i \leq \langle z, x\rangle \leq P$. Let $j \in \mathbb{N}$ be minimal such that $S_j \leq P$. Then $S_1 \cap P = S_j$ and $[P\colon S_j] = p$. In particular $|S_j| = p^{m-1}$ and so $j = n-(m-1)$ and $S_j = \mathrm{Z}_{m-1}(S)$. Using the fact that x is in neither S_1 nor $\mathrm{C}_S(\mathrm{Z}_2(S))$, we conclude that $P_k = S_{n-m+k}$ for every $k \geq 1$ and so P has maximal nilpotency class.

Suppose $P \leq \mathrm{C}_S(\mathrm{Z}_2(S))$. Consider the group $\overline{S} = S/\mathrm{Z}(S)$. It is a p-group of maximal nilpotency class and $\overline{S}_i = S_i/\mathrm{Z}(S)$ for every i and $\mathrm{Z}_2(\overline{S}) = \mathrm{Z}_3(S)/\mathrm{Z}(S)$. Thus $\overline{S_1} = \mathrm{C}_{\overline{S}}(\mathrm{Z}_2(\overline{S}))$. By assumption $\mathrm{Z}(S) \leq P$ so we can consider the group $\overline{P} = P/\mathrm{Z}(S) \leq \overline{S}$. Note that $\overline{P}$ is not contained in $\overline{S_1}$ so by what was proved above we conclude that $[\overline{P}\colon \overline{S}_j] = p$ for some j and either $|\overline{P}| = p$ or $\overline{P}$ has maximal nilpotency class. In particular if $|\overline{P}| > p^2$ then $\mathrm{Z}(\overline{P}) = \mathrm{Z}_2(S)/\mathrm{Z}(S)$. Since $\mathrm{Z}(S) \leq \mathrm{Z}(P)$ we conclude that if $|P| > p^3$ then $\mathrm{Z}(P) = \mathrm{Z}_2(S)$. □

If the group S_1 is neither abelian nor extraspecial, then we prove that the order of the automorphism group of S divides $p^m(p-1)$ for some $m \in \mathbb{N}$. This result will be crucial in the classification of fusion systems on p-groups of maximal nilpotency class.

Theorem 3.11. *If S_1 is neither abelian nor extraspecial, then*

$$\mathrm{Aut}(S) \cong P\colon H$$

where $P \in \mathrm{Syl}_p(\mathrm{Aut}(S))$ and H is a cyclic group whose order divides $p-1$.

Proof. Note that S_1 char S and $S/S_2 \cong \mathrm{C}_p \times \mathrm{C}_p$ since S has maximal class. Let $\varphi \in \mathrm{Aut}(S)$ be a morphism of order prime to p. Then φ acts non-trivially on S/S_2 and it normalizes $S_1/S_2 = \langle s_1 \rangle S_2/S_2$ for some $s_1 \in S_1 \backslash S_2$. Therefore by Maschke's Theorem there exists $x \in S \backslash S_1$ such that $\langle x \rangle S_2/S_2$ is normalized by φ. Let $1 \leq \lambda, \mu \leq p-1$ be such that

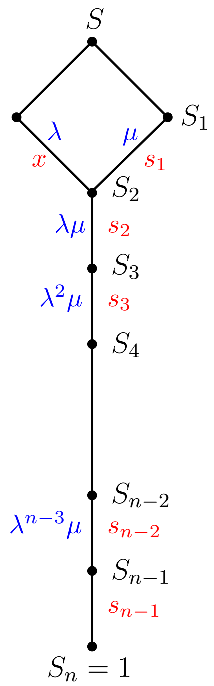
$$x\varphi = x^{\lambda} \mod S_2 \quad \text{and} \quad s_1\varphi = s_1^{\mu} \mod S_2.$$

In other words the morphism φ acts on S/S_2 as $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ with respect to the basis $\{xS_2, s_1S_2\}$. We want to prove that $\mu = \lambda^t$ for some $1 \leq t \leq p-1$.

Define

$$s_i \colon = [x, s_{i-1}] \text{ for every } 2 \leq i \leq n-1 \text{ and}$$

$$s_{n-1} = \begin{cases} [x, s_{n-2}] & \text{if } S_1 = \mathrm{C}_S(\mathrm{Z}_2(S)) \\ [s_1, s_{n-2}] & \text{otherwise} \end{cases}$$



Note that by Theorem 3.6 and the fact that $x \notin S_1$ we deduce $S_i = \langle s_i \rangle S_{i+1}$ for every $2 \leq i \leq n-1$.

The morphism φ acts on every quotient S_i/S_{i+1}. We will show by induction on i that

$$s_i\varphi = s_i^{\lambda^{i-1}\mu} \mod S_{i+1} \text{ for every } 1 \leq i \leq n-2 \text{ and}$$

$$s_{n-1}\varphi = \begin{cases} s_{n-1}^{\lambda^{n-2}\mu} & \text{if } S_1 = \mathrm{C}_S(\mathrm{Z}_2(S)) \\ s_{n-1}^{\lambda^{n-3}\mu^2} & \text{otherwise} \end{cases} \tag{3.1}$$

If $i = 1$, then the statement is true by definition of μ. Assume $1 < i < n-2$. Then by the inductive hypothesis we have

$$s_i\varphi = [x, s_{i-1}]\varphi = [x^\lambda u, s_{i-1}^{\lambda^{i-2}\mu} v] \quad \text{for some} \quad u \in S_2, v \in S_i.$$

Thus

$$s_i\varphi = [x^\lambda, s_{i-1}^{\lambda^{i-2}\mu}] \mod S_{i+1} = s_i^{\lambda^{i-1}\mu} \mod S_{i+1}.$$

The same argument works for $i = n-1$ when $S_1 = \mathrm{C}_S(\mathrm{Z}_2(S))$. If $S \neq \mathrm{C}_S(\mathrm{Z}_2(S))$ and $i = n-1$ then we have

$$s_{n-1}\varphi = [s_1, s_{n-2}]\varphi = [s_1^\mu u, s_{n-2}^{\lambda^{n-3}\mu} v] = s_{n-1}^{\lambda^{n-3}\mu^2},$$

for some $u \in S_2$ and $v \in \mathrm{Z}(S)$.

We can now show that μ depends on λ. By assumption S_1 is non-abelian, so there exists $i < j \leq n-2$ such that $[s_i, s_j] \neq 1$. Then $[s_i, s_j] \in S_r \backslash S_{r+1}$ for some $1 \leq r \leq n-1$. So $[s_i, s_j] = s_r^k \mod S_{r+1}$ for some $1 \leq k \leq p-1$. Suppose $S_1 \neq \mathrm{C}_S(\mathrm{Z}_2(S))$. Then $\mathrm{Z}(S) = S_{n-1} = \mathrm{Z}(S_1)$. Since S_1 is not extraspecial by assumption and $S_1^p \leq \mathrm{Z}(S)$ by Theorem 3.7 (2) and Lemma 3.8 (1), we have $S_{n-1} < [S_1, S_1]$. Thus if $S_1 \neq \mathrm{C}_S(\mathrm{Z}_2(S))$ we may assume $r < n-1$.

Returning to the general case, by equation 3.1 we have $s_r^k\varphi = (s_r\varphi)^k = s_r^{k(\lambda^{r-1}\mu)} \mod S_{r+1}$. On the other hand,

$$s_r^k\varphi = [s_i, s_j]\varphi = [s_i\varphi, s_j\varphi] = [s_i^{\lambda^{i-1}\mu}, s_j^{\lambda^{j-1}\mu}] \mod S_{r+1} = (s_r^k)^{\lambda^{i+j-2}\mu^2} \mod S_{r+1}.$$

Hence

$$\mu = \lambda^{r+1-i-j} \mod p.$$

We proved that the morphism φ is completely determined by its action on $\langle x\rangle S_2$. Thus every morphism $\varphi \in \mathrm{Aut}(S)$ of order prime to p is completely determined by the maximal subgroup $M/S_2 \leq S/S_2$ distinct from S_1/S_2 that it normalizes and by its action on it. Since there are p maximal subgroups of S/S_2 distinct from S_1/S_2 and $\mathrm{Aut}(M/S_2) \cong \mathrm{C}_{p-1}$, we get

$$|\{\varphi \mathrm{C}_{\mathrm{Aut}(S)}(S/\Phi(S)) \mid \varphi \in \mathrm{Aut}(S) \text{ has order prime to } p\}| \leq p(p-1).$$

Note that the quotient $\mathrm{Aut}(S)/\mathrm{C}_{\mathrm{Aut}(S)}(S/\Phi(S))$ is isomorphic to a subgroup of $\mathrm{GL}_2(p)$ (because $S/\Phi(S) = S/S_2 \cong \mathrm{C}_p \times \mathrm{C}_p$). Since S_1 is a characteristic subgroup of S, we deduce that

$$\mathrm{Aut}(S)/\mathrm{C}_{\mathrm{Aut}(S)}(S/\Phi(S)) \cong U \leq \left\langle \begin{pmatrix} a & 0 \\ c & b \end{pmatrix} \mid a, b \in \mathrm{GF}(p)^*, c \in \mathrm{GF}(p) \right\rangle.$$

Since $\mathrm{C}_{\mathrm{Aut}(S)}(S/\Phi(S))$ is a p-group (for example by Theorem 1.10) and U has a normal Sylow p-subgroup, we deduce that $\mathrm{Aut}(S)$ has a unique normal Sylow p-subgroup P. Using what we proved above we conclude that $\mathrm{Aut}(S) \cong P \colon H$, where H is a cyclic group whose order divides $p-1$. $\square$

If S_1 is extraspecial, then the conclusion of Theorem 3.11 is not true. As an example, the group $G_2(p)$ has Sylow p-subgroups S of order p^6 and of maximal nilpotency class such that S_1 is extraspecial and $|\mathrm{Aut}(S)|$ is divisible by $(p-1)^2$.

We end this section with a bound on the order of S depending on its sectional rank.

Theorem 3.12. *Let S be a p-group of maximal nilpotency class and sectional rank $k \geq 2$ such that $p \geq k+2$. Then $|S| \leq p^{2k}$ (with strict inequality if $S_1 = \mathrm{C}_S(\mathrm{Z}_2(S))$).*

Proof. Clearly the statement is true if $|S| \leq p^3$, so suppose $|S| \geq p^4$.

Note that $[S_i, S_j] \leq S_{i+j}$ for every $i, j \geq 1$. This can be proven by induction on j using the three-subgroup lemma ([Gor80, Theorem 2.2.3]).

1. Assume $|S| = p^n \leq p^{p+1}$. Then for every $2 \leq i \leq n$ the group S_i has exponent p by Lemma 3.8. Note that $[S_{\lceil n/2 \rceil}, S_{\lceil n/2 \rceil}] \leq S_n = 1$, so $S_{\lceil n/2 \rceil}$ is abelian. Therefore it is elementary abelian and by definition of sectional rank we get $\lfloor n/2 \rfloor \leq k$ and so $n \leq 2k+1$. Note that $[S_k, S_k] = [S_k, S_{k+1}] \leq S_{2k+1} = 1$; so S_k is an elementary abelian group of order p^{n-k}. Thus we have $n \leq 2k$. Finally suppose $S_1 = \mathrm{C}_S(\mathrm{Z}_2)$. Then for every i, j we have $[S_i, S_j] \leq S_{i+j+1}$ by [LGM02, Theorem 3.2.6]. Assume for a contradiction that $|S| = p^{2k}$. Then $[S_{k-1}, S_{k-1}] = [S_{k-1}, S_k] \leq S_{2k} = 1$. Thus S_{k-1} is an elementary abelian group of order p^{k+1}, contradicting the assumptions.

2. Assume $|S| = p^n > p^{p+1}$. By Theorem 3.7 we have $S_1 = \mathrm{C}_S(\mathrm{Z}_2(S))$. Thus by [LGM02, Theorem 3.2.6] we have $[S_i, S_j] \leq S_{i+j+1}$ for every $1 \leq i, j \leq n$. In particular
$$[S_{k-1}, S_{k-1}] = [S_{k-1}, S_k] \leq S_{2k}.$$
Suppose for a contradiction $n \geq 2k$, then $[S_{k-1} \colon S_{2k}] = p^{k+1}$. Since S has sectional rank k, we have $S_{2k} < \Phi(S_{k-1}) = [S_{k-1}, S_{k-1}]S_{k-1}^p$. Since $[S_{k-1}, S_{k-1}] \leq S_{2k}$, we have $S_{2k} < S_{k-1}^p$. By Lemma 3.8 either $S_{k-1}^p = 1$ or $S_{k-1}^p = S_{k+p-2}$. Therefore we have $S_{2k} < S_{k+p-2}$ that implies $k+p-2 < 2k$. So $p < k+2$, contradicting the assumption that $p \geq k+2$. Hence $|S| < p^{2k}$.

$\square$

The previous theorem guarantees that if S is a p-group of sectional rank 3 and maximal nilpotency class then $|S| \leq p^6$ whenever $p \geq 5$. If $p = 3$ then $|S| \leq 3^4$, as the next lemma shows.

Lemma 3.13. *Let S be a 3-group of maximal nilpotency class. If $|S| > 3^4$ then S has sectional rank 2.*

Proof. Since $p = 3$, by [LGM02, Theorem 3.4.3] the group S_2 is abelian and $|[S_1, S_1]| \leq 3$. Thus either S_1 is abelian or $[S_1, S_1] = \mathrm{Z}(S)$ is the unique normal subgroup of S having order 3.

Suppose $|S| = p^n > 3^{3+1}$. Then by Lemma 3.8 we have $S_i^3 = S_{i+2}$ for every $1 \leq i \leq n-2$. Let $s_1, s_2 \in S$ be such that $S_1 = \langle s_1 \rangle S_2$ and $S_2 = \langle s_2 \rangle S_3$. Then for every $1 \leq i \leq n-1$ we have

$$S_i = \begin{cases} \langle\ s_1^{a_i}\ \rangle S_{i+1} \text{ with } a_i = 3^{(i-1)/2} & \text{if } i \text{ is odd} \\ \langle\ s_2^{b_i}\ \rangle S_{i+1} \text{ with } b_i = 3^{(i-2)/2} & \text{if } i \text{ is even} \end{cases}$$

In particular $[S_1, S_1] \leq \mathrm{Z}(S) \leq \langle s_j \rangle$ for some $j \in \{1, 2\}$. Thus $X = \langle s_j \rangle$ is a normal subgroup of S_1 and S_1/X is cyclic. Therefore the p-group S_1 is metacyclic. In particular every subgroup of S_1 is metacyclic and has rank at most 2.

Since $|S| > 3^{3+1}$ we also have $S_1 = \mathrm{C}_S(\mathrm{Z}_2(S))$ by Theorem 3.7. Thus by Lemma 3.10 every subgroup of S not contained in S_1 has maximal nilpotency class, and so it has rank at most 2.

Hence every subgroup of S has rank at most 2 and so S has sectional rank 2. $\square$

Lemma 3.14. *Let p be an odd prime and let S be a p-group of order p^4 and maximal nilpotency class. If S has sectional rank* 3 *and $S_1 \neq \Omega_1(S)$ then S is isomorphic to a Sylow p-subgroup of the group* $\mathrm{Sp}_4(p)$.

Proof. Since $|S| = p^4$ we have $S_1 = \mathrm{C}_S(\mathrm{Z}_2(S))$ by Theorem 3.7. Since $[S_1 \colon \mathrm{Z}_2(S)] = p$ and $\mathrm{Z}_2(S) \leq \mathrm{Z}(S_1)$, we deduce that the group S_1 is abelian. Also, the quotient $S/\mathrm{Z}(S)$ is not abelian and every maximal subgroup of S distinct from S_1 has nilpotency class 2 by Theorem 3.6. Since S has sectional rank 3 this implies that the group S_1 is elementary abelian, $S_1 \cong \mathrm{C}_p \times \mathrm{C}_p \times \mathrm{C}_p$. By assumption $S_1 \neq \Omega_1(S)$, so there exists an element $x \in S$ of order p such that $S = \langle x \rangle S_1$. Since $x \notin S_1$, we have $[S_1, x] = S_2$ and $[S_2, x] = S_3$. Hence x acts on S_1 as the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus $S \cong S_1 \colon \langle x \rangle$ is isomorphic to a Sylow p-subgroup of the group $\mathrm{Sp}_4(p)$. $\square$

3.2 Essential subgroups of p-groups containing pearls

Let p be an odd prime and let $\mathcal{F}$ be a fusion system on a p-group S. We start showing that (quotients of) $\mathcal{F}$-essential subgroups having maximal nilpotency class are isomorphic to pearls.

Lemma 3.15. *Let $E \leq S$ be an $\mathcal{F}$-essential subgroup of S. Suppose there exists a subgroup K of E such that*

- *K is $\mathcal{F}$-characteristic in E;*
- *E/K has maximal nilpotency class; and*
- *$E < \mathrm{C}_{\mathrm{N}_S(E)}(\ K/(K \cap \Phi(E))\ )$.*

Then E/K is isomorphic to either $\mathrm{C}_p \times \mathrm{C}_p$ or p_+^{1+2}.

Remark 3.16. Note that $[E, K] \leq K \cap [E, E] \leq K \cap \Phi(E)$, so we always have $E \leq \mathrm{C}_{\mathrm{N}_S(E)}(K/(K \cap \Phi(E))$. The third condition of the previous lemma says that there exists an element $g \in \mathrm{N}_S(E)$ such that the conjugation map c_g is *not* an inner automorphism of E and acts trivially on $K/(K \cap \Phi(E))$. Note in particular that this is true when $K \cap \Phi(E) = K$ (that is $K \leq \Phi(E)$).

Proof. Set $\overline{E} = E/K$. Aiming for a contradiction, suppose $|\overline{E}| = p^m > p^3$. Let Z_i be the preimage in E of $\mathrm{Z}_i(\overline{E})$ for every $i \geq 1$ and let C be the preimage in E of $\mathrm{C}_{\overline{E}}(\mathrm{Z}_2(\overline{E}))$. Consider the following sequence of subgroups of E:

$$K \cap \Phi(E) \leq K < Z_1 < Z_2 < \cdots < Z_{m-2} < C < E.$$

All the subgroups in the sequence are $\mathcal{F}$-characteristic in E (because K is $\mathcal{F}$-characteristic in E) and since $\overline{E}$ has maximal nilpotency class every quotient of consecutive members of the sequence, except $K/(K \cap \Phi(E))$, has order p.

Let $g \in \mathrm{C}_{\mathrm{N}_S(E)}(\ K/(K \cap \Phi(E))\ )$ be such that $g \notin E$ (the existence of g is guaranteed by hypothesis). Then c_g acts trivially on every quotient of consecutive subgroups in the sequence. Hence $c_g \in \mathrm{Inn}(E)$ by Lemma 2.24, that is a contradiction.

Hence we have $|\overline{E}| \leq p^3$. If $|\overline{E}| = p$ then $\Phi(E) \leq K$ and by assumption the map c_g centralizes every quotient of consecutive subgroups in the sequence $\Phi(E) < K < E$, giving again a contradiction by Lemma 2.24. Thus $p^2 \leq |\overline{E}| \leq p^3$.

Since $\overline{E}$ has maximal nilpotency class, then either $\overline{E}$ is abelian of order p^2 or $\overline{E}$ is extraspecial of order p^3. Moreover $\overline{E}$ has exponent p, otherwise we can consider the sequence $K \cap \Phi(E) \leq K \leq K\Phi(E) < K\Omega_1(E) < E$ and we get a contradiction by Lemma 2.24. Thus either $\overline{E} \cong \mathrm{C}_p \times \mathrm{C}_p$ or $\overline{E} \cong p_+^{1+2}$. □

A direct consequence of Lemma 3.15 applied with $K = 1$ is the following

Corollary 3.17. *Let $E \leq S$ be an $\mathcal{F}$-essential subgroup of S. If E has maximal nilpotency class then E is a pearl.*

Note that if $E \leq S$ is a pearl then E has rank 2 and so by Theorem 1 we have

$$\mathrm{SL}_2(p) \leq \mathrm{Out}_{\mathcal{F}}(E) \leq \mathrm{GL}_2(p).$$

Lemma 3.18. *Let $E \leq S$ be a pearl. Then E has maximal normalizer tower, the members of such tower are the only subgroups of S containing E, and every morphism in $\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(E)}(\mathrm{Aut}_S(E))$ is the restriction of a morphism in $\mathrm{Aut}(S)$ that normalizes each member of the normalizer tower.*

Proof. It follows from Lemma 2.41 with $K = 1$ if $E \cong \mathrm{C}_p \times \mathrm{C}_p$ and $K = \mathrm{Z}(E)$ otherwise (note that in the second case K is the center of every member N^i of the normalizer tower of E in S, hence it is characteristic in every N^i). □

Recall that by Lemma 3.3 every p-group containing a pearl has maximal nilpotency class. From now on suppose that S has maximal nilpotency class and order $|S| = p^n > p^3$.

Lemma 3.19. *Suppose S has maximal nilpotency class and $|S| > p^3$ and let $E \leq S$ be an $\mathcal{F}$-essential subgroup of S. Then the following are equivalent:*

1. *E is a pearl;*

2. *E is contained in neither S_1 nor $\mathrm{C}_S(\mathrm{Z}_2(S))$;*

3. *there exists an element $x \in S \backslash \mathrm{C}_S(\mathrm{Z}_2(S))$ of order p such that*

$$\textit{either} \quad E = \langle x \rangle \mathrm{Z}(S) \quad \textit{or} \quad E = \langle x \rangle \mathrm{Z}_2(S);$$

Proof. Note that the group $\mathrm{Z}_2(S) \cong \mathrm{C}_p \times \mathrm{C}_p$ is self-centralizing in S if and only if $|S| = p^3$. Since we assume that $|S| > p^3$ we have $\mathrm{Z}_2(S) < \mathrm{C}_S(\mathrm{Z}_2(S))$ (recall that $[S \colon \mathrm{C}_S(\mathrm{Z}_2(S))] = p$). In particular $\mathrm{Z}_2(S)$ is not a pearl.

(1 $\Rightarrow$ 2) Suppose E is a pearl. Then either $E \cong \mathrm{C}_p \times \mathrm{C}_p$ and $E \neq \mathrm{Z}_2(S)$ or E is non-abelian of order p^3. Thus $E \not\leq \mathrm{C}_S(\mathrm{Z}_2(S))$. Note that $\Phi(E)$ is either 1 or equal to $\mathrm{Z}(S)$, hence we can consider the group $S/\Phi(E)$. Also $\mathrm{C}_S(E) \leq E$, since E is $\mathcal{F}$-centric, and $\Phi(E) < [\mathrm{N}_S(E), E]\Phi(E)$ by Lemma 2.35. Thus $E/\Phi(E)$ is a soft subgroup of $S/\Phi(E)$. Let $M \leq S$ be the preimage in S of the unique maximal subgroup of $S/\Phi(E)$ containing $E/\Phi(E)$. By Theorem 1.27 (2) the group $M/\Phi(E)$ has maximal nilpotency class. If $\Phi(E) = 1$ then M has maximal nilpotency class. If $\Phi(E) = \mathrm{Z}(S)$ then $\Phi(E) = \mathrm{Z}(M)$ and since $|\Phi(E)| = p$ we conclude that M has maximal nilpotency class by Lemma 3.4. Hence in both cases the group M has maximal nilpotency class. Thus by Theorem 3.6 we get $M \neq S_1$. Hence E is contained in neither S_1 nor $\mathrm{C}_S(\mathrm{Z}_2(S))$.

(2 $\Rightarrow$ 3) Suppose E is contained in neither S_1 nor $\mathrm{C}_S(\mathrm{Z}_2(S))$. By Lemma 3.10 and the fact that E is not cyclic by Lemma 2.25, we get that E has maximal nilpotency class and so E is a pearl by Corollary 3.17. Lemma 3.10 also tells us that if $|E| = p^m$ then $[E\colon Z_{m-1}] = p$. Thus there exists an element $x \in S$ such that either $E = \langle x \rangle \mathrm{Z}(S) \cong \mathrm{C}_p \times \mathrm{C}_p$ or $E = \langle x \rangle \mathrm{Z}_2(S) \cong p_+^{1+2}$. Note in particular that x has order p (since pearls have exponent p) and that $x \notin \mathrm{C}_S(\mathrm{Z}_2(S))$ because $E \nleq \mathrm{C}_S(\mathrm{Z}_2(S))$.

(3 $\Rightarrow$ 1) Suppose statement 3 holds. Since S has maximal nilpotency class we have $\mathrm{Z}(S) \cong \mathrm{C}_p$ and $\mathrm{Z}_2(S) \cong \mathrm{C}_p \times \mathrm{C}_p$. If x is an element of S having order p and $x \notin \mathrm{C}_S(\mathrm{Z}_2(S))$, then $\langle x \rangle \mathrm{Z}(S) \cong \mathrm{C}_p \times \mathrm{C}_p$ and $\langle x \rangle \mathrm{Z}_2(S) \cong p_+^{1+2}$. Thus E is a pearl.

□

We now want to investigate the nature of $\mathcal{F}$-essential subgroups of S that are not pearls.

Lemma 3.20. *Suppose S has maximal nilpotency class and $|S| > p^3$. Let $E \leq \mathrm{C}_S(\mathrm{Z}_2(S))$ be an $\mathcal{F}$-essential subgroup of S such that $E \nleq S_1$. Then one of the following holds:*

1. *$[E\colon \mathrm{Z}_2(S)] = p$ and $E \cong \mathrm{C}_p \times \mathrm{C}_p \times \mathrm{C}_p$;*
2. *$[E\colon \mathrm{Z}_3(S)] = p$ and $E/\mathrm{Z}(S) \cong p_+^{1+2}$;*
3. *$[E\colon \mathrm{Z}_4(S)] = p$ and $E/\mathrm{Z}_2(S) \cong p_+^{1+2}$ (and $\mathrm{Z}(S)$ is not $\mathcal{F}$-characteristic in E);*

In particular if $E = \mathrm{C}_S(\mathrm{Z}_2(S))$ then $|S| = p^6$ and E is as in case 3.

Proof. Note that $\mathrm{Z}_2(S) \leq \mathrm{Z}(E)$ and $\mathrm{Z}_2(S)$ is not $\mathcal{F}$-essential because $|S| > p^3$. So $\mathrm{Z}_2(S) < E$. In particular $|E| \geq p^3$. If $|E| = p^3$ then E is abelian ($\mathrm{Z}_2(S) \leq \mathrm{Z}(E)$ and $[E\colon \mathrm{Z}_2(S)] = p$) and so either $E \cong \mathrm{C}_p \times \mathrm{C}_p \times \mathrm{C}_p$ or $E \cong \mathrm{C}_{p^2} \times \mathrm{C}_p$. In the second case we can consider the sequence $\Phi(E) < \Omega_1(E) < E$ of $\mathcal{F}$-characteristic subgroups of E and we deduce that $\mathrm{Aut}_S(E) = \mathrm{Inn}(E)$ by Lemma 2.24, which is a contradiction. Thus the only possibility is $E \cong \mathrm{C}_p \times \mathrm{C}_p \times \mathrm{C}_p$.

Suppose $|E| = p^m > p^3$. Since $E \not\leq S_1$, by Lemma 3.10 the group $E/\mathrm{Z}(S)$ has maximal nilpotency class and $[E\colon \mathrm{Z}_{m-1}(S)] = p$. Also, since E does not centralize the quotient $\mathrm{Z}_3(S)/\mathrm{Z}(S)$, we conclude that $\mathrm{Z}_2(S) = \mathrm{Z}(E)$. In particular $\mathrm{Z}_2(S)$ is an $\mathcal{F}$-characteristic subgroup of E.

- Assume $\mathrm{Z}(S)$ is $\mathcal{F}$-characteristic in E. Note that $\mathrm{Z}(S)$ is centralized by $\mathrm{Aut}_S(E)$. Then by Lemma 3.15 the quotient $E/\mathrm{Z}(S)$ is isomorphic to either $\mathrm{C}_p \times \mathrm{C}_p$ or p_+^{1+2}. Since $|E| > p^3$ we have $E/\mathrm{Z}(S) \cong p_+^{1+2}$, $|E| = p^4$ and $[E\colon \mathrm{Z}_3(S)] = p$.

- Assume $\mathrm{Z}(S)$ is not $\mathcal{F}$-characteristic in E. Note that the group $E/\mathrm{Z}_2(S)$ has maximal nilpotency class (for example applying Lemma 3.10 to $S/\mathrm{Z}(S)$). Let $C = \mathrm{C}_S(\mathrm{Z}_2(S))$.

  If $E < C$ then $E < \mathrm{N}_C(E) \leq \mathrm{N}_S(E)$ and $\mathrm{N}_C(E)$ centralizes $\mathrm{Z}_2(S)$. Hence by Lemma 3.15 the group $E/\mathrm{Z}_2(S)$ is isomorphic to either $\mathrm{C}_p \times \mathrm{C}_p$ or p_+^{1+2}. If $E/\mathrm{Z}_2(S) \cong \mathrm{C}_p \times \mathrm{C}_p$ then $|E| = p^4$ and $[E\colon \mathrm{Z}_3(S)] = p$. If $E/\mathrm{Z}_2(S) \cong p_+^{1+2}$ then $|E| = p^5$ and $[E\colon \mathrm{Z}_4(S)] = p$.

  Suppose $E = C$. Then $\Phi(E) \cap \mathrm{Z}_2(S)$ is a characteristic subgroup of S and since $\mathrm{Z}_2(S) = \mathrm{Z}(E)$ we have $\Phi(E) \cap \mathrm{Z}_2(S) \neq 1$. If $\Phi(E) \cap \mathrm{Z}_2(S) < \mathrm{Z}_2(S)$, then $\Phi(E) \cap \mathrm{Z}_2(S) = \mathrm{Z}(S)$ (by Lemma 3.4(2)), contradicting the fact that $\mathrm{Z}(S)$ is not $\mathcal{F}$-characteristic in E. So $\Phi(E) \cap \mathrm{Z}_2(S) = \mathrm{Z}_2(S)$ and we conclude by Lemma 3.15.

Finally, since $\mathrm{C}_S(\mathrm{Z}_2(S)) \neq S_1$ we have $p^6 \leq |S| \leq p^{p+1}$ by Theorem 3.7. By what was proved above we have $p^3 \leq |E| \leq p^5$ (and $|E| = p^5$ only if $\mathrm{Z}(S)$ is not $\mathcal{F}$-characteristic in E). Thus if $E = \mathrm{C}_S(\mathrm{Z}_2(S))$ then, since $|S| = p|E|$, we conclude that $|E| = p^5$, $|S| = p^6$, $E/\mathrm{Z}_2(S) \cong p_+^{1+2}$ and $\mathrm{Z}(S)$ is not $\mathcal{F}$-characteristic in E.

□

Lemma 3.21. *Suppose S has maximal nilpotency class and the subgroup S_1 of S is extraspecial. Let E be an $\mathcal{F}$-essential subgroup of S. If $E \leq S_1$ then $E = S_1$.*

Proof. Let $m \in \mathbb{N}$ be such that $|S_1| = p^{2m+1}$. Notice that $[E, S_1] \leq \Phi(S_1) = \mathrm{Z}(S) \leq E$, so $E \trianglelefteq S_1$. Suppose for a contradiction that $E < S_1$. If $\Phi(E) \neq 1$ then $\Phi(E) = \Phi(S_1)$ and S_1/E centralizes $E/\Phi(E)$. Hence S_1/E is isomorphic to a subgroup of $O_p(\mathrm{Out}_{\mathcal{F}}(E))$ by Lemma 1.34, contradicting the fact that E is $\mathcal{F}$-essential (and so $\mathcal{F}$-radical). Therefore we have $\Phi(E) = 1$. Thus E is a maximal elementary abelian subgroup of S_1. Since S_1 is extraspecial, by Lemma 1.20 we deduce that $|E| \leq p^{m+1}$. Since $S_1 \leq \mathrm{N}_S(E)$ we have $[\mathrm{N}_S(E) \colon E] \geq p^m$. By Theorem 2.40 we also have $[\mathrm{N}_S(E) \colon E] \leq p^{\lfloor \frac{m+1}{2} \rfloor}$. Hence $m \leq 1$, $|S_1| = p^3$ and $|S| = p^4$. However by Theorem 3.7 this implies $S_1 = \mathrm{C}_S(\mathrm{Z}_2(S))$. So S_1 is abelian and we get a contradiction. □

We can now prove the first part of Theorem 3:

Theorem 3.22. *Let $\mathcal{F}$ be a saturated fusion system on a p-group S, that has maximal nilpotency class. Let E be an $\mathcal{F}$-essential subgroup of S. Then one of the following holds:*

1. *E is a pearl;*
2. *$E \leq S_1$ (and if S_1 is extraspecial or abelian then $E = S_1$);*
3. *$E \leq \mathrm{C}_S(\mathrm{Z}_2(S))$, $E \not\leq S_1$, $[E \colon \mathrm{Z}_i(S)] = p$ for some $i \in \{2,3,4\}$ and either $E \cong \mathrm{C}_p \times \mathrm{C}_p \times \mathrm{C}_p$ or $E/\mathrm{Z}_2(S)$ is isomorphic to either $\mathrm{C}_p \times \mathrm{C}_p$ or p_+^{1+2}.*

Proof. Note that if S_1 is abelian then none of its proper subgroups is $\mathcal{F}$-centric. The statement follows from Lemmas 3.19, 3.20 and 3.21. □

To prove the second part of Theorem 3 we first have to investigate fusion systems containing pearls.

If E is a pearl of $\mathcal{F}$ then the group $\mathrm{Out}_{\mathcal{F}}(E)$ has a subgroup isomorphic to $\mathrm{SL}_2(p)$ (by Theorem 1). More precisely, the quotient $E/\Phi(E)$ is a natural $\mathrm{SL}_2(p)$-module for $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(p)$. In particular we can consider a morphism $\varphi \in \mathrm{Aut}_{\mathcal{F}}(E)$ of order $p-1$ acting on $E/\Phi(E)$ as

$$\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$$

for some $1 \neq \lambda \in \mathrm{GF}(p)^*$ (and centralizing $\Phi(E)$). Note that $\varphi \in \mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(E)}(\mathrm{Aut}_S(E))$ so by Lemma 3.18 it is a restriction of a morphism in $\mathrm{Aut}_{\mathcal{F}}(S)$. In other words, when E is a pearl and p is odd there is a *non-trivial* automorphism $\varphi \in \mathrm{Aut}_{\mathcal{F}}(S)$ acting on $E/\Phi(E)$ as $\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$ and normalizing every member of the normalizer tower of E (that is maximal) and every member of the lower central series of S. For this reason the assumption of p being an *odd* prime becomes necessary.

Lemma 3.23. *Let $E \leq S$ be a pearl. Let $\varphi \in \mathrm{Aut}_{\mathcal{F}}(S)$ be a morphism of order $p-1$ that normalizes E, centralizes $\Phi(E)$ and acts as λ^{-1} on $E/(E \cap S_1)$ and as λ on $(E \cap S_1)/\Phi(E)$ for some $\lambda \in \mathrm{GF}(p)^*$. Then for every $1 \leq i \leq n-1$, and every $s_i \in S_i \backslash S_{i+1}$ we have*

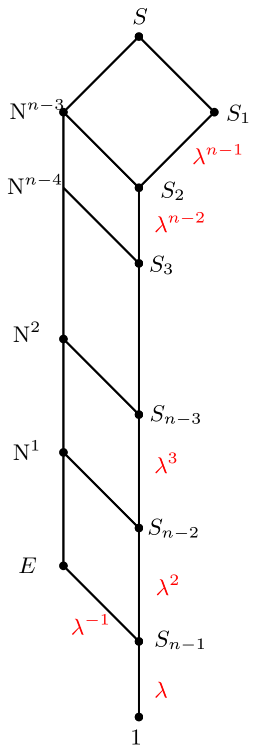
$$s_i\varphi = s_i^{a_i} \mod S_{i+1} \quad \textit{with} \quad a_i = \lambda^{n-i-\epsilon},$$

where $\epsilon = 0$ if E is abelian and $\epsilon = 1$ otherwise.

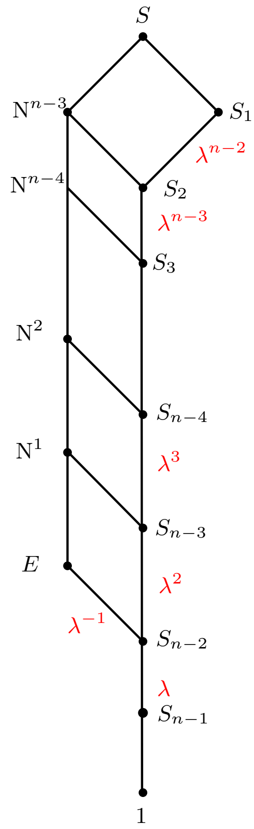
Remark 3.24. The fact that the morphism φ centrlalizes $\Phi(E)$ is a consequence of its action on $E/\Phi(E)$. Indeed, if $x, y \in E$ are such that $E = \langle x, y \rangle$ and $x\varphi = x^{\lambda^{-1}}$ and $y\varphi = y^{\lambda}$, then by Lemma 1.4 we get

$$[x, y]\varphi = [x\varphi, y\varphi] = [x^{\lambda^{-1}}, y^{\lambda}] = [x, y].$$

Also note that either $\Phi(E) = 1$ or $\Phi(E) = \mathrm{Z}(S)$. So in both cases $\Phi(E) \leq E \cap S_1$.



Action of φ for $E \cong \mathrm{C}_p \times \mathrm{C}_p$.



Action of φ for $E \cong p_+^{1+2}$.

Proof. By Lemma 3.19 there exists an element $x \in S$ of order p such that either $E = \langle x \rangle \mathrm{Z}(S)$ or $E = \langle x \rangle \mathrm{Z}_2(S)$ and x is contained neither in S_1 nor in $\mathrm{C}_S(\mathrm{Z}_2(S))$. Let s_1 be an element of S_1 not contained in S_2. Set $s_i = [x, s_{i-1}]$ for every $i \geq 2$. Then $s_i \in S_i \backslash S_{i+1}$ and $S_i = \langle s_i \rangle S_{i+1}$. Note that φ normalizes every quotient S_i/S_{i+1}. By assumption

$$s_{n-1}\varphi = s_{n-1}^{\lambda^{1-\epsilon}}.$$

We prove the statement by induction, showing that if it holds for s_{i+1} then it holds for s_i. Suppose

$$s_{i+1}\varphi = s_{i+1}^{\lambda^{n-(i+1)-\epsilon}} \mod S_{i+2}. \tag{3.2}$$

Let $a \in \mathrm{GF}(p)^*$ and $h \in S_{i+1}$ be such that

$$s_i\varphi = s_i^a h.$$

Note that $[x, h] \in S_{i+2}$ and S_i/S_{i+2} is elementary abelian. Therefore by Lemma 1.4 we get

$$[x, s_i]\varphi = [x, s_i]^{\lambda^{-1}a} = s_{i+1}^{\lambda^{-1}a} \mod S_{i+2}. \tag{3.3}$$

On the other hand, $[x, s_i] = s_{i+1}$, so comparing equations 3.2 and 3.3 we get

$$s_{i+1}^{\lambda^{n-(i+1)-\epsilon}} = s_{i+1}^{\lambda^{-1}a} \mod S_{i+2}.$$

Hence

$$s_i\varphi = s_i^a \mod S_{i+1} \quad \text{with} \quad a = \lambda^{n-i-\epsilon}.$$

□

In the next result we see that when $\mathcal{F}$ contains a pearl, the charcateristic subgroup S_1 of S is extraspecial if and only if it does not centralize $\mathrm{Z}_2(S)$, and we can determine the order of S and its exponent.

Theorem 3.25. *Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on a p-group S containing a pearl E. Then the following are equivalent:*

1. *S_1 is extraspecial;*

2. *$S_1 \neq \mathrm{C}_S(\mathrm{Z}_2(S))$;*

3. *$E \cong \mathrm{C}_p \times \mathrm{C}_p$, $|S| = p^{p-1}$ and S_1 is not abelian.*

Also, if one (and then each) of the cases above occurs, then $p \geq 7$, $S_1 \cong p_+^{1+(p-3)}$ and S has exponent p.

Proof. By Lemma 3.3 the group S has maximal nilpotency class.

(1 ⇒ 2) If S_1 is extraspecial then $\mathrm{Z}(S_1) = \mathrm{Z}(S)$ so $S_1 \neq \mathrm{C}_S(\mathrm{Z}_2(S))$.

(2 ⇒ 3) Since S_1 does not centralize $\mathrm{Z}_2(S)$, it cannot be abelian. Let $\varphi \in \mathrm{Aut}_{\mathcal{F}}(S)$ be the morphism described in Lemma 3.23. Then φ normalizes E and every member of the normalizer tower of E in S. Let M be the unique maximal subgroup of S containing E. Then $M \neq S_1$ and $M \neq \mathrm{C}_S(\mathrm{Z}_2(S))$ by Lemma 3.19. So the morphism φ normalizes the distinct groups S_1, $\mathrm{C}_S(\mathrm{Z}_2(S))$ and M and since $S/S_2 \cong \mathrm{C}_p \times \mathrm{C}_p$ we deduce that φ acts as a scalar on S/S_2. If $E \cong p_+^{1+2}$ then by Lemma 3.23 the morphism φ acts on S_1/S_2 as λ^{n-2}. Thus we have $n-2 = -1 \mod (p-1)$, that is $n = 1 \mod (p-1)$. In particular n is odd, as $p-1$ is even. Therefore by Theorem 3.7 we conclude $S_1 = \mathrm{C}_S(\mathrm{Z}_2(S))$, contradicting the assumptions. Hence we have $E \cong \mathrm{C}_p \times \mathrm{C}_p$. In this case, the morphism φ acts on S_1/S_2 as λ^{n-1}. So $n-1 = -1 \mod (p-1)$, that is $n = 0 \mod (p-1)$. So the group S has order $p^{\alpha(p-1)}$ for some $\alpha \in \mathbb{N}$. By Theorem 3.7 we also have $6 \leq \alpha(p-1) \leq p+1$. Thus $\alpha = 1$ and $|S| = p^{p-1}$.

(3 ⇒ 1) Since $|S| = p^{p-1}$, by Lemma 3.8 we have $S^p \leq \mathrm{Z}(S)$ and $S_2^p = 1$. Let $\varphi \in \mathrm{Aut}_{\mathcal{F}}(S)$ be the $\mathcal{F}$-automorphism of S normalizing E described in Lemma 3.23. We first show that for every $i \geq 1$ we can choose $s_i \in S_i \backslash S_{i+1}$ such that $s_i\varphi = s_i^{\lambda^{-i}}$.

For every $1 \leq i \leq n-2$ the morphism φ acts on the quotient $S_i/S_{i+2} \cong \mathrm{C}_p \times \mathrm{C}_p$, normalizing the maximal subgroup S_{i+1}/S_{i+2}. Hence by Theorem 1.15 there exists a subgroup $V_i \leq S_i$ containing S_{i+2} and distinct from S_{i+1}, such that V_i/S_{i+2} is normalized by φ. More precisely, φ acts on V_i/S_{i+2} as on S_i/S_{i+1} (so as $\lambda^{p-1-i} = \lambda^{-i}$). If $i \neq 1$, then $V_i/S_{i+3} \cong \mathrm{C}_p \times \mathrm{C}_p$ (because S_2 has exponent p) and we can find a subgroup $W_i \leq V_i$ distinct from S_{i+2} such that φ normalizes W_i/S_{i+3} and acts on it as λ^{-i}. Iterating this process we find a subgroup U of S_i having order p such that $S_i = US_{i+1}$ and φ acts on U as λ^{-i}. In other words, for every $i \geq 2$ we can find $s_i \in S_i \backslash S_{i+1}$ such that $s_i\varphi = s_i^{\lambda^{-i}}$.

If $i = 1$ then from $S_1^p \leq \mathrm{Z}(S)$ we get that we can repeat the same argument to find a subgroup U of S_1 of order p^2 not contained in S_2 and containing $\mathrm{Z}(S)$ such that φ normalizes U and acts on $U/\mathrm{Z}(S)$ as λ^{-1}. Recall that φ acts as λ on $\mathrm{Z}(S)$. Hence U cannot be cyclic and so it is elementary abelian. In particular $S_1 = US_2$ has exponent p (by Lemma 1.9) and we can find $s_1 \in S_1 \backslash S_2$ such that $s_1\varphi = s_1^{\lambda^{-1}}$.

We now want to prove that $[s_1, s_i] = 1$ for every $i < p-3$ (so for every i such that $s_i \notin S_{p-3} = \mathrm{Z}_2(S)$). Assume for a contradiction $[s_1, s_i] \neq 1$ for some $i < p-3$ and let $k \leq p-2$ be such that $[s_1, s_i] \in S_k \backslash S_{k+1}$. Since $[S_1, S_i] \leq S_{i+2}$ by Theorem 3.6(2), we also have $i + 2 \leq k$. By Lemma 1.4 we have

$$[s_1, s_i]\varphi = [s_1^{\lambda^{-1}}, s_i^{\lambda^{-i}}] = [s_1, s_i]^{\lambda^{-1-i}}.$$

On the other hand, we have $[s_1, s_i]\varphi = [s_1, s_i]^{\lambda^{-k}} \mod S_{k+1}$. Since $[s_1, s_i] \neq 1 \mod S_{k+1}$, we get $-k = -1 - i \mod p-1$. So $k = 1 + i \mod p-1$, contradicting the fact that $i + 2 \leq k \leq p-2$. Thus we deduce $[s_1, s_i] = 1$.

If $S_1 = \mathrm{C}_S(\mathrm{Z}_2(S))$ then $[s_1, s_{p-3}] = 1$ (note $S_{p-3} = \mathrm{Z}_2(S)$) and so $s_1 \in \mathrm{Z}(S_1)$. Since $\mathrm{Z}(S_1) = S_i$ for some i and $s_1 \notin S_2$ we get $S_1 = \mathrm{Z}(S_1)$, contradicting the assumption that S_1 is not abelian. Hence we have $S_1 \neq \mathrm{C}_S(\mathrm{Z}_2(S))$ and $[s_1, s_{p-3}] \neq 1$. In particular $s_{p-3} \notin \mathrm{Z}(S_1)$ and so $\mathrm{Z}(S_1) = \mathrm{Z}(S)$.

Consider the group $S/\mathrm{Z}(S)$, which has maximal nilpotency class by Lemma 3.4. Let Z be the preimage in S of $\mathrm{Z}(S_1/\mathrm{Z}(S))$. Then $Z \leq S_1$ and $Z \trianglelefteq S$, so $Z = S_i$ for some i. Also, $s_1 \in Z$ by what was proved above. Since $s_1 \notin S_2$ we conclude $Z = S_1$.

Therefore the group $S_1/\mathrm{Z}(S)$ is abelian. Hence $[S_1, S_1] = \mathrm{Z}(S) = \mathrm{Z}(S_1)$ and since S_1 has exponent p we get $[S_1, S_1] = \Phi(S_1)$. Thus S_1 is extraspecial.

Note that $S = S_1E$ and if S_1 is extraspecial then both E and S_1 have exponent p (as we saw in the proof of $(3 \Rightarrow 1)$). Since $|S| = p^{p-1}$, by Lemma 1.9 we conclude that S has exponent p. Also, by Lemma 3.7 we have $|S| \geq p^6$, so $p \geq 7$. $\square$

In order to prove the second part of Theorem 3, we need the next lemma.

Lemma 3.26. *Let $p \in \{5, 7\}$ and let S be a p-group of maximal nilpotency class and order p^6 such that $S_1 \cong p_+^{1+4}$ and $\mathrm{C}_S(\mathrm{Z}_2(S))$ has exponent p. Then S is isomorphic to a Sylow p-subgroup of the group $\mathrm{G}_2(p)$.*

Proof. We use *Magma* to establish the isomorphism type of S.

Suppose $p = 5$ and consider the code in Table 3.1. Then S is uniquely determined and isomorphic to the group SmallGroup(5^6, 643).

```
N:=[];
for i in [1..NumberOfSmallGroups(5^6)] do S:=SmallGroup(5^6,i);
  if NilpotencyClass(S) eq 5 and
  ♯[M : M in MaximalSubgroups(S) | IsExtraSpecial(M‘subgroup) eq true
    and Exponent(M‘subgroup) eq 5] ne 0 and
  Exponent(Centralizer(S, UpperCentralSeries(S)[3])) eq 5 then
  Append(~N,i);
  end if; end for; N;

Output:  [643]
```

Table 3.1

Suppose $p = 7$. Then $|S| < 7^7$ and $S = \mathrm{C}_S(\mathrm{Z}_2(S))S_1$ is generated by elements of order 7. Therefore by Lemma 1.9 we deduce that S has exponent 7. Consider the code in Table 3.2.

```
N:=[];
for i in [1..NumberOfSmallGroups(7^6)] do S:=SmallGroup(7^6,i);
  if NilpotencyClass(S) eq 5 and
  Exponent(S) eq 7 and
  ♯[M : M in MaximalSubgroups(S) |
      IsExtraSpecial(M`subgroup) eq true] ne 0 then
  Append(~N,i);
  end if; end for; N;

Output:  [807]
```

Table 3.2

This shows that S is uniquely determined and isomorphic to the group SmallGroup(7^6,807).

It is now easy to check (with the command `IsIsomorphic`) that in both cases S is isomorphic to a Sylow p-subgroup of $G_2(p)$. $\square$

Theorem 3.27. *Let $\mathcal{F}$ be a saturated fusion system on a p-group S, that has maximal nilpotency class. Suppose that $O_p(\mathcal{F}) = 1$, S_1 is extraspecial and that the group $\mathrm{C}_S(\mathrm{Z}_2(S))$ is $\mathcal{F}$-essential. Then $p \geq 5$ and S is isomorphic to a Sylow p-subgroup of the group $\mathrm{G}_2(p)$ (with $p = 7$ if there is a pearl) and $\mathcal{F}$ is one of the fusion systems determined by Parker and Semeraro in [PS16].*

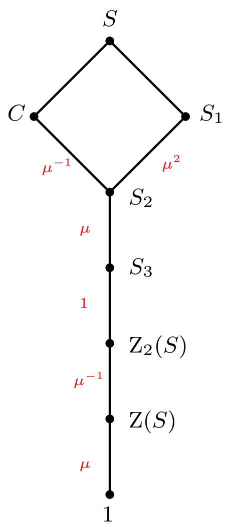
Proof. Note that since S_1 is extraspecial we have $\mathrm{Z}(S_1) = \mathrm{Z}(S)$ and so $\mathrm{C}_S(\mathrm{Z}_2(S)) \neq S_1$. Thus $p^6 \leq |S| \leq p^{p+1}$ by Lemma 3.7 and so $p \geq 5$. By Lemma 3.20 we have $|S| = p^6$ and $\mathrm{C}_S(\mathrm{Z}_2(S))/\mathrm{Z}_2(S) \cong p_+^{1+2}$. Also the group S_2 and the quotient $S/\mathrm{Z}(S)$ have exponent p by Lemma 3.8.

Suppose S contains a pearl. By Theorem 3.25 we get that $|S| = p^{p-1}$ and S has exponent p. Since $|S| = p^6$ we deduce that $p = 7$. Therefore by Lemma 3.26 the group S is isomorphic to a Sylow 7-subgroup of the group $\mathrm{G}_2(7)$.

Suppose that none of the $\mathcal{F}$-essential subgroups of S is a pearl. By Lemma 2.28 the assumption $O_p(\mathcal{F}) = 1$ implies that there exists an $\mathcal{F}$-essential subgroup E of S such that either $\mathrm{Z}_2(S) \nleq E$ or $\mathrm{Z}_2(S)$ is not $\mathcal{F}$-characteristic in E. By Theorem 3.22 and the fact that S_1 is extraspecial, either $E = S_1$ or $E \leq \mathrm{C}_S(\mathrm{Z}_2(S))$ and $[E\colon \mathrm{Z}_2(S)] = p$. In the second case E is abelian of rank at most 3 and by Lemma 2.41 it is not contained in any other $\mathcal{F}$-essential subgroup of S, contradicting the fact that $\mathrm{C}_S(\mathrm{Z}_2(S))$ is $\mathcal{F}$-essential. Therefore $\mathrm{C}_S(\mathrm{Z}_2(S))$ and S_1 are the only $\mathcal{F}$-essential subgroups of S.

Set $C = \mathrm{C}_S(\mathrm{Z}_2(S))$. Since $S/\mathrm{Z}(S)$ has exponent p, we have $C^p \leq \mathrm{Z}(S) = \mathrm{Z}(S_1)$. The assumption $O_p(\mathcal{F}) = 1$ then implies $C^p = 1$. Also, since $C/\mathrm{Z}_2(S)$ is not elementary abelian and $\Phi(C) = S_j$ for some $j \geq 2$ by Theorem 3.4, the only possibility is $\Phi(C) = S_3$. Thus C has rank 2. In particular by Theorem 1 we deduce

$$\mathrm{SL}_2(p) \leq \mathrm{Out}_{\mathcal{F}}(C) \leq \mathrm{GL}_2(p).$$



Let $\psi \in \mathrm{Out}_{\mathcal{F}}(S)$ be the $\mathcal{F}$-automorphism that normalizes C and acts on $C/\Phi(C)$ as $\begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix}$, for some $\mu \in \mathrm{GF}(p)$ of order $p-1$. Then ψ acts on S_1, which is characteristic in S. Note that $[C, S_i] = S_{i-1}$, for every $1 \leq i \leq 3$ and $[S_1, \mathrm{Z}_2(S)] = \mathrm{Z}(S)$. Hence by Lemma 1.4 we deduce that ψ acts as μ^2 on S_1/S_2, centralizes $S_3/\mathrm{Z}_2(S)$ and acts as $\begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix}$ on $\mathrm{Z}_2(S)$.

Since $S_1^p \leq \mathrm{Z}(S)$ and $\mu^2 \neq \mu \mod p$, we deduce that $s_1^p = 1$. Since S_2 is elementary abelian, $S_1 = \langle s_1 \rangle S_2$ and $|S_1| = p^5 \leq p^p$, by Lemma 1.9 we deduce that S_1 has exponent p. So $S_1 \cong p_+^{1+4}$.

If $p = 5$ or $p = 7$ then by Lemma 3.26 we conclude that S is isomorphic to a Sylow p-subgroup of the group $\mathrm{G}_2(p)$.

Suppose $p \geq 11$. Let G_1 and G_2 be models for $\mathrm{N}_{\mathcal{F}}(C)$ and $\mathrm{N}_{\mathcal{F}}(S_1)$, respectively (whose existence is guaranteed by Theorem 2.30), and set $A_i = \langle S^{G_i} \rangle$. Let G_{12} be a model for $\mathrm{N}_{\mathcal{F}}(S)$. Then by Lemma 2.32 the triple of groups $\mathcal{A} = \mathcal{A}(G_1, G_2, G_{12})$ is an amalgam of rank 2. We want to prove that $\mathcal{A} = \mathcal{A}(G_1, G_2, G_{12}, A_1, A_2)$ is a weak BN-pair of rank 2.

Let $P \leq C \cap S_1$ be such that P is normal in both G_1 and G_2. Then P is normal in S and since S has maximal nilpotency class we have $P = S_i$, for some $i \geq 2$. Thus P is characteristic in S. Since S_1 and C are the only $\mathcal{F}$-essential subgroups of S, by Lemma 2.28 we deduce that $P \trianglelefteq \mathcal{F}$. Hence $P = 1$ by assumption. Thus no non-trivial subgroup of $S_1 \cap C$ is normal in both G_1 and G_2.

We first show that $\mathcal{A}$ is a symplectic amalgam.

We have $\langle (C \cap S_1)^{A_1} \rangle = \langle S_2^{A_1} \rangle = C$ and following the axioms of Definition 1.43 we get

1. If $H \leq G_{12}$ is normal in both G_1 and G_2 then $H \cap S = 1$ and so $[H, S] \leq H \cap S = 1$. Thus H acts trivially on S, which implies $H = 1$.

2. $S \in \mathrm{Syl}_p(G_1) \cap \mathrm{Syl}_p(G_2)$;

3. $\mathrm{C}_{G_1}(C) \leq C$ and $\mathrm{C}_{G_2}(S_1) \leq S_1$, by Theorem 2.30;

4. $A_1/C \cong \mathrm{SL}_2(p)$;

5. $G_{12} \cong \mathrm{N}_{G_i}(S)$ by Lemma 2.31;

6. $G_2 = \mathrm{N}_{G_2}(S)A_2 = \mathrm{N}_{G_2}(S)\mathrm{N}_{A_2}(S)\langle (S)^{A_2} \rangle = \mathrm{N}_{G_2}(S)\langle C^{A_2} \rangle$ and $A_2/\langle C^{A_2} \rangle \cong S_1/(\langle C^{A_2} \rangle \cap S_1)$ is a p-group, so $O^p(A_2) \leq \langle C^{A_2} \rangle$;

7. $\mathrm{Z}(S) = \mathrm{Z}(S_1)$ is normalized by G_2 and so it is centralized by $A_2 = \langle S^{G_2} \rangle$;

8. $\mathrm{Z}_2(S) \leq S_1$. Suppose for a contradiction that for every $g \in G_2$ we have $\mathrm{Z}_2(S) \leq C^g$. This is equivalent to $\mathrm{Z}_2(S)^g \leq C$ for every $g \in G_2$. In particular the group $X = \langle \mathrm{Z}_2(S)^{G_2} \rangle$ is a subgroup of $C \cap S_1 = S_2$ properly containing $\mathrm{Z}_2(S)$ and normalized by G_2. Note that X is normal in S, so by Lemma 3.4 the group X is characteristic in S. Since $G_2/S_1 \cong \mathrm{Out}_{\mathcal{F}}(S_1)$ we deduce that X is $\mathcal{F}$-characteristic in S_1. However, $\mathrm{Z}(S)$ is the only non-trivial proper $\mathcal{F}$-characteristic subgroup of S_1 and we have a contradiction. So there exists $g \in G_2$ such that $\mathrm{Z}_2(S) \not\leq C^g$.

Therefore $\mathcal{A}$ is a symplectic amalgam. Also, the only subgroups of C that are normal in G_1 are $\mathrm{Z}_2(S)$, S_3 and C. Since $[S_3 \colon \mathrm{Z}_2(S)] = p$, we deduce that $\mathrm{Z}_2(S)$ and C are the only non-central chief factors of G_1 in C. Hence, recalling that $p \geq 11$, by Theorem 1.44 we get that $A_2 \cong p_+^{1+4}.\mathrm{SL}_2(p)$. Hence $A_2/S_1 \cong A_1/C \cong \mathrm{SL}_2(p)$. Thus $\mathcal{A}(G, H, G \cap H, A_G, A_H)$ is a weak BN-pair of rank 2 by Theorem 2.34 and so S is isomorphic to a Sylow p-subgroup of one of the groups listed in by Theorem 1.42. Recalling that $|S| = p^6$ and the maximal subgroup S_1 of S is extraspecial, we conclude that S is isomorphic to a Sylow p-subgroup of the group $\mathrm{G}_2(p)$. □

We now prove Theorem 5, which characterizes the $\mathcal{F}$-essential subgroups of a p-group S that contains a pearl depending on the nature of the group S_1.

Proof of Theorem 5. The group S has maximal nilpotency class by Lemma 3.3. When S_1 is neither abelian nor extraspecial, then the order of $\mathrm{Aut}_{\mathcal{F}}(S)$ is at most $p^{n-1}(p-1)$ as a consequence of Theorem 3.11 and the fact that $S/\mathrm{Z}(S) \cong \mathrm{Inn}(S) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(S))$. By assumption there exists an $\mathcal{F}$-essential subgroup E of S that is a pearl. In particular $p-1$ divides the order of $\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(E)}(\mathrm{Aut}_S(E))$. Hence $p-1$ divides the order of $\mathrm{Aut}_{\mathcal{F}}(S)$ by Lemma 3.18 and we conclude that $|\mathrm{Aut}_{\mathcal{F}}(S)| = p^{n-1}(p-1)$.

The remaining statements follow by Theorems 3.22, 3.25 and 3.27 and Lemma 3.26. □

3.3 Fusion systems on p-groups of sectional rank 3 containing pearls

Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on a p-group S. Suppose that $\mathcal{F}$ contains a pearl and S has sectional rank 3. By Lemma 3.13 and Theorem 3.12 we get that either $p = 3$ and $|S| = 3^4$ or $p \geq 5$ and $|S| \leq p^6$, with strict inequality if $S_1 = \mathrm{C}_S(\mathrm{Z}_2(S))$. Thus we start studying p-groups containing pearls and having order at most p^6.

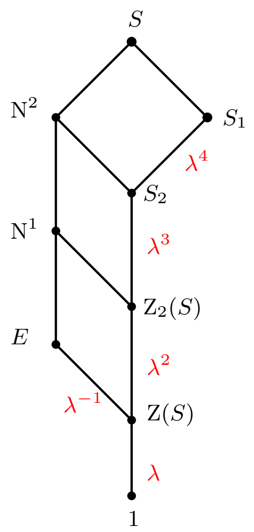
Theorem 3.28. *Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on a p-group S containing a pearl. Suppose that S has order p^4 and sectional rank greater than 2. Then S is isomorphic to a Sylow p-subgroup of the group $\mathrm{Sp}_4(p)$.*

Proof. Let $E \leq S$ be a pearl. Since E has exponent p and is not contained in S_1 by Lemma 3.19, we deduce that $S_1 \neq \Omega_1(S)$. Hence the statement follows from Lemma 3.14. □

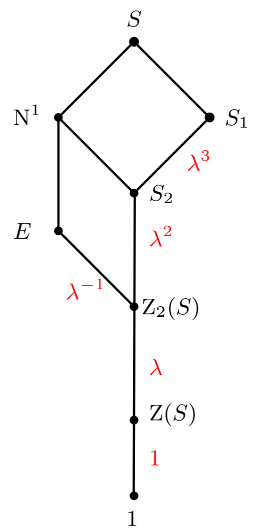
Theorem 3.29. *Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on a p-group S containing a pearl E. Suppose that S has order p^5 and sectional rank greater than 2. Then $p \geq 5$, S has exponent p, $S_1 = \mathrm{C}_S(\mathrm{Z}_2(S))$ and one of the following holds:*

1. *S_1 is elementary abelian and S has sectional rank 4;*
2. *$p = 7$, $E \cong \mathrm{C}_7 \times \mathrm{C}_7$ and S is isomorphic to the group indexed in* Magma *as SmallGroup($7^5, 37$), which has sectional rank 3.*

Proof. The p-group S has maximal nilpotency class by Lemma 3.3 and the fact that S_1 centralizes $\mathrm{Z}_2(S)$ follows from Theorem 3.7. Let $\varphi \in \mathrm{Aut}_{\mathcal{F}}(S)$ be the $\mathcal{F}$-automorphism of S normalizing E described in Lemma 3.23.



Action of φ for $E \cong \mathrm{C}_p \times \mathrm{C}_p$



Action of φ for $E \cong p_+^{1+2}$

Since S has sectional rank at least 3, by Lemma 3.13 we have $p \geq 5$. In particular by Lemma 3.8 the group S_2 and the quotient $S/\mathrm{Z}(S)$ have exponent p. Let $x \in S_1 \backslash S_2$. Then $x\varphi = x^a z$, with $z \in \mathrm{Z}(S)$ and $a = \lambda^4$ if $E \cong \mathrm{C}_p \times \mathrm{C}_p$ and $a = \lambda^3$ otherwise. Thus by Lemma 1.3 we have

$$(x^p)\varphi = (x\varphi)^p = (x^a)^p z^p [x^a, z]^{p\frac{p-1}{2}} = (x^p)^a.$$

However, $x^p \in \mathrm{Z}(S)$, so if $x^p \neq 1$ then

- either $E \cong \mathrm{C}_p \times \mathrm{C}_p$ and $\lambda^4 = \lambda \mod p$;
- or $E \cong p_+^{1+2}$ and $\lambda^3 = 1 \mod p$.

Since neither of the cases above can occur, we get $x^p = 1$. Thus $S = ES_2\langle x \rangle$ is generated by elements of order p and so it has exponent p by Lemma 1.9.

Let $y \in S_2 \backslash \mathrm{Z}_2(S)$. Since S_2 is elementary abelian we have $y\varphi = y^b$ for $b = \lambda^3$ if $E \cong \mathrm{C}_p \times \mathrm{C}_p$ and $b = \lambda^2$ otherwise. Note that $[x, y] \in \mathrm{Z}(S)$ by definition of S_1. Hence we can apply Lemma 1.4 and we get

$$[x, y]\varphi = [x^a z, y^b] = [x, y]^{ab}.$$

If $[x, y] \neq 1$ then:

- either $E \cong \mathrm{C}_p \times \mathrm{C}_p$ and $\lambda^7 = \lambda \mod p$;
- or $E \cong p_+^{1+2}$ and $\lambda^5 = 1 \mod p$.

Hence either $[x, y] = 1$ and S_1 is elementary abelian or $p = 7$ and $E \cong \mathrm{C}_7 \times \mathrm{C}_7$. If S_1 is elementary abelian then, since $|S_1| = p^4$ and S is not abelian of order p^5, we deduce that S has sectional rank 4. In the second case, S is a 7-group having order 7^5, nilpotency class 4, exponent 7, and such that the group $S_1 = \mathrm{C}_S(\mathrm{Z}_2(S))$ is non abelian. We can use this information to prove, using *Magma*, that S is uniquely determined (up to isomorphism) and isomorphic to the group SmallGroup$(7^5, 37)$.

```
N:=[];
for i in [1..NumberOfSmallGroups(7^5)] do S:=SmallGroup(7^5,i);
if NilpotencyClass(S) eq 4 and
  Exponent(S) eq 7 and
  IsAbelian(Centralizer(S,UpperCentralSeries(S)[3])) eq false then
  Append(~N,i);
end if; end for; N;


Ouput:  [37]
```

□

Theorem 3.30. *Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on a p-group S containing a pearl E. Suppose that S has order p^6 and sectional rank greater than 2. Then*

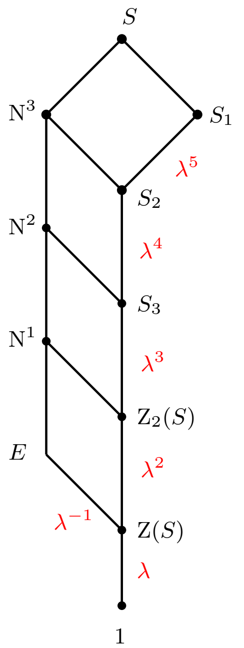
1. *either S_1 is extraspecial, $p = 7$, $E \cong \mathrm{C}_7 \times \mathrm{C}_7$ and S isomorphic to a Sylow 7-subgroup of $\mathrm{G}_2(7)$;*
2. *or S_2 is elementary abelian of order p^4 and one of the following holds:*
   - *(a) S_1 is abelian;*
   - *(b) $p = 5$, $E \cong \mathrm{C}_5 \times \mathrm{C}_5$, $S_3 = \mathrm{Z}(S_1)$ and $S \cong$ SmallGroup($5^6, 636$),*
   - *(c) $p = 5$, $E \cong \mathrm{C}_5 \times \mathrm{C}_5$, $S_3 = \mathrm{Z}(S_1)$, $S \cong$ SmallGroup($5^6, i$), for $i \in \{639, 640, 641, 642\}$ and if $P \in \mathcal{P}(\mathcal{F})$ then $P \in E^{\mathcal{F}}$ (also S has exponent 25 and S_1 has exponent 5 if and only if $i = 639$);*
   - *(d) $p = 7$, $E \cong 7_+^{1+2}$, $\mathrm{Z}_2(S) = \mathrm{Z}(S_1)$, S_1 has exponent 7 and $S \cong$ SmallGroup($7^6, 813$);*
   - *(e) $p = 7$, $E \cong 7_+^{1+2}$, $\mathrm{Z}_3(S) = \mathrm{Z}(S_1)$, S_1 has exponent 7 and $S \cong$ SmallGroup($7^6, 798$).*

In particular S has sectional rank at least 4.

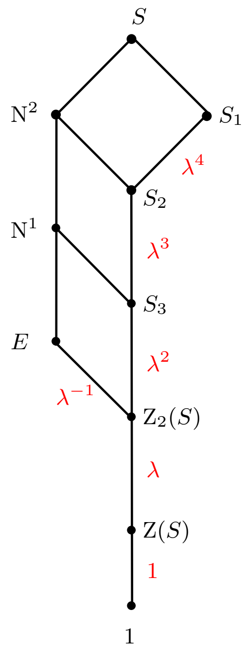
Proof. The p-group S has maximal nilpotency class by Lemma 3.3 and since the sectional rank of S is at least 3, by Lemma 3.13 we conclude that $p \geq 5$.

If S_1 is extraspecial, then Theorem 3.25 implies that $E \cong \mathrm{C}_p \times \mathrm{C}_p$ and that S has order p^{p-1} and exponent p. So $p = 7$ and by Lemma 3.26 the group S is isomorphic to a Sylow 7-subgroup of $G_2(7)$.

Suppose S_1 is not extraspecial. In particular $S_1 = \mathrm{C}_S(\mathrm{Z}_2(S))$ by Theorem 3.25. Let $\varphi \in \mathrm{Aut}_{\mathcal{F}}(S)$ be the automorphism of S normalizing E described in Lemma 3.23.



Action of φ for $E \cong \mathrm{C}_p \times \mathrm{C}_p$



Action of φ for $E \cong p_+^{1+2}$

Note that the group S_3 is abelian. Since $|S| = p^6$ and $p \geq 5$, the group S_2 has exponent p by Lemma 3.8. Thus S_3 is elementary abelian.

Let M be the maximal subgroup of S containing E. Then M has sectional rank at least 3 (since $S_3 \leq S_2 \leq M$) and order p^5. Also $\varphi|_M \in \mathrm{Aut}_{\mathcal{F}}(M)$. This is enough to allow us to apply Theorem 3.29 to M. Let M_i denote the i-th term of the lower central series of M and let M_1 be the centralizer in M of $M_2/M_4 = S_3/\mathrm{Z}(S)$. Then $M_1 = S_2$ and either S_2 is elementary abelian or $p = 7$ and $E \cong \mathrm{C}_7 \times \mathrm{C}_7$. In the latter case, since $|S| = 7^6$, by Theorem 3.25 we conclude that S_1 is extraspecial, contradicting the assumption. Therefore if S_1 is not extraspecial then the group S_2 is elementary abelian of order p^4.

Let $u \in S_2$ be such that $s_1\varphi = s_1^b u$, where $b = \lambda^5$ if $E \cong \mathrm{C}_p \times \mathrm{C}_p$ and $b = \lambda^4$ otherwise. Then by Lemma 1.4 we get

$$[s_1, s_3]\varphi = [s_1^b u, s_3^a] = [s_1, s_3]^{ab}.$$

Since $[s_1, s_3] \in \mathrm{Z}(S)$ by the definition of S_1, if $[s_1, s_3] \neq 1$ then

- either $E \cong \mathrm{C}_p \times \mathrm{C}_p$ and $\lambda^8 = \lambda \mod p$;
- or $E \cong p_+^{1+2}$ and $\lambda^6 = 1 \mod p$.

Thus either $S_3 \leq \mathrm{Z}(S_1)$ or $p = 7$ and $E \cong 7_+^{1+2}$ (and in the latter we are in case 2(d)). Suppose $S_3 \leq \mathrm{Z}(S_1)$.

We now study $[s_1, s_2]$. Set $c = \lambda^4$ if $E \cong \mathrm{C}_p \times \mathrm{C}_p$ and $c = \lambda^3$ otherwise. Then

$$[s_1, s_2]\varphi = [s_1^b u, s_2^c] = [s_1, s_2]^{bc}.$$

Thus either $[s_1, s_2] = 1$ or, since $[s_1, s_2] \in \mathrm{Z}_2(S)$ one of the following holds:

- $E \cong \mathrm{C}_p \times \mathrm{C}_p$ and λ^9 modulo p is equal to either λ^2 or λ;
- $E \cong p_+^{1+2}$ and λ^7 modulo p is equal to either λ or 1.

Therefore either S_1 is abelian (and case 2(a) holds), or $S_3 = \mathrm{Z}(S_1)$ and either $E \cong \mathrm{C}_5 \times \mathrm{C}_5$ (and we are in cases 2(b) and 2(c)) or $E \cong 7_+^{1+2}$ (and case 2(e) holds).

If $p = 5$ and $E \cong \mathrm{C}_5 \times \mathrm{C}_5$ then $\lambda^5 = \lambda \mod 5$ and so the group S_1 can have exponent 25. If $p = 7$ and $E \cong 7_+^{1+2}$ then $\lambda^4 \neq 1 \mod 7$ and so the group S_1 has exponent 7.

Note that we proved that either S_2 is elementary abelian (in case 2) or $S_1/\mathrm{Z}(S)$ is elementary abelian (in case 1). Since $|S_2| = [S_1 \colon \mathrm{Z}(S)] = p^4$, we deduce that S has sectional rank at least 4.

We now use *Magma* to determine the isomorphism type of S when $p = 5$, $E \cong \mathrm{C}_5 \times \mathrm{C}_5$ and $S_3 = \mathrm{Z}(S_1)$ or $p = 7$, $E \cong 7_+^{1+2}$, $\mathrm{Z}_2(S) \leq \mathrm{Z}(S_1) \leq S_3$ and S_1 has exponent 7.

If $p = 5$ then $S \cong \mathrm{SmallGroup}(5^6, i)$ for $i \in \{636, 639, 640, 641, 642\}$ (see Table 3.3).

```
N:=[];
for i in [1..NumberOfSmallGroups(5^6)] do S:=SmallGroup(5^6,i);
L:=LowerCentralSeries(S);
if NilpotencyClass(S) eq 5 and
  IsAbelian(L[2]) eq true and
  Exponent(L[2]) eq 5 and
  Center(Centralizer(S,L[4])) eq L[3] and
  ♯[E : E in Subgroups(S)| ♯E`subgroup eq 25 and Exponent(E`subgroup) eq 5
    and Centralizer(S,E`subgroup) eq E`subgroup] ne 0 then
  Append(~N,i);
end if; end for; N;


Output:
[636, 639, 640, 641, 642]
```

Table 3.3: Isomorphism type of S for $p = 5$

We can also check that if $i \neq 636$ then there is a unique S-conjugacy class of groups isomorphic to $\mathrm{C}_5 \times \mathrm{C}_5$ that are self-centralizing in S (i.e. `♯H eq 1`). Thus if $P \in \mathcal{P}(\mathcal{F})$ then $P \cong \mathrm{C}_5 \times \mathrm{C}_5$ by what we proved above and so $P \in E^{\mathcal{F}}$.

Moreover the group $S_1 = \mathrm{C}_S(\mathrm{Z}_2(S))$ has exponent 5 if and only if $i = 639$.

If $p = 7$ then either $S \cong \mathrm{SmallGroup}(7^6, 798)$, and in this case we have $\mathrm{Z}(S_1) = S_3$, or $S \cong \mathrm{SmallGroup}(7^6, 813)$ and $\mathrm{Z}(S_1) = \mathrm{Z}_2(S)$ (see Table 3.4).

```
N:=[];
for i in [1..NumberOfSmallGroups(7^6)] do S:=SmallGroup(7^6,i);
L:=LowerCentralSeries(S);
if NilpotencyClass(S) eq 5 and
  IsAbelian(L[2]) eq true and
  IsAbelian(Centralizer(S,L[4])) eq false and
  Exponent(Centralizer(S,L[4])) eq 7 and
  ♯[E : E in Subgroups(S)| IsExtraSpecial(E‘subgroup) eq true
    and Exponent(E‘subgroup) eq 7 and ♯E‘subgroup eq 343
    and Centralizer(S,E‘subgroup) subset E‘subgroup] ne 0
  then Append(~N,i);
end if; end for; N;


Output:
[798, 813]
```

Table 3.4: Isomorphism type of S for $p = 7$

$\square$

Proof of Theorem 6. Suppose that S contains a pearl, has sectional rank greater than 2 and $|S| \leq p^6$. If $|S| = p^4$ then S is isomorphic to a Sylow p-subgroup of the group $\mathrm{Sp}_4(p)$ by Theorem 3.28. If $p^5 \leq |S| \leq p^6$ then the statement follows by Theorem 3.29 and Theorem 3.30. $\square$

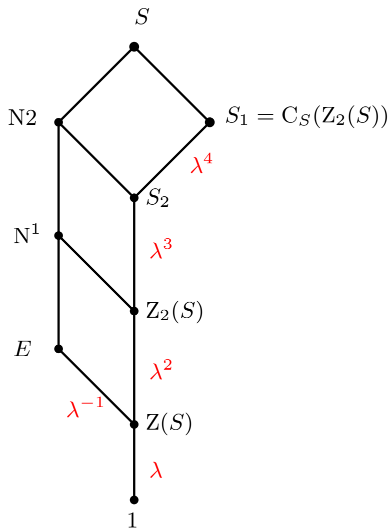
We end this chapter with the classification of the saturated fusion systems on p-groups of sectional rank 3 containing a pearl (Theorem 7).

Theorem 3.31. *Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on a p-group S. Suppose that S has sectional rank 3 and there exists a subgroup E of S that is a pearl. Then either S is isomorphic to a Sylow p-subgroup of $\mathrm{Sp}_4(p)$ or the following hold:*

1. *$p = 7$, $E \cong \mathrm{C}_7 \times \mathrm{C}_7$ and S is isomorphic to the group indexed in* Magma *as SmallGroup($7^5, 37$) (in particular S has order 7^5, exponent 7 and none of its maximal subgroups is abelian);*
2. *$\mathrm{Aut}_{\mathcal{F}}(E) \cong \mathrm{SL}_2(7)$;*
3. *$\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Inn}(S)\langle\varphi\rangle$, where φ is a morphism of order 6 normalizing E;*
4. *$\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(E)\rangle_S$;*
5. *$\mathcal{F}$ is simple;*
6. *$\mathcal{F}$ is exotic.*

Proof. By Lemma 3.3 the group S has maximal nilpotency class and by Lemma 3.13 and Theorem 3.12 we have that either $p = 3$ and $|S| = 3^4$ or $p \geq 5$ and $|S| \leq p^6$. Hence by Theorems 3.28, 3.29 and 3.30 either S is isomorphic to a Sylow p-subgroup of $\mathrm{Sp}_4(p)$ or S is isomorphic to the group indexed in *Magma* as SmallGroup($7^5, 37$) and $E \cong \mathrm{C}_7 \times \mathrm{C}_7$.

Suppose we are in the second case. Note that $|\mathrm{Aut}(S)| = 7^7 \cdot 6$ (as expected because of Theorem 3.11). Since E has rank 2 and $O_7(\mathrm{Aut}_{\mathcal{F}}(E)) = 1$, by Lemma 2.26 we get $\mathrm{SL}_2(7) \leq \mathrm{Aut}_{\mathcal{F}}(E) \leq \mathrm{GL}_2(7)$. By Lemma 3.18 every morphism in $\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(E)}(\mathrm{Aut}_S(E))$ is the restriction of a morphism in $\mathrm{Aut}_{\mathcal{F}}(S)$. Since 6^2 does not divide the order of $\mathrm{Aut}(S)$, we get $\mathrm{Aut}_{\mathcal{F}}(E) \cong \mathrm{SL}_2(7)$ and $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Inn}(S)\langle\varphi\rangle$ where φ is a morphism of order 6 such that $\varphi|_E \in \mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(E)}(\mathrm{Aut}_S(E))$. Thus we can choose φ as the morphism that acts on E as $\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$ for some $\lambda \in \mathrm{GF}(7)$ of order 6. In particular the action of φ is described in Lemma 3.23.



Action of φ on S

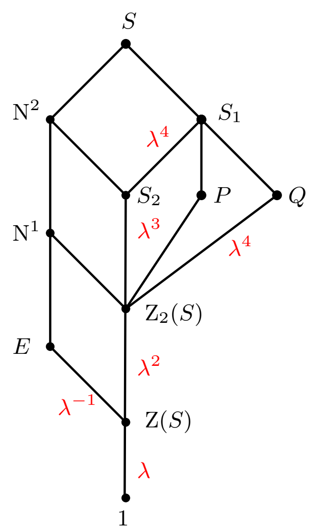
By the Alperin-Goldschmidt Fusion Theorem, to show that $\mathcal{F}$ is completely determined by $\mathrm{Aut}_{\mathcal{F}}(S)$ and $\mathrm{Aut}_{\mathcal{F}}(E)$, we have to prove that E is the unique $\mathcal{F}$-essential subgroup of S up to $\mathcal{F}$-conjugation. Suppose there exists an $\mathcal{F}$-essential subgroup P of S not $\mathcal{F}$-conjugate to E. Note that S has sectional rank 3, so P has rank at most 3 and $[\mathrm{N}_S(P) \colon P] = 7$ by Theorem 2.40. By Theorem 3.22 either $P \leq S_1$ or P is a pearl.

- Suppose $P \leq S_1$. If $P \trianglelefteq S$ then $|P| = 7^4$ and so $P = S_1$. Note that $\mathrm{Z}_2(S) = \mathrm{Z}(P)$ and $\mathrm{Z}(S) = \Phi(P)$. Thus $\mathrm{Out}_{\mathcal{F}}(P)$ acts reducibly on $P/\Phi(P)$ and by Theorem 1 we get

$$\mathrm{SL}_2(7) \leq \mathrm{Out}_{\mathcal{F}}(P) \leq \mathrm{GL}_2(7) \times \mathrm{GL}_1(7).$$

  Note that the morphism φ acts on $P/\mathrm{Z}_2(S)$ as $\begin{pmatrix} \lambda^4 & 0 \\ 0 & \lambda^3 \end{pmatrix}$. Since $\lambda^7 = \lambda \neq 1 \mod 7$, we deduce that $\mathrm{GL}_2(7) \leq \mathrm{Out}_{\mathcal{F}}(P)$. In particular $|\mathrm{N}_{\mathrm{Out}_{\mathcal{F}}(P)}(\mathrm{Out}_S(P))| = 7 \cdot 6^2$. Since P is receptive and $S = \mathrm{N}_S(P)$, every morphism in $\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_S(P))$ is the restriction of a morphism in $\mathrm{Aut}_{\mathcal{F}}(S)$. Therefore 6^2 divides the order of $\mathrm{Aut}_{\mathcal{F}}(S)$, which is a contradiction. Hence P is not normal in S.

  Since $P \leq \mathrm{C}_S(\mathrm{Z}_2(S))$ and P is $\mathcal{F}$-centric we have $\mathrm{Z}_2(S) < P < S_1$. Thus the only option is $P \cong \mathrm{C}_7 \times \mathrm{C}_7 \times \mathrm{C}_7$ (recall that S has exponent 7) and $P \neq S_2$. Note that φ normalizes the quotient $S_1/\mathrm{Z}_2(S) \cong \mathrm{C}_7 \times \mathrm{C}_7$ and its maximal subgroup $S_2/\mathrm{Z}_2(S)$. Hence by Theorem 1.15 there exists a maximal subgroup Q of S_1 such that $Q/\mathrm{Z}_2(S)$ is normalized by φ. Since S_2 is the unique normal subgroup of S of order 7^3 and the group $S_1/\mathrm{Z}_2(S)$ has 8 proper non-trivial subgroups, the group S acts transitively on the maximal subgroups of S_1 containing $\mathrm{Z}_2(S)$ that are distinct from S_2. Thus $Q \in P^{\mathcal{F}}$ and by Lemma 2.26 the group Q is $\mathcal{F}$-essential.



By Theorem 1 we have that either $O^{7'}(\mathrm{Out}_{\mathcal{F}}(Q)) \cong \mathrm{SL}_2(7)$ or $O^{7'}(\mathrm{Out}_{\mathcal{F}}(Q)) \cong \mathrm{PSL}_2(7)$. Since $|\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(Q)}(\mathrm{Aut}_S(Q))| \leq 6$ we conclude that $\mathrm{Out}_{\mathcal{F}}(Q) = O^{7'}(\mathrm{Out}_{\mathcal{F}}(Q))$. Note that the morphism φ acts on Q as the matrix

$$D = \begin{pmatrix} \lambda^4 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Since none of the sections of Q is centralized by φ we deduce that $\mathrm{Out}_{\mathcal{F}}(Q) \cong \mathrm{PSL}_2(7)$. However the determinant of the matrix D is $\lambda^7 = \lambda \mod 7$, that is not congruent to 1 modulo 7, and we reach a contradiction.

- Suppose P is a pearl. Then by Theorem 3.31 we have $P \cong \mathrm{C}_7 \times \mathrm{C}_7$. Also, since P is not $\mathcal{F}$-conjugate to E, by Theorem 1.27 the maximal subgroup of S containing E is distinct from the maximal subgroup of S containing P, i.e. $\mathrm{N}^2(E) \neq \mathrm{N}^2(P)$. Let $\psi \in \mathrm{Aut}_{\mathcal{F}}(S)$ be an $\mathcal{F}$-automorphism of S of order 6 normalizing P (whose existence is guaranteed by Lemma 3.18). Hence $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Inn}(S)\langle\psi\rangle$. So there exists $x \in S$ and $1 \leq i \leq 5$ such that $\varphi = c_x \cdot \psi^i$. In particular φ normalizes $\mathrm{N}^2(E)$, $\mathrm{N}^2(P)$ and S_1, so it acts as scalar on S/S_2. Thus $\lambda^4 = \lambda^{-1} \mod 7$, which is a contradiction.

Therefore E is the unique $\mathcal{F}$-essential subgroup of S up to $\mathcal{F}$-conjugation and $\mathcal{F}$ is completely determined by $\mathrm{Aut}_{\mathcal{F}}(S)$ and $\mathrm{Aut}_{\mathcal{F}}(E)$. Furthermore, $\mathrm{Aut}_{\mathcal{F}}(S)$ is determined by the inner automorphisms of S and by the lifts of elements in $\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(E)}(\mathrm{Aut}_S(E))$.

We now prove that $\mathcal{F}$ is simple. Assume for a contradiction that there exists a proper normal fusion subsystem $\mathcal{E} \subseteq \mathcal{F}$ on some subgroup $P \leq S$. Since $E^{\mathcal{F}}$ is the unique class of $\mathcal{F}$-essential subgroups of S, if $P = S$ then either $\mathcal{E} = \mathcal{F}$ (and we get a contradiction) or $\mathcal{E} = \langle \mathrm{Aut}_{\mathcal{E}}(S)\rangle_S = \mathrm{N}_{\mathcal{E}}(S)$. Let $\psi \in \mathrm{Aut}_{\mathcal{F}}(E)$ be an $\mathcal{F}$-automorphism of E such that $\mathrm{Z}(S)\psi \neq \mathrm{Z}(S)$. Then we cannot write ψ as the composition of a morphism $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ and a morphism $\beta \in \mathrm{Hom}_{\mathcal{E}}(E, S)$. Hence $\mathcal{E}$ does not satisfy the Frattini condition and is therefore not weakly normal. Thus we have $P < S$. By definition P must be strongly $\mathcal{F}$-closed. So it has to be normal in S. Note that every normal subgroup of S contains the center $\mathrm{Z}(S)$ and

$$\langle \mathrm{Z}(S)\alpha \mid \alpha \in \mathrm{Hom}_{\mathcal{F}}(\mathrm{Z}(S), S)\rangle = E.$$

Therefore P is a normal subgroup of S containing E. So $P = \mathrm{N}^2$, that is the unique maximal subgroup of S containing E.

The Frattini condition in the definition of normal fusion subsystem (definition 2.13) implies that $\mathrm{Aut}_{\mathcal{E}}(E) = \mathrm{Aut}_{\mathcal{F}}(E) \cong \mathrm{SL}_2(7)$ and so E is an $\mathcal{E}$-essential subgroup of N^2.

Let $h \in S \backslash \mathrm{N}^2$. Then $c_h \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^2) \backslash \mathrm{Aut}_{\mathcal{E}}(\mathrm{N}^2)$ (since $\mathcal{E}$ is defined on N^2 and so $\mathrm{Inn}(\mathrm{N}^2) \in \mathrm{Syl}_7(\mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^2))$). We have

$$(\mathrm{Aut}_{\mathcal{E}}(E))^{c_h} \leq \mathrm{Aut}_{\mathcal{E}}(E^h).$$

Also, E^h is $\mathcal{E}$-centric and fully normalized, therefore $\mathcal{E}$-essential. Hence $E^h \in \mathcal{P}(\mathcal{E})$ and $E^h \notin E^{\mathcal{E}}$. In particular by Theorem 1.27 we have $\mathrm{N}^1(E) \neq \mathrm{N}^1(E^h)$ and $\mathrm{N}^1(E^h) \neq S_2$ (because $S_2^h = S_2$). Note that $|\mathrm{Aut}_{\mathcal{E}}(\mathrm{N}^2)| = 7^3 \cdot 6$. Let $\varphi \in \mathrm{Aut}_{\mathcal{F}}(S)$ be the morphism of order 6 described above. Since $\mathcal{E}$ is a saturated fusion system on N^2, with the same argument used to show that E is the unique $\mathcal{F}$-essential subgroup of S (up to $\mathcal{F}$-conjugation) we can prove that the morphism φ normalizes $\mathrm{N}^1(E)$, $\mathrm{N}^1(E^h)$ and S_2. However φ does not act as scalar on $\mathrm{N}^2/\mathrm{Z}_2(S)$ and we get a contradiction.

Therefore $\mathcal{F}$ is a simple fusion system.

Finally, we show that $\mathcal{F}$ is exotic using the Classification of Finite Simple Groups.. Suppose for a contradiction that $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G such that $S \in \mathrm{Syl}_7(G)$. Since $\mathcal{F}$ is simple, we may assume that G is simple ([Cra11, Theorem 5.71]). Thus we are looking for a finite simple group having a non-abelian Sylow 7-subgroup of order 7^5.

The only sporadic group whose order is divisible by 7^5 is the Monster group, that has Sylow 7-subgroups of order 7^6 (indeed our 7-group S is isomorphic to a maximal subgroup of such groups). Thus G is not sporadic.

Suppose G is alternating. Since 7^5 is the larger power of 7 dividing the order of G, we have $G = \mathrm{Alt}(n)$ for $35 \leq n \leq 41$. Consider the following subgroup of G:

$$P = \langle (1\ 2\ 3\ 4\ 5\ 6\ 7),\ (8\ 9\ 10\ 11\ 12\ 13\ 14),\ (15\ 16\ 17\ 18\ 19\ 20\ 21),$$
$$(22\ 23\ 24\ 25\ 26\ 27\ 28),\ (29\ 30\ 31\ 32\ 33\ 34\ 35)\rangle.$$

Then $P \cong \mathrm{C}_7 \times \mathrm{C}_7 \times \mathrm{C}_7 \times \mathrm{C}_7 \times \mathrm{C}_7$ and P is a Sylow 7-subgroup of G. Since P is abelian it is not isomorphic to S. Hence S is not a Sylow 7-subgroup of G and we get a contradiction. Thus G is not an alternating group.

Suppose G is of Lie type in characteristic $r \neq 7$. Note that S has 7-rank 3. Therefore by [GLS98, Theorem 4.10.3(c)] the group S haves to have a unique elementary abelian subgroup of rank 3. However, every proper subgroup of S_1 containing $\mathrm{Z}_2(S)$ is elementary abelian, since S_1 has exponent 7, and we get a contradiction.

Therefore G is a group of Lie Type in characteristic 7 having a Sylow 7-subgroup of order 7^5, and this is a contradiction by [GLS98, Theorem 2.2.9 and Table 2.2]. $\square$

It remains to show that the fusion system described in the previous theorem exists. To do this, we need the next result.

Lemma 3.32. *[BLO06, Proposition 5.1] Let G be a finite group, let S be a Sylow p-subgroup of G and let $E_1, \dots E_m$ be subgroups of S such that no E_i is G-conjugate to a subgroup of E_j for $i \neq j$. For each i, set $K_i = \mathrm{Out}_G(E_i)$, and fix subgroups $\Delta_i \leq \mathrm{Out}(E_i)$ which contain K_i. Set $\mathcal{F} = \langle \mathrm{Mor}(\mathcal{F}_S(G)), \Delta_1, \dots, \Delta_m \rangle_S$. Assume for each i that*

1. *$|\Delta_i \colon K_i|_p = 1$;*
2. *E_i is p-centric in G but no proper subgroup $P < E_i$ is $\mathcal{F}$-centric or an essential p-subgroup of G; and*
3. *for all $\alpha \in \Delta_i \backslash K_i$ we have $|K_i \cap \alpha^{-1} K_i \alpha|_p = 1$.*

Then $\mathcal{F}$ is a saturated fusion system on S.

Lemma 3.33 (Existence of $\mathcal{F}$)**.** *Let S be the 7-group indexed in* Magma *as SmallGroup($7^5, 37$). Then there exists a saturated fusion system $\mathcal{F}$ on S and a subgroup $E \leq S$ isomorphic to $\mathrm{C}_7 \times \mathrm{C}_7$ such that E is $\mathcal{F}$-essential.*

Proof. Using *Magma* we prove that there exists a subgroup E of S isomorphic to $\mathrm{C}_7 \times \mathrm{C}_7$ that is self-centralizing in S and an automorphism φ of S that acts on E as $\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$.

Input:	Output:
S:=SmallGroup(7^5, 37);	

S;	GrpPC : S of order 16807 = 7^5
	PC-Relations:
	  S.2^S.1 = S.2 * S.3,
	  S.3^S.1 = S.3 * S.4,
	  S.3^S.2 = S.3 * S.5,
	  S.4^S.1 = S.4 * S.5

E:=sub<S | S.1, S.5 >;	

E;	GrpPC : E of order 49 = 7^2
	PC-Relations:
	  E.1^7 = Id(E),
	  E.2^7 = Id(E)

Centralizer(S,E) eq E;	true
A:=AutomorphismGroup(S);	

A.1;	Automorphism of GrpPC : S which maps:
	  S.1 |-> S.1^3
	  S.2 |-> S.2^2
	  S.3 |-> S.3^6 * S.4^6 * S.5^5
	  S.4 |-> S.4^4 * S.5
	  S.5 |-> S.5^5

Order(A.1);	6

Table 3.5

Set $G = S\colon \langle\varphi\rangle$ and let Δ be a subgroup of $\mathrm{Out}(E) \cong \mathrm{GL}_2(7)$ containing $\mathrm{Out}_G(E)$ and isomorphic to $\mathrm{SL}_2(7)$. Note that $|\mathrm{Out}_G(E)| = 7 \cdot 6$ and $\mathrm{Out}_G(E) = \mathrm{N}_\Delta(\mathrm{Out}_S(E))$. Then by Lemma 3.32 the fusion system $\mathcal{F} = \langle \mathrm{Mor}(\mathcal{F}_S(G)), \Delta\rangle_S$ is saturated and E is $\mathcal{F}$-essential (because $\mathrm{Out}_G(E)$ is a strongly 7-embedded subgroup of $\mathrm{Out}_{\mathcal{F}}(E) = \Delta \cong \mathrm{SL}_2(7)$). $\square$

CHAPTER 4

CHARACTERIZATION OF THE $\mathcal{F}$-ESSENTIAL SUBGROUPS.

'It is only with the heart that one can see rightly;

what is essential is invisible to the eyes.'

[Antoine de Saint Exupéry]

Let p be an *odd* prime, let S be a p-group having sectional rank 3 and let $\mathcal{F}$ be a saturated fusion system on S.

In this chapter we investigate the structure of the $\mathcal{F}$-essential subgroups of S and we determine their $\mathcal{F}$-automorphism groups.

Since S has sectional rank 3, every $\mathcal{F}$-essential subgroup E of S has rank 2 or 3 (recall that an $\mathcal{F}$-essential subgroup cannot be cyclic). Therefore Theorem 1 tells us that the group $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E))$ is isomorphic to either $\mathrm{SL}_2(p)$, or $\mathrm{PSL}_2(p)$ or $13\colon 3$ (and $p = 3$ in the last case). In Section 4.1 we improve this result, proving the following theorem.

Theorem 8. *Let $E \leq S$ be an $\mathcal{F}$-essential subgroup.*

- *If E is $\mathcal{F}$-characteristic in S and there exists an $\mathcal{F}$-essential subgroup P of S, $P \neq E$, such that every morphism in $\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_S(P))$ is a restriction of an $\mathcal{F}$-automorphism of S, then*
  - *either $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(p)$*
  - *or $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{PSL}_2(p)$, $O^{p'}(\mathrm{Out}_{\mathcal{F}}(P)) \cong \mathrm{SL}_2(p)$ and S has rank 2.*
- *If E is not $\mathcal{F}$-characteristic in S then $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(p)$ and*
  - *either $[E\colon \Phi(E)] = p^2$ and $\mathrm{SL}_2(p) \leq \mathrm{Out}_{\mathcal{F}}(E) \leq \mathrm{GL}_2(p)$;*
  - *or $[E\colon \Phi(E)] = p^3$ and $\mathrm{SL}_2(p) \leq \mathrm{Out}_{\mathcal{F}}(E) \leq \mathrm{GL}_2(p) \times \mathrm{GL}_1(p)$.*

Note that the assumptions on P are always satisfied when P is $\mathcal{F}$-characteristic in S, by Lemma 2.8.

If $E \leq S$ is $\mathcal{F}$-essential and E is $\mathcal{F}$-characteristic in $\mathrm{N}_S(E)$, then $E \trianglelefteq \mathrm{N}_S(\mathrm{N}_S(E))$. So $\mathrm{N}_S(E) = S$ and E is $\mathcal{F}$-characteristic in S. Thus for every $\mathcal{F}$-essential subgroup E that is not $\mathcal{F}$-characteristic in S there exists an $\mathcal{F}$-automorphism α of $\mathrm{N}_S(E)$ such that $E \neq E\alpha$. Note that $E\alpha$ is an $\mathcal{F}$-essential subgroup of S by Theorem 2.26(5).

To prove Theorem 8 we study the interplay of two distinct $\mathcal{F}$-essential subgroups E_1 and E_2 satisfying $\mathrm{N}_S(E_1) = \mathrm{N}_S(E_2)$, where either $E_2 = E_1\alpha$ for some $\alpha \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}_S(E_1))$ or both E_1 and E_2 are $\mathcal{F}$-characteristic in S.

Note that if T and U are subgroups of $E_1 \cap E_2$ that are $\mathcal{F}$-characteristic in E_1, E_2 and $\mathrm{N}_S(E_1)$ then $UT \leq E_1 \cap E_2$ is $\mathcal{F}$-characteristic in E_1, E_2 and $\mathrm{N}_S(E_1)$. Thus we can talk about the *largest* subgroup of $E_1 \cap E_2$ that is $\mathcal{F}$-characteristic in E_1, E_2 and $\mathrm{N}_S(E_1)$. This subgroup plays a key role in most of the results proved in this chapter.

Definition 4.1. Let $E_1 \leq S$ and $E_2 \leq S$ be $\mathcal{F}$-essential subgroups of S such that $\mathrm{N}_S(E_1) = \mathrm{N}_S(E_2)$. We define the *$\mathcal{F}$-core of E_1 and E_2*, denoted $\mathrm{core}_{\mathcal{F}}(E_1, E_2)$, as the largest subgroup of $E_1 \cap E_2$ that is $\mathcal{F}$-characteristic in E_1, E_2 and $\mathrm{N}_S(E_1)$.

We set $\mathrm{core}_{\mathcal{F}}(E_1) = \mathrm{core}_{\mathcal{F}}(E_1, E_1)$ and we call it the *$\mathcal{F}$-core of E_1*.

Note that in general the $\mathcal{F}$-core of an $\mathcal{F}$-essential subgroup E is not normal in the fusion system $\mathcal{F}$ (but it is normal in the fusion subsystems $\mathrm{N}_{\mathcal{F}}(E)$ and $\mathrm{N}_{\mathcal{F}}(\mathrm{N}_S(E))$).

Section 4.2 aims to study the properties of the $\mathcal{F}$-core of two distinct $\mathcal{F}$-essential subgroups of S.

Note that an $\mathcal{F}$-essential subgroup $E \leq S$ is $\mathcal{F}$-characteristic in S if and only if $\mathrm{core}_{\mathcal{F}}(E) = E$. In Lemma 4.11 we prove that if E is not $\mathcal{F}$-characteristic in S then $\mathrm{core}_{\mathcal{F}}(E) = \mathrm{core}_{\mathcal{F}}(E, E\alpha)$, for every $\alpha \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}_S(E))$. In particular the $\mathcal{F}$-core of an $\mathcal{F}$-essential subgroup E not $\mathcal{F}$-characteristic in S can always be seen as the $\mathcal{F}$-core of two distinct $\mathcal{F}$-essential subgroups of S.

Theorem 9. *Let E_1 and E_2 be distinct $\mathcal{F}$-essential subgroups of S such that $\mathrm{N}_S(E_1) = \mathrm{N}_S(E_2)$. Set $N = \mathrm{N}_S(E_1)$, $E_{12} = E_1 \cap E_2$ and $T = \mathrm{core}_{\mathcal{F}}(E_1, E_2)$. Then for every $i \in \{1, 2\}$ we have*

1. $\mathrm{C}_{E_i}(T) \nleq T$*;*
2. $\mathrm{C}_N(T) \nleq E_{12}$*;*
3. *if* $\mathrm{C}_N(T) \nleq E_i$ *then* $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_i))$ *centralizes* T*;*
4. *either* $T \leq \Phi(E_i)$ *or* $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_i)) \cong \mathrm{SL}_2(p)$, $[T\Phi(E_i) \colon \Phi(E_i)] = p$ *and*

$$T\Phi(E_i)/\Phi(E_i) = \mathrm{C}_{E_i/\Phi(E_i)}(O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_i))).$$

We also prove that whenever E is not $\mathcal{F}$-characteristic in S and $T = \mathrm{core}_{\mathcal{F}}(E)$, we have $\mathrm{C}_{\mathrm{N}^1}(T) \nleq E$ and so $N = E\mathrm{C}_N(T)$ and $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E))$ centralizes T (Lemma 4.14).

We close this section by proving that under certain assumptions on the $\mathcal{F}$-essential subgroups E_1 and E_2 of S, always satisfied when E_1 is not $\mathcal{F}$-characteristic in S and $\mathrm{core}_{\mathcal{F}}(E_1, E_2) = \mathrm{core}_{\mathcal{F}}(E_1)$, the $\mathcal{F}$-core T of E_1 and E_2 is abelian and there exists a subgroup $T_1 \leq T$ of order p such that T/T_1 is a cyclic group.

Theorem 10. *Let E_1 and E_2 be distinct $\mathcal{F}$-essential subgroups of S such that $\mathrm{N}_S(E_1) = \mathrm{N}_S(E_2)$. Set $T = \mathrm{core}_{\mathcal{F}}(E_1, E_2)$. Suppose that $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_1))$ is isomorphic to $\mathrm{SL}_2(p)$ and centralizes T, and there exists a subgroup $V \leq E_1$ such that V is $\mathcal{F}$-characteristic in E_1, V/T is a natural $\mathrm{SL}_2(p)$-module for $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_1))$ and $V \leq \mathrm{C}_{E_1}(T)T$. Then*

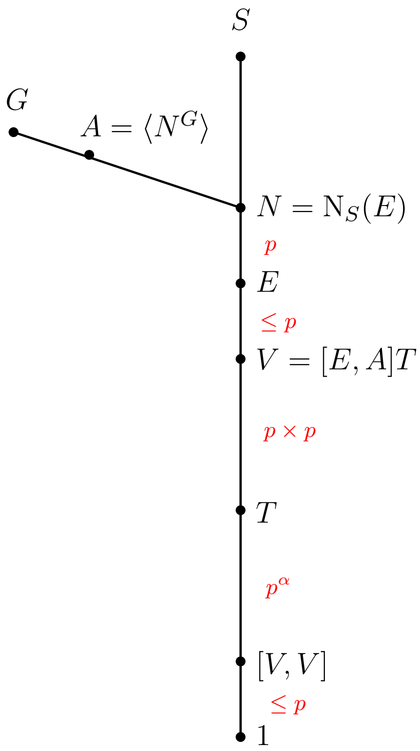
$$T \text{ is abelian, } T \leq \mathrm{Z}(V), |[V,V]| \leq p \text{ and the group } T/[V,V] \text{ is cyclic.}$$

In Section 4.3 we focus on $\mathcal{F}$-essential subgroups that are not $\mathcal{F}$-characteristic in S. We first apply Stellmacher's Theorem (Theorem 1.26) and the results of Section 4.2 to describe the shape of such $\mathcal{F}$-essential subgroups.

Theorem 11. *Let E be an $\mathcal{F}$-essential subgroup of S not $\mathcal{F}$-characteristic in S and let $T = \mathrm{core}_{\mathcal{F}}(E)$. Set $V = [E, O^{p'}(\mathrm{Out}_{\mathcal{F}}(E))]T$. Then*

1. *V/T is a natural $\mathrm{SL}_2(p)$-module for $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E))$;*
2. *$\mathrm{N}_S(E)/T$ has exponent p;*
3. *E/T is elementary abelian and $p^2 \leq [E\colon T] \leq p^3$;*
4. *$[E/T\colon \mathrm{Z}(\mathrm{N}_S(E)/T)] = p$;*
5. *T is abelian, $T \leq \mathrm{Z}(V)$, $|[V,V]| \leq p$ and $T/[V,V]$ is a cyclic group.*

Moreover, if $[E\colon T] = p^2$, then $T \leq \mathrm{Z}(\mathrm{N}_S(E))$.



Structure of an $\mathcal{F}$-essential subgroup E of S not $\mathcal{F}$-characteristic in S,
where G is a model for $\mathrm{N}_{\mathcal{F}}(E)$.

The characterization of $\mathcal{F}$-essential subgroups given by Theorem 4.17 allows us to prove that $\mathcal{F}$-essential subgroups of S of rank 2 (and not $\mathcal{F}$-characteristic in S) are pearls.

Theorem 12. *Let $E \leq S$ be an $\mathcal{F}$-essential subgroup of S not $\mathcal{F}$-characteristic in S. If E has rank 2 then E is a pearl and $\mathcal{F}$ is one of the fusion systems described in Theorem 7.*

For the rest of this section we focus on $\mathcal{F}$-essential subgroups of rank 3.

By Theorem 11 we know the shape of E. Also, we get that the $\mathcal{F}$-core T of E has index either p^2 or p^3 in E. We study these two cases separately, aiming to find a bound for the index of E in S.

Theorem 13. *Suppose that $E \leq S$ is an $\mathcal{F}$-essential subgroup of S and let $T = \mathrm{core}_{\mathcal{F}}(E)$. If $[E\colon T] = p^2$ and $|T| = p^a$ then*

- *either* $E \cong \mathrm{C}_p \times \mathrm{C}_p \times \mathrm{C}_{p^a}$*;*
- *or* $E \cong \frac{\Omega_1(E)\times T}{(\mathrm{Z}(\Omega_1(E))=\Omega_1(T))} \cong p_+^{1+2} \circ \mathrm{C}_{p^a}$*;*
- *or* $E \cong p_+^{1+2} \times \mathrm{C}_{p^{a-1}}$*.*

Lemma 4.2. *Suppose that $E \leq S$ is an $\mathcal{F}$-essential subgroup of S not $\mathcal{F}$-characteristic in S and let $T = \mathrm{core}_{\mathcal{F}}(E)$. If E has rank 3 and $[E\colon T] = p^2$ then either $E \trianglelefteq S$ or $p = 3$ and if we set $\mathrm{N}^1 = \mathrm{N}_S(E)$ and $\mathrm{N}^2 = \mathrm{N}_S(\mathrm{N}^1)$, then T is $\mathcal{F}$-characteristic in N^2, $T \leq \mathrm{Z}(\mathrm{N}^2)$, $\Phi(E) \trianglelefteq \mathrm{N}^2$, the quotient $\mathrm{N}^2/\Phi(E)$ has exponent 9 and one of the following holds*

1. *either $[S\colon E] = 3^2$, N^1 $\mathrm{char}_{\mathcal{F}}$ S and $S/\Phi(E) \cong$ SmallGroup($3^5, 52$);*
2. *or $\mathrm{N}^2/\Phi(E) \cong$ SmallGroup($3^5, 53$).*

Note that the group SmallGroup($3^5, 53$) is isomorphic to a section of a Sylow 3-subgroup of the group $\mathrm{SL}_4(19)$ and of a Sylow 3-subgroup of the group $\mathrm{SL}_4(109)$. This fact suggests to look at the Sylow 3-subgroups of the group $\mathrm{SL}_4(q)$ where q is an odd prime power such that $q \equiv 1 \mod 3$.

Lemma 4.3. *Let q be an odd prime power such that $q \equiv 1 \mod 3$ and let S be a Sylow 3 of the group $G = \mathrm{SL}_4(q)$. Let $k \geq 1$ is such that 3^k is the largest power of 3 dividing $q-1$. Then there exist $\mathcal{F}_S(G)$-essential subgroups A and E of S such that*

- *$A \cong \mathrm{C}_{3^k} \times \mathrm{C}_{3^k} \times \mathrm{C}_{3^k}$ is the unique abelian subgroup of S having index 3 in S, and is therefore characteristic in S;*
- *$E \cong \frac{\mathrm{C}_{3^k}\times 3_+^{1+2}}{(\Omega_1(\mathrm{C}_{3^k})=\mathrm{Z}(3_+^{1+2}))} \cong \mathrm{C}_{3^k} \circ 3_+^{1+2}$ is such that $\mathrm{core}_{\mathcal{F}_{\mathrm{S}}(\mathrm{G})}(E) = \mathrm{Z}(E) = \mathrm{Z}(S) \cong \mathrm{C}_{3^k}$ has index 3^2 in E.*

Also, every $\mathcal{F}_S(G)$-essential subgroup of S is of this form.

As a consequence of the previous theorem we conclude that there is no hope to bound the index of an $\mathcal{F}$-essential subgroup E in S when the group $\mathrm{N}_S(\mathrm{N}_S(E))$ is isomorphic to the group $\mathrm{SmallGroup}(3^5, 53)$.

As for $\mathcal{F}$-essential subgroups having $\mathcal{F}$-core of index p^3, we prove the following result.

Lemma 4.4. *Suppose that $E \leq S$ is an $\mathcal{F}$-essential subgroup of S not $\mathcal{F}$-characteristic in S and let $T = \mathrm{core}_{\mathcal{F}}(E)$. If $[E\colon T] = p^3$ then either $E \trianglelefteq S$ or $p = 3$ and if we set $\mathrm{N}^1 = \mathrm{N}_S(E)$ and $\mathrm{N}^2 = \mathrm{N}_S(\mathrm{N}^1)$ then T is $\mathcal{F}$-characteristic in N^2 and one of the following holds:*

1. *$[S\colon E] = 3^2, \mathrm{N}^1$ $\mathrm{char}_{\mathcal{F}}$ S, $\mathrm{C}_{S/T}(\Phi(S/T))$ has exponent 3 and*

$$S/T \quad \cong \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ y & a & 1 & 0 \\ z & b & a & 1 \end{pmatrix} \mid a, b, x, y, z \in \mathbb{F}_3 \right\} \cong \mathrm{SmallGroup}(3^5, 56);$$

2. *$[S\colon E] = 3^2$, N^1 $\mathrm{char}_{\mathcal{F}}$ S, $\mathrm{C}_{S/T}(\Phi(S/T))$ has exponent 9 and $S/T \cong \mathrm{SmallGroup}(3^5, 57)$;*
3. *$\mathrm{C}_{\mathrm{N}^2/T}(\Phi(\mathrm{N}^2/T))$ has exponent 9, $\mathrm{N}^2/T \cong \mathrm{SmallGroup}(3^5, 58)$ and if $\mathrm{N}^2 < S$ then $\mathrm{N}_S(\mathrm{N}^2)/T$ is isomorphic to a Sylow 3-subgroup of the group $\mathrm{P\Gamma L}_3(64)$.*

We also show that if $\mathrm{N}^2 < S$ then the group $\mathrm{N}_S(\mathrm{N}^2)$ is the normalizer in S of T (Lemma 4.30). In particular, if $T = 1$ and E is not normal in S, then we deduce that $p = 3$, $3^5 \leq |S| \leq 3^6$ and if $|S| = 3^6$ then S is isomorphic to a Sylow 3-subgroup of the group $\mathrm{P\Gamma L}_3(64)$.

This fact suggests to look at the Sylow 3-subgroups of the group $G = \mathrm{P\Gamma L}_3(q^{3^k})$, where $q \equiv 1 \mod 3$. We claim that if $P \in \mathrm{Syl}_3(G)$ then there exists a unique G-conjugacy class of $\mathcal{F}_P(G)$-essential subgroups of P and that the groups belonging to this class are isomorphic to the group $\mathrm{C}_3 \times \mathrm{C}_3 \times \mathrm{C}_{3^k}$.

In particular, in analogy with what we saw for $\mathcal{F}$-essential subgroups with $\mathcal{F}$-core of index p^2, given an $\mathcal{F}$-essential subgroup E with $\mathcal{F}$-core of index p^3, if $p = 3$ and $\mathrm{N}_S(\mathrm{N}_S(E)) < S$ then we cannot bound the index of E in S.

As a corollary of Lemmas 4.2 and 4.4 we get the following

Theorem 14. *Suppose that $E \leq S$ is an $\mathcal{F}$-essential subgroup of S having rank 3. If $p \geq 5$ then $E \trianglelefteq S$.*

Proof. If E is $\mathcal{F}$-characteristic in S then E is normal in S. Suppose that $p \geq 5$ and E is not $\mathcal{F}$-characteristic in S. Then Lemmas 4.2 and 4.4 imply that $E \trianglelefteq S$. $\square$

In Section 4.4 we suppose there are two $\mathcal{F}$-essential subgroups E_1 and E_2 of S that are $\mathcal{F}$-characteristic in S and we study their interplay to describe the isomorphism type of S.

We first note that either the quotient $S/\mathrm{core}_{\mathcal{F}}(E_1, E_2)$ is extraspecial of exponent p, or there exists a weak BN-pair associated to the fusion system $\mathcal{F}$ and the $\mathcal{F}$-essential subgroups E_1 and E_2, as we saw in Theorem 2.34. This enables us to determine the isomorphism type of the quotient $S/\mathrm{core}_{\mathcal{F}}(E_1, E_2)$ and some properties of E_1 and E_2.

Theorem 15. *Let $E_1 \leq S$ and $E_2 \leq S$ be distinct $\mathcal{F}$-essential subgroups $\mathcal{F}$-characteristic in S and let $T = \mathrm{core}_{\mathcal{F}}(E_1, E_2)$. Then*

- *either $S/T \cong p_+^{1+2}$ and for every $1 \leq i \leq 2$ the group E_i is abelian and $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_i)) \cong \mathrm{SL}_2(p)$;*
- *or S/T is isomorphic to a Sylow p-subgroup of $\mathrm{Sp}_4(p)$ and there exist $1 \leq i, j \leq 2$ such that $i \neq j$, $\mathrm{Z}(S) = \mathrm{Z}(E_i)$ is the preimage in S of $\mathrm{Z}(S/T)$ and the following hold:*
    1. *$E_i/T \cong p_+^{1+2}$ and $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_i)) \cong \mathrm{SL}_2(p)$;*

2. E_j is abelian, $T = \Phi(E_j)$ and $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_j)) \cong \mathrm{PSL}_2(p)$.

Using the characterization given by Theorem 15 we then show that the group $\mathrm{core}_{\mathcal{F}}(E_1, E_2)$ is normal in the fusion system $\mathcal{F}$. In particular if $O_p(\mathcal{F}) = 1$ then we have $T = 1$ and since the group p_+^{1+2} has sectional rank 2, the only possibility is that S is isomorphic to a Sylow p-subgroup of the group $\mathrm{Sp}_4(p)$. Also, we get $E_i \cong p_+^{1+2}$, so E_i is a pearl.

Theorem 16. *Let $E_1 \leq S$ and $E_2 \leq S$ be distinct $\mathcal{F}$-essential subgroups $\mathcal{F}$-characteristic in S. Then the group $\mathrm{core}_{\mathcal{F}}(E_1, E_2)$ is normal in $\mathcal{F}$. In particular, if $O_p(\mathcal{F}) = 1$ then S is isomorphic to a Sylow p-subgroup of the group $\mathrm{Sp}_4(p)$, $E_i \cong p_+^{1+2}$ and $E_j \cong \mathrm{C}_p \times \mathrm{C}_p \times \mathrm{C}_p$ for some $i, j \in \{1, 2\}$, $i \neq j$, and $\mathcal{F}$ is one of the fusion systems classified in [COS16].*

4.1 Automorphism groups of the $\mathcal{F}$-essential subgroups

In this section we determine the automorphism groups of the $\mathcal{F}$-essential subgroups of a p-group S that has sectional rank 3.

Recall that if E is a subgroup of S then $\mathrm{N}^i(E)$ denotes the i-th term of the normalizer tower of E in S.

We start with some results implying that the group $\mathrm{SL}_2(p)$ is a subgroup of the outer $\mathcal{F}$-automorphism group of certain $\mathcal{F}$-essential subgroups of S.

Lemma 4.5. *Let $E \leq S$ be an $\mathcal{F}$-essential subgroup. If $\mathrm{N}^1 = \mathrm{N}^1(E)$ has rank 3 then*

$$O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(p).$$

Proof. By Lemma 2.35 we have $\Phi(E) < [E, \mathrm{N}^1]\Phi(E) \leq \Phi(\mathrm{N}^1)$. Since S has sectional rank 3, the group E has rank at most 3 and by Theorem 1 we get $[\mathrm{N}^1 \colon E] = p$. So $[E \colon \Phi(\mathrm{N}^1)] = p^2$, that implies $[\Phi(\mathrm{N}^1) \colon \Phi(E)] = p$. Thus

$$[E, \mathrm{N}^1]\Phi(E) = \Phi(\mathrm{N}^1) \quad \text{and} \quad [E, \mathrm{N}^1, \mathrm{N}^1] \leq \Phi(E).$$

Hence the group $\mathrm{Out}_S(E) \cong \mathrm{N}^1/E$ acts quadratically on the elementary abelian p-group $E/\Phi(E)$. Also, $\mathrm{Out}_{\mathcal{F}}(E)$ acts faithfully on $E/\Phi(E)$ by Theorem 2.26 and $O_p(\mathrm{Out}_{\mathcal{F}}(E)) = 1$ since E is $\mathcal{F}$-essential. Therefore by Theorem 1.23 we get that the group $\mathrm{Out}_{\mathcal{F}}(E)$ involves $\mathrm{SL}_2(p)$ and by Theorem 1 we conclude $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(p)$. □

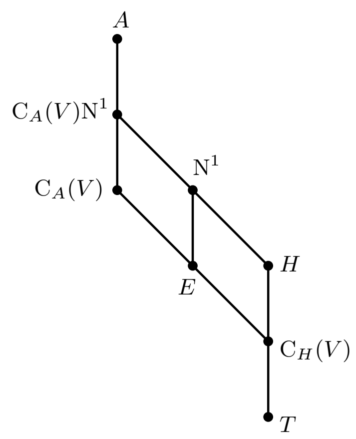
Let E be an $\mathcal{F}$-essential subgroup of S. Then by Theorem 2.30 there exists a finite group G that is a model for the saturated fusion system $\mathrm{N}_{\mathcal{F}}(E)$ defined on $\mathrm{N}^1(E)$. In particular $\mathrm{N}^1(E) \in \mathrm{Syl}_p(G)$, $E = O_p(G)$ and $G/E \cong \mathrm{Out}_{\mathcal{F}}(E)$.

Lemma 4.6. *Let $E \leq S$ be an $\mathcal{F}$-essential subgroup, let G be a model for $\mathrm{N}_{\mathcal{F}}(E)$ and set $\mathrm{N}^1 = \mathrm{N}^1(E)$ and $A = \langle (\mathrm{N}^1)^G \rangle$. Let T be a subgroup of E that is $\mathcal{F}$-characteristic in E and N^1 and set $\overline{\mathrm{N}^1} = \mathrm{N}^1/T$ and $\overline{E} = E/T$. Assume that there exist subgroups $\overline{H} \leq \overline{\mathrm{N}^1}$ and $\overline{V} \leq \Omega_1(\mathrm{Z}(\overline{E}))$ such that $\overline{H} \nleq \overline{E}$, $\overline{V}$ is normal in $\overline{G} = G/T$ and $\overline{H}$ acts quadratically on $\overline{V}$. Then*

$$O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(p).$$

Proof. To simplify notation we assume $T = 1$. First notice that the group V is elementary abelian. By Theorem 1 the quotient N^1/E has order p, so from $H \nleq E$ we get $\mathrm{N}^1 = EH$. Since H acts quadratically on V, we have $[V, H] \neq 1$. In particular $H \nleq \mathrm{Z}(\mathrm{N}^1)$ and $E = \mathrm{C}_{\mathrm{N}^1}(V)$. Thus

$$\mathrm{N}^1/E \cong H/(E \cap H) = H/\mathrm{C}_H(V) \quad \text{and} \quad \mathrm{N}^1/E \cong \mathrm{C}_A(V)\mathrm{N}^1/\mathrm{C}_A(V).$$



Therefore N^1/E is isomorphic to a p-subgroup of $A/\mathrm{C}_A(V)$ that acts quadratically on V. Note that $A/\mathrm{C}_A(V)$ acts faithfully on V. Since E is $\mathcal{F}$-essential, the group G/E has a strongly p-embedded subgroup. Thus by Lemmas 1.32 and 1.36 we get $O_p(A/E) = 1$. Also, $\mathrm{N}^1 \in \mathrm{Syl}_p(A)$, thus $O_p(A/\mathrm{C}_A(V)) = 1$. By Theorem 1.23 we deduce that $A/\mathrm{C}_A(V)$ involves $\mathrm{SL}_2(p)$. Hence $A/E \cong O^{p'}(\mathrm{Out}_{\mathcal{F}}(E))$ involves $\mathrm{SL}_2(p)$ and by Theorem 1 we conclude that $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(p)$. $\square$

Lemma 4.7. *Let $E \leq S$ be an $\mathcal{F}$-essential subgroup of S and set $\mathrm{N}^1 = \mathrm{N}^1(E)$. Let T be a subgroup of E that is $\mathcal{F}$-characteristic in E and N^1 and set $\overline{\mathrm{N}^1} = \mathrm{N}^1/T$ and $\overline{E} = E/T$. If $\mathrm{J}(\overline{\mathrm{N}^1}) \nleq \overline{E}$ and $\Omega_1(\mathrm{Z}(\overline{E})) \neq \Omega_1(\mathrm{Z}(\overline{\mathrm{N}^1}))$ then*

$$O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(p).$$

Proof. To simplify notation assume $T = 1$. Recall that $\mathrm{J}(\mathrm{N}^1) = \langle \mathbf{A}(\mathrm{N}^1) \rangle$, where $\mathbf{A}(\mathrm{N}^1)$ is the set of abelian subgroups of N^1 having maximal order. Set $V = \Omega_1(\mathrm{Z}(E))$ and let $H \in \mathbf{A}(\mathrm{N}^1)$ be such that $H \nleq E$ and $|V \cap H|$ is maximal. By Theorem 1 we have $[\mathrm{N}^1 \colon E] = p$. So $\mathrm{N}^1 = EH$ and since $V \neq \Omega_1(\mathrm{Z}(\mathrm{N}^1))$ and H is abelian we deduce that $[V, H] \neq 1$. In particular $V \nleq H$.

Note that V is normal in N^1, so it is normalized by H. If V normalizes H then $[V, H, H] \leq [H, H] = 1$. So H acts quadratically on V and by Lemma 4.6 we conclude $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(p)$.

Suppose for a contradiction that V does not normalize H. Then by the Thompson replacement theorem (Theorem 1.18) there exists an abelian subgroup $H^* \in \mathbf{A}(\mathrm{N}^1)$ such that $V \cap H < V \cap H^*$ and H^* normalizes H. Since $|V \cap H|$ is maximal by the choice of H, we have $H^* \leq E$. Therefore $V \leq H^*$, by maximality of $|H^*|$, and so V normalizes H, that is a contradiction. $\square$

We can now characterize the $\mathcal{F}$-automorphism group of every $\mathcal{F}$-essential subgroup of S not $\mathcal{F}$-characteristic in S. Recall that when we write $A \leq \mathrm{Out}_{\mathcal{F}}(E) \leq C$ we mean that $\mathrm{Out}_{\mathcal{F}}(E)$ is isomorphic to a group B such that $A \leq B \leq C$.

Theorem 4.8. *Let $E \leq S$ be an $\mathcal{F}$-essential subgroup and set* $\mathrm{N}^1 = \mathrm{N}^1(E)$. *Suppose that E is not $\mathcal{F}$-characteristic in S. Then* $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(p)$ *and*

- *either* $[E\colon \Phi(E)] = p^2$ *and* $\mathrm{SL}_2(p) \leq \mathrm{Out}_{\mathcal{F}}(E) \leq \mathrm{GL}_2(p)$*;*
- *or* $[E\colon \Phi(E)] = p^3$ *and* $\mathrm{SL}_2(p) \leq \mathrm{Out}_{\mathcal{F}}(E) \leq \mathrm{GL}_2(p) \times \mathrm{GL}_1(p)$.

Proof. Let $\alpha \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^1)$ be such that $E \neq E\alpha$. By Theorem 2.26(6) the group $E\alpha$ is an $\mathcal{F}$-essential subgroup of S. Set $T = \mathrm{core}_{\mathcal{F}}(E, E\alpha)$ and $\overline{\mathrm{N}^1} = \mathrm{N}^1/T$. If the Thompson subgroup $\mathrm{J}(\overline{\mathrm{N}^1})$ is contained in $\overline{E} \cap \overline{E}\alpha$ then $\mathrm{J}(\overline{\mathrm{N}^1}) = \mathrm{J}(\overline{E}) = \mathrm{J}(\overline{E\alpha})$ and so $\mathrm{J}(\overline{\mathrm{N}^1}) = 1$ by the maximality of T, which is a contradiction. Hence $\mathrm{J}(\overline{\mathrm{N}^1}) \nleq \overline{E} \cap \overline{E\alpha}$ and since $\mathrm{J}(\overline{\mathrm{N}^1}) = \mathrm{J}(\overline{\mathrm{N}^1})\alpha$, we deduce that $\mathrm{J}(\overline{\mathrm{N}^1}) \nleq \overline{E}$. Note that $\Omega_1(\mathrm{Z}(\overline{E})) \neq \Omega_1(\mathrm{Z}(\overline{\mathrm{N}^1}))$ by maximality of T. Therefore by Lemma 4.7 we get that $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(p)$ and we conclude by Theorem 1. $\square$

Next, we prove that when there are two $\mathcal{F}$-essential subgroups of S that are $\mathcal{F}$-characteristic in S, then the $\mathcal{F}$-automorphism group of at least one of them contains a subgroup isomorphic to $\mathrm{SL}_2(p)$.

Theorem 4.9. *Let $E_1 \leq S$ and $E_2 \leq S$ be distinct $\mathcal{F}$-essential subgroups of S that are $\mathcal{F}$-characteristic in S. Then there exists $i \in \{1, 2\}$ such that*

$$O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_i)) \cong \mathrm{SL}_2(p).$$

Proof. Let G_i be a model for $\mathrm{N}_{\mathcal{F}}(E_i)$, whose existence is guaranteed by Theorem 2.30. Set $T = \mathrm{core}_{\mathcal{F}}(E_1, E_2)$. To simplify notation we assume $T = 1$. Set

$$Z = \Omega_1(\mathrm{Z}(S)) \quad \text{and} \quad V_i = \langle Z^{G_i} \rangle \leq \Omega_1(\mathrm{Z}(E_i)).$$

Let $\mathrm{J}(S)$ be the Thompson subgroup of S. If $\mathrm{J}(S) \leq E_1 \cap E_2$ then $\mathrm{J}(S) = \mathrm{J}(E_1) =$

$\mathrm{J}(E_2) = 1$ by maximality of T, giving a contradiction. Thus we may assume $\mathrm{J}(S) \nleq E_1$. Hence by Lemma 4.7 if $\Omega_1(\mathrm{Z}(E_1)) \neq Z$ then $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_1)) \cong \mathrm{SL}_2(p)$ and we are done.

Suppose $\Omega_1(\mathrm{Z}(E_1)) = Z$. By maximality of T the group Z is not $\mathcal{F}$-characteristic in E_2. In particular $Z < V_2$. Note that V_2 is an elementary abelian subgroup of S, that has sectional rank 3. Therefore either $|V_2| = p^2$ (and $|Z| = p$) or $|V_2| = p^3$.

Suppose $|V_2| = p^2$. Then $G_2/\mathrm{C}_{G_2}(V_2)$ is isomorphic to a subgroup of $\mathrm{GL}_2(p)$. Note that $S \nleq \mathrm{C}_{G_2}(V_2)$ (otherwise $V \leq Z$) and $E_2 \in \mathrm{Syl}_p(\mathrm{C}_{G_2}(V_2))$. So $\langle (S)^{G_2} \rangle / E_2$ acts non-trivially on V_2 and by Theorem 1 we deduce that $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_2)) \cong \langle (S)^{G_2} \rangle / E_2 \cong \mathrm{SL}_2(p)$.

Suppose $|V_2| = p^3$.

- If $V_2 \nleq E_1$ then $S = E_1 V_2$, since $[S \colon E_1] = p$. Also $[E_1, S] \nleq \Phi(E)$ by Lemma 2.35, so $[E_1, V_2] \nleq \Phi(E_1)$. On the other hand, since V_2 is abelian and normal in S we get $[E_1, V_2, V_2] = 1$. Thus $V_2\Phi(E_1)/\Phi(E_1)$ acts quadratically on $E_1/\Phi(E_1)$ and by Lemma 4.6 we conclude that $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_1)) \cong \mathrm{SL}_2(p)$.

- Assume $V_2 \leq E_1$. By the maximality of T the group V_2 is not normalized by G_1. Hence there exists $g \in G_1 \backslash \mathrm{N}_{G_1}(S)$ such that $V_2 \neq V_2^g$. Note that $V_2^g \leq E_1$. If $[V_2^g, V_2] = 1$ then $V_2 V_2^g$ is an elementary abelian subgroup of E_1 and so $|V_2 V_2^g| \leq p^3$ because S has sectional rank 3, contradicting the fact that $V_2 < V_2 V_2^g$ and $|V_2| = p^3$. Thus $[V_2^g, V_2] \neq 1$. In particular $V_2^g \nleq E_2$. On the other hand $[V_2, V_2^g] \leq V_2 \cap V_2^g$, so $[V_2, V_2^g, V_2^g] = 1$. Hence V_2^g acts quadratically on V_2 and by Lemma 4.6 we conclude that $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_2)) \cong \mathrm{SL}_2(p)$.

□

We can finally prove the first part of Theorem 8.

Theorem 4.10. *Let $E_1 \leq S$ and $E_2 \leq S$ be distinct $\mathcal{F}$-essential subgroups of S. Suppose that E_1 is $\mathcal{F}$-characteristic in S and E_2 is such that every morphism in $\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(E_2)}(\mathrm{Aut}_S(E_2))$ is a restriction of an $\mathcal{F}$-automorphism of S. Then*

1. either $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_1)) \cong \mathrm{SL}_2(p)$*;*

2. or $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_1)) \cong \mathrm{PSL}_2(p)$*,* $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_2)) \cong \mathrm{SL}_2(p)$ *and* S *has rank* 2*.*

Proof. If S has rank 3 then $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_1)) \cong \mathrm{SL}_2(p)$ by Lemma 4.5. Suppose S has rank 2 and $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_1))$ is not isomorphic to $\mathrm{SL}_2(p)$. By Theorems 4.8 (if E_2 is not $\mathcal{F}$-characteristic in S) and 4.9 (if E_2 is $\mathcal{F}$-characteristic in S) we deduce that $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_2)) \cong \mathrm{SL}_2(p)$. By Theorem 1 we have $[E_1 \colon \Phi(E_1)] = p^3$ and either $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_1)) \cong \mathrm{PSL}_2(p)$ or $p = 3$ and $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_1)) \cong 13\colon 3$.

Suppose for a contradiction that the latter holds. Let $\tau \in \mathrm{Z}(O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_2)))$ be an involution and let $\overline{\tau} \in \mathrm{Out}_{\mathcal{F}}(S)$ be such that $\overline{\tau}|_{E_2} = \tau$ (that exists by assumption). Since E_1 is $\mathcal{F}$-characteristic in S, we deduce that $\overline{\tau}|_{E_1} \in \mathrm{Out}_{\mathcal{F}}(E_1)$. So $\mathrm{Out}_{\mathcal{F}}(E_1) \cong (13\colon 3) \times \mathrm{C}_2$ (as $13\colon \mathrm{C}_6$ is not a subgroup of $\mathrm{GL}_3(3)$). In particular $\overline{\tau} \in \mathrm{Z}(\mathrm{Out}_{\mathcal{F}}(E_1))$ and so it acts trivially on the quotient S/E_1. By assumption $\overline{\tau}$ acts trivially on $S/E_2 \cong E_1/(E_1 \cap E_2)$. Consider the following sequence of $\mathcal{F}$-characteristic subgroups of S:

$$\Phi(S) \leq E_1 \cap E_2 \leq E_1 < S.$$

Since S has rank 2 we have $\Phi(S) = E_1 \cap E_2$ and by Lemma 1.34 we deduce that $\overline{\tau} \in O_3(\mathrm{Out}_{\mathcal{F}}(S))$. However, $O_3(\mathrm{Out}_{\mathcal{F}}(S)) = \mathrm{Inn}(S)$ since S is fully normalized, and we get a contradiction. $\square$

Note that the assumptions of the previous theorem are always satisfied when there are two $\mathcal{F}$-essential subgroups that are $\mathcal{F}$-characteristic in S. If there is only one $\mathcal{F}$-essential subgroup E that is $\mathcal{F}$-characteristic in S and $\mathcal{F}$-essential subgroups of S not $\mathcal{F}$-characteristic in S that do not satisfy the hypothesis of the theorem, then a priori the group $\mathrm{Out}_{\mathcal{F}}(E)$ might involve the group $13\colon 3$.

4.2 Properties of the $\mathcal{F}$-core

Lemma 4.11. *Let E be an $\mathcal{F}$-essential subgroup of S not $\mathcal{F}$-characteristic in S, set $T = \mathrm{core}_{\mathcal{F}}(E)$ and let $\alpha \in \mathrm{Hom}_{\mathcal{F}}(\mathrm{N}^1(E), S)$. Then $T\alpha = \mathrm{core}_{\mathcal{F}}(E\alpha)$.*

In particular $\mathrm{core}_{\mathcal{F}}(E) = \mathrm{core}_{\mathcal{F}}(E, E\alpha) = \mathrm{core}_{\mathcal{F}}(E\alpha)$ for every $\alpha \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^1(E))$.

Proof. If $E = E\alpha$ then $\alpha|_E \in \mathrm{Aut}_{\mathcal{F}}(E)$ and so $T\alpha = T = \mathrm{core}_{\mathcal{F}}(E)$.

Suppose $E \neq E\alpha$. Clearly $T\alpha$ is a subgroup of $E\alpha$. Note that $\mathrm{Aut}_{\mathcal{F}}(E\alpha) = \alpha^{-1}\mathrm{Aut}_{\mathcal{F}}(E)\alpha$ (by Lemma 2.7). Since $[\mathrm{N}^1(E) \colon E] = p$, by Theorem 2.26(5) the group $E\alpha$ is $\mathcal{F}$-essential in S (not $\mathcal{F}$-characteristic in S) and we have $\mathrm{N}^1(E)\alpha = \mathrm{N}^1(E\alpha)$. Thus $\mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^1(E)\alpha) = \alpha^{-1}\mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^1(E))\alpha$. It's now easy to see that $T\alpha = \mathrm{core}_{\mathcal{F}}(E\alpha)$.

Assume $\alpha \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^1(E))$. Since $\mathrm{core}_{\mathcal{F}}(E, E\alpha) \leq E$, by maximality of T we have $\mathrm{core}_{\mathcal{F}}(E, E\alpha) \leq T$. On the other hand, $T = T\alpha = \mathrm{core}_{\mathcal{F}}(E\alpha)$, so T is contained in $E \cap E\alpha$ and is $\mathcal{F}$-characteristic in E, $E\alpha$ and $\mathrm{N}^1(E)$. Hence $T \leq \mathrm{core}_{\mathcal{F}}(E, E\alpha)$, which implies $T = \mathrm{core}_{\mathcal{F}}(E, E\alpha)$. $\square$

In particular, whenever E is an $\mathcal{F}$-essential subgroup of S not $\mathcal{F}$-characteristic in S, we can write $\mathrm{core}_{\mathcal{F}}(E)$ as the $\mathcal{F}$-core of two *distinct* $\mathcal{F}$-essential subgroups of S.

Lemma 4.12. *Let E_1 and E_2 be distinct $\mathcal{F}$-essential subgroups of S such that $\mathrm{N}^1(E_1) = \mathrm{N}^1(E_2)$. Set $\mathrm{N}^1 = \mathrm{N}^1(E_1) = \mathrm{N}^1(E_2)$, $E_{12} = E_1 \cap E_2$ and $T = \mathrm{core}_{\mathcal{F}}(E_1, E_2)$. Then for every $i \in \{1, 2\}$ the following hold:*

1. $\mathrm{C}_{E_i}(T) \nleq T$;
2. $\mathrm{C}_{\mathrm{N}^1}(T) \nleq E_{12}$;
3. *if* $\mathrm{C}_{\mathrm{N}^1}(T) \nleq E_i$ *then* $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_i))$ *centralizes* T.

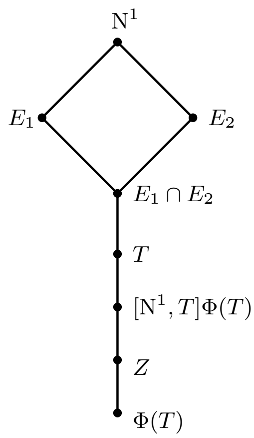
Proof. Let G_i be a model for $\mathrm{N}_{\mathcal{F}}(E_i)$ (whose existence is guaranteed by Theorem 2.30). As an intermediate step we show that $\mathrm{C}_{G_i}(T) \nleq T$ for every $i \in \{1, 2\}$.

Suppose for a contradiction that $\mathrm{C}_{G_1}(T) \leq T$. Then $\mathrm{C}_{\mathrm{N}^1}(T) \leq T$. In particular $\mathrm{C}_{\mathrm{N}^1}(T) \trianglelefteq \mathrm{C}_{G_i}(T)$ for every $i \in \{1,2\}$. Let $g \in \mathrm{C}_{G_2}(T)$. Note that $[E_2, g] \leq E_2 \cap \mathrm{C}_{G_2}(T) = \mathrm{C}_{E_2}(T) \leq T$. Thus g centralizes every quotient of consecutive subgroups in the sequence $1 < T < E_2$. Hence by Lemma 2.24 we deduce that $g \in E_2$, and so $g \in \mathrm{C}_{E_2}(T) \leq T$. Therefore $\mathrm{C}_{G_2}(T) \leq T$. Hence we have $\mathrm{C}_{G_i}(T) \leq T$ for every $i \in \{1,2\}$.

Let $g \in G_1$ be such that $[T, g] \in \Phi(T)$. Then the order of g is a power of p (otherwise g centralizes T by Theorem 1.10 contradicting the fact that $\mathrm{C}_{G_1}(T) \leq T$). Hence the group $\mathrm{C}_{G_1}(T/\Phi(T))$ is a normal p-subgroup of G_1 and so $\mathrm{C}_{G_1}(T/\Phi(T)) \leq E_1$. With the same argument we can prove that $\mathrm{C}_{G_2}(T/\Phi(T)) \leq E_2$. Therefore $\mathrm{C}_{\mathrm{N}^1}(T/\Phi(T)) \leq E_1 \cap E_2$ and $\mathrm{C}_{\mathrm{N}^1}(T/\Phi(T)) = \mathrm{C}_{G_i}(T/\Phi(T))$ for every i. By the maximality of T and the fact that T centralizes $T/\Phi(T)$ we conclude that

$$T = \mathrm{C}_{G_1}(T/\Phi(T)) = \mathrm{C}_{G_2}(T/\Phi(T)).$$

Thus the quotient $\mathrm{N}^1/(E_1 \cap E_2)$ acts non-trivially on $T/\Phi(T)$. Since $[\mathrm{N}^1 \colon (E_1 \cap E_2)] = p^2$ and S has sectional rank 3, we deduce that $[T \colon \Phi(T)] = p^3$. Also for every $i \in \{1,2\}$ the quotient G_i/T is isomorphic to a subgroup of $\mathrm{Aut}(T/\Phi(T)) \cong \mathrm{GL}_3(p)$, and so $\mathrm{N}^1/T \cong p_+^{1+2}$. In particular $[(E_1 \cap E_2) \colon T] = p$ and $E_i/T \cong \mathrm{C}_p \times \mathrm{C}_p$ for every $i \in \{1,2\}$. Note that $[T \colon [\mathrm{N}^1, T]\Phi(T)] = p$ and if $Z/\Phi(T) = \mathrm{Z}(\mathrm{N}^1/\Phi(T))$ then $[Z \colon \Phi(T)] = p$.



Note that $[E_i, T]\Phi(T) \leq [\mathrm{N}^1, T]\Phi(T)$ for every $i \in \{1, 2\}$ and $[\mathrm{N}^1, T] = [E_1, T][E_2, T]$. If $[E_1, T]\Phi(T) = [E_2, T]\Phi(T)$ then $[E_1, T]\Phi(T) = [\mathrm{N}^1, T]\Phi(T)$ and $\mathrm{N}^1/(E_1 \cap E_2)$ is isomorphic to a subgroup of $\mathrm{Aut}([\mathrm{N}^1, T]\Phi(T)/\Phi(T))$, that is a contradiction.

Thus $[E_1, T]\Phi(T) \neq [E_2, T]\Phi(T)$. We have $Z \leq [E_i, T]\Phi(T) \leq [\mathrm{N}^1, T]\Phi(T)$ for every i. So we may assume that

$$[E_1, T]\Phi(T) = Z \text{ and } [E_2, T]\Phi(T) = [\mathrm{N}^1, T]\Phi(T).$$

Let $x \in (E_1 \cap E_2)\backslash T$ and let $t \in T$. Note that $[x, t] \in [E_1, T]\Phi(T) = Z$, so $[x, t]$ commutes with t and x modulo $\Phi(T)$. Hence by Theorem 1.3 we have

$$(xt)^p = t^p x^p [x, t]^{\frac{p(p-1)}{2}} = x^p \mod \Phi(T).$$

Since $E_1 \cap E_2 = \langle x \rangle T$ we deduce that $(E_1 \cap E_2)^p\Phi(T) = \langle x^p \rangle \Phi(T)$. Thus $(E_1 \cap E_2)^p\Phi(T) = Z$ and the quotient $(E_1 \cap E_2)/Z$ is elementary abelian of order p^3.

Note that $\Phi(E_1) \leq T$ and so either E_1 has rank 2 (and $T = \Phi(E_1)$) or $[T \colon \Phi(E_1)] = p$. In particular by Theorem 1 we have $\langle (\mathrm{N}^1)^{G_1} \rangle / E_1 \cong \mathrm{SL}_2(p)$. Also, G_1 acts transitively on the maximal subgroups of E_1 containing T and normalizes $[T, E_1]\Phi(T)$. Hence we conclude that $E_1/[T, E_1]\Phi(T) = E_1/Z$ has exponent p.

Let $\tau \in \langle (\mathrm{N}^1)^{G_1} \rangle$ be an involution that inverts E_1/T. Note that T/Z is a natural $\mathrm{SL}_2(p)$-module for $\langle (\mathrm{N}^1)^{G_1} \rangle / E_1$ (otherwise $\langle (\mathrm{N}^1)^{G_1} \rangle$ would centralize every quotient of two consecutive subgroups in the sequence $\Phi(T) < Z < T$ and so $\langle (\mathrm{N}^1)^{G_1} \rangle$ would be a p-group by Lemma 1.34, that is a contradiction). Hence τ inverts the quotient T/Z. In other words τ acts as -1 on every quotient of two consecutive subgroups in the sequence

$$\mathrm{Z} < [\mathrm{N}^1, T]\Phi(T) < T < E_1 \cap E_2 < E_1.$$

Therefore the group E_1/Z is abelian and so elementary abelian of order p^4, contradicting the fact that S has sectional rank 3.

Hence we have $\mathrm{C}_{G_i}(T) \nleq T$ for every i. Now suppose for a contradiction that $\mathrm{C}_{E_i}(T) \leq T$ for some i. Then $\mathrm{C}_{G_i}(T)$ is a normal subgroup of G_i not contained in $E_i = O_p(G_i)$. Hence $\mathrm{C}_{G_i}(T)$ is not a p-group and there exists a non trivial element $g \in \mathrm{C}_{G_i}(T)$ of order prime to p. Note that the direct product $\langle g \rangle \times T$ acts by conjugation on E_i. Then by Lemma 1.14 we get $[g, \mathrm{C}_{E_i}(T)] \neq 1$, contradicting the fact that $\mathrm{C}_{E_i}(T) \leq T \leq \mathrm{C}_{G_i}(g)$. Thus $\mathrm{C}_{E_i}(T) \nleq T$ for every i.

Suppose for a contradiction that $\mathrm{C}_{\mathrm{N}^1}(T) \leq E_1 \cap E_2$. Then $\mathrm{C}_{\mathrm{N}^1}(T) = \mathrm{C}_{E_1}(T) = \mathrm{C}_{E_2}(T)$ and by maximality of T we conclude $\mathrm{C}_{E_i}(T) \leq T$, contradicting what was proved above.

Finally, assume that $\mathrm{C}_{\mathrm{N}^1}(T) \nleq E_i$ for some i. Then $\mathrm{N}^1 = E_i \mathrm{C}_{\mathrm{N}^1}(T)$, since $[\mathrm{N}^1 \colon E_i] = p$ by Theorem 1. In particular $\mathrm{Out}_S(E_i) \cong \mathrm{N}^1/E_i \cong \mathrm{C}_{\mathrm{N}^1}(T)/\mathrm{C}_{E_i}(T)$ centralizes T. Hence $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_i)) = \langle \mathrm{Out}_S(E_i)^{\mathrm{Out}_{\mathcal{F}}(E_i)} \rangle$ centralizes T. □

Lemma 4.13. *Let E_1 and E_2 be distinct $\mathcal{F}$-essential subgroups of S such that $\mathrm{N}^1(E_1) = \mathrm{N}^1(E_2)$. Set $T = \mathrm{core}_{\mathcal{F}}(E_1, E_2)$. Then for every $1 \leq i \leq 2$ either $T \leq \Phi(E_i)$ or $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_i)) \cong \mathrm{SL}_2(p)$, $[T\Phi(E_i) \colon \Phi(E_i)] = p$ and*

$$T\Phi(E_i)/\Phi(E_i) = \mathrm{C}_{E_i/\Phi(E_i)}(O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_i))).$$

Proof. Fix i and set $E = E_i$ and $\mathrm{N}^1 = \mathrm{N}^1(E)$. Note that $\Phi(E)T$ is a proper $\mathcal{F}$-characteristic subgroup of E. If the action of $\mathrm{Out}_{\mathcal{F}}(E)$ on $E/\Phi(E)$ is irreducible, then we have $\Phi(E)T = \Phi(E)$, and so $T \leq \Phi(E)$. Suppose the action is reducible. Then $[E \colon \Phi(E)] = p^3$ and by Theorem 1 we get that $\mathrm{Out}_{\mathcal{F}}(E)$ is isomorphic to a subgroup of $\mathrm{GL}_2(p) \times \mathrm{GL}_1(p)$ containing $\mathrm{SL}_2(p)$. Let $\tau \in O^{p'}(\mathrm{Out}_{\mathcal{F}}(E))$ be an involution. Then by coprime action (Theorem 1.11) we have

$$E/\Phi(E) \cong \mathrm{C}_{E/\Phi(E)}(\tau) \times [E/\Phi(E), \tau].$$

Note that the groups $\mathrm{C}_{E/\Phi(E)}(\tau)$ and $[E/\Phi(E), \tau]$ are the only subgroups of $E/\Phi(E)$ that are normalized by $\mathrm{Out}_{\mathcal{F}}(E)$. Thus $\mathrm{C}_{E/\Phi(E)}(\tau) = \mathrm{C}_{E/\Phi(E)}(O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)))$ and $[E/\Phi(E), \tau] = [E/\Phi(E), O^{p'}(\mathrm{Out}_{\mathcal{F}}(E))]$. Also, either $T \leq \Phi(E)$ or $T\Phi(E)$ is the preimage in E of one of these two subgroups of $E/\Phi(E)$.

It remains to prove that $T\Phi(E)$ cannot be the preimage in E of $[E/\Phi(E), O^{p'}(\mathrm{Out}_{\mathcal{F}}(E))]$. Suppose for a contradiction that it is. Then $T/(T \cap \Phi(E)) \cong T\Phi(E)/\Phi(E)$ is a natural $\mathrm{SL}_2(p)$-module for $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E))$. So $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E))$ does not centralize T and, by Lemma 4.12, we have $\mathrm{C}_{\mathrm{N}^1}(T) \leq E$. Since $T\Phi(E) \leq E_1 \cap E_2$ and $[E\colon \Phi(E)] = p^3$, we deduce that $T\Phi(E) = E_1 \cap E_2$. Let $j \neq i$, $1 \leq j \leq 2$. Then $\mathrm{C}_{E_j}(T) \leq \mathrm{C}_{\mathrm{N}^1}(T) \leq E$ and

$$\mathrm{C}_{E_j}(T) = \mathrm{C}_{\mathrm{N}^1}(T) \cap E_j = \mathrm{C}_E(T) \cap T\Phi(E).$$

Thus $\mathrm{C}_{E_j}(T)$ is $\mathcal{F}$-characteristic in E. Moreover $\Phi(E)T = \Phi(\mathrm{N}^1)T$, so $\mathrm{C}_{E_j}(T) = \mathrm{C}_{\mathrm{N}^1}(T) \cap \Phi(\mathrm{N}^1)T$ is $\mathcal{F}$-characteristic in N^1. Clearly $\mathrm{C}_{E_j}(T)$ is $\mathcal{F}$-characteristic in E_j and we get $\mathrm{C}_{E_j}(T) \leq T$ by the maximality of T, contradicting Lemma 4.12.

Thus either $T \leq \Phi(E)$ or $T\Phi(E)$ is the preimage in E of $\mathrm{C}_{E/\Phi(E)}(O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)))$. $\square$

All properties listed in Lemmas 4.12 and 4.13 hold when E is not $\mathcal{F}$-characteristic in S and $T = \mathrm{core}_{\mathcal{F}}(E)$. We collect them in the next lemma.

Lemma 4.14. *Let $E \leq S$ be an $\mathcal{F}$-essential subgroup not $\mathcal{F}$-characteristic in S and set $\mathrm{N}^1 = \mathrm{N}^1(E)$ and $T = \mathrm{core}_{\mathcal{F}}(E)$. Then*

1. $\mathrm{C}_E(T) \nleq T$;

2. $\mathrm{C}_{\mathrm{N}^1}(T) \nleq E$ *and* $\mathrm{N}^1 = E\mathrm{C}_{\mathrm{N}^1}(T)$;

3. $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E))$ *centralizes* T;

4. *either* $T \leq \Phi(E)$ *or* $T/\Phi(E) = \mathrm{C}_{E/\Phi(E)}(O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)))$ *and* $[T\Phi(E)\colon \Phi(E)] = p$.

Proof. Note that $T = \mathrm{core}_{\mathcal{F}}(E, E\alpha)$, for some $\alpha \in \mathrm{Aut}_F(\mathrm{N}_S(E))$ such that $E \neq E\alpha$. Therefore we can apply Lemmas 4.12 and 4.13.

For point 2, note that $\mathrm{C}_{\mathrm{N}^1}(T) \nleq E \cap E\alpha$ and the fact that $\mathrm{C}_{\mathrm{N}^1}(T)$ is $\mathcal{F}$-characteristic in N^1 implies that $\mathrm{C}_{\mathrm{N}^1}(T) \nleq E$. From $[\mathrm{N}^1\colon E] = p$, we then deduce $\mathrm{N}^1 = E\mathrm{C}_{\mathrm{N}^1}(T)$. □

We end this section proving that under certain conditions the $\mathcal{F}$-core T of E_1 and E_2 is an abelian group of rank at most 2, and if it has rank 2 then $T \cong \mathrm{C}_{p^a} \times \mathrm{C}_p$, for some $a \in \mathbb{N}$.

Theorem 4.15. *Let E_1 and E_2 be distinct $\mathcal{F}$-essential subgroups of S such that $\mathrm{N}^1(E_1) = \mathrm{N}^1(E_2)$. Set $\mathrm{N}^1 = \mathrm{N}^1(E_1) = \mathrm{N}^1(E_2)$, $E_{12} = E_1 \cap E_2$ and $T = \mathrm{core}_{\mathcal{F}}(E_1, E_2)$. Suppose that $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_1))$ is isomorphic to $\mathrm{SL}_2(p)$ and centralizes T, and there exists a subgroup $V \leq E_1$ such that V is $\mathcal{F}$-characteristic in E_1, V/T is a natural $\mathrm{SL}_2(p)$-module for $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_1))$ and $V \leq \mathrm{C}_{E_1}(T)T$. Then T is abelian, $T \leq \mathrm{Z}(V)$, $|[V,V]| \leq p$ and the group $T/[V,V]$ is cyclic.*

Proof. Set $E = E_1$. Since $V \leq \mathrm{C}_E(T)T$, we get

$$\mathrm{C}_V(T)T = (V \cap \mathrm{C}_E(T))T = V \cap \mathrm{C}_E(T)T = V.$$

Note that $\mathrm{C}_V(T) \cap T = \mathrm{Z}(T)$ and so

$$V/\mathrm{Z}(T) \cong T/\mathrm{Z}(T) \times \mathrm{C}_V(T)/\mathrm{Z}(T).$$

Since $\mathrm{C}_V(T)/\mathrm{Z}(T) \cong V/T \cong \mathrm{C}_p \times \mathrm{C}_p$ and S has sectional rank 3, we deduce that $T/\mathrm{Z}(T)$ has to be cyclic. Thus by Lemma 1.6 the group T is abelian. In particular $T \leq \mathrm{C}_V(T)$ and so $V = \mathrm{C}_V(T)$. Hence $T \leq \mathrm{Z}(V)$ and since $[V\colon T] = p^2$ by Lemma 1.6 we get that $|[V,V]| \leq p$.

Let $\tau \in O^{p'}(\mathrm{Out}_{\mathcal{F}}(E))$ be an involution. Then by assumption τ acts on V and T is the centralizer in V of τ. Thus by coprime action (Theorem 1.11) we get

$$V/[V,V] \cong T/[V,V] \times [V/[V,V],\tau].$$

Since $[V/[V,V],\tau] \cong \mathrm{C}_p \times \mathrm{C}_p$ and S has sectional rank 3, we deduce that the group $T/[V,V]$ is cyclic. $\square$

4.3 Structure of the $\mathcal{F}$-essential subgroups that are not $\mathcal{F}$-characteristic in S

Throughout this section we work under the following assumptions.

Main Hypothesis A. Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on the p-group S, that has sectional rank 3. Let E be an $\mathcal{F}$-essential subgroup of S, **not** $\mathcal{F}$-characteristic in S, and let $T = \mathrm{core}_{\mathcal{F}}(E)$.

We write N^i for the i-th term of the normalizer tower of E in S. Since S has sectional rank 3 and E is not $\mathcal{F}$-characteristic in S, by Theorem 4.8 we have that

- either E has rank 2 and $\mathrm{SL}_2(p) \leq \mathrm{Out}_{\mathcal{F}}(E) \leq \mathrm{GL}_2(p)$;
- or E has rank 3 and $\mathrm{SL}_2(p) \leq \mathrm{Out}_{\mathcal{F}}(E) \leq \mathrm{GL}_2(p) \times \mathrm{GL}_1(p)$.

In particular $[\mathrm{N}^1 \colon E] = p$. Also recall that we can always find a finite group G that is a model for the fusion system $\mathrm{N}_{\mathcal{F}}(E)$ on N^1 (by Theorem 2.34). For the rest of this section, we are going to use these important facts without referencing the relevant theorems.

In order to describe the structure of E, we intend to apply Stellmacher's Pushing Up Theorem (Theorem 1.26). We first show that the quotient group N^1/T is non-abelian.

Lemma 4.16. *The quotient group* $\mathrm{N}^1/T\Phi(E)$ *is non-abelian.*

Proof. Consider the following sequence of $\mathcal{F}$-characteristic subgroups of E:

$$\Phi(E) \leq T\Phi(E) < E.$$

By Lemma 4.14 we have $[T\Phi(E) \colon \Phi(E)] \leq p$. So N^1 centralizes the quotient $T\Phi(E)/\Phi(E)$. Since $\mathrm{Inn}(E) \neq \mathrm{Aut}_S(E)$, by Lemma 2.24 the group N^1 cannot centralize the quotient $E/T\Phi(E)$. Thus the quotient group $\mathrm{N}^1/T\Phi(E)$ is not abelian. □

Theorem 4.17. *Set $V = [E, O^{p'}(\mathrm{Aut}_{\mathcal{F}}(E))]T$. Then*

1. *V/T is a natural $\mathrm{SL}_2(p)$-module for $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E))$;*

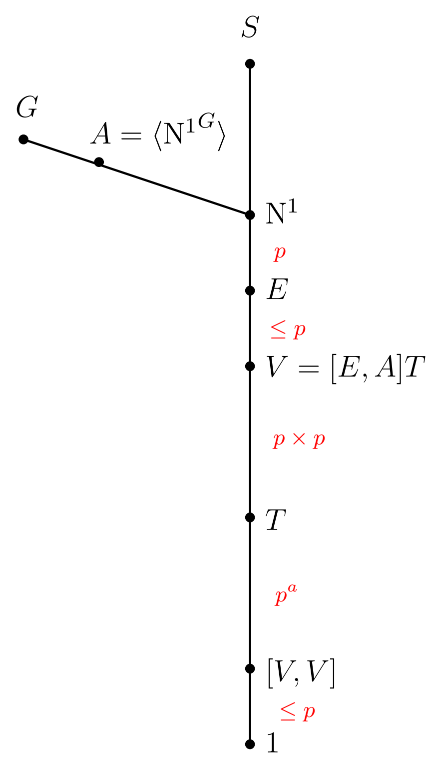
2. *N^1/T has exponent p;*

3. *E/T is elementary abelian and $p^2 \leq [E\colon T] \leq p^3$;*

4. *$[E/T\colon \mathrm{Z}(\mathrm{N}^1/T)] = p$;*

5. *T is abelian, $T \leq \mathrm{Z}(V)$, $|[V, V]| \leq p$ and $T/[V, V]$ is a cyclic group.*

Moreover, if $[E\colon T] = p^2$, then $T \leq \mathrm{Z}(\mathrm{N}^1)$.



Structure of an $\mathcal{F}$-essential subgroup E of S not $\mathcal{F}$-characteristic in S, where G is a model for $\mathrm{N}_{\mathcal{F}}(E)$.

Proof. Let G be a model for $\mathrm{N}_{\mathcal{F}}(E)$ and let $A = \langle \mathrm{N}^{1^G} \rangle \leq G$. Then $A/E \cong O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(p)$. We intend to apply Stellmacher's Pushing Up Theorem (Theorem 1.26) to the group A/T and to its Sylow p-subgroup N^1/T. Note that the quotient N^1/T is non-abelian by Lemma 4.16.

Let $T \leq W \leq \mathrm{N}^1$ be such that W/T char N^1/T and $W/T \trianglelefteq A/T$. Then $W \leq E = O_p(A)$, W $\mathrm{char}_{\mathcal{F}}$ N^1 and $W \trianglelefteq \mathrm{N}_G(\mathrm{N}^1)A = G$. So $W = T$ by maximality of T and $W/T = 1$. By Theorem 4.8 the quotient A/E is isomorphic to $\mathrm{SL}_2(p)$. Thus by Theorem 1.26 and the fact that S has sectional rank 3, we get that $V/T \leq \mathrm{Z}(E/T)$ and V/T is a natural $\mathrm{SL}_2(p)$-module for A/E. In particular $[E\colon T] \geq p^2$.

Let $\Omega_N \leq \mathrm{N}^1$ be the preimage in N^1 of $\Omega_1(\mathrm{Z}(\mathrm{N}^1/T))$ and let Ω_E be the preimage in E of $\Omega_1(\mathrm{Z}(E/T))$. Then Theorem 1.26 tells us that N^1/Ω_N is elementary abelian. Since S has sectional rank 3 we deduce $[\mathrm{N}^1\colon \Omega_N] \leq p^3$.

If $\Omega_N \not\leq E$ then $\mathrm{N}^1 = E\Omega_N$ and

$$[N, N] = [E, N] = [E, E][E, \Omega_N] \leq \Phi(E)T,$$

contradicting the fact that $N/T\Phi(E)$ is non-abelian by Lemma 4.16. Therefore $\Omega_N \leq E$ and so $\Omega_N \leq \Omega_E$. By maximality of T, we also have $\Omega_N \neq \Omega_E$. In particular

$$[E\colon \Omega_E] < [E\colon \Omega_N] = p^{-1}[\mathrm{N}^1\colon \Omega_N] \leq p^2.$$

Therefore $[E\colon \Omega_E] \leq p$, which implies that E/T is abelian.

Let $\tau \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(E)) \cong A/\mathrm{Z}(E)$ be an involution and let $C \leq E$ be the preimage of $\mathrm{C}_{E/T}(\tau)$. Then by coprime action (Theorem 1.11) we get

$$E/T \cong C/T \times [E/T, \tau].$$

Note that $[E/T, \tau] \leq V/T$ and since V/T is an $\mathrm{SL}_2(p)$-module for $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E))$, we deduce that $E/T \cong C/T \times V/T$. Thus N^1/C is isomorphic to a Sylow p-subgroup of the group $(\mathrm{C}_p \times \mathrm{C}_p)\colon \mathrm{SL}_2(p)$. Thus $\mathrm{N}^1/C \cong p_+^{1+2}$ and so $(\mathrm{N}^1)^p \leq C$. Hence $(\mathrm{N}^1)^p T$ is a subgroup of E that is $\mathcal{F}$-characteristic in N^1 and normalized by $G = A\mathrm{N}_G(\mathrm{N}^1)$. By maximality of T we get $(\mathrm{N}^1)^p \leq T$. Hence N^1/T has exponent p and E/T is elementary abelian. In particular $[E\colon T] \leq p^3$.

Since E/T is elementary abelian we have $\Omega_N/T = \Omega_1(\mathrm{Z}(\mathrm{N}^1/T)) = \mathrm{Z}(\mathrm{N}^1/T)$. Let α be an $\mathcal{F}$-automorphism of N^1 such that $E \neq E\alpha$. Then $\mathrm{N}^1 = EE\alpha$ and $E\alpha/T \cong E/T$ is abelian. Hence $\Omega_N = E \cap E\alpha$ and $[E\colon \Omega_N] = p$.

Point 5. is a consequence of Theorem 4.15, once we have shown that $V \leq \mathrm{C}_E(T)T$. Note that the group $\mathrm{C}_E(T)T$ is a $\mathcal{F}$-characteristic subgroup of E not contained in T (by Lemma 4.14). Since $\Phi(E) \leq T$ and $\mathrm{Out}_{\mathcal{F}}(E) \leq \mathrm{GL}_2(p) \times \mathrm{GL}_1(p)$, either $V \leq \mathrm{C}_E(T)T$ or $T = \Phi(E)$ and $[\mathrm{C}_E(T)T\colon T] = p$. Suppose for a contradiction that the latter holds. Since $\mathrm{C}_E(T)T \trianglelefteq \mathrm{N}^1$ we deduce $\mathrm{C}_E(T)T \leq \Omega_N$. Also $\mathrm{C}_E(T)T = (\mathrm{C}_{\mathrm{N}^1}(T) \cap E)T = \mathrm{C}_{\mathrm{N}^1}(T)T \cap E$. Therefore $\mathrm{C}_E(T)T = \mathrm{C}_{\mathrm{N}^1}(T)T \cap \Omega_N$. In particular $\mathrm{C}_E(T)T$ is normalized by $\mathrm{Aut}_{\mathcal{F}}(E)$ and $\mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^1)$, contradicting the maximality of T. Therefore $V \leq \mathrm{C}_E(T)T$ and we conclude by Theorem 4.15.

Finally, if $[E\colon T] = p^2$ then $V = E$ so $E = \mathrm{C}_E(T)$. Since $\mathrm{N}^1 = E\mathrm{C}_{\mathrm{N}^1}(T)$ by Lemma 4.14, we deduce that $\mathrm{N}^1 = \mathrm{C}_{\mathrm{N}^1}(T)$ and so $T \leq \mathrm{Z}(\mathrm{N}^1)$. □

Lemma 4.18. *Let $V = [E, O^{p'}(\mathrm{Aut}_{\mathcal{F}}(E))]T$. Then $V^p = T^p$ and $V = \Omega_1(V)T$. In particular $\Omega_1(\mathrm{N}^1) \nleq E$.*

Proof. Let $\tau \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(E))$ be the involution that acts on V/T as $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Additionally τ centralizes T by Lemma 4.14. Let $x \in V \backslash T$. Then $x\tau = x^{-1}t$ for some $t \in T$. Also by Theorem 4.17 we have $x^p \in T$ and $T \leq \mathrm{Z}(V)$. So by Theorem 1.3 we get

$$x^p = (x^p)\tau = (x\tau)^p = (x^{-1}t)^p = (x^p)^{-1}t^p.$$

Hence $x^p \in T^p$ and since $T^p \leq V^p$, we conclude $V^p = T^p$.

Let $M = T\langle x\rangle \leq V$. Thus M is abelian, since $[M : T] = p$, and

$$|\Omega_1(M)| = |M : M^p| = |M : T^p| = p|T : T^p| = p|\Omega_1(T)|.$$

Therefore $\Omega_1(T) < \Omega_1(M)$. Thus in every maximal subgroup of M containing T there are elements of order p not belonging to T. In particular $V = \Omega_1(V)T$.

Note that $[E : V] \leq p$, so the group $\Omega_1(E)T$ is either equal to V or to E. If $\Omega_1(\mathrm{N}^1) \leq E$ then $\Omega_1(E)T = \Omega_1(\mathrm{N}^1)T$ is $\mathcal{F}$-characteristic in E and N^1, contradicting the maximality of T. Thus $\Omega_1(\mathrm{N}^1) \not\leq E$. $\square$

We can now prove that $\mathcal{F}$-essential subgroups of rank 2 of p-groups having sectional rank 3 are pearls.

Theorem 4.19. *Suppose that E has rank 2. Then E is a pearl.*

Proof. By Theorem 4.17 we have $T = \Phi(E)$. By Lemma 4.18 we get $E^p = T^p$ and $T^p = \Phi(T)$ since T is abelian. Therefore

$$T = \Phi(E) = E^p[E, E] = \Phi(T)[E, E].$$

Hence $T = [E, E]$. In particular $|T| \leq p$ and either $E \cong \mathrm{C}_p \times \mathrm{C}_p$ or $E \cong p_+^{1+2}$. So E is a pearl. $\square$

Fusion systems on p-groups containing $\mathcal{F}$-essential subgroups that are pearls have been studied in Chapter 3. Therefore from now on we focus on $\mathcal{F}$-essential subgroups that have rank 3.

We see another consequence of Lemma 4.18.

Lemma 4.20. *Let Z be the preimage in N^1 of $\mathrm{Z}(\mathrm{N}^1/\Phi(E))$. Then $[E : Z] = p$, the group $\mathrm{N}^1/\Phi(E)$ has exponent p and $\Phi(E)$ $\mathrm{char}_{\mathcal{F}}$ N^1. Also, if E has rank 3 then N^1 has rank 3.*

Proof. If $\Phi(E) = T$ then the first statement follows from Theorem 4.17.

Suppose $\Phi(E) \neq T$. Then E has rank 3, $[E\colon T] = p^2$ and $T/\Phi(E) = \mathrm{C}_{E/\Phi(E)}(O^{p'}(\mathrm{Aut}_{\mathcal{F}}(E)))$ (Lemma 4.14(4)). Let $\tau \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(E))$ be an involution. Then by coprime action (Theorem 1.11) we have

$$E/\Phi(E) \cong \mathrm{C}_{E/\Phi(E)}(\tau) \times [E/\Phi(E), \tau] \cong T/\Phi(E) \times [E/\Phi(E), O^{p'}(\mathrm{Aut}_{\mathcal{F}}(E))].$$

Note that $T/\Phi(E)$ and $[E/\Phi(E), O^{p'}(\mathrm{Aut}_{\mathcal{F}}(E))]$ are normal subgroups of $\mathrm{N}^1/\Phi(E)$, so

$$[E/\Phi(E), O^{p'}(\mathrm{Aut}_{\mathcal{F}}(E))] \cap \mathrm{Z}(\mathrm{N}^1/\Phi(E)) \;\neq\; 1 \;\neq\; T/\Phi(E) \cap \mathrm{Z}(\mathrm{N}^1/\Phi(E)).$$

Hence we have $|\mathrm{Z}(\mathrm{N}^1/\Phi(E))| \geq p^2$ and since $[\mathrm{N}^1, E] \not\leq \Phi(E)$ by Lemma 2.35, we conclude that $|\mathrm{Z}(\mathrm{N}^1/\Phi(E))| = p^2$. In other words $[E\colon Z] = p$.

Since $[\mathrm{N}^1\colon Z] = p^2$ and the group N^1/Z is not cyclic by Lemma 1.6, we get $\Phi(\mathrm{N}^1) \leq Z$. By Lemma 4.18 there exists an element $h \in \mathrm{N}^1$ of order p such that $h \notin E$. Since $[\mathrm{N}^1\colon E] = p$ we deduce that $\mathrm{N}^1 = E\langle h\rangle$. Thus every element g of N^1 can be written as a product eh^i for some $e \in E$ and $1 \leq i \leq p$. Since $[e, h^i] \in \Phi(\mathrm{N}^1) \leq Z$, by Theorem 1.3 we get

$$g^p = (eh^i)^p = (h^i)^p e^p [h^i, e]^{\frac{p(p-1)}{2}} = 1 \mod \Phi(E).$$

Hence the group $\mathrm{N}^1/\Phi(E)$ has exponent p.

Suppose that E has rank 3. We now prove that N^1 has rank 3. By Lemma 1.6 the group $[\mathrm{N}^1/\Phi(E), \mathrm{N}^1/\Phi(E)]$ has order p and since $\mathrm{N}^1/\Phi(E)$ has exponent p we have $\Phi(\mathrm{N}^1/\Phi(E)) = [\mathrm{N}^1/\Phi(E), \mathrm{N}^1/\Phi(E)]$. Therefore $[\mathrm{N}^1/\Phi(E)\colon \Phi(\mathrm{N}^1/\Phi(E))] = p^3$. Since the group $\Phi(\mathrm{N}^1)$ is contained in the preimage in N^1 of $\Phi(\mathrm{N}^1/\Phi(E))$ and S has sectional rank 3, we deduce that $[\mathrm{N}^1\colon \Phi(\mathrm{N}^1)] = p^3$ and so N^1 has rank 3.

Finally we show that $\Phi(E)$ $\mathrm{char}_{\mathcal{F}}$ N^1. If $\Phi(E) = T$ then this follows from the definition of T. Thus we may assume $\Phi(E) < T$ and so $[E\colon T] = p^2$ and E has rank 3. Suppose for a contradiction that there exists $\alpha \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^1)$ such that $\Phi(E)\alpha \neq \Phi(E)$. Thus $T = \Phi(E)\Phi(E)\alpha$. In particular $T \leq \Phi(\mathrm{N}^1)$ and since N^1/T is non abelian by Lemma 4.16, we deduce $T < \Phi(\mathrm{N}^1)$. Hence $[\mathrm{N}^1\colon \Phi(\mathrm{N}^1)] \leq p^2$, contradicting the fact that N^1 has rank 3. Therefore $\Phi(E)$ is $\mathcal{F}$-characteristic in N^1. □

Thanks to Theorems 4.17 and 4.22 we have a better understanding of the shape of E. The next step toward the classification of saturated fusion systems on p-groups of sectional rank 3 is *to bound the index of E in S.*

The key idea to achieve this is to study the action of an $\mathcal{F}$-automorphism φ of E of order $p-1$ that is a restriction of an $\mathcal{F}$-automorphism of N^i, for some $i \geq 2$ (ideally φ will be a restriction of an $\mathcal{F}$-automorphism of S).

We already know that every morphism φ in $\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(E)}(\mathrm{Aut}_S(E))$ is the restriction of an $\mathcal{F}$-automorphism of N^1 (Lemma 2.8). Indeed, Lemma 2.41 applied to $\mathrm{N}^j = \mathrm{N}^2$ and $K = T$, implies the next result.

Lemma 4.21. *Suppose* $\mathrm{N}^1 < S$. *Then* $[\mathrm{N}^2\colon \mathrm{N}^1] = p$, $\mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^1) = \mathrm{Aut}_S(\mathrm{N}^1)\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^1)}(E)$ *and every morphism in* $\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(E)}(\mathrm{Aut}_S(E))$ *is a restriction of an $\mathcal{F}$-automorphism of* N^2.

We now consider separately the cases in which T has index p^2 or p^3 in E.

4.3.1 $\mathcal{F}$-essential subgroups with $\mathcal{F}$-core of index p^2.

In this subsection we work under the assumption of Hypothesis B.

Main Hypothesis B. Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on the p-group S, that has sectional rank 3. Let E be an $\mathcal{F}$-essential subgroup of S, let $T = \mathrm{core}_{\mathcal{F}}(E)$ and suppose that $[E\colon T] = p^2$.

Note that the assumption $[E\colon T] = p^2$ implies that E is not $\mathcal{F}$-characteristic in S. In particular, by Theorem 4.17 we have $T \leq \mathrm{Z}(E)$. We start determining the isomorphism type of E, proving Theorem 13.

Theorem 4.22. *Suppose that $E \leq S$ is an $\mathcal{F}$-essential subgroup of S and let $T = \mathrm{core}_{\mathcal{F}}(E)$. If $[E\colon T] = p^2$ and $|T| = p^a$ then*

- *either $E \cong \mathrm{C}_p \times \mathrm{C}_p \times \mathrm{C}_{p^a}$;*
- *or $E \cong \frac{\Omega_1(E)\times T}{(\mathrm{Z}(\Omega_1(E)=\Omega_1(T))} \cong p_+^{1+2} \circ \mathrm{C}_{p^a}$;*
- *or $E \cong p_+^{1+2} \times \mathrm{C}_{p^{a-1}}$.*

Proof. By Lemma 4.18 we have $E = T\Omega_1(E)$. Thus $\Omega_1(E)/\Omega_1(T) \cong E/T \cong \mathrm{C}_p \times \mathrm{C}_p$. Let $x, y \in E$ be such that $\Omega_1(E) = \langle x, y\rangle\Omega_1(T)$. By Theorem 4.17 we have $T \leq \mathrm{Z}(E)$.

Suppose E is abelian. Then $\langle x, y\rangle \cong \mathrm{C}_p \times \mathrm{C}_p$ and

$$E \cong \langle x, y\rangle \times T \cong \mathrm{C}_p \times \mathrm{C}_p \times \mathrm{C}_{p^a}.$$

Suppose E is non-abelian. Then $[x, y] \neq 1$, $[E, E] = \langle [x, y]\rangle$ and $\langle x, y\rangle \cong p_+^{1+2}$.

By Theorem 4.17 the group T is either cyclic or isomorphic to the group $C \times [E, E]$, for some cyclic group C of order p^{a-1}. If T is cyclic then $\langle [x, y]\rangle = [E, E] = \Omega_1(T)$ and so

$$E \cong \frac{\Omega_1(E) \times T}{(\ \mathrm{Z}(\Omega_1(E)) = \Omega_1(T)\ )} \cong p_+^{1+2} \circ \mathrm{C}_{p^a}.$$

If $T \cong C \times [E, E]$ then $\langle x, y\rangle \cap T = [E, E]$ and

$$E \cong p_+^{1+2} \times \mathrm{C}_{p^{a-1}}.$$

□

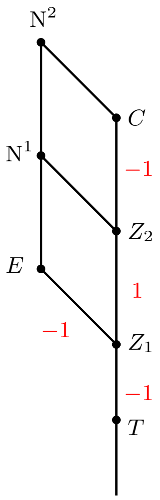
Note that the group E/T is a self-centralizing subgroup of the group $\mathrm{N}_S(T)/T$, isomorphic to the group $\mathrm{C}_p \times \mathrm{C}_p$. Thus the group $\mathrm{N}_S(T)/T$ has maximal nilpotency class by Lemma 3.2. Also E/T is a natural $\mathrm{SL}_2(p)$-module for $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E))$, so there exists an $\mathcal{F}$-automorphism φ of E that acts on E/T as $\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$, for some $\lambda \in \mathrm{GF}(p)$ of order $p-1$. Let Z_i be the preimage in $\mathrm{N}_S(T)$ of the group $\mathrm{Z}_i(\mathrm{N}^i/T)$. If the morphism φ is a restriction of a morphism of N^i, then it acts on the quotient $\mathrm{Z}_j/\mathrm{Z}_{j-1}$ for every $j \leq i$ and the action is the one described in Lemma 3.23. We will use this fact to find elementary abelian sections of S and so bound the index of E in S.

Recall that $T \trianglelefteq \mathrm{N}^2$ by the definition of T.

Lemma 4.23. *Suppose that* $[E\colon T] = p^2$ *and* $\mathrm{N}^1 < S$. *Then* $T \leq \mathrm{Z}(\mathrm{N}^2)$.

Proof. Let $Z_i \leq E$ be the preimage in E of $\mathrm{Z}_i(\mathrm{N}^2/T)$ for $i \in \{1,2\}$. The group N^2/T has maximal nilpotency class (since $E/T \cong \mathrm{C}_p \times \mathrm{C}_p$ is self-centralizing in N^2/T) and so $[Z_2\colon Z_1] = [Z_1\colon T] = p$ and $Z_2 \leq \mathrm{N}^1$.

By Lemma 4.21 and the fact that $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong \mathrm{SL}_2(p)$, there exists a morphism $\tau \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^2)$ such that $\tau|_E \in \mathrm{Aut}_{\mathcal{F}}(E)$ and τ acts on E/T as $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then τ acts as 1 on $\mathrm{N}^1/E \cong Z_2/Z_1$ and as -1 on $\mathrm{N}^2/\mathrm{N}^1$. Also, by Lemma 4.14 the morphism τ centralizes T.



Let $C \leq \mathrm{N}^2$ be the preimage in N^2 of $\mathrm{C}_{\mathrm{N}^2/T}(Z_2/T)$. By Theorem 4.17 we have $T \leq \mathrm{Z}(\mathrm{N}^1)$, so it is enough to prove that $[C,T] = 1$.

Let $x \in C$ be such that $C = \langle x \rangle Z_2$. Note that $x\tau = x^{-1}y$ for some $y \in Z_2$ and for every $t \in T$ we have $[x,t] \in T$ and $[y,t] = 1$. So by Theorem 1.3 we get

$$[x,t] = [x,t]\tau = [x\tau, t\tau] = [x^{-1}y, t] = [x^{-1}, t].$$

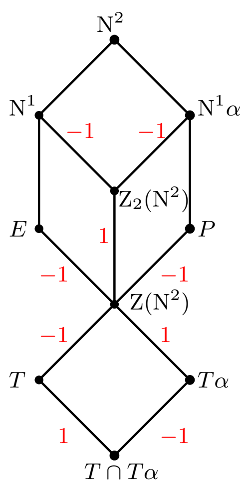
Therefore by Lemma 1.5 we deduce $[x,t] = 1$. Since this is true for every $t \in T$, we conclude that $T \leq \mathrm{Z}(C)$ and so $T \leq \mathrm{Z}(\mathrm{N}^2)$. $\square$

The next lemma shows that if $[E\colon T] = p^2$ then T is $\mathcal{F}$-characteristic in N^2. As a consequence we get $\mathrm{N}^3 \leq \mathrm{N}_S(T)$ and by Lemma 2.41 every morphism in $\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(E)}(\mathrm{Aut}_S(E))$ is a restriction of an $\mathcal{F}$-automorphism of the group N^3.

Lemma 4.24. *If* $[E\colon T] = p^2$ *and* $\mathrm{N}^1 < S$ *then* T $\mathrm{char}_{\mathcal{F}}$ N^2.

Proof. By Lemma 4.23 we have $T \leq \mathrm{Z}(\mathrm{N}^2)$. If $T = 1$ or $T = \mathrm{Z}(\mathrm{N}^2)$ then T char N^2. Suppose $1 \neq T < \mathrm{Z}(\mathrm{N}^2)$. Then $[E\colon \mathrm{Z}(\mathrm{N}^2)] = p$ and so E is abelian. Hence by Corollary 2.42 the group E has maximal normalizer tower in S and every morphism in $\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(E)}(\mathrm{Aut}_S(E))$ is a restriction of an $\mathcal{F}$-automorphism of S. Also, E satisfies all the properties listed in Theorem 1.27.

Suppose for a contradiction that there exists $\alpha \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^2)$ such that $T\alpha \neq T$. Then $\mathrm{N}^1\alpha \neq \mathrm{N}^1$, $\mathrm{N}^1(\mathrm{N}^1\alpha) = \mathrm{N}^2$, $\mathrm{N}^1 \cap \mathrm{N}^1\alpha = \mathrm{Z}_2(\mathrm{N}^2)$ and $TT\alpha = \mathrm{Z}(\mathrm{N}^2)$.



Let $\tau \in \mathrm{Aut}_{\mathcal{F}}(S)$ be morphism that acts on E/T as $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Note that the action of τ is the one described by Lemma 3.23. In particular τ acts as a scalar on $\mathrm{N}^2/\mathrm{Z}_2(\mathrm{N}^2)$. Hence τ normalizes $\mathrm{N}^1\alpha$, $T\alpha$ and $T \cap T\alpha$. By Lemma 4.14, the morphism τ centralizes T. Let $x \in \mathrm{N}^1\alpha \backslash \mathrm{Z}_2(\mathrm{N}^2)$ and $y \in \mathrm{Z}_2(\mathrm{N}^2) \backslash \mathrm{Z}(\mathrm{N}^2)$. Then $x\tau = x^{-1}z_1$ and $y\tau = yz_2$ for some $z_1, z_2 \in \mathrm{Z}(S)$. Hence by Lemma 1.4 we get

$$[x,y]\tau = [x^{-1}z_1, yz_2] = [x,y]^{-1}.$$

In particular, since τ centralizes $\mathrm{Z}(\mathrm{N}^2)/T\alpha$, we conclude that $[x,y] \in T\alpha$ and so $\mathrm{N}^1\alpha/T\alpha$ is abelian.

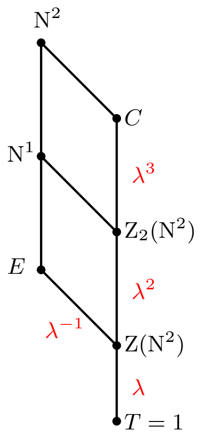
By Theorem 1.15 there exists a maximal subgroup P of $\mathrm{N}^1\alpha$ containing $\mathrm{Z}(\mathrm{N}^2)$ and distinct from $Z_2(N^2)$ that is normalized by τ. Note that such P is conjugate in N^2 to $E\alpha$ and so is $\mathcal{F}$-conjugate to E. Thus P is $\mathcal{F}$-essential by Theorem 2.26(5). Note that $N^1\alpha = N_S(E\alpha) = N_S(P)$ and $T\alpha = \mathrm{core}_{\mathcal{F}}(E\alpha) = \mathrm{core}_{\mathcal{F}}(P)$ by Lemma 4.11. Thus the group $N^1\alpha/T\alpha = N_S(P)/\mathrm{core}_{\mathcal{F}}(P)$ cannot be abelian by Lemma 4.16 and we have a contradiction. Therefore $T = T\alpha$ and $T \operatorname{char}_{\mathcal{F}} N^2$. $\square$

A priori the group N^2/T might have sectional rank 2. In the next lemma we show that if $p \geq 5$ and $N^1 < S$ then the group N^2/T has sectional rank 3 and is isomorphic to a Sylow p-subgroup of the group $\mathrm{Sp}_4(p)$.

Lemma 4.25. *Suppose $[E\colon T] = p^2$ and $N^1 < S$. Let C be the preimage of $C_{N^2/T}(Z_2(N^2/T))$. If $p \geq 5$ then C/T has exponent p and N^2/T is isomorphic to a Sylow p-subgroup of the group $\mathrm{Sp}_4(p)$.*

Proof. To simplify notation, suppose $T = 1$ (this is equivalent to work with the quotient group N^2/T). Note $E \cong C_p \times C_p$ is self-centralizing in N^2, so N^2 has maximal nilpotency class (by Lemma 3.2) and order p^4. Set $C = C_{N^2}(Z_2(N^2))$.

Let $\varphi \in \mathrm{Aut}_{\mathcal{F}}(N^2)$ be the morphism that acts on E as $\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$ for some $\lambda \in \mathrm{GF}(p)$ of order $p-1$. Then the action of φ is described in Lemma 3.23.



Action of φ on N^2

By Lemma 3.8 we have $\mathrm{Z}_2(\mathrm{N}^2)^p = 1$ and $C^p \leq \mathrm{Z}(\mathrm{N}^2)$. Also notice that the group C is abelian. Let $c \in C \backslash \mathrm{Z}_2(\mathrm{N}^2)$ and let $z \in \mathrm{Z}_2(\mathrm{N}^2)$ be such that $c\varphi = c^{\lambda^3} z$. Then by Theorem 1.3 we have

$$(c^p)\varphi = (c^{\lambda^3} z)^p = (c^p)^{\lambda^3} z^p [c^{\lambda^3}, z]^{\frac{p(p-1)}{2}} = (c^p)^{\lambda^3}.$$

If $c^p \neq 1$ then we have $\lambda^3 = \lambda \mod p$ and so $p = 3$. Hence if $p \geq 5$ the group C is elementary abelian. In particular the group N^2 has sectional rank 3 and since $E \nleq C$, by Lemma 3.14 we deduce that N^2 is isomorphic to a Sylow p-subgroup of the group $\mathrm{Sp}_4(p)$. □

We are now ready to determine a bound for the index of E in S when $p \geq 5$ and the isomorphism type of the group $\mathrm{N}^2/\Phi(E)$ when $p = 3$ and N^2/T has sectional rank 2 (completing the proof of Lemma 4.2).

Lemma 4.26. *Suppose $[E \colon T] = p^2$ and E has rank 3. Then either $E \trianglelefteq S$ or $p = 3$, N^2/T has exponent 9 and one of the following holds:*

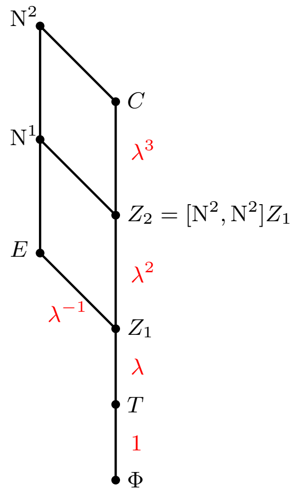
1. *either $\mathrm{N}^2 = S$, N^1 $\mathrm{char}_{\mathcal{F}}$ S and $S/\Phi(E) \cong \mathrm{SmallGroup}(3^5, 52)$;*
2. *or $\mathrm{N}^2/\Phi(E) \cong \mathrm{SmallGroup}(3^5, 53)$.*

Proof. Suppose that $\mathrm{N}^1 < S$ and set $\Phi = \Phi(E)$. By Lemma 4.20 we have $\Phi \trianglelefteq \mathrm{N}^2$ and we can consider the group N^2/Φ.

Note that E/Φ is a soft subgroup of N^2/Φ, so E/Φ satisfies the properties of Theorem 1.27 as a subgroup of N^2/Φ. In particular, if we denote by Z_1 the preimage in N^2 of $\mathrm{Z}(\mathrm{N}^1/\Phi)$ and we set $Z_2 = [\mathrm{N}^2, \mathrm{N}^2]Z_1$, then $\mathrm{N}^1/Z_1 \cong \mathrm{N}^2/Z_2 \cong \mathrm{C}_p \times \mathrm{C}_p$.

Let $\varphi \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^2)$ be the morphism that acts on E/T as $\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$ for some $\lambda \in \mathrm{GF}(p)$ of order $p-1$. Such a morphism exists by Lemma 4.21 and the fact that $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E)) \cong$

$\mathrm{SL}_2(p)$. Then φ acts on N^2/T as described in Lemma 4.25 and centralizes T by Lemma 4.14.



Action of φ on $\mathrm{N}^2/\Phi(E)$

Let $C \leq \mathrm{N}^2$ be the preimage in N^2 of $\mathrm{C}_{\mathrm{N}^2/T}(\mathrm{Z}_2(\mathrm{N}^2/T))$. We aim to prove that $C^p \not\leq T$.

By Lemma 4.20 the group N^1/Φ has exponent p. In particular the quotient Z_2/Φ is elementary abelian of order p^3.

From $[\mathrm{N}^1\colon E] = p$ and the fact that E/Φ is abelian, we deduce that $[[\mathrm{N}^1, \mathrm{N}^1]\Phi\colon \Phi] = p$. Also, $[\mathrm{N}^1, \mathrm{N}^1]\Phi = \Phi(\mathrm{N}^1)$, since N^1/Φ has exponent p and $T \neq \Phi(\mathrm{N}^1)$ by Lemma 4.16. Therefore T/Φ and $\Phi(\mathrm{N}^1)/\Phi$ are normal subgroup of N^2/Φ of order p and we get $T\Phi(\mathrm{N}^1)/\Phi \leq \mathrm{Z}(\mathrm{N}^2/\Phi) \leq Z_1/\Phi$. Therefore Z_1 is the preimage of $\mathrm{Z}(\mathrm{N}^2/\Phi)$. In particular Z_1/Φ is in the center of C/Φ. By definition of C, we have $[C, Z_2] \leq T$. Let $c \in C \backslash Z_2$ and $z \in Z_2 \backslash Z_1$. Then, recalling that C/Z_1 and Z_2/Φ are elementary abelian, we may assume $c\varphi = c^{\lambda^3}v$ for some $v \in Z_1$ and $z\varphi = z^{\lambda^2}u$ for some $u \in \Phi$. Since $[c, z] \in T \leq \mathrm{Z}(\mathrm{N}^2)$ and φ centralizes T, by Lemma 1.4 we get

$$[c, z] = [c, z]\varphi = [c^{\lambda^3}v, z^{\lambda^2}u] = [c, z]^{\lambda^5} \mod \Phi.$$

Since $\lambda^5 \neq 1 \mod p$, we deduce that $[c, z] \in \Phi$.

Therefore the group C/Φ is abelian of order p^4. Also,

$$(c^p)\varphi = (c^{\lambda^3} v)^p = v^p (c^p)^{\lambda^3} [v, c^{\lambda^3}]^{\frac{p(p-1)}{2}} = (c^p)^{\lambda^3} \mod \Phi.$$

Suppose that $C^p \leq T$. Since $\lambda^3 \neq 1 \mod p$, we deduce that $c^p \in \Phi$ and so the group C/T is elementary abelian of order p^4, contradicting the fact that S has sectional rank 3. Hence $C^p \nleq T$.

In particular by Lemma 4.25 we conclude that if $\mathrm{N}^1 < S$ then $p = 3$.

Suppose $\mathrm{N}^1 < S$ and $p = 3$ and consider the group $\overline{\mathrm{N}^2} = \mathrm{N}^2/\Phi$. Note that $\overline{C}$ is abelian and $\Omega_1(\overline{C}) = \overline{Z_2}$. So $|\overline{C}^3| = 3$. Since $C^3 \neq T$ and the groups $\overline{T}, \overline{C}^3$ and $\overline{\Phi(\mathrm{N}^1)}$ are subgroups of $\overline{Z_1}$ normalized by φ, we have $C^3\Phi = \Phi(\mathrm{N}^1)$. Since S has sectional rank 3 we deduce $\Phi(C) = \Phi(\mathrm{N}^1)$. Let $g \in \mathrm{N}^2$. Then $g = xc$ for some $x \in \mathrm{N}^1$ and $c \in C$ and by Theorem 1.3 we have

$$g^3 = (xc)^3 = c^3 x^3 [x, c]^3 = 1 \mod \Phi(C).$$

Hence the group $\mathrm{N}^2/\Phi(C)$ has exponent 3.

Recall that we showed that $\overline{Z_1} = \mathrm{Z}(\overline{\mathrm{N}^2})$ and since N^2/Z_1 is non-abelian (otherwise $E \trianglelefteq \mathrm{N}^2$), we conclude that the group $\overline{\mathrm{N}^2}$ has nilpotency class 3, that $\overline{Z_2} = \mathrm{Z}_2(\overline{\mathrm{N}^2})$ and $\overline{C} = \mathrm{C}_{\overline{\mathrm{N}^2}}(\overline{Z_2})$. Finally note that the group $\overline{\mathrm{N}^1}$ is a maximal subgroup of $\overline{\mathrm{N}^2}$ having exponent 3.

We enter this information in the computer program *Magma* to prove that $\overline{\mathrm{N}^2}$ is isomorphic to either SmallGroup$(3^5, 52)$ or SmallGroup$(3^5, 53)$.

```
N:=[];
for i in [1..NumberOfSmallGroups(3^5)] do S:=SmallGroup(3^5,i);
  if NilpotencyClass(S) eq 3 then
    C:=Centralizer(S,UpperCentralSeries(S)[3]);
  if Exponent(C) eq 9 and
  IsAbelian(C) eq true and
  Exponent(S/FrattiniSubgroup(C)) eq 3 then
    M:=[M : M in MaximalSubgroups(S) | Exponent(M`subgroup) eq 3];
  if ♯M ne 0 then
    Append(~N,i);
  end if; end if; end if;
end for; N;

Ouput:  [52, 53]
```

Suppose that $\overline{\mathrm{N}^2}$ is isomorphic to the group $\mathrm{SmallGroup}(3^5, 52)$. Then there is a unique maximal subgroup of $\overline{\mathrm{N}^2}$ having exponent 3, namely $\overline{\mathrm{N}^1}$. Note that if M is a maximal subgroup of $\overline{\mathrm{N}^2}$ and M/T has exponent 3, then $M^3 \leq T$ and since $\overline{\mathrm{N}^2}^3$ has order 3 and is not contained in T, we deduce that $M^3 = 1$ and so $M = \overline{\mathrm{N}^1}$. Thus the group N^1/Φ is the unique maximal subgroup of N^2/Φ having exponent 3, and is therefore characteristic in N^2/T. Since T $\mathrm{char}_{\mathcal{F}}$ N^2 by Lemma 4.24, we conclude that N^1 $\mathrm{char}_{\mathcal{F}}$ N^2 and so $\mathrm{N}^2 = S$. $\square$

We end this subsection showing that if we are in case 2 of Theorem 4.26 then we cannot bound the index of E in S.

Note that the group $\mathrm{SmallGroup}(3^5, 53)$ is isomorphic to a section of a Sylow 3-subgroup of the group $\mathrm{SL}_4(19)$ and of a Sylow 3-subgroup of the group $\mathrm{SL}_4(109)$ (this can be checked with a computer program, for example *Magma*). In general, let q be an odd prime power such that $q \equiv 1 \mod 3$ and let S be a Sylow 3-subgroup of the group

$G = \mathrm{SL}_4(q)$. Then $S \cong \mathrm{C}_{3^k} \wr \mathrm{C}_3$ is a 3-group of sectional rank 3 and order 3^{3k+1}, where $k \geq 1$ is such that 3^k is the largest power of 3 dividing $q-1$. In the next lemma we characterize the $\mathcal{F}_S(G)$-essential subgroups of S.

Lemma 4.27. *Let q be a prime power such that $q \equiv 1 \mod 3$ and let S be a Sylow 3-subgroup of the group $G = \mathrm{SL}_4(q)$. Let $k \geq 1$ is such that 3^k is the largest power of 3 dividing $q-1$. Then there exist $\mathcal{F}_S(G)$-essential subgroups A and E of S such that*

- *$A \cong \mathrm{C}_{3^k} \times \mathrm{C}_{3^k} \times \mathrm{C}_{3^k}$ is the unique abelian subgroup of S having index 3 in S, and is therefore characteristic in S;*
- *$E \cong \mathrm{C}_{3^k} \circ 3^{1+2}_+ = \frac{\mathrm{C}_{3^k} \times 3^{1+2}_+}{(\Omega_1(\mathrm{C}_{3^k}) = \mathrm{Z}(3^{1+2}_+))}$ is such that $\mathrm{core}_{\mathcal{F}_{\mathrm{S}}(\mathrm{G})}(E) = \mathrm{Z}(E) = \mathrm{Z}(S) \cong \mathrm{C}_{3^k}$ has index 3^2 in E.*

Also, every $\mathcal{F}_S(G)$-essential subgroup of S is of this form.

Proof. Let $\lambda \in \mathrm{GF}(q)$ be an element of order 3^k and consider the following elements of G:

$$a_1 := \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda^{-1} \end{pmatrix} \quad a_2 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda^{-1} \end{pmatrix} \quad a_3 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda^{-1} \end{pmatrix} \quad x := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Set $A := \langle a_1, a_2, a_3 \rangle$. Then $A \cong \mathrm{C}_{3^k} \times \mathrm{C}_{3^k} \times \mathrm{C}_{3^k}$ and we may assume $S = A \colon \langle x \rangle$.

If $u \in \mathrm{Z}(S)$, then u commutes with x and since every matrix in G has determinant 1, we deduce that $u = \begin{pmatrix} \mu & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu^{-3} \end{pmatrix}$, for some $\mu \in \mathrm{GF}(q)$ having order a power of 3.

Thus the group $\mathrm{Z}(S)$ is cyclic and generated by $z = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda^{-3} \end{pmatrix}$.

Finally, let

$$y = \begin{pmatrix} \mu^{-1} & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ where } \mu = \lambda^{3^{k-1}}, \text{ and set } E := \langle x, y, z \rangle.$$

Then $\mathrm{Z}(S) = \langle z \rangle = \mathrm{Z}(E)$, $[x, y] = z^{3^{k-1}}$ and $E \cong \mathrm{C}_{3^k} \circ 3^{1+2}_{+}$.

It is easy to see that $\mathrm{C}_S(A) = A$ (since A has index 3 in S and S is not abelian) and $\mathrm{C}_S(E) = \mathrm{Z}(E)$ (because every element of S centralizing x is a power of z, as shown before).

We can also prove that

$$\mathrm{Out}_G(A) \cong \mathrm{Sym}(4) \cong \mathrm{PGL}_2(3) \text{ and } \mathrm{Out}_G(E) \cong \mathrm{SL}_2(3).$$

Thus $\mathrm{Out}_G(A)$ and $\mathrm{Out}_G(E)$ have a strongly 3-embedded subgroup.

Finally, A is normal in S and so is fully normalized in $\mathcal{F}_S(G)$. As for E, it is G-conjugate to a fully normalized subgroup P of S (since $\mathcal{F}_S(G)$ is saturated) and we can show $[\mathrm{N}_S(P) \colon P] = 3$, so E is fully normalized in $\mathcal{F}_S(G)$.

Hence A and E are $\mathcal{F}_S(G)$-essential subgroups of S.

Also note that A is $\mathcal{F}$-characteristic in S and $\mathrm{Z}(E) = \mathrm{Z}(S) = \mathrm{core}_{\mathcal{F}_{\mathrm{S}}(\mathrm{G})}(E)$ has index 3^2 in E.

We now prove that if P is an $\mathcal{F}_S(G)$-essential subgroup of S, then P is either A or it is isomorphic to E. The proof of this statement is based on the results and terminology used in[AF90].

Suppose $P \leq S$ is $\mathcal{F}_S(G)$-essential. Then P is $\mathcal{F}_S(G)$-radical and so $P = O_3(\mathrm{N}_G(P))$. Thus P is a 3-radical subgroup of G, according to the definition given in [AF90]. Let $X = O_3(\mathrm{Z}(\mathrm{GL}_4(q))) \cong \mathrm{C}_{3^k}$. Note that there is a bijection:

$$\begin{array}{rcl}\{\text{3-radical subgroups of } \mathrm{GL}_4(q)\} & \to & \{\text{3-radical subgroups of } \mathrm{SL}_4(q)\} \\ R & \mapsto & R \cap \mathrm{SL}_4(q) \\ Q \times X & \leftarrow\!\shortmid & Q\end{array}$$

We first determine the 3-radical subgroups of $\mathrm{GL}_4(q)$. Set $R = P \times X$ and let V be a vector space of dimension 4 over $\mathrm{GF}(q)$ so that $\mathrm{GL}_4(q) = \mathrm{GL}_4(V)$. By [AF90, Theorem (4A)] there exist decompositions

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_s \text{ and } R = R_0 \times R_1 \times \cdots \times R_s,$$

such that $V_0 = \mathrm{C}_V(R)$, R_0 is the trivial subgroup of $\mathrm{GL}(V_0)$ and R_i is a basic subgroup of $\mathrm{GL}(V_i)$. Note that $\mathrm{GL}(V_i) = \mathrm{GL}_d(q)$ with $d \leq 4$.

Following the notation of [AF90], let $B = R_{m,\alpha,\gamma,c}$ be a basic subgroup of $\mathrm{GL}(d,q)$. Then $d = m3^{\alpha+\gamma+c}$. Since $d \leq 4$, one of the following holds:

- $\alpha = \gamma = c = 0$ and $B \cong \mathrm{C}_{3^k}$;
- $d = 3$, $m = \alpha = 1$, $\gamma = c = 0$ and $B \cong \mathrm{C}_{3^{k+1}}$;
- $d = 3$, $m = \gamma = 1$, $\alpha = c = 0$ and $B \cong \mathrm{C}_{3^k} \circ 3^{1+2}_+$;
- $d = 3$, $m = c = 1$, $\alpha = \gamma = 0$ and $B \cong \mathrm{C}_{3^k} \wr \mathrm{C}_3$.

Note that $\mathrm{Z}(S \times X) = \mathrm{Z}(S) \times X \leq R$. In particular $V_0 = \mathrm{C}_V(R) \leq \mathrm{C}_V(\mathrm{Z}(S) \times X) = 0$. Thus $V = V_1 \oplus \cdots \oplus V_s$ with $s \leq 4$.

We write $4 = a_1 + a_2 + a_3 + a_4$ where $a_i = \dim(V_i)$ and we get the following characterizations of R depending on the partition of $\dim(V) = 4$.

- $4 = 1 + 1 + 1 + 1$ and $R \cong \mathrm{C}_{3^k} \times \mathrm{C}_{3^k} \times \mathrm{C}_{3^k} \times \mathrm{C}_{3^k}$;
- $4 = 1 + 1 + 2$ and $R \cong \mathrm{C}_{3^k} \times \mathrm{C}_{3^k} \times \mathrm{C}_{3^k}$;
- $4 = 2 + 2$ and $R \cong \mathrm{C}_{3^k} \times \mathrm{C}_{3^k}$;
- $4 = 1 + 3$ and
    1. $R \cong \mathrm{C}_{3^k} \times \mathrm{C}_{3^{k+1}}$ or
    2. $R \cong \mathrm{C}_{3^k} \times (\mathrm{C}_{3^k} \circ 3_+^{1+2})$ or
    3. $R \cong \mathrm{C}_{3^k} \times \mathrm{C}_{3^k} \wr \mathrm{C}_3 \cong X \times S$.

By assumption $P \cong R/\mathrm{C}_{3^k}$ and $P < S$. Also, P has rank 3 (since S is not a 3-group of maximal nilpotency class). Therefore either $P \cong \mathrm{C}_{3^k} \times \mathrm{C}_{3^k} \times \mathrm{C}_{3^k}$ (and so $P = A$) or $P \cong \mathrm{C}_{3^k} \circ 3_+^{1+2} \cong E$.

$\square$

4.3.2 $\mathcal{F}$-essential subgroups with $\mathcal{F}$-core of index p^3.

We continue our analysis considering the $\mathcal{F}$-essential subgroups E with $\mathcal{F}$-core T of index p^3. Note that this implies that E has rank 3, $T = \Phi(E)$ and E is not $\mathcal{F}$-characteristic in S.

For the rest of this subsection we assume the following.

Main Hypothesis C. Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on the p-group S, that has sectional rank 3. Let E be an $\mathcal{F}$-essential subgroup of S, let $T = \mathrm{core}_{\mathcal{F}}(E)$ and suppose that $[E\colon T] = p^3$.

We first assume that E is not normal in S and we describe the structure of the group N^2/T, showing that the only possibility is $p = 3$ and proving the first part of Lemma 4.4.

Theorem 4.28. *Suppose that $\mathrm{N}^1 < S$ and write Z_i for the preimage in N^2 of the group $\mathrm{Z}_i(\mathrm{N}^2/T)$. Then $p = 3$ and if C is the preimage in N^2 of $\mathrm{C}_{\mathrm{N}^2/T}(\Phi(\mathrm{N}^2))$, then the following holds*

1. $Z_1 = \Phi(\mathrm{N}^1)$ *and* $[Z_1\colon T] = 3$;
2. $\mathrm{N}^2 = \mathrm{N}^1 C$ *and* $Z_2 = \mathrm{N}^1 \cap C$;
3. Z_2/T *is elementary abelian of order* 3^3;
4. $\Phi(\mathrm{N}^2)/T = \mathrm{Z}(C/T)$ *and* $|\mathrm{Z}(C/T)| = 3^2$;
5. N^2/T *has exponent* 9 *and* $(\mathrm{N}^2)^3 T = Z_1$;
6. *there exists* $\varphi \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^2)$ *such that* $\varphi|_E \in \mathrm{Aut}_{\mathcal{F}}(E)$ *and* φ *normalizes* N^1 *and* C *and acts as in Figure 4.1.*

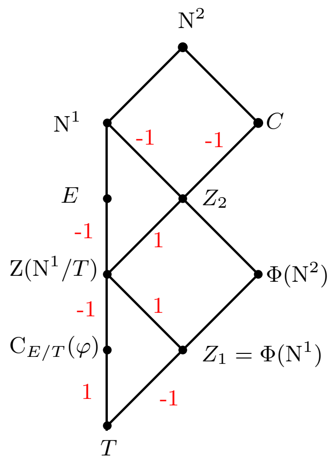


Figure 4.1

The proof of this theorem is long and mostly technical. Again, the key idea is that the structure of the group N^2/T is determined by the action of an $\mathcal{F}$-automorphism φ of

N^2 that acts on E/T as

$$\begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ for some } \lambda \in \mathrm{GF}(p) \text{ of order } p-1.$$

Proof. First notice that $T \trianglelefteq \mathrm{N}^2$ so we can consider the group N^2/T. To simplify notation we assume $T = 1$ (this will not create any problem since we are going to work on the quotient N^2/T, considering normal subgroups and groups normalized by an $\mathcal{F}$-automorphism of N^2 that centralizes T).

Recall that by Theorem 4.8 we have

$$\mathrm{SL}_2(p) \leq \mathrm{Out}_{\mathcal{F}}(E) \leq \mathrm{GL}_2(p) \times \mathrm{GL}_1(p).$$

Set $C = \mathrm{C}_E(O^{p'}(\mathrm{Aut}_{\mathcal{F}}(E)))$. Then $|C| = p$ and by maximality of T the group C is not $\mathcal{F}$-characteristic in N^1. By Lemma 4.21 we have $\mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^1) = \mathrm{Aut}_S(\mathrm{N}^1)\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^1)}(E)$. Since $\mathrm{Aut}_S(\mathrm{N}^1) \cong \mathrm{N}^2/\mathrm{Z}(\mathrm{N}^1)$ and C is $\mathcal{F}$-characteristic in E, we deduce that C is not normal in N^2. In particular $C \not\leq Z_1$. By Theorem 4.17 we have $|Z(\mathrm{N}^1)| = p^2$. Also, $C \leq \mathrm{Z}(\mathrm{N}^1)$. Hence we have $Z_1 < \mathrm{Z}(\mathrm{N}^1)$ and so $|Z_1| = p$.

Let $\varphi \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^2)$ be the morphism that centralizes C and acts on E/C as

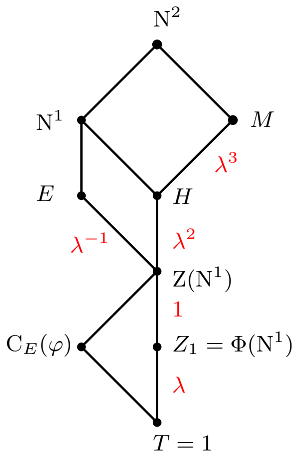
$$\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}, \text{ for some } \lambda \in \mathrm{GF}(p) \text{ of order } p-1,$$

with respect to the basis $\{xC, zC\}$, where $x \in E \backslash \mathrm{Z}(\mathrm{N}^1)$ and $z \in \mathrm{Z}(\mathrm{N}^1) \backslash C$. Note that φ centralizes T by Theorem 4.14, so every characteristic subgroup of N^2/T is normalized by φ. Thus we can keep working under the assumption $T = 1$.

Since E is abelian and $[\mathrm{N}^1 \colon E] = p$, the group E is a soft subgroup of N^2. In particular if we set $H = \mathrm{Z}(\mathrm{N}^1)[\mathrm{N}^2, \mathrm{N}^2]$, then by Theorem 1.27 we have $H \leq \mathrm{N}^1$, $\mathrm{N}^2/H \cong \mathrm{C}_p \times \mathrm{C}_p$

and H is normalized by φ. Hence by Theorem 1.15 there exists a maximal subgroup M of N^2 containing H and distinct from N^1 that is normalized by φ.

We now study the action of φ on N^2.



Since the action of φ on $\mathrm{Z}(\mathrm{N}^1)$ is not scalar, we deduce that C and Z_1 are the only maximal subgroups of $\mathrm{Z}(\mathrm{N}^1)$ normalized by φ. Note that N^1 has exponent p by Theorem 4.17 and $[\mathrm{N}^1 \colon \mathrm{Z}(\mathrm{N}^1)] = p^2$. Thus by Lemma 1.6 we have $|\Phi(\mathrm{N}^1)| = p$. Also, $\Phi(\mathrm{N}^1) \leq \mathrm{Z}(\mathrm{N}^1)$ is normalized by φ and normal in N^2. Thus we have $\Phi(\mathrm{N}^1) = Z_1$. In particular $[E, H] = Z_1$ (since N^1 is non-abelian by Lemma 4.16).

We first prove that φ acts as λ^2 on $H/\mathrm{Z}(\mathrm{N}^1)$ and as λ^3 on M/H.

Let $x \in E\backslash\mathrm{Z}(\mathrm{N}^1)$ and $h \in H\backslash\mathrm{Z}(\mathrm{N}^1)$. Then $x\varphi = x^{\lambda^{-1}}$ and $h\varphi = h^a z$ for some $z \in \mathrm{Z}(\mathrm{N}^1)$. Thus by Lemma 1.4 we have

$$[x,h]^{\lambda} = [x,h]\varphi = [x^{\lambda^{-1}}, h^a z] = [x,h]^{\lambda^{-1}a}.$$

Hence $a = \lambda^2$.

Note that $H = \mathrm{Z}(\mathrm{N}^1)[\mathrm{N}^1, M]$ and $H/\mathrm{Z}(\mathrm{N}^1) = \mathrm{Z}(\mathrm{N}^2/\mathrm{Z}(\mathrm{N}^1))$. Let $y \in \mathrm{N}^1\backslash H$ and $g \in M\backslash H$. Then $y\varphi = y^{\lambda^{-1}}z$ for some $z \in \mathrm{Z}(\mathrm{N}^1)$ and $g\varphi = g^b h$ for some $h \in H$. Hence by Lemma 1.4 and the fact that $[y,g] \nleq \mathrm{Z}(\mathrm{N}^1)$, we have

$$[y,g]^{\lambda^2} = [y^{\lambda^{-1}}z, g^b h] = [y,g]^{\lambda^{-1}b} \mod \mathrm{Z}(\mathrm{N}^1).$$

Hence $b = \lambda^3$.

Therefore φ acts as λ^2 on $H/\mathrm{Z}(\mathrm{N}^1)$ and as λ^3 on M/H.

Consider the quotient M/Z_1. If $h \in H\backslash \mathrm{Z}(\mathrm{N}^1)$, $g \in M\backslash H$ and $k \in H$ is such that $g\varphi = g^{\lambda^3}k$, then we have

$$[h,g] = [h,g]\varphi = [h^{\lambda^2}, g^{\lambda^3}k] = [h,g]^{\lambda^5} \mod Z_1.$$

Since $\lambda^5 \neq 1 \mod p$, we deduce that $[g,h] \in Z_1$ and the group M/Z_1 is abelian. In particular $H/Z_1 = \mathrm{Z}(\mathrm{N}^2/Z_1)$ and so $H = Z_2$. Also, the group Z_2 is elementary abelian (since N^1 has exponent p and $[H\colon \mathrm{Z}(\mathrm{N}^1)] = p$).

Next step is to prove that $p = 3$ (and so we can choose $\lambda = -1$). Let $c \in C$ and $g \in M\backslash H$. Then $c\varphi = c$ and $g\varphi = g^{\lambda^3}k$ for some $k \in H$ and we have

$$[c,g]\varphi = [c, g^{\lambda^3}k] = [c,g]^{\lambda^3}.$$

Note that $[c,g] \in [\mathrm{Z}(\mathrm{N}^1), M] = Z_1$ (since $[\mathrm{Z}(\mathrm{N}^1), M]$ is a proper subgroup of $\mathrm{Z}(\mathrm{N}^1)$ normalized by φ and C is not normal in M). If $\lambda^3 \neq \lambda \mod p$, then $[c,g] = 1$ and $C \leq \mathrm{Z}(M)$, contradicting the fact that C is not normal in $\mathrm{N}^2 = \mathrm{N}^1 M$. Thus we have $\lambda^3 = \lambda \mod p$, that implies $p = 3$. In particular we can choose $\lambda = -1$.

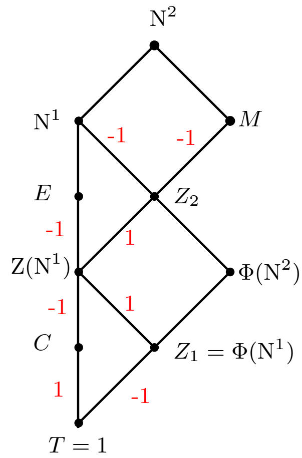
Recall that the group M/Z_1 is abelian and Z_2/Z_1 is elementary abelian. Since φ inverts M/H and centralizes $\mathrm{Z}(\mathrm{N}^1)/Z_1$, we deduce that M/Z_1 is elementary abelian. Every element u of N^2 can be written as a product xg, where $x \in \mathrm{N}^1$ and $g \in M$. So by Lemma 1.3 we have

$$u^3 = (xg)^3 = g^3x^3[g,x]^3 = 1 \mod Z_1.$$

Thus N^2/Z_1 has exponent 3.

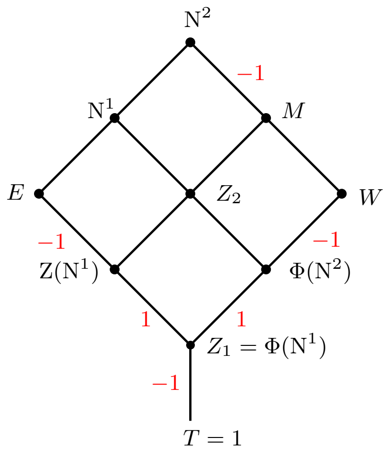
In particular $\Phi(\mathrm{N}^2/Z_1) = [\mathrm{N}^2/Z_1, \mathrm{N}^2/Z_1]$ and since $[\mathrm{N}^2\colon Z_2] = 3^2$, by Lemma 1.6 we deduce that $|\Phi(\mathrm{N}^2/Z_1)| = 3$. Since $Z_1 = \Phi(\mathrm{N}^1) \leq \Phi(\mathrm{N}^2)$ we deduce that $\Phi(\mathrm{N}^2/Z_1) =$

$\Phi(\mathrm{N}^2)/Z_2$ and so $\Phi(\mathrm{N}^2)/Z_1 = 3$. Also, $\Phi(\mathrm{N}^2) \neq \mathrm{Z}(\mathrm{N}^1)$ because E is not normal in N^2.



Note that $\Phi(\mathrm{N}^2) \cong \mathrm{C}_3 \times \mathrm{C}_3$ is a normal subgroup of N^2 not centralized by N^2. Hence $\mathrm{N}^2/\mathrm{C}_{\mathrm{N}^2}(\Phi(\mathrm{N}^2))$ is isomorphic to a subgroup of $\mathrm{Aut}(\Phi(\mathrm{N}^2)) \cong \mathrm{GL}_2(3)$. Therefore the group $\mathrm{C}_{\mathrm{N}^2}(\Phi(\mathrm{N}^2))$ has index 3 in N^2, is distinct from N^1 and contains Z_2. Also, $\mathrm{C}_{\mathrm{N}^2}(\Phi(\mathrm{N}^2))$ is characteristic in N^2 and so is normalized by φ. Therefore we can assume $M = \mathrm{C}_{\mathrm{N}^2}(\Phi(\mathrm{N}^2))$. Since M is non-abelian (we saw that $C \not\leq \mathrm{Z}(M)$), we conclude that $\Phi(\mathrm{N}^2) = \mathrm{Z}(M)$.

Note that $M^3 \leq (\mathrm{N}^2)^3 \leq Z_1$. It remains to prove that N^2 has exponent 9.
If M has exponent 9 then $M^3 = Z_1$ and N^2 has exponent 9.



Suppose that M has exponent 3. Since $M/\Phi(\mathrm{N}^2) \cong \mathrm{C}_p \times \mathrm{C}_p$, by Theorem 1.15 there exists a maximal subgroup W of M such that $\Phi(\mathrm{N}^2) \leq W$, $W \neq Z_2$ and W is normalized by φ. From $[W : \mathrm{Z}(M)] = 3$, we deduce that W is abelian. Hence W is elementary abelian. Also, $W = \mathrm{C}_{\mathrm{N}^2}(W)$ since $M = \mathrm{C}_{\mathrm{N}^2}(\Phi(\mathrm{N}^2))$ is non-abelian, and W is normal in N^2 because it contains $\Phi(\mathrm{N}^2)$.

Thus N^2/W acts faithfully on W and is therefore isomorphic to a subgroup of a Sylow

3-subgroup of the group $\mathrm{GL}_3(3)$. Hence every element of N^2/W can be written in the following form:

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \qquad \text{for some } a, b, c \in \mathbb{F}_3.$$

Note that both $\Phi(\mathrm{N}^2)$ and W/Z_1 are not centralized by N^2/M. Since N^2/W has order 3^2, we have

$$\mathrm{N}^2/W \quad \cong \langle \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \rangle.$$

Finally notice that if we take $e_1, e_2 \in E$ such that $E = \langle e_1, e_2 \rangle Z_1$, then $\langle e_1, e_2 \rangle \cong \mathrm{C}_3 \times \mathrm{C}_3$ is a complement for W in N^2, acting on W as described above. Therefore

$$\mathrm{N}^2 \quad \cong \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ y & a & 1 & 0 \\ z & b & a & 1 \end{pmatrix} \mid a, b, x, y, z \in \mathbb{F}_3 \right\}.$$

In particular N^2 has exponent 9 (take for example the matrix with $a = b = x = y = z = 1$). $\square$

For the rest of this subsection we focus on 3-groups.

Lemma 4.29. *Suppose that $p = 3$, $[E \colon T] = 3^3$ and $\mathrm{N}^1 < S$. Then T $\mathrm{char}_{\mathcal{F}}$ N^2.*

Proof. By Theorem 4.28 we know almost all the structure of the group N^2/T and the action on N^2/T of the $\mathcal{F}$-automorphism φ of N^2 that acts on E/T as

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

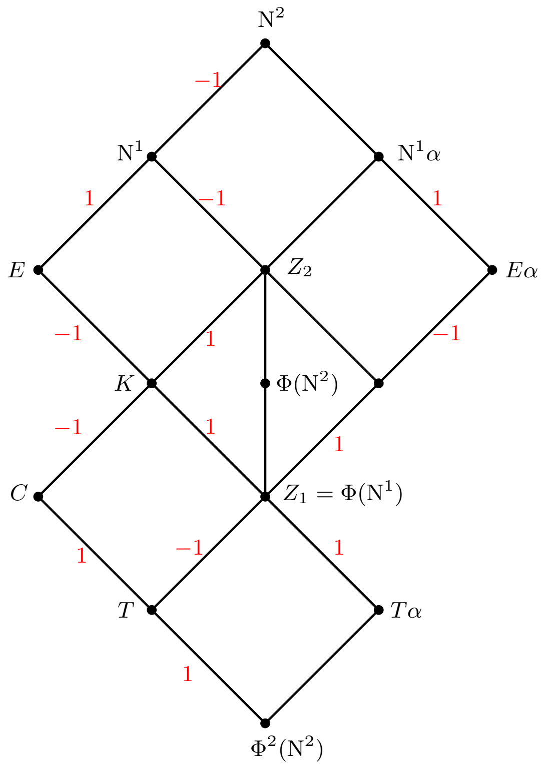
Note in particular that $\Phi(\mathrm{N}^2)/T \cong \mathrm{C}_3 \times \mathrm{C}_3$.

Suppose for a contradiction that T is not $\mathcal{F}$-characteristic in N^2. Set

$$\Phi^2(\mathrm{N}^2) = \Phi(\Phi(\mathrm{N}^2)).$$

Then $\Phi^2(\mathrm{N}^2) < T$ and since S has sectional rank 3 we have $[T\colon \Phi^2(\mathrm{N}^2)] = 3$. Let $\alpha \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^2)$ be such that $T \neq T\alpha$. Then $\mathrm{N}^1\alpha \neq \mathrm{N}^1$, $E\alpha \neq E$ and $T \cap T\alpha = \Phi^2(\mathrm{N}^2)$.

Write Z_i for the preimage in N^2 of $\mathrm{Z}_i(\mathrm{N}^2/T)$. Then by Theorem 4.28(1,3) we have $[Z_1\colon T] = 3$ and $[\mathrm{N}^2\colon Z_2] = 3^2$. Since $T \trianglelefteq \mathrm{N}^2$ and is not $\mathcal{F}$-characteristic in N^2, the group $T/\Phi^2(\mathrm{N}^2)$ is properly contained in the center of $\mathrm{N}^2/\Phi^2(\mathrm{N}^2)$. Hence we deduce that Z_1 is the preimage of $\mathrm{Z}(\mathrm{N}^2/\Phi^2(\mathrm{N}^2))$ and Z_2 is the preimage of $\mathrm{Z}_2(\mathrm{N}^2/\Phi^2(\mathrm{N}^2))$. In particular $Z_1 = Z_1\alpha = TT\alpha$ and $Z_2 = Z_2\alpha = \mathrm{N}^1 \cap \mathrm{N}^1\alpha$.



Since φ acts as scalar on N^2/Z_2, it normalizes the group $\mathrm{N}^1\alpha$. Note that $E\alpha$ is an $\mathcal{F}$-essential subgroup of S by Lemma 2.26(5), $\mathrm{N}^1\alpha = \mathrm{N}_S(E\alpha)$ and $T\alpha = \mathrm{core}_{\mathcal{F}}(E\alpha)$ by Lemma 4.11. Thus $T\alpha$ is $\mathcal{F}$-characteristic in $\mathrm{N}^1\alpha$ and so it is normalized by φ.

Let K be the preimage in N^1 of $\mathrm{Z}(\mathrm{N}^1/T)$. Then $K\alpha$ is the preimage in $\mathrm{N}^1\alpha$ of $\mathrm{Z}(\mathrm{N}^1\alpha/T\alpha)$ and both K/Z_1 and $K\alpha/Z_1$ are centralized by φ (not that we might have $K = K\alpha$). Since $\mathrm{N}^1\alpha/K\alpha \cong \mathrm{N}^1/K \cong \mathrm{C}_3 \times \mathrm{C}_3$ and φ normalizes Z_2, by Theorem 1.15 there exists a maximal subgroups Q of $\mathrm{N}^1\alpha$ containing $K\alpha$ and distinct from Z_2 that is normalized by φ. Note that Q is conjugate to $E\alpha$ by an element g of N^2. So Q is $\mathcal{F}$-essential by Lemma 2.26(5) and $T\alpha = \mathrm{core}_{\mathcal{F}}(Q)$ by Lemma 4.11. Thus we may assume $Q = E\alpha$ (by replacing α by $c_g\alpha$). Hence $E\alpha$ is normalized by φ. Looking at the action of φ on $\mathrm{N}^1\alpha$ and using Lemma 1.4, we deduce that the group $\mathrm{N}^1\alpha/T\alpha$ is abelian, contradicting Lemma 4.16. Therefore T has to be $\mathcal{F}$-characteristic in N^2. □

In the next lemma we show that the group N^3 is the normalizer in S of T.

Lemma 4.30. *Suppose that $p = 3$ and $[E\colon T] = 3^3$. If $\mathrm{N}^2 < S$ then $\mathrm{N}^3 = \mathrm{N}_S(T)$.*

Proof. Note that $T \trianglelefteq \mathrm{N}^3$ by Lemma 4.29. If $\mathrm{N}^3 = S$ then $\mathrm{N}^3 = \mathrm{N}_S(T)$. Suppose $\mathrm{N}^3 < S$. By Lemma 2.41 applied to $\mathrm{N}^j = \mathrm{N}^3$ and $K = T$ we get that $[\mathrm{N}^3\colon \mathrm{N}^2] = 3$ and every morphism of $\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(E)}(\mathrm{Aut}_S(E))$ is a restriction of an $\mathcal{F}$-automorphism of N^3.

To simplify notation we assume $T = 1$, taking care to not use the fact that 1 is characteristic in N^3.

Set $Z_i = \mathrm{Z}_i(\mathrm{N}^2)$. By Theorem 4.28 we know the structure of the group N^2. In particular $|Z_1| = 3$, $Z_2 \cong \mathrm{C}_3 \times \mathrm{C}_3 \times \mathrm{C}_3$ and $\mathrm{C}_{\mathrm{N}^2}(Z_2) = Z_2$.

Set $H = \mathrm{Z}(\mathrm{N}^1)[\mathrm{N}^3, \mathrm{N}^3]$. Since E is abelian and $[\mathrm{N}^1\colon E] = 3$, by Theorem 1.27 we have $\mathrm{N}^3/H \cong \mathrm{C}_3 \times \mathrm{C}_3$ and $Z_2 \le H$.

Since T $\mathrm{char}_{\mathcal{F}}$ N^2 by Lemma 4.29, the group Z_2 is $\mathcal{F}$-characteristic in N^2 and therefore normal in N^3. Hence $\mathrm{C}_{\mathrm{N}^3}(Z_2) \trianglelefteq \mathrm{N}^3$ and

$$[\mathrm{C}_{\mathrm{N}^3}(Z_2), \mathrm{N}^1] \leq \mathrm{C}_{\mathrm{N}^3}(Z_2) \cap [\mathrm{N}^3, \mathrm{N}^2] \leq \mathrm{C}_{\mathrm{N}^3}(Z_2) \cap \mathrm{N}^2 = \mathrm{Z}_2 \leq \mathrm{N}^1.$$

Therefore $\mathrm{C}_{\mathrm{N}^3}(Z_2) \leq \mathrm{N}_S(\mathrm{N}^1) = \mathrm{N}^2$ and so $\mathrm{C}_{\mathrm{N}^3}(Z_2) = Z_2$.

Suppose that $Z_2 \trianglelefteq \mathrm{N}^4$. Then

$$[\mathrm{C}_{\mathrm{N}^4}(Z_2), \mathrm{N}^1] \leq \mathrm{C}_{\mathrm{N}^4}(Z_2) \cap [\mathrm{N}^4, \mathrm{N}^3] \leq \mathrm{C}_{\mathrm{N}^4}(Z_2) \cap \mathrm{N}^3 = \mathrm{Z}_2 \leq \mathrm{N}^1.$$

So $\mathrm{C}_{\mathrm{N}^4}(Z_2) = Z_2$. Hence $\mathrm{N}^4/\mathrm{Z}_2$ is isomorphic to a subgroup of $\mathrm{Aut}(Z_2)$. However $Z_2 \cong \mathrm{C}_3 \times \mathrm{C}_3 \times \mathrm{C}_3$ and $[\mathrm{N}^4 \colon Z_2] \geq 3^4$, giving a contradiction.

Therefore we have $Z_2 \neq Z_2^g$ for some $g \in \mathrm{N}^4$.

Suppose that $Z_2 Z_2^g = H$ and let $h \in H$. Then $h = xy$ for some $x \in Z_2$ and $y \in Z_2^g$. Note that Z_2 and Z_2^g are elementary abelian and $[Z_2, Z_2^g] \leq Z_1 = \mathrm{Z}(\mathrm{N}^3)$ by definition of Z_2. Hence

$$h^3 = (xy)^3 = y^3 x^3 [y, x]^3 = 1.$$

Thus the group H has exponent 3. Note that the maximal subgroups of N^2 containing Z_2 are H and the conjugates of N^1, all having of exponent 3. Therefore N^2 has exponent 3, contradicting Lemma 4.28(5).

Thus $Z_2 Z_2^g \neq H$. In particular H is not normal in N^4. By Theorem 1.27 the group H/T is characteristic in N^3/T, so the group T is not normal in N^4.

□

We can now complete the proof of Lemma 4.4.

Lemma 4.31. *Suppose that $p = 3$ and $[E\colon T] = 3^3$. Then either $E \trianglelefteq S$ or one of the following holds:*

1. $\mathrm{N}^2 = S$, N^1 $\mathrm{char}_{\mathcal{F}}$ S, $\mathrm{C}_{S/T}(\Phi(S/T))$ *has exponent* 3 *and*

$$S/T \quad \cong \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ y & a & 1 & 0 \\ z & b & a & 1 \end{pmatrix} \mid a, b, x, y, z \in \mathbb{F}_3 \right\} \cong \mathrm{SmallGroup}(3^5, 56);$$

2. $\mathrm{N}^2 = S$, N^1 $\mathrm{char}_{\mathcal{F}}$ S, $\mathrm{C}_{S/T}(\Phi(S/T))$ *has exponent* 9 *and* $S/T \cong \mathrm{SmallGroup}(3^5, 57)$*;*

3. $\mathrm{C}_{\mathrm{N}^2/T}(\Phi(\mathrm{N}^2/T))$ *has exponent* 9, $\mathrm{N}^2/T \cong \mathrm{SmallGroup}(3^5, 58)$ *and if* $\mathrm{N}^2 < S$ *then* N^3/T *is isomorphic to a Sylow* 3*-subgroup of the group* $\mathrm{P}\Gamma\mathrm{L}_3(64)$.

Proof. Suppose that E is not normal in S. Hence $\mathrm{N}^1 < S$. We enter the information collected in Theorem 4.28 into the computer program *Magma* to determine the isomorphism type of the group N^2/T. More precisely, we look for a group of order 3^5 having nilpotency class 3, exponent 9, center of order 3 and at least one maximal subgroup (N^1/T) having exponent 3.

```
N:=[];
for i in [1..NumberOfSmallGroups(3^5)] do S:=SmallGroup(3^5,i);
if NilpotencyClass(S) eq 3 and
  Exponent(S) eq 9 and
  ♯Center(S) eq 3 and
  ♯[M : M in MaximalSubgroups(S)| Exponent(M‘subgroup) eq 3] ne 0 then
  Append(~N,i);
end if; end for; N;


Ouput:  [56,57,58]
```

Thus $\mathrm{N}^2/T \cong \mathrm{SmallGroup}(3^5, i)$ for $i \in \{56, 57, 58\}$.

- If $i = 56$ then the centralizer in N^2/T of the Frattini subgroup $\Phi(\mathrm{N}^2/T)$ has exponent 3 and N^2/T is isomorphic to the matrix group presented in the statement of the corollary, as we saw at the end of the proof of Theorem 4.28. Also N^1/T is the only other maximal subgroup of N^2/T containing $\mathrm{Z}_2(\mathrm{N}^2/T)$ and having exponent 3. Thus N^1/T is characteristic in N^2/T and since T is $\mathcal{F}$-characteristic in N^2 by Lemma 4.29, we deduce that $\mathrm{N}^1 \operatorname{char}_{\mathcal{F}} \mathrm{N}^2$ and $\mathrm{N}^2 = S$.

- If $i = 57$ then N^1/T is the unique maximal subgroup of N^2/T containing $\mathrm{Z}_2(\mathrm{N}^2/T)$ and having exponent 3. So, as in the previous case, we have $\mathrm{N}^1 \operatorname{char}_{\mathcal{F}} \mathrm{N}^2$ and $\mathrm{N}^2 = S$. Also, the centralizer in N^2/T of the Frattini subgroup $\Phi(\mathrm{N}^2/T)$ has exponent 9.

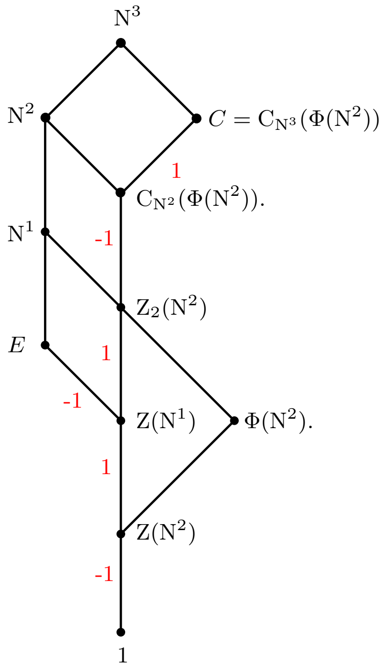
- If $i = 58$ then there are 3 maximal subgroups of N^2/T having exponent 3 so the group N^1/T is not necessarily characteristic in N^2/T. Finally, the centralizer in N^2/T of the Frattini subgroup $\Phi(\mathrm{N}^2/T)$ has exponent 9.

Now suppose that $\mathrm{N}^2 < S$. Then by what was proved above we have $p = 3$ and $\mathrm{N}^2/T \cong \mathrm{SmallGroup}(3^5, 58)$. Also, T is $\mathcal{F}$-characteristic in N^2 by Lemma 4.29, so it is normal in N^3 and we can consider the group N^3/T. By Lemma 2.41 applied to $\mathrm{N}^j = \mathrm{N}^3$ and $K = T$, we get that $[\mathrm{N}^3 \colon \mathrm{N}^2] = 3$ and every morphism in $\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(E)}(\mathrm{Aut}_S(E))$ is a restriction of an $\mathcal{F}$-automorphism of N^3. To simplify notation, we assume $T = 1$.

Let $\tau \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{N}^3)$ be the morphism that acts on E as

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the structure of N^2 and the action of τ on N^2 are described in Theorem 4.28.



The Frattini subgroup $\Phi(N^2)$ of N^2 is normal in N^3 and has order 3^2. Thus $[N^3 \colon C_{N^3}(\Phi(N^2))] = 3$. Set $C = C_{N^3}(\Phi(N^2))$. Then C is a maximal subgroup of N^3, distinct form N^2 and normalized by τ. Also, $[N^2, C] \leq (N^2 \cap C)\backslash N^1$, since N^1 is not normal in N^3. Hence using Lemma 1.4 on the quotient $N^3/Z_2(N^2)$ we deduce that τ centralizes the quotient $C/C_{N^2}(\Phi(N^2))$.

Note that $Z_2(N^3) \leq Z_2(N^2)$. In particular we have

$$[Z_3(N^3), N^1] \leq Z_2(N^3) \leq N^1,$$

so $Z_3(N^3) \leq N_S(N^1) = N^2$. Thus N^3 has nilpotency class at least 4.

We now use the computer program *Magma* to identify the groups of order 3^6 containing a maximal subgroup isomorphic to the group SmallGroup$(3^5, 58)$ and having nilpotency class at least 4. We find that $N^3 \cong$ SmallGroup$(3^6, \mathrm{i})$ for $i \in \{411, 413\}$.

The only difference between the presentations of SmallGroup$(3^6, 411)$ and SmallGroup$(3^6, 413)$ given by *Magma* is the description of the 3-rd power of the element $x = S.2$ (see Table 4.1). It is easy to check that $x \in C$, $h = S.5 \in \Phi(N^2)\backslash Z(N^2)$ and $z = S.6 \in Z(N^2)$.

We have $x^3 = h^2 z^j$ for $j = 0$ if $S \cong$ SmallGroup$(3^6, 411)$ and $j = 1$ otherwise. Then

$$(x^3)\tau = (h\tau)^2 (z\tau)^j = h^2 (z^j)^{-1} = x^3 z^j.$$

`S:=SmallGroup(3^6,411);`
`S;`
 	
`GrpPC : S of order 729 = 3^6`
`PC-Relations:`
`(S.2)^3 = (S.5)^2,`
`(S.4)^3 = (S.6)^2,`
`(S.1,S.2) = S.4,`
`(S.2,S.3) = S.5,`
`(S.1,S.4) = S.5,`
`(S.3, S.4) = S.6^2,`
`(S.1,S.5) = S.6`
	`S:=SmallGroup(3^6,413);`
	`S;`
	
	`GrpPC : S of order 729 = 3^6`
	`PC-Relations:`
	`(S.2)^3 = (S.5)^2 * S.6,`
	`(S.4)^3 = (S.6)^2,`
	`(S.1,S.2) = S.4,`
	`(S.2,S.3) = S.5,`
	`(S.1,S.4) = S.5,`
	`(S.3, S.4) = S.6^2,`
	`(S.1,S.5) = S.6`

Table 4.1: Presentations of the groups `SmallGroup(3^6,411)` and `SmallGroup(3^6,413)`

On the other hand $x\tau = xy$ for some $y \in \mathrm{Z}_2(\mathrm{N}^2)$ and since $\mathrm{Z}(C) = \Phi(\mathrm{N}^2) = \Phi(C)$ (we can check this with *Magma*), by Theorem 1.3 we get

$$(x\tau)^3 = (xy)^3 = y^3x^3[y,x]^3 = x^3.$$

Thus we have $x^3 = x^3z^j$, which implies $j = 0$ and so $S \cong \mathrm{SmallGroup}(3^6, 411)$.

Finally notice that the group $\mathrm{SmallGroup}(3^6, 411)$ is isomorphic to a Sylow 3-subgroup of the group $\mathrm{P\Gamma L}_3(64)$. □

Let P be a Sylow 3-subgroup of the group $G = \mathrm{P\Gamma L}_3(64)$. If $E \cong \mathrm{C}_3 \times \mathrm{C}_3 \times \mathrm{C}_3$ and $T = 1$ then either $S \cong P$ or S is isomorphic to a maximal subgroup of P (the unique one isomorphic to the group indexed in *Magma* as $\mathrm{SmallGroup}(3^5, 58)$). If $S \cong P$, then we can check with *Magma* that there exists a unique $\mathcal{F}_P(G)$-essential subgroup of P (up to G-conjugation) and that such group is isomorphic to the group $\mathrm{C}_3 \times \mathrm{C}_3 \times \mathrm{C}_3$ and has trivial $\mathcal{F}_P(G)$-core (as we expected).

Let q be a prime power such that $q \equiv 1 \mod 3$. We claim that the fusion category of the group $G = \mathrm{P\Gamma L}_3(q^{3^k})$ on one of its Sylow 3-subgroups P contains a unique $\mathcal{F}_P(G)$-essential subgroup (up to G-conjugation) and that such subgroup is isomorphic to the group $\mathrm{C}_3 \times \mathrm{C}_3 \times \mathrm{C}_{3^k}$.

Set $\underline{X} = \mathrm{PGL}_3(q)$ and let $S_{\underline{X}}$ be a Sylow 3-subgroup of $\underline{X}$. Take $\gamma \in G$ such that γ centralizes $S_{\underline{X}}$ and $G = \mathrm{PGL}(q^{3^k})\langle\gamma\rangle$. Such γ exists because the iterate of the Frobenius morphism acts trivially on $S_{\underline{X}}$.

Note that there exist a natural projection π and a map ι defined as follows:

$$\mathrm{GL}_3(q^{3^k}) \xrightarrow{\pi} \mathrm{PGL}_3(q^{3^k}) \xrightarrow{\iota} \mathrm{PGL}_3(q^{3^k})\langle\gamma\rangle = G.$$

$$\mathrm{Z}(\mathrm{GL}_3(q^{3^k}))\mathrm{GL}_3(q) \xrightarrow{\pi} \mathrm{PGL}_3(q) = \underline{X} \xrightarrow{\iota} \mathrm{PGL}_3(q)\langle\gamma\rangle.$$

Let $H = \mathrm{GL}_3(q^{3^k})$ and let $S_H \in \mathrm{Syl}_3(H)$. With a technique similar to the one used in the proof of Lemma 4.27 we can show that there exists an $\mathcal{F}_{S_H}(H)$-essential subgroup Q of H isomorphic to the group $\mathrm{C}_{3^t} \circ 3^{1+2}_+$, where 3^t is the largest power of 3 dividing q^{3^k}. Note that $Q \in \mathrm{Z}(\mathrm{GL}_3(q^{3^k}))\mathrm{GL}_3(q)$. Hence $Q\pi \in \underline{X}$ and we may assume $Q\pi \leq S_{\underline{X}}$. Since γ was chosen to centralize $S_{\underline{X}}$, we conclude that $(Q\pi)\iota \cong Q\pi \times C$, where $C \leq \langle\gamma\rangle$ is a cyclic group of order 3^k. Hence

$$(Q\pi)\iota \cong \mathrm{C}_3 \times \mathrm{C}_3 \times \mathrm{C}_{3^k}.$$

We claim that $E = (Q\pi)\iota$ is an $\mathcal{F}_P(G)$-essential subgroup of P and that its G-conjugacy class is the unique G-conjugacy class of $\mathcal{F}_P(G)$-essential subgroups of P.

The previous example suggests that when $p = 3$ and E is an $\mathcal{F}$-essential subgroup of the 3-group S having $\mathcal{F}$-core of index 3^3, then we cannot bound the index of E in S.

4.4 Interplay of $\mathcal{F}$-characteristic $\mathcal{F}$-essential subgroups

In this last section of Chapter 4 we suppose that there are two $\mathcal{F}$-essential subgroup E_1 and E_2 of S that are $\mathcal{F}$-characteristic in S and we determine the isomorphism type of the quotient $S/\mathrm{core}_{\mathcal{F}}(E_1, E_2)$.

Main Hypothesis D. Let p be an odd prime, let S be a p-group having sectional rank 3 and let $\mathcal{F}$ be a saturated fusion system on S. Let E_1 and E_2 be distinct $\mathcal{F}$-essential subgroups of S that are $\mathcal{F}$-characteristic in S and let $T = \mathrm{core}_{\mathcal{F}}(E_1, E_2)$.

In this section we assume Main Hypothesis D holds.

Lemma 4.32. *Let G_i be a model for $\mathrm{N}_{\mathcal{F}}(E_i)$ and let $A_i = \langle S^{G_i} \rangle$. Then*

$$\mathrm{C}_{A_i}(E_i/T) \leq E_i/T \quad \textit{for every } 1 \leq i \leq 2$$

and

1. *either $\mathrm{C}_{G_i}(E_i/T) \leq E_i/T$ for every i;*
2. *or $\Phi(E_1) = \Phi(E_2)$, $\Phi(E_1) < T$, $S/T \cong p_+^{1+2}$ and for every $1 \leq i \leq 2$ the group E_i has rank 3 and $\mathrm{Out}_{\mathcal{F}}(E_i)$ contains a subgroup isomorphic to $\mathrm{SL}_2(p) \times \langle \theta \rangle$, where $\theta \in \mathrm{Out}_{\mathcal{F}}(S)$ is a morphism of order dividing $p-1$ that centralizes S/T and acts non trivially on $T/\Phi(E_1) \cong \mathrm{C}_p$.*

An example of the situation described in part 2 is given by the fusion category of the group $G = (\mathrm{C}_p \colon \mathrm{C}_{p-1}) \times \mathrm{PSL}_3(p)$ on one of its Sylow p-subgroups $S \cong \mathrm{C}_p \times p_+^{1+2}$.

Proof. Fix $1 \leq i \leq 2$ and set $E = E_i$, $G = G_i$ and $A = A_i$. Note that $E/T = O_p(G/T)$. Hence if $\mathrm{C}_G(E/T)$ is a p-group then $\mathrm{C}_G(E/T) \leq E/T$.

Suppose there exists a non-trivial element $g \in \mathrm{C}_G(E/T)$ of order coprime to p. If $T \leq \Phi(E)$ then g centralizes $E/\Phi(E)$ and so $g = 1$ by Theorem 1.10, which is a contradiction.

Thus we have $T \not\leq \Phi(E)$. By Lemma 4.13 we have $A/E \cong \mathrm{SL}_2(p)$, $[T\Phi(E)\colon \Phi(E)] = p$ and $T\Phi(E)/\Phi(E) = \mathrm{C}_{E/\Phi(E)}(A/E)$. If $g \in A$ then g centralizes every quotient in the sequence of subgroups:

$$\Phi(E) < T\Phi(E) < E.$$

Thus $g \in \mathrm{Inn}(E)$ by Lemma 2.24, which is a contradiction.

Therefore $\mathrm{C}_A(E/T) \leq E/T$ and $g \notin A$. By the Frattini argument we have $G = A\mathrm{N}_G(S)$. Thus we may assume that $g \in \mathrm{N}_G(S) \backslash \mathrm{N}_A(S)$. Also, g does not centralize $E/\Phi(E)$, thus it acts non-trivially on $T\Phi(E)/\Phi(E) \cong \mathrm{C}_p$. In particular g has order dividing $p-1$.

Suppose that g does not centralize S/T. Then $E/T = \mathrm{C}_{S/T}(g)$ and since $[S\colon E] = p$ by Theorem 1, by Lemma 1.13 we get

$$S/T \cong E/T \times [S/T, g].$$

Thus $[S,g]T$ is a subgroup of S centralizing E/T. Since $T\Phi(E)/\Phi(E) \cong \mathrm{C}_p$ and $[S,g]T$ is a p-group, we deduce that $[S,g]T$ centralizes every quotient in the sequence of subgroups $\Phi(E) < T\Phi(E) < E$, all $\mathcal{F}$-characteristic in E. Hence by Lemma 2.24 we have $[S,g]T \leq E$, which is a contradiction.

Thus g centralizes the group S/T. Note that $S/T\Phi(E)$ is a Sylow p-subgroup of the group $A/T\Phi(E) \cong (\mathrm{C}_p \times \mathrm{C}_p)\colon \mathrm{SL}_2(p)$. Thus $S/T\Phi(E) \cong p_+^{1+2}$. Also, since g centralizes S/T but acts non-trivially on $T\Phi(E)/\Phi(E)$, every element of $T\Phi(E)/\Phi(E)$ is not a p-th power of an element in $S/\Phi(E)$. Hence the group $S/\Phi(E)$ has exponent p.

Let $P = E_j$ for $j \neq i$ and consider the group $P/\Phi(E)$, that has order p^3 and exponent p. Since g centralizes $P/T\Phi(E)$ and acts non-trivially on $T\Phi(E)/\Phi(E)$, we deduce that $P/\Phi(E)$ is elementary abelian. Since S has sectional rank 3, we conclude $\Phi(E) = \Phi(P)$. In particular $\Phi(E) \leq T$ by maximality of T and $S/T \cong p_+^{1+2}$. Also, by Lemma 4.13 we have $O^{p'}(\mathrm{Out}_{\mathcal{F}}(P)) \cong \mathrm{SL}_2(p)$.

Set $\theta = c_g\mathrm{Inn}(S) \in \mathrm{Out}_{\mathcal{F}}(S)$. Then $\theta|_{E_i} \in \mathrm{Out}_{\mathcal{F}}(E_i) \backslash O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_i))$ for every i. By Theorem 1 we have $\mathrm{SL}_2(p) \leq \mathrm{Out}_{\mathcal{F}}(E) \leq \mathrm{GL}_2(p) \times \mathrm{C}_{p-1}$, so we conclude that $\mathrm{Out}_{\mathcal{F}}(E_i)$ contains a subgroup isomorphic to the group $\mathrm{SL}_2(p) \times \langle \theta \rangle$. □

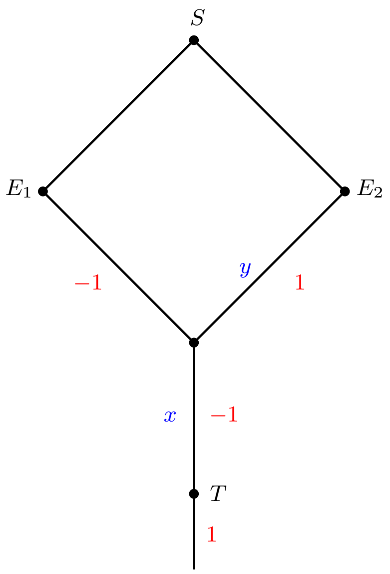
Theorem 4.33. *Either $S/T \cong p_+^{1+2}$ or S/T is isomorphic to a Sylow p-subgroup of the group $\mathrm{Sp}_4(p)$.*

Proof. Let G_i be a model for $\mathrm{N}_{\mathcal{F}}(E_i)$ and let G_{12} be a model for $\mathrm{N}_{\mathcal{F}}(S)$. Set $A_i = \langle S^{G_i} \rangle$. By Lemma 4.32 either $S/T \cong p_+^{1+2}$ or $\mathrm{C}_{G_i}(E_i/T) \leq E_i/T$ for every i. Suppose we are in the second case. Then by Lemma 2.34 the amalgam $\mathcal{A} = \mathcal{A}(A_1/T, A_2/T, (A_1 \cap A_2)/T)$ is a weak BN-pair of rank 2 and the quotient S/T is isomorphic to a Sylow p-subgroup of one of the groups listed in Theorem 1.42. Since p is odd and S/T has sectional rank at most 3, by [GLS98, Theorem 3.3.3] we deduce that S/T is isomorphic to a Sylow p-subgroup of either $\mathrm{PSL}_3(p)$ or $\mathrm{PSp}_4(p)$. Finally notice that every Sylow p-subgroup of $\mathrm{PSL}_3(p)$ is isomorphic to the group p_+^{1+2} and that the Sylow p-subgroups of $\mathrm{PSp}_4(p)$ are isomorphic to the Sylow p-subgroups of $\mathrm{Sp}_4(p)$. □

Theorem 4.34. *If $S/T \cong p_+^{1+2}$ then E_1 and E_2 are abelian, $E_1 \cap E_2 = \mathrm{Z}(S)$ and T is the centralizer in $\mathrm{Z}(S)$ of $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(E_i))$, for every $1 \leq i \leq 2$.*

Proof. Note that $E_1/T \cong E_2/T \cong \mathrm{C}_p \times \mathrm{C}_p$. Thus $\Phi(E_i) \leq T$ and by Theorem 1 and Lemma 4.13 we have $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_i)) \cong \mathrm{SL}_2(p)$. In particular E_i/T is a natural $\mathrm{SL}_2(p)$-module for $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_i))$. By Lemma 4.12 we have $E_i = \mathrm{C}_{E_i}(T)T$ and $\mathrm{C}_S(T) \not\leq E_1 \cap E_2$. Thus we may assume $\mathrm{C}_S(T) \not\leq E_1$ and so $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_1))$ centralizes T (again by Lemma 4.12). Therefore by Theorem 4.15 we deduce that T is abelian, $T \leq \mathrm{Z}(E_1)$, $|[E_1, E_1]| \leq p$ and $T/[E_1, E_1]$ is cyclic. Note that $E_2 = \mathrm{C}_{E_2}(T)$, so $T \leq \mathrm{Z}(E_2)$ and since $S = E_1E_2$ we conclude $T \leq \mathrm{Z}(S)$.

Let $\tau \in \mathrm{Aut}_{\mathcal{F}}(S)$ be a morphism that acts on E_1/T as the involution $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Since $S/T \cong p_+^{1+2}$, we have $[E_1, E_2]T = E_1 \cap E_2$. Thus the morphism τ centralizes the quotient $E_2/(E_1 \cap E_2)$.



Action of τ on S

Let $x \in (E_1 \cap E_2)\backslash T$ and $y \in E_2\backslash(E_1 \cap E_2)$. Then $x\tau = x^{-1}t_1$ and $y\tau = yt_2$, for some $t_1, t_2 \in T$. Since $T \leq \mathrm{Z}(S)$ and τ centralizes T, by Lemma 1.4 we have

$$[x, y] = [x, y]\tau = [x^{-1}t_1, yt_2] = [x, y]^{-1}.$$

Thus $[x, y] = 1$ and the group E_2 is abelian.

Note that $\mathrm{C}_S(T) = S \nleq E_2$ so $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(E_2))$ centralizes T by Lemma 4.12. Hence we can repeat the same argument with E_2 in place of E_1 to prove that E_1 is abelian.

Since E_1 and E_2 are abelian, we have $E_1 \cap E_2 = \mathrm{Z}(S)$. Also, since there exists an involution of $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_i))$ that inverts the quotient $\mathrm{Z}(S)/T$, we conclude that T is the centralizer in $\mathrm{Z}(S)$ of $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(E_i))$, for every $1 \leq i \leq 2$.

□

Theorem 4.35. *If S/T is isomorphic to a Sylow p-subgroup of $\mathrm{Sp}_4(p)$ then there exist $1 \leq i, j \leq 2$ with $i \neq j$ such that $\mathrm{Z}(S) = \mathrm{Z}(E_i)$ is the preimage in S of $\mathrm{Z}(S/T)$ and the following hold:*

1. *$E_i/T \cong p_+^{1+2}$ and $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_i)) \cong \mathrm{SL}_2(p)$;*
2. *E_j is abelian, $T = \Phi(E_j)$ and $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_j)) \cong \mathrm{PSL}_2(p)$.*

Proof. Let Z be the preimage in S of $\mathrm{Z}(S/T)$. Then $[Z\colon T] = p$. In particular, there exists i such that E_i/T is not abelian. Thus E_i/T is extraspecial. Also, $T \not\leq \Phi(E_i)$ and so by Lemma 4.13 we get $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_i)) \cong \mathrm{SL}_2(p)$ and $[T\Phi(E_i)\colon T] = p$. Note that $\Phi(E_i)T$ is normal in S, so $T\Phi(E_i) = Z$. If E_i/T has exponent p^2, then there exists an $\mathcal{F}$-characteristic subgroup of E_i containing $T\Phi(E_i)$ and of index p in E_i. In particular $\mathrm{Aut}_S(E) = \mathrm{Inn}(E)$ by Lemma 2.24, giving a contradiction. Thus $E_i/T \cong p_+^{1+2}$.

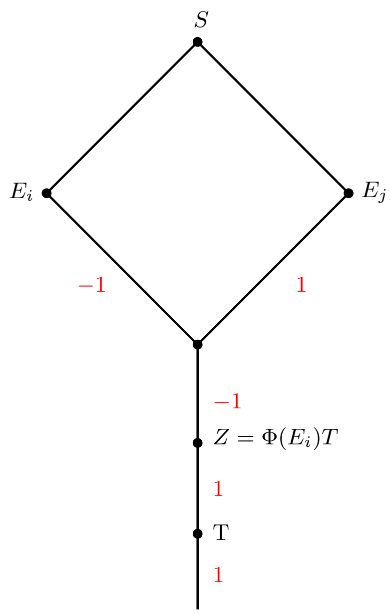
Let $j \neq i$. If $T \not\leq \Phi(E_j)$ then by Lemma 4.13 we have $[T\Phi(E_j)\colon T] = p$. Thus $T\Phi(E_j) = Z = T\Phi(E_j)$, contradicting the maximality of T. Therefore $T \leq \Phi(E_j) < Z$ and since S has sectional rank 3 we deduce that $T = \Phi(E_j)$.

Suppose that $\mathrm{Out}_{\mathcal{F}}(E_j)$ acts reducibly on E_j/T. Then by Theorem 1 the group $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_j))$ is isomorphic to $\mathrm{SL}_2(p)$ and if $C \leq E_j$ is the preimage in E_j of the group $\mathrm{C}_{E_j/T}(O^{p'}(\mathrm{Aut}_{\mathcal{F}}(E_j)))$, then $[C\colon T] = p$. Note that $C \trianglelefteq S$ so $C = Z$. Hence C is $\mathcal{F}$-characteristic in E_1, E_2 and S, contradicting the maximality of T. Therefore $\mathrm{Out}_{\mathcal{F}}(E_j)$ acts irreducibly on E_j/T and by Theorem 4.10 we deduce that $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_j)) \cong \mathrm{PSL}_2(p)$. In particular, since $\mathrm{C}_{E_j}(T) \not\leq T$ by Lemma 4.12 and $T\mathrm{C}_{E_j}(T)$ is $\mathcal{F}$-characteristic in E_j, we have $T\mathrm{C}_{E_j}(T) = E_j$.

If $\mathrm{C}_S(T) \leq E_i$ then $E_j \leq T\mathrm{C}_S(T) \leq E_i$, that is a contradiction. Thus $\mathrm{C}_S(T) \not\leq E_i$ and $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_i))$ centralizes T by Lemma 4.12. Also, $E_i \cap E_j \leq E_j \leq T\mathrm{C}_S(T)$. So $E_i \cap E_j \leq T\mathrm{C}_{E_i}(T)$ and recalling that $Z < E_i \cap E_j$ is $\mathcal{F}$-characteristic in E_i we conclude that $E_i = T\mathrm{C}_{E_i}(T)$ and $S = T\mathrm{C}_S(T)$.

Note that $E_i/\mathrm{Z}(T) \cong T/\mathrm{Z}(T) \times \mathrm{C}_{E_i}(T)/\mathrm{Z}(T)$ and $\mathrm{C}_{E_i}/\mathrm{Z}(T) \cong E_i/T \cong p_+^{1+2}$. Since S has sectional rank 3, we deduce that the group $T/\mathrm{Z}(T)$ is cyclic and so T is abelian by Lemma 1.6. Hence $S = \mathrm{C}_S(T)$ and $T \leq \mathrm{Z}(S)$.

Let $\tau \in \mathrm{Aut}_{\mathcal{F}}(S)$ be the morphism that acts on E_i/Z as the involution $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then τ centralizes Z/T and since $[E_i, E_j]T = E_i \cap E_j$, the morphism τ centralizes the quotient $E_j/(E_i \cap E_j)$.



Action of τ on S

Let $x \in (E_i \cap E_j) \backslash Z$. Since $\langle x \rangle Z/Z$ is the only section of E_j that is not centralized by τ, we deduce that $x \in \mathrm{Z}(E_j)$ (for example using Lemma 1.4). Since $\mathrm{Z}(E_j)$ is $\mathcal{F}$-characteristic in E_j and $\mathrm{Out}_{\mathcal{F}}(E_j)$ acts irreducibly on E_j/T, the only possibility is $E_j = \mathrm{Z}(E_j)$. Thus E_j is abelian. Similarly, since $T \leq \mathrm{Z}(S)$ and Z/T is the only section of E_i/T not inverted by τ, we deduce that $Z \leq \mathrm{Z}(E_i)$. Finally, the group E_i is non abelian ($E_i/T \cong p_+^{1+2}$) so $Z = \mathrm{Z}(E_i) = \mathrm{Z}(S)$. $\square$

We end this section proving that if there are two $\mathcal{F}$-essential subgroups of S that are $\mathcal{F}$-characteristic in S and $O_p(\mathcal{F}) = 1$ then S is isomorphic to a Sylow p-subgroup of the group $\mathrm{Sp}_4(p)$.

Theorem 4.36. *Let E_1 and E_2 be distinct $\mathcal{F}$-essential subgroups of S that are $\mathcal{F}$-characteristic in S and set $T := \mathrm{core}_{\mathcal{F}}(E_1, E_2)$. Then T is normal in $\mathcal{F}$. In particular, if $O_p(\mathcal{F}) = 1$ then S is isomorphic to a Sylow p-subgroup of the group $\mathrm{Sp}_4(p)$.*

Proof. By Theorem 4.33 either $S/T \cong p_+^{1+2}$ or S/T is isomorphic to a Sylow p-subgroup of the group $\mathrm{Sp}_4(p)$. Note that the group p_+^{1+2} has sectional rank 2, so if $O_p(\mathcal{F}) = 1$ and $T \trianglelefteq \mathcal{F}$ then $T = 1$ and the second statement follows from the fact that S has sectional rank 3. Our goal is to prove that T is normal in $\mathcal{F}$. Note that by Theorems 4.34 and 4.35 we have $T \leq \mathrm{Z}(S)$. So T is contained in every $\mathcal{F}$-essential subgroup of S. By Lemma 2.28 we have to show that T is $\mathcal{F}$-characteristic in every $\mathcal{F}$-essential subgroup of S.

- Suppose $S/T \cong p_+^{1+2}$. Then by Theorem 4.34 the group T is the centralizer is S of $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(E_i))$, for every $1 \leq i \leq 2$. Let E_3 be an $\mathcal{F}$-essential subgroup of S distinct from E_1. Then $\mathrm{Z}(S) < E_3$, so E_3 is abelian.

  Suppose E_3 is $\mathcal{F}$-characteristic in S and set $T_3 = \mathrm{core}_{\mathcal{F}}(E_1, E_3)$. Since both E_1 and E_3 are abelian, by Theorems 4.33 and 4.35 we have $S/T_3 \cong p_+^{1+2}$. Thus T_3 is the centralizer in $\mathrm{Z}(S)$ of $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_1))$, which implies $T_3 = T$. Therefore T is $\mathcal{F}$-characteristic in E_3.

  Suppose E_3 is not $\mathcal{F}$-characteristic in S (so $E_3 \neq E_2$). Then $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_3)) \cong \mathrm{SL}_2(p)$ by Theorem 4.8 and there exists a morphism $\varphi \in \mathrm{Aut}_{\mathcal{F}}(S)$ that inverts $E_3/\mathrm{Z}(S)$ and centralizes S/E_3. In particular the action of φ on $S/\mathrm{Z}(S)$ is not scalar. However, φ normalizes E_1, E_2 and E_3 and we get a contradiction.

  Hence T is $\mathcal{F}$-characteristic in every $\mathcal{F}$-essential subgroup of S and is therefore normal in $\mathcal{F}$.

- Suppose S/T is isomorphic to a Sylow p-subgroup of the group $\mathrm{Sp}_4(p)$. Then by Theorem 4.35 we can assume that $E_1/T \cong p_+^{1+2}$ and E_2 is abelian. Let E_3 be an $\mathcal{F}$-essential subgroup of S distinct from E_1 and E_2. Note that $[E_3 \colon \mathrm{Z}(S)] \leq p^2$.

  Suppose $[E_3 \colon \mathrm{Z}(S)] = p^2$. Then E_3 is normal in S. If $\mathrm{Z}(S)$ is not $\mathcal{F}$-characteristic in E_3, then $\mathrm{Z}(S) < \mathrm{Z}(E_3)$ and so E_3 is abelian. In particular $\mathrm{Z}(S) = E_2 \cap E_3$ has index p in E_3, which is a contradiction. Thus $\mathrm{Z}(S)$ is $\mathcal{F}$-characteristic in E_3. Let G_3 be a model for $\mathrm{N}_{\mathcal{F}}(E_3)$. Then $T^g \leq \mathrm{Z}(S)$ for every $g \in G_3$. Since T is $\mathcal{F}$-characteristic in S and $G_3 = \langle S^{G_3} \rangle \mathrm{N}_{G_3}(S)$ by the Frattini argument, we conclude that G_3 normalizes T and so T is $\mathcal{F}$-characteristic in E_3.

  Suppose $[E_3 \colon \mathrm{Z}(S)] = p$. Then E_3 is abelian and not normal in S. Set $T_3 = \mathrm{core}_{\mathcal{F}}(E_3)$ and suppose $T \neq T_3$. Note that $\mathrm{Z}(S)/T_3 = \mathrm{Z}(S/T_3)$. In particular by Theorem 4.28(1) we have $[E_3 \colon T_3] = p^2$. So E_3/T_3 is a self-centralizing subgroup of the p-group S/T_3 isomorphic to the group $\mathrm{C}_p \times \mathrm{C}_p$.

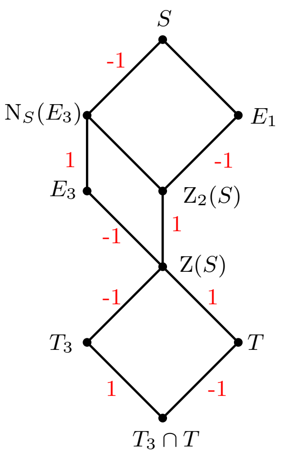
  

  Figure 4.2

  Let $\varphi \in \mathrm{Aut}_{\mathcal{F}}(E_3)$ be a morphism that inverts the quotient E_3/T_3 and centralizes T_3. Such morphism exists because $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_3)) \cong \mathrm{SL}_2(p)$ (Theorem 4.8) and T_3 is centralized by $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E_3))$ (Lemma 4.14). Note that φ is a restriction of an $\mathcal{F}$-automorphism of S by Lemma 4.21. With abuse of notation, we assume that φ acts on S. Then the action of φ is the one described in Lemma 3.23 (see Figure 4.2).

  Looking at the action of φ on E_1/T we deduce that E_1/T is abelian, contradicting the assumption that E_1/T is extraspecial.

  Hence T is $\mathcal{F}$-characteristic in every $\mathcal{F}$-essential subgroup of S and so $T \trianglelefteq \mathcal{F}$.

$\square$

CHAPTER 5

CLASSIFICATION OF SIMPLE FUSION SYSTEMS ON p-GROUPS OF SECTIONAL RANK 3

'Little by little does the trick.'

[Aesop]

We are ready to use the information patiently collected in the previous chapters to determine the saturated fusion systems $\mathcal{F}$ on p-groups of sectional rank 3 satisfying $O_p(\mathcal{F}) = 1$ when p is an odd prime.

Let p be an *odd* prime, let S be a p-group of sectional rank 3 and let $\mathcal{F}$ be a saturated fusion system on S such that $O_p(\mathcal{F}) = 1$.

For every subgroup $P \leq S$ containing $\mathrm{Z}(S)$ we define

$$Z_P = \langle\ \Omega_1(\mathrm{Z}(S))^{\mathrm{Aut}_{\mathcal{F}}(P)}\ \rangle.$$

Note that $Z_S = \Omega_1(\mathrm{Z}(S))$ and $Z_S \leq Z_P \leq \Omega_1(\mathrm{Z}(P))$. In particular Z_P is elementary abelian and since S has sectional rank 3 we deduce $|Z_P| \leq p^3$.

Since $O_p(F) = 1$ and Z_S is an $\mathcal{F}$-characteristic subgroup of S contained in every $\mathcal{F}$-essential subgroup of S (recall that every $\mathcal{F}$-essential subgroup is $\mathcal{F}$-centric), then by Lemma 2.28 there exists and $\mathcal{F}$-essential subgroup E of S such that $Z_S < Z_E$ (when this happens we say that E *moves* Z_S).

In Section 5.1 we characterize non-$\mathcal{F}$-characteristic $\mathcal{F}$-essential subgroups of S that move the group Z_S.

Theorem 17. *Let $E \leq S$ be an $\mathcal{F}$-essential subgroup of S not $\mathcal{F}$-characteristic in S. Then*

$$Z_S = Z_E \text{ if and only if } Z_S \leq \mathrm{core}_{\mathcal{F}}(E).$$

We also prove that if E moves Z_S then E is abelian. Note that since E is not $\mathcal{F}$-characteristic in S, we deduce that if $E \trianglelefteq S$ then S has at least two abelian subgroups of index p. We show that when $O_p(\mathcal{F}) = 1$ this implies $S \cong p_+^{1+2}$, contradicting the fact that S has sectional rank 3. Therefore we have the following result.

Theorem 18. *Let $E \leq S$ be an $\mathcal{F}$-essential subgroup of S not $\mathcal{F}$-characteristic in S. If $Z_S < Z_E$ then E is abelian and if $O_p(\mathcal{F}) = 1$ then E is not normal in S.*

In Section 5.2 we prove our final results concerning simple fusion systems on p-groups of sectional rank 3, for p odd.

Theorem 19. *Suppose that $O_p(\mathcal{F}) = 1$. Then one of the following holds:*

1. *S is isomorphic to a Sylow p-subgroup of the group $\mathrm{Sp}_4(p)$;*

2. *there exists an $\mathcal{F}$-essential subgroup of S that is not normal in S.*

Theorem 20. *Let $p \geq 5$ be a prime, let S be a p-group having sectional rank 3 and let $\mathcal{F}$ be a saturated fusion system on S such that $O_p(\mathcal{F}) = 1$. Then $\mathcal{F}$ contains a pearl.*

As we saw on page xvii this is the last ingredient required to prove our Main Theorem.

5.1 Essential subgroups moving the center of S

In this section we characterize the $\mathcal{F}$-essential subgroups of S whose automorphism group does not normalize the center of S.

We start proving Theorem 17.

Theorem 5.1. *Let $E \leq S$ be an $\mathcal{F}$-essential subgroup of S not $\mathcal{F}$-characteristic in S. Then*

$$Z_S = Z_E \text{ if and only if } Z_S \leq \mathrm{core}_{\mathcal{F}}(E).$$

Proof.

- Suppose $Z_S = Z_E$ and set $T = \mathrm{core}_{\mathcal{F}}(E)$. Then TZ_S is a subgroup of E containing T and normalized by $\mathrm{Aut}_{\mathcal{F}}(E)$. Note that Z_S is an $\mathcal{F}$-characteristic subgroup of S.

  If $E \trianglelefteq S$ then TZ_S is $\mathcal{F}$-characteristic in S and by maximality of T we have $Z_S \leq T$. Suppose E is not normal in S. Note that $TZ_S \trianglelefteq \mathrm{N}^2$, so $TZ_S \neq E$. If $[E\colon T] = p^2$ this implies $TZ_S = T$ and so $Z_S \leq T$. Suppose $[E\colon T] = p^3$. Since $(TZ_S)/T \leq \mathrm{Z}(\mathrm{N}^2/T)$, by Theorem 4.28(1) we get $[TZ_S\colon T] \leq p$ and if $[TZ_S\colon T] = p$ then $TZ_S = \Phi(\mathrm{N}^1)$. Since $\Phi(\mathrm{N}^1)$ $\mathrm{char}_{\mathcal{F}}$ N^1, by maximality of T we deduce that $TZ_S = T$ and so $Z_S \leq T$.

- We want to prove that if $Z_S < Z_E$ then $Z_S \nleq \mathrm{core}_{\mathcal{F}}(E)$, for every $\mathcal{F}$-essential subgroup E not $\mathcal{F}$-characteristic in S. Aiming for a contradiction, assume there exists an $\mathcal{F}$-essential subgroup E of S, not $\mathcal{F}$-characteirstic in S, such that $Z_S < Z_E$ and $Z_S \leq \mathrm{core}_{\mathcal{F}}(E)$. We can choose E such that if $E < P$ and P is an $\mathcal{F}$-essential subgroup of S moving Z_S then either P is $\mathcal{F}$-characteristic in S or $Z_S \nleq \mathrm{core}_{\mathcal{F}}(P)$.

  Set $T = \mathrm{core}_{\mathcal{F}}(E)$. From $Z_S \leq T$ we get $Z_E \leq T$. So $Z_E \leq \Omega_1(T)$ and by Theorem 11 and the fact that $Z_S < Z_E$ we conclude $|Z_E| = p^2$ and $|Z_S| = p$.

  By Theorem 4.14 the group $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E))$ centralizes T. Note that $\mathrm{Inn}(S)$ acts trivially on Z_S, so the group $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(E))$ centralizes Z_S. By the Frattini argument we have

$$\mathrm{Aut}_{\mathcal{F}}(E) = O^{p'}(\mathrm{Aut}_{\mathcal{F}}(E))\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(E)}(\mathrm{Aut}_S(E)).$$

Then we may assume that there exists $\alpha \in \mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(E)}(\mathrm{Aut}_S(E))$ of order prime to p such that $Z_S\alpha \neq Z_S$. Note that α can be viewed as an $\mathcal{F}$-automorphism of $\mathrm{N}_S(E)$ (by Lemma 2.8) but it is not a restriction of an $\mathcal{F}$-automorphism of S (otherwise it normalizes Z_S). In particular the group E is not abelian (Corollary 2.42).

By Alperin's Fusion Theorem there exist subgroups $P_1, P_2, \dots, P_n$ of S and morphisms $\phi_i \in \mathrm{Aut}_{\mathcal{F}}(P_i)$ for every $1 \leq i \leq n$ such that

- every P_i is either $\mathcal{F}$-essential or equal to S,
- $\mathrm{N}_S(E) \leq P_1 \cap P_n$,
- and $\phi_1 \cdot \phi_2 \cdots \phi_n|_{\mathrm{N}_S(E)} = \alpha|_{\mathrm{N}_S(E)(E)}$.

Suppose $Z_S\phi_1 = Z_S$. Note that $E\phi_1$ is an $\mathcal{F}$-essential subgroup of S isomorphic to E (by Theorem 2.26(5)) and $\mathrm{Aut}_{\mathcal{F}}(E\phi_1) = \phi_1^{-1}\mathrm{Aut}_{\mathcal{F}}(E)\phi$. In particular Z_S is not normalized by $\mathrm{Aut}_{\mathcal{F}}(E\phi_1)$ and we can replace E by $E\phi_1$. Thus we may assume $Z_S\phi_1 \neq Z_S$. In particular $P = P_1$ is an $\mathcal{F}$-essential subgroup of S containing $\mathrm{N}_S(E)$ such that $Z_S < Z_P$.

Suppose P is not $\mathcal{F}$-characteristic in S and set $T_P = \mathrm{core}_{\mathcal{F}}(P)$. Then by the choice of E we have $Z_S \not\leq T_P$. In particular $T \not\leq T_P$ and since $\Phi(E) \leq \Phi(P) \leq T_P$, we deduce that $[E\colon T] = p^2$. Since E is not abelian and $T \leq \mathrm{Z}(E)$ by Theorem 11, we conclude $T = \mathrm{Z}(E)$. If $[P\colon T_P] = p^2$, then $T_P \leq \mathrm{Z}(P)$ and since E is $\mathcal{F}$-centric we get $T_P \leq \mathrm{Z}(E) = T$. Hence $[P\colon T] \leq [P\colon T_P] = p^2$, contradicting the fact that P contains $\mathrm{N}_S(E)$. Thus we have $[P\colon T_P] = p^3$. Note that $[T_PZ_S\colon T_P] = p$ and by maximality of T_P we deduce $[T_PZ_P\colon T_P] \geq p^2$. Since $Z_P \leq Z_E \leq T$ we conclude $[TT_P\colon T_P] \geq p^2$. However, $T \cap T_P = \Phi(E)$ and so $[T\colon T \cap T_P] = p$, giving a contradiction.

Hence the $\mathcal{F}$-essential subgroup P has to be $\mathcal{F}$-characteristic in S. If $Z_P \leq T$ then $Z_P = \Omega_1(T)$ (since $Z_S < Z_P$ and $|\Omega_1(T)| = p^2$). So $[E, E] \leq Z_P$ and by Lemma 2.41 with $K = Z_P$ we conclude that E has maximal normalizer tower in S, P is the maximal subgroup of S containing E and P is not $\mathcal{F}$-essential, which is a contradiction. Thus $Z_P \nleq T$. In particular $\Omega_1(\mathrm{Z}(E)) \nleq T$ and so $Z_E < \Omega_1(\mathrm{Z}(E))$. Since $|Z_E| = p^2$, we get $|\Omega_1(\mathrm{Z}(E))| = p^3$ and

$$[T\Omega_1(\mathrm{Z}(E))\colon T] = [\Omega_1(\mathrm{Z}(E))\colon T \cap \Omega_1(\mathrm{Z}(E))] = [\Omega_1(\mathrm{Z}(E))\colon Z_E] = p.$$

Recall that by Theorem 11 either $E = \mathrm{C}_E(T)$ or $[\mathrm{C}_E(T)\colon T] = p^2$. Since $T\Omega_1(\mathrm{Z}(E)) < \mathrm{C}_E(T)$ and it is $\mathcal{F}$-characteristic in E, we deduce that $T \leq \mathrm{Z}(E)$. Also $T \neq \mathrm{Z}(E)$ (otherwise $\Omega_1(\mathrm{Z}(E)) \leq T$) and E is not abelian, so $[E\colon \mathrm{Z}(E)] = p^2$. Note that $\mathrm{N}_S(E) = E\mathrm{C}_E(T)$ by Lemma 4.14 so $T \leq \mathrm{Z}(\mathrm{N}_S(E))$. Also, $\mathrm{Z}(\mathrm{N}_S(E)) < \mathrm{Z}(E)$ by maximality of T and we conclude $T = \mathrm{Z}(\mathrm{N}_S(E))$. In particular $Z_P \leq \mathrm{Z}(P) \leq \mathrm{Z}(\mathrm{N}_S(E)) \leq T$, and we get a contradiction.

Therefore whenever E is not $\mathcal{F}$-characteristic in S and $Z_S < Z_E$ the group Z_S is not contained in the $\mathcal{F}$-core of E.

$\square$

We now prove the first part of Theorem 18.

Theorem 5.2. *Let $E \leq S$ be an $\mathcal{F}$-essential subgroup of S not $\mathcal{F}$-characteristic in S. If $Z_S < Z_E$ then E is abelian.*

Proof. Set $T = \mathrm{core}_{\mathcal{F}}(E)$. If $T = 1$ then E is elementary abelian, so we can assume $T \neq 1$. By Theorem 5.1 we have $Z_S \nleq T$. So $Z_E \nleq T$.

Suppose $Z_S T = Z_E T$. If $\mathrm{N}_S(E) = S$ then $Z_S T$ is $\mathcal{F}$-characteristic in $\mathrm{N}_S(E)$ and so $Z_S T = Z_E T = T$ by the maximality of T. Thus $Z_S \leq T$, that is a contradiction. If $\mathrm{N}_S(E) < S$ then $\mathrm{Aut}_{\mathcal{F}}(\mathrm{N}_S(E)) = \mathrm{Aut}_S(E)\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(\mathrm{N}_S(E))}(E)$ by Lemma 4.21 and since $Z_S T$ is normal in $\mathrm{N}_S(\mathrm{N}_S(E))$ we deduce that $Z_E T$ is $\mathcal{F}$-characteristic in $\mathrm{N}_S(E)$. Hence $Z_S T = Z_E T = T$ by the maximality of T and $Z_S \leq T$, giving a contradiction.

So we have $T < Z_S T < Z_E T$. In particular $[Z_E T \colon T] \geq p^2$. If $T \leq \mathrm{Z}(E)$ then $Z_E T \leq \mathrm{Z}(E)$ and so E is abelian. Suppose $T \nleq \mathrm{Z}(E)$. Hence $Z_E T = \mathrm{C}_E(T) = \Omega_1(\mathrm{Z}(E))T$ and $[Z_E T \colon T] = p^2$. In particular, since S has sectional rank 3 and $T \cap \Omega_1(\mathrm{Z}(E))) \neq 1$, we deduce that $|\Omega_1(\mathrm{Z}(E))| = p^3$, $\Omega_1(\mathrm{Z}(E)) = \Omega_1(E)$ and $|\Omega_1(T)| = p$, so T is cyclic.

Let $y \in E$ be of minimal order such that $E = \langle y \rangle \Omega_1(E)T$. We want to show that y commutes with T, contradicting the fact that $T \nleq \mathrm{Z}(E)$. Note that $y^p \in \Phi(E) = T$. Suppose that $\langle y \rangle T$ has rank 2. Then there exists a normal subgroup of $\langle y \rangle T$ isomorphic to the group $C_p \times C_p$. In particular y has order p and so $y \in \Omega_1(E)$. Thus $E = \Omega_1(E)T = \mathrm{C}_E(T)$ contradicting the assumptions. Thus the group $\langle y \rangle T$ has to be cyclic. In particular y commutes with T and so $E = \mathrm{C}_E(T)$, which is a contradiction. $\square$

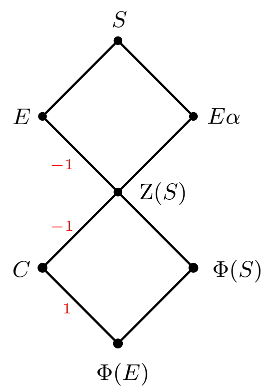
The second part of Theorem 18 is a consequence of the following lemma.

Lemma 5.3. *Let $E \leq S$ be an $\mathcal{F}$-essential subgroup of S not $\mathcal{F}$-characteristic in S. Suppose that E is abelian and normal in S. Then E has rank 3 and if C is the preimage in E of the group $\mathrm{C}_{E/\Phi(E)}(O^{p'}(\mathrm{Aut}_{\mathcal{F}}(E)))$ then $S/C \cong p_+^{1+2}$ and for every $\mathcal{F}$-essential subgroup P of S we have $C = \mathrm{core}_{\mathcal{F}}(P)$. In particular $C \trianglelefteq \mathcal{F}$ and $O_p(\mathcal{F}) \neq 1$.*

Proof. Since E is normal in S we have $[S\colon E] = p$ by Theorem 1. If E has rank 2 then E is a pearl by Theorem 12 so $E \cong \mathrm{C}_p \times \mathrm{C}_p$ and $|S| = p^3$, contradicting the fact that S has sectional rank 3. Therefore E has rank 3. Let $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ be such that $E\alpha \neq E$. Then $S = EE\alpha$ and since E is abelian we deduce that $E\alpha$ is abelian and $\mathrm{Z}(S) = E \cap E\alpha$. Thus $[S\colon \mathrm{Z}(S)] = p^2$ and by Lemma 1.6 we get $|[S, S]| = p$. Also by Lemma 4.20 the group $S/\Phi(E)$ has exponent p, $\Phi(E)$ $\mathrm{char}_{\mathcal{F}}$ S and S has rank 3 ($\Phi(S) = \Phi(E)[S, S]$).

Let $C \leq E$ be the preimage in E of $\mathrm{C}_{E/\Phi(E)}(O^{p'}(\mathrm{Aut}_{\mathcal{F}}(E)))$. Then $C/\Phi(E) \leq \mathrm{Z}(S/\Phi(E)) = \mathrm{Z}(S)/\Phi(E)$ and so $C \leq \mathrm{Z}(S)$. In particular C is contained in every $\mathcal{F}$-essential subgroup of S. Also, $C \neq \Phi(S)$ otherwise $\mathrm{Aut}_S(E)$ centralizes every quotient of consecutive subgroups in the sequence $\Phi(E) < C < E$, contradicting Lemma 2.24. Since S/C has exponent p and order p^3, we deduce that $S/C \cong p_+^{1+2}$.

Let $\tau \in \mathrm{Aut}_{\mathcal{F}}(S)$ be the morphism that acts as $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ on E/C and centralizes $C/\Phi(E)$. Then τ normalizes $\Phi(S)/\Phi(E)$ and does not act as scalar on $\mathrm{Z}(S)/\Phi(E)$. In particular $C/\Phi(E)$ and $\Phi(S)/\Phi(E)$ are the only maximal subgroups of $\mathrm{Z}(S)/\Phi(E)$ that can be $\mathcal{F}$-characteristic in S. Since the inner automorphisms of S act trivially on $\mathrm{Z}(S)/\Phi(E)$ and S is fully automized, the group $\mathrm{Aut}_{\mathcal{F}}(S)/\mathrm{C}_{\mathrm{Aut}_{\mathcal{F}}(S)}(\mathrm{Z}(S)/\Phi(E))$ has order prime to p. Since $\Phi(S)$ is $\mathcal{F}$-characteristic in S, by Theorem 1.15 there exists a maximal subgroup of $\mathrm{Z}(S)/\Phi(E)$ distinct from $\Phi(S)/\Phi(E)$ that is $\mathcal{F}$-characteristic in S. Hence C and $\Phi(S)$ are the only maximal subgroups of $\mathrm{Z}(S)$ containing $\Phi(E)$ that are $\mathcal{F}$-characteristic in S. By the definition of the $\mathcal{F}$-core we get $C = \mathrm{core}_{\mathcal{F}}(E)$.



Let P be an $\mathcal{F}$-essential subgroup of S. Then $\mathrm{Z}(S) < P < S$. So $[P\colon \mathrm{Z}(S)] = p$ and $P/\Phi(E)$ is elementary abelian (since $S/\Phi(E)$ has exponent p). Since S has sectional rank 3 we deduce that $\Phi(E) = \Phi(P)$. Thus $\Phi(E)$ is $\mathcal{F}$-characteristic in P. By Theorem 8, since S has rank 3, we deduce that $O^{p'}(\mathrm{Out}_{\mathcal{F}}(P)) \cong \mathrm{SL}_2(p)$. In particular if H is the preimage in P of $\mathrm{C}_{P/\Phi(P)}(O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P)))$, then $[H\colon \Phi(P)] = p$. So $H/\Phi(E)$ is a maximal subgroup of $\mathrm{Z}(S)/\Phi(E)$. Let $\mu \in \mathrm{Aut}_{\mathcal{F}}(S)$ be the morphism that acts on P/H as $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and centralizes $H/\Phi(E)$. Then μ does not act as scalar on $\mathrm{Z}(S)/\Phi(E)$ and normalizes $C/\Phi(E), H/\Phi(E)$ and $\Phi(S)/\Phi(E)$. Since P is $\mathcal{F}$-essential, the group S/H is not abelian and so $H \neq \Phi(S)$. Hence $H = C$. In particular $C = \mathrm{core}_{\mathcal{F}}(P)$.

Therefore the group C is $\mathcal{F}$-characteristic in S and in every $\mathcal{F}$-essential subgroup of S and by Lemma 2.28 we conclude that $C \trianglelefteq \mathcal{F}$. Also, since E has rank 3 we have $|C| \geq p$, and so $O_p(\mathcal{F}) \neq 1$. $\square$

5.2 Final results

We show that if $p \geq 5$ and $O_p(\mathcal{F}) = 1$ then $\mathcal{F}$ contains a pearl.

Lemma 5.4. *Let $E \leq S$ be a normal $\mathcal{F}$-essential subgroup of S such that E has rank 3 and is not $\mathcal{F}$-characteristic in S. Let $P \leq S$ be an $\mathcal{F}$-characteristic $\mathcal{F}$-essential subgroup of S. Then $\Phi(P) = \Phi(E)$.*

Proof. Let $C \leq E$ be the preimage in E of $\mathrm{C}_{E/\Phi(E)}(O^{p'}(\mathrm{Aut}_{\mathcal{F}}(E)))$. Then $[C\colon \Phi(E)] = p$ and there exists an $\mathcal{F}$-automorphism τ of S that acts as $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ on E/C and centralizes S/E and $C/\Phi(E)$. Also note that $C \operatorname{char}_{\mathcal{F}} E \trianglelefteq S$, so $C \trianglelefteq S$, and since $|C/\Phi(E)| = p$ we conclude $C/\Phi(E) \leq \mathrm{Z}(S/\Phi(E))$.

Case 1: suppose $C \leq P$. Then $C < E \cap P$ and τ acts on $P/\Phi(E)$ as illustrated in Figure 5.1. Let $x \in P \backslash E$ and $y \in (E \cap P) \backslash C$. Then $x\tau = xc$ for some $c \in C$ and $y\tau = y^{-1}u$ for some $u \in \Phi(E)$. Hence by Lemma 1.4 we get

$$[x, y]\tau = [xc, y^{-1}u] = [x, y]^{-1} \mod \Phi(E).$$

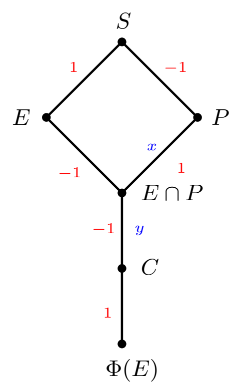


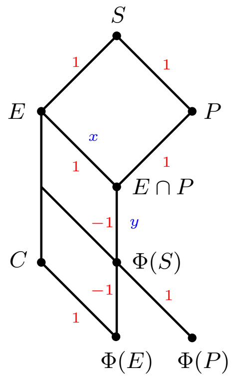
Figure 5.1

Since τ centralizes $C/\Phi(E)$ and $[x, y] \in C$ (note that $S/C \cong p_+^{1+2}$), we deduce that $[x, y] = 1 \mod \Phi(E)$ and so the group $P/\Phi(E)$ is abelian. Since S has sectional rank 3 and the group $S/\Phi(E)$ has exponent p by Lemma 4.20, we conclude that $\Phi(E) = \Phi(P)$.

Case 2: suppose $C \not\leq P$. Then $E/\Phi(E) \cong C/\Phi(E) \times (E \cap P)/\Phi(E)$ and so $(E \cap P)/\Phi(E)$ is an $\mathrm{SL}_2(p)$-module for $O^{p'}(\mathrm{Out}_{\mathcal{F}}(E))$. Suppose for a contradiction that $\Phi(E) \neq \Phi(P)$. By Lemma 4.20 the group S has rank 3. Since P is $\mathcal{F}$-essential, by Lemma 2.35 we have

$$\Phi(P) < [S, S]\Phi(P) \leq \Phi(S).$$

Thus $S/\Phi(P)$ is non-abelian, P has rank 3 and $\Phi(S) = \Phi(E)\Phi(P)$. By Lemma 4.14, the morphism τ centralizes $\Phi(E)$. Hence it centralizes $\Phi(S)/\Phi(P)$.



Let $x \in E \backslash P$ and $y \in (E \cap P) \backslash \Phi(S)$. Then $x\tau = xu$ for some $u \in \Phi(S)$ and $y\tau = y^{-1}v$ for some $v \in \Phi(P)$. Therefore by Lemma 1.4 we get

$$[x, y]\tau = [xu, y^{-1}v] = [x, y]^{-1} \mod \Phi(P).$$

Since $[x, y] \in \Phi(S)$, we deduce that $[x, y] \in \Phi(P)$ and so the group $E/\Phi(P)$ is abelian. In particular $(E \cap P)/\Phi(P) \leq \mathrm{Z}(S/\Phi(P))$.

Since $S/\Phi(P)$ is non-abelian, we get $(E \cap P)\Phi(P) = \mathrm{Z}(S/\Phi(P))$ and since P is $\mathcal{F}$-characteristic in S, we deduce that $(E \cap P)$ $\mathrm{char}_{\mathcal{F}}$ S. Thus $E \cap P \leq \mathrm{core}_{\mathcal{F}}(E) \leq C$ and we get a contradiction. Therefore we have $\Phi(E) = \Phi(P)$. □

Proof of Theorem 19. Suppose that all the $\mathcal{F}$-essential subgroups of S are normal in S. Set $Z_S = \Omega_1(\mathrm{Z}(S))$. Then Z_S is $\mathcal{F}$-characteristic in S and contained in every $\mathcal{F}$-essential subgroup of S. Since $O_p(\mathcal{F}) = 1$, by Lemma 2.28 there exists an $\mathcal{F}$-essential subgroup P of S such that Z_S is not $\mathcal{F}$-characteristic in P.

By Theorem 18 the group P has to be $\mathcal{F}$-characteristic in S. Also, since $O_p(\mathcal{F}) = 1$, there exists an $\mathcal{F}$-essential subgroup of S distinct from P.

Suppose that every $\mathcal{F}$-essential subgroup of S distinct from P is not $\mathcal{F}$-characteristic in S and has rank 3. Then by Lemma 5.4 the group $\Phi(P)$ is the Frattini subgroup of every $\mathcal{F}$-essential subgroup of S. Since $O_p(\mathcal{F}) = 1$, by Lemma 2.28 we get $\Phi(P) = 1$.

Thus there is an elementary abelian $\mathcal{F}$-essential subgroup of S that is normal in S but not $\mathcal{F}$-characteristic in S and by Lemma 5.3 we get $O_p(\mathcal{F}) \neq 1$, which is a contradiction.

Hence there exists an $\mathcal{F}$-essential subgroup E of S, distinct from P, that is either $\mathcal{F}$-characteristic in S or has rank 2. Therefore S is isomorphic to a Sylow p-subgroup of $\mathrm{Sp}_4(p)$ (by Theorem 16 if E is $\mathcal{F}$-characteristic in S and by Theorems 12 and 7 if E has rank 2). □

Proof of Theorem 20. Note that the Sylow p-subgroups of the group $\mathrm{Sp}_4(p)$ have sectional rank 3.

- If S is isomorphic to a Sylow p-subgroup of $\mathrm{Sp}_4(p)$ then S has maximal nilpotency class. Also, the group $S_1 = \mathrm{C}_S(\mathrm{Z}_2(S))$ is abelian. Since $O_p(\mathcal{F}) = 1$, the group S_1, if $\mathcal{F}$-essential, cannot be the only $\mathcal{F}$-essential subgroup of S (by Lemma 2.28). Thus there exists an $\mathcal{F}$-essential subgroup E of S distinct from S_1, and by Theorem 3 we deduce that E is a pearl.

- If S is not isomorphic to a Sylow p-subgroup of $\mathrm{Sp}_4(p)$, then by Theorem 19 there exists a subgroup E of S not normal in S. Thus E has rank 2 by Theorem 14 and so it is a pearl by Theorem 12.

□

Remark 5.5. As a consequence of Theorem 20, if $p \geq 5$, S has sectional rank 3, $O_p(\mathcal{F}) = 1$ and S is not isomorphic to a Sylow p-subgroup of $\mathrm{Sp}_4(p)$, then $\mathcal{F}$ is the simple exotic fusion system described in Theorem 7.

If S is isomorphic to a Sylow p-subgroup of $\mathrm{Sp}_4(p)$ and $\mathcal{F}$ is a simple fusion system on S, then $\mathcal{F}$ is reduced (as defined in [AKO11, Definition III.6.2]) and $\mathcal{F}$ is among the fusion systems described in [Oli14] and [COS16].

CHAPTER 6

CONCLUSION AND FUTURE PROJECTS

'We shall not cease from exploration, and the end of all our exploring will be to arrive where we started and know the place for the first time.'

[T. S. Eliot]

This work aimed to investigate the saturated fusion systems on p-groups of sectional rank 3, for p odd. Let $\mathcal{F}$ be a saturated fusion system on a p-group S having sectional rank 3. We showed that if $O_p(\mathcal{F}) = 1$ and $p \geq 5$ then $\mathcal{F}$ contains a pearl and so either S is isomorphic to a Sylow p-subgroup of the group $\mathrm{Sp}_4(p)$ and $\mathcal{F}$, if reduced, is one of the fusion systems classified in [Oli14] and [COS16], or $p = 7$, S is isomorphic to a maximal subgroup of the Sylow 7-subgroup of the group $\mathrm{G}_2(7)$ and $\mathcal{F}$ is a simple exotic fusion system completely determined by $\mathrm{Inn}(S)$, $\mathrm{Out}_{\mathcal{F}}(S) \cong \mathrm{C}_6$ and $\mathrm{Out}_{\mathcal{F}}(E) \cong \mathrm{SL}_2(7)$, where E is an $\mathcal{F}$-essential subgroup of S isomorphic to the group $\mathrm{C}_7 \times \mathrm{C}_7$.

Let S be a 3-group of sectional rank 3 and let $\mathcal{F}$ be a saturated fusion system on S such that $O_3(\mathcal{F}) = 1$. Since every $\mathcal{F}$-essential subgroup of S of rank 2 is a pearl (Theorem 12), if S contains an $\mathcal{F}$-essential subgroup of rank 2 then S is isomorphic to a Sylow 3-subgroup of the group $\mathrm{Sp}_4(3)$ (Theorem 7).

Suppose that all the $\mathcal{F}$-essential subgroups of S have rank 3 (in particular S is not isomorphic to a Sylow 3-subgroup of $\mathrm{Sp}_4(3)$, that contains a pearl). Then by Theorem 19 there exists an $\mathcal{F}$-essential subgroup of S that is not normal in S (and so $|S| > 3^4$).

If there exists an $\mathcal{F}$-essential subgroup E of S such that $[E\colon \mathrm{core}_{\mathcal{F}}(E)] = 3^3$ and $\mathrm{core}_{\mathcal{F}}(E) \trianglelefteq S$, then $[S\colon E] \leq 3^3$ (Theorem 4.30). In particular, if $|S| > 3^4$, $E \cong \mathrm{C}_3 \times \mathrm{C}_3 \times \mathrm{C}_3$ and $\mathrm{core}_{\mathcal{F}}(E) = 1$ then S is isomorphic to either a Sylow 3-subgroup of the group $\mathrm{P\Gamma L}_3(64)$ or to the group indexed in *Magma* as $\mathrm{SmallGroup}(3^5, 58)$ (that is isomorphic to a maximal subgroup of a Sylow 3-subgroup of the group $\mathrm{P\Gamma L}_3(64)$).

If every $\mathcal{F}$-essential subgroup of S has index at most 3^2 in S then the results presented in the Appendix (in particular Theorem C) show that $|S| \leq 3^7$ and the isomorphism type of S can be determined using the computer program *Magma*.

The case in which the $\mathcal{F}$-essential subgroups of S have arbitrary index in S is still open. We know that if $E \leq S$ is an $\mathcal{F}$-essential subgroup of S having rank 3 and $[S\colon E] \geq 3^3$ then, if we set $\mathrm{N}^2 = \mathrm{N}_S(\mathrm{N}_S(E))$ and $\mathrm{N}^3 = \mathrm{N}_S(\mathrm{N}^2)$, either the quotient group $\mathrm{N}^2/\Phi(E)$ is isomorphic to the group $\mathrm{SmallGroup}(3^5, 53)$ or the quotient group $\mathrm{N}^3/\Phi(E)$ is isomorphic to a Sylow 3-subgroup of the group $\mathrm{P\Gamma L}_3(64)$ (Theorems 4.2 and 4.4). Examples of this situations are given by the fusion categories of the groups $\mathrm{SL}_4(q)$ and $\mathrm{P\Gamma L}_3(q^{3^k})$ (with $q = 1 \mod 3$) on one of their Sylow 3-subgroups. In particular, neither the order of S nor the index of the $\mathcal{F}$-essential subgroups in S can be bound.

The methodology developed in this thesis gives a general approach to the classification of simple fusion systems on p-groups of small sectional rank. The natural continuation of this project is the determination of the simple fusion systems on p-groups having sectional rank 4, for p odd. We will start working on this project during a 6 months PostDoc at the University of Aberdeen, supported by the LMS Postgraduate Mobility Grant 2016-2017.

One of the main differences between sectional rank 3 and sectional rank 4 groups is that the automorphism group of $\mathcal{F}$-essential subgroups of p-groups having sectional rank 4 can contain a subgroup isomorphic to the group $\mathrm{SL}_2(p^2)$. In particular, we can find $\mathcal{F}$-essential subgroups having index p^2 in their normalizer (note that this is in accordance with Theorem 2.40).

In the fortuitous case in which the p-group S considered has sectional rank 4 and $\mathcal{F}$ contains a pearl, then Theorems 2 and 4 assure that $p \neq 3$ and either $p = 5$ or S has order at most p^7. Moreover, if $|S| \leq p^6$ then either S contains an abelian subgroup of index p (and $\mathcal{F}$ is one of the fusion systems studied in [Oli14] and [COS16]) or the isomorphism type of S is known (Theorem 6). This gives a very good starting point to classify simple fusion systems on p-groups having sectional rank 4 and containing pearls.

The classification of simple fusion systems containing pearls on p-groups of arbitrary sectional rank, for p odd, is another subject that we wish to investigate, using the results and the theory developed in Chapter 3.

GUIDE FOR THE PROOFS OF THE THEOREMS PRESENTED IN THE INTRODUCTION

Proof of Theorem 1: proof of Theorem 2.37 on page 52.

Proof of Theorem 2: combination of Theorem 3.12 and Lemma 3.13 on pages 71–72.

Proof of Theorem 3: combination of Theorems 3.22 and 3.27 on pages 79 and 86.

Proof of Theorem 4: proof of Theorem 3.25 on page 82.

Proof of Theorem 5: on page 89.

Proof of Theorem 6: on page 97.

Proof of Theorem 7: proof of Theorem 3.31 on page 98.

Proof of Theorem 8: combination of Theorems 4.8 and 4.10 on pages 117– 118.

Proof of Theorem 9: combination of Lemmas 4.12 and 4.13 on page 120–123.

Proof of Theorem 10: proof of Theorem 4.15 on page 125.

Proof of Theorem 11: proof of Theorem 4.17 on page 128.

Proof of Theorem 12: proof of Theorem 4.19 on page 131.

Proof of Theorem 13: proof of Theorem 4.22 on page 134.

Proof of Theorem 14: on page 112.

Proof of Theorem 15: combination of Theorems 4.33, 4.34 and 4.35 on pages 162–164.

Proof of Theorem 16: proof of Theorem 4.36 on page 166.

Proof of Theorem 17: proof of Theorem 5.1 on page 170.

Proof of Theorem 18: combination of Theorem 5.2 and Lemma 5.3 on pages 173–174.

Proof of Theorem 19: on page 177.

Proof of Theorem 20: on page 178.

APPENDIX: SOME RESULTS FOR $p = 3$

We present here some results about saturated fusion systems on 3-groups having sectional rank 3, that might be used for future research projects.

Let $p = 3$, let S be a 3-group having sectional rank 3 and let $\mathcal{F}$ be a saturated fusion system on S such that $O_3(\mathcal{F}) = 1$.

If all the $\mathcal{F}$-essential subgroups of S are normal in S, then by Theorem 19 we conclude that S is isomorphic to a Sylow 3-subgroup of the group $\mathrm{Sp}_4(3)$.

If $|S| = 3^4$ and there exists an $\mathcal{F}$-essential subgroup E of S not normal in S, then E is a pearl and so by Theorem 7 the group S is isomorphic to a Sylow 3-subgroup of the group $\mathrm{Sp}_4(3)$.

If $|S| \geq 3^5$ then the situation is more complicated. As we saw in Chapter 4, there is no hope to bound the order of S when $p = 3$. However, we can find a bound for $|S|$ if all the $\mathcal{F}$-essential subgroups of S have index at most 3^2 in S.

Lemma A. *Suppose $p = 3$ and all the $\mathcal{F}$-essential subgroups of S have rank 3. Let $E, P \leq S$ be $\mathcal{F}$-essential subgroups of S such that $[S\colon E] = 3^2$ and $P \trianglelefteq S$. Set $\mathrm{N}^1 = \mathrm{N}_S(E)$. Then $\Phi(\mathrm{N}^1) = \Phi(P)$.*

Proof. By Theorems 4.2 and 4.4 we know that the group $\overline{S} = S/\Phi(E)$ is isomorphic to the group indexed in *Magma* as SmallGroup$(3^5, j)$, where $j \in \{52, 53\}$ if $[E\colon \mathrm{core}_{\mathcal{F}}(E)] = 3^2$ and $j \in \{56, 57, 58\}$ otherwise. Using *Magma* we can also check that there exists a subgroup $\overline{H} \leq \overline{S}$ of order 3 such that for every maximal subgroup $\overline{M}$ of $\overline{S}$ either $\Phi(\overline{M}) = \Phi(\overline{S})$ or $\Phi(\overline{M}) = \overline{H}$. Note that $\Phi(E) \leq \Phi(S) \leq M$ for every maximal subgroup M of S. Since S has sectional rank 3 we deduce that for every maximal subgroup M of S either $\Phi(M) = \Phi(S)$ or $\Phi(M) = H$, where $H \leq S$ is the preimage in S of $\overline{H}$.

Note that the group N^1 has rank 3 by Lemma 4.20. In particular $\Phi(\mathrm{N}^1) < \Phi(S)$ and so $\Phi(\mathrm{N}^1) = H$. Since P is $\mathcal{F}$-essential, by Lemma 2.35 we have $\Phi(P) < \Phi(S)$. Therefore $\Phi(P) = H = \Phi(\mathrm{N}^1)$. □

Theorem B. *Suppose that* $p = 3$, $O_3(\mathcal{F}) = 1$ *and that there exists an* $\mathcal{F}$*-essential subgroup* E *of* S *of rank* 3 *such that* $Z_S < Z_E$ *and* $[S\colon E] = 3^2$. *Set* $T = \mathrm{core}_{\mathcal{F}}(E)$ *and suppose* $T \neq 1$. *Then* E *is abelian and*

1. *either* $T \cong \mathrm{C}_3$; *or*

2. $\Phi(E) < T$, $T \cong \mathrm{C}_9$ *and there exists an* $\mathcal{F}$*-essential subgroup* P *of* S *that is abelian and* $\mathcal{F}$*-characteristic in* S *and such that* $\Omega_1(T)$ *is not* $\mathcal{F}$*-characteristic in* P.

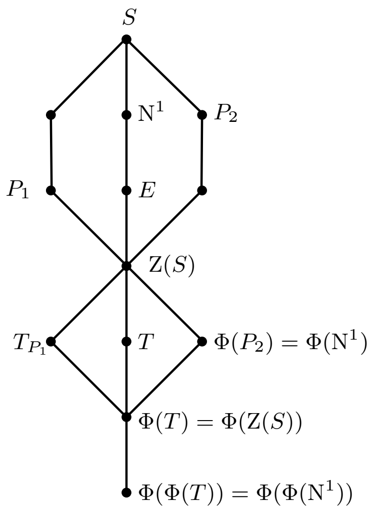
In particular $|S| \leq 3^6$.

Proof. By Theorem 5.2 the group E is abelian and so T is cyclic by Theorem 11. By Lemmas 4.24 and 4.29 the group T is $\mathcal{F}$-characteristic in S. Since $|\Omega_1(T)| = 3$ we conclude that the group $\Omega_1(T)$ is an $\mathcal{F}$-characteristic subgroup of S contained in $\mathrm{Z}(S)$, and so in every $\mathcal{F}$-centric subgroup of S. By Lemma 2.28, there exists an $\mathcal{F}$-essential subgroup P of S such that $\Omega_1(T)$ is not $\mathcal{F}$-characteristic in P. In particular, no non-trivial subgroup of T is $\mathcal{F}$-characteristic in P. Set $T_P = \mathrm{core}_{\mathcal{F}}(P)$ and $\mathrm{N}^1 = \mathrm{N}_S(E)$. Let $C \leq E$ be the preimage in E of $\mathrm{C}_{E/\Phi(E)}(O^{p'}(\mathrm{Aut}_{\mathcal{F}}(E))$ and let τ be the $\mathcal{F}$-automorphism of S that centralizes $C/\Phi(E)$ and acts on E/C as $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then τ acts as -1 on S/N^1 and centralizes T (by Lemma 4.14).

- Suppose $[E\colon T] = 3^2$. Then by Lemma 4.23 and the fact that $Z_S < Z_E$ we get $T < \mathrm{Z}(S)$. Thus $[S\colon \mathrm{Z}(S)] = 3^3$ which implies $3 \leq [S\colon P] \leq 3^2$.

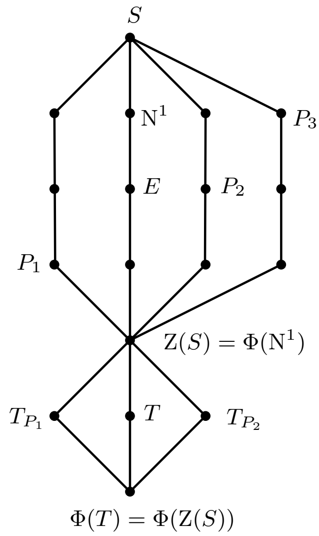
Assume $[S\colon P] = 3^2$. Thus $[P\colon \mathrm{Z}(S)] = 3$, P is abelian, T_P is cyclic by Theorem 11 and we have $T \cap T_P = 1$. Since P is not normal in S, we have $\mathrm{Z}(S)T_P < P$ and so $T_P \leq \mathrm{Z}(S)$. Also, by Theorem 4.28(1) we deduce $[P\colon T_P] = 3^2$. From $[\mathrm{Z}(S)\colon T] = 3$ we conclude $\mathrm{Z}(S) = TT_P$ and so $|T| = [\mathrm{Z}(S)\colon T_P] = 3$.

Assume $[S\colon P] = 3$, so $[P\colon \mathrm{Z}(S)] = 3^2$. If $\mathrm{Z}(P) = \mathrm{Z}(S)$ then $\Phi(T) = \Phi(\mathrm{Z}(S)) = 1$, so $|T| = 3$. Suppose $\mathrm{Z}(S) < \mathrm{Z}(P)$. Then P is abelian. If there exists $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ such that $P \neq P\alpha$, then $\mathrm{Z}(S) = P \cap P\alpha$ and $[S\colon \mathrm{Z}(S)] = 3^2$, that is a contradiction. Thus P is $\mathcal{F}$-characteristic in S and by Lemma A we have $\Phi(\mathrm{N}^1) = \Phi(P)$. Thus $\Phi(\Phi(P)) = \Phi(\Phi(\mathrm{N}^1)) = \Phi(\Phi(T)) = 1$. Therefore $|T| \leq 3^2$.



Structure of S for $[E\colon T] = 3^2$ and P_1, P_2 $\mathcal{F}$-essential subgroups.

- Suppose $[E\colon T] = 3^3$. By Lemma 4.14 we have $\mathrm{N}^1 = E\mathrm{C}_{\mathrm{N}^1}(T)$. Since E is abelian we conclude $T \leq \mathrm{Z}(\mathrm{N}^1)$. Since τ centralizes T and inverts S/N^1 and $T \trianglelefteq S$, we can show (using Lemma 1.4) that $T \leq \mathrm{Z}(S)$. By assumption Z_S is not $\mathcal{F}$-characteristic in E. Therefore $T < \mathrm{Z}(S)$ and by Theorem 4.28(1) we get $[\mathrm{Z}(S)\colon T] = 3$. Thus $[S\colon \mathrm{Z}(S)] = 3^4$ and so $3 \leq [S\colon P] \leq 3^3$.



Structure of S for $[E \colon T] = 3^3$ and P_1, P_2, P_3 $\mathcal{F}$-essential subgroups.

Assume $[S \colon P] = 3^3$. Then P is abelian, T_P is cyclic and $T \cap T_P = 1$. Since P is not normal in S we have $\mathrm{Z}(S)T_P < P$, and so $T_P < \mathrm{Z}(S)$. Therefore $\mathrm{Z}(S) = TT_P$ and since $[P \colon \mathrm{Z}(S)] = p$, by Theorem 4.28(1) we get $[P \colon T_P] = 3^2$. Hence $|T| = [\mathrm{Z}(S) \colon T_P] = 3$.

Assume $[S \colon P] = 3^2$. If $[P \colon T_P] = 3^2$ then by Lemma 4.23 we have $T_P = \mathrm{Z}(S)$. Hence $\Phi(T) = \Phi(\mathrm{Z}(S)) = \Phi(T_P) = 1$ and $|T| = 3$.

Suppose $[P \colon T_P] = 3^3$. By Lemma 4.28(1) we have $[P \colon \mathrm{Z}(S)T_P] \geq 3^2$, so $T_P < \mathrm{Z}(S)$. By maximality of T_p we have $\mathrm{Z}(S) < \mathrm{Z}(P)$, so P is abelian and T_P is cyclic. Thus $T \cap T_P = 1$ and since $\mathrm{Z}(S) = TT_p$ we conclude $|T| = 3$.

Assume $[S \colon P] = 3$. Then $P \trianglelefteq S$ and by Lemma A we have $\Phi(P) = \Phi(\mathrm{N}^1)$. Note that $\Phi(\mathrm{N}^1) = \mathrm{Z}(S)$. So $\Phi(\Phi(P)) = \Phi(\mathrm{Z}(S)) = \Phi(T) = 1$ and $|T| = 3$.

$\square$

Theorem C. *Suppose that $p = 3$, $O_3(\mathcal{F}) = 1$ and all the $\mathcal{F}$-essential subgroups of S have index at most 3^2 in S. Then $|S| \leq 3^7$.*

Moreover, if $|S| = 3^7$ then there exists an $\mathcal{F}$-characteristic $\mathcal{F}$-essential subgroup P of S such that $Z_S < Z_P$ and $\Phi(\Phi(P)) = 1$, and $Z_S = Z_E$ for every $\mathcal{F}$-essential subgroup E of S distinct from P.

Proof. If there exists an $\mathcal{F}$-essential subgroup of S having rank 2, then S contains a pearl by Theorem 12 and $|S| = 3^4$ by Theorem 7.

Suppose that all the $\mathcal{F}$-essential subgroups of S have rank 3. Since $O_3(\mathcal{F}) = 1$ there exists an $\mathcal{F}$-essential subgroup P of S such that Z_S is not $\mathcal{F}$-characteristic in P.

Case 1: suppose P is not $\mathcal{F}$-characteristic in S. Then by Theorem 18 the group P is not normal in S and by assumption we deduce $[S\colon P] = 3^2$. Hence by Theorem B we conclude $|S| \leq 3^6$.

Case 2: suppose P is $\mathcal{F}$-characteristic in S. Since $O_3(\mathcal{F}) = 1$, there exists an $\mathcal{F}$-essential subgroup $E \leq S$ distinct from P. If E is $\mathcal{F}$-characteristic in S then S is isomorphic to a Sylow 3-subgroup of $\mathrm{Sp}_4(3)$ by Theorem 16, and so $|S| = 3^4$.

Suppose E is not $\mathcal{F}$-characteristic in S. If $E \trianglelefteq S$ then $\Phi(E) = \Phi(P)$ by Lemma 5.4 and if $[S\colon E] = 3^2$ then $\Phi(\mathrm{N}_S(E)) = \Phi(P)$ by Lemma A. Set $T = \mathrm{core}_{\mathcal{F}}(E)$. Then either $\Phi(\Phi(P)) = \Phi(T)$ or $\Phi(\Phi(P)) = \Phi(\Phi(T))$. Therefore in any case the group $\Phi(\Phi(P))$ is $\mathcal{F}$-characteristic in E. Note that this holds for every $\mathcal{F}$-essential subgroup of S not $\mathcal{F}$-characteristic in S. Since $O_3(\mathcal{F}) = 1$ we deduce that $\Phi(\Phi(P)) = 1$. Hence $|P| \leq 3^6$ and $|S| \leq 3^7$, with equality only if P is the only $\mathcal{F}$-essential subgroup of S moving Z_S. □

We end this Appendix identifying the isomorphism type of S when $p = 3$, $O_3(\mathcal{F}) = 1$ and for every $\mathcal{F}$-essential subgroup E of S we have $[S\colon E] \leq 3^2$.

Lemma D. *Suppose that* $p = 3$, $O_3(\mathcal{F}) = 1$, $|S| > 3^4$ *and all the* $\mathcal{F}$*-essential subgroups of* S *have rank* 3 *and index at most* 3^2 *in* S*. Let* $i \in \mathbb{N}$ *be such that* $S \cong \mathrm{SmallGroup}(|\mathrm{S}|, \mathrm{i})$*. Then* S *has order at most* 3^7 *and if* $E \leq S$ *is an* $\mathcal{F}$*-essential subgroup of* S *not normal in* S *and* $j \in \mathbb{N}$ *is such that* $S/\Phi(E) \cong \mathrm{SmallGroup}(3^5, \mathrm{j})$*, then one of the following holds:*

1. $|S| = 3^5$, $E \cong \mathrm{C}_3 \times \mathrm{C}_3 \times \mathrm{C}_3$, $j = i$ *and* $i \in \{52, 53, 56, 57, 58\}$.
2. $|S| = 3^6$ *and one of the following holds:*
   (a) $E \cong \mathrm{C}_9 \times \mathrm{C}_3 \times \mathrm{C}_3, [E\colon \mathrm{core}_{\mathcal{F}}(E)] = 3^2, Z_S < Z_E$ *and* $[j, i] \in \{[52, 277], [53, 278]\}$.
   (b) $E \cong \mathrm{C}_9 \times \mathrm{C}_9 \times \mathrm{C}_9, [E\colon \mathrm{core}_{\mathcal{F}}(E) = 3^3, Z_S < Z_E$ *and* $[j, i]$ *is one of the following*

$$[56, 183], [56, 210],$$
$$[57, 184], [57, 212], [57, 281], [57, 356],$$
$$[58, 182], [58, 214], [58, 282], [58, 355]$$

   (c) $E \cong 3_+^{1+2} \times \mathrm{C}_3$, $\mathrm{core}_{\mathcal{F}}(E) = \mathrm{Z}(S) \cong \mathrm{C}_3 \times \mathrm{C}_3$ *and* $[j, i] \in \{[53, 394], [53, 395]\}$*;*
   (d) $E \cong 3_+^{1+2} \circ \mathrm{C}_9$, $\mathrm{core}_{\mathcal{F}}(E) = \mathrm{Z}(S) \cong \mathrm{C}_9$ *and* $[j, i] = [53, 397]$ *(thus* S *isomorphic to a maximal subgroup of a Sylow* 3*-subgroup of the group* $\mathrm{SL}_4(19)$*).*
   (e) $\mathrm{core}_{\mathcal{F}}(E) = \Phi(E) = \mathrm{Z}(S) \cong \mathrm{C}_3$ *and* $[j, i] \in \{[58, 411], [58, 412], [58, 413], [58, 414]\}$.
3. $|S| = 3^7$, $E \cong \mathrm{C}_9 \times 3_+^{1+2}$, $\mathrm{Z}(S) = \mathrm{core}_{\mathcal{F}}(E)$*, there exists an* $\mathcal{F}$*-essential subgroup* P *of* S *that is* $\mathcal{F}$*-characteristic in* S *and such that* $Z_S < Z_P$*, and*
   (a) *either* $\mathrm{Z}(S) \cong \mathrm{C}_3 \times \mathrm{C}_9$, $P \cong \mathrm{C}_9 \times \mathrm{C}_9 \times \mathrm{C}_9$, $i \in \{5402, 5403\}$ *and* $j = 53$*;*
   (b) *or* $\mathrm{Z}(S) \cong \mathrm{C}_3 \times \mathrm{C}_3$, $j = 58$ *and there are 66 possibilities for* i.

Also, when $j \in \{52, 53\}$ *there exists an abelian* $\mathcal{F}$*-essential subgroup of* S *having index* 3 *in* S *and so* $\mathcal{F}$ *is among the fusion systems described in [COS16].*

Proof. By Theorem C we have $|S| \leq 3^7$. Let E be an $\mathcal{F}$-essential subgroup E of S that is not normal in S (whose existence is guaranteed by Theorem 5.2). By assumption we have $[S\colon E] = 3^2$.

If $\Phi(E) = 1$ then $|S| = 3^5$ and by Theorems 4.2 and 4.4 we deduce that the group S is isomorphic to SmallGroup(3^5,i) for $i \in \{52, 53, 56, 57, 58\}$. In particular if $i = 52$ or $i = 53$ then there exists an abelian subgroup A of S having index 3 in S and if A is not $\mathcal{F}$-essential then by [Oli14, Theorem 2.8] we have $|\mathrm{Z}(S)| = 3$, that is false. Thus A is an $\mathcal{F}$-essential subgroup of S and $\mathcal{F}$ is among the fusion systems described in [COS16].

Set $T = \mathrm{core}_{\mathcal{F}}(E)$ and assume $\Phi(E) \neq 1$.

Case 1: suppose $|S| = 3^6$. Thus $|\Phi(E)| = 3$ and $|E| = 3^4$.

Case 1a: suppose $[E\colon T] = 3^2$ and $Z_S < Z_E$. Then by Theorem B we have $T \cong \mathrm{C}_9$, $E \cong \mathrm{C}_3 \times \mathrm{C}_3 \times \mathrm{C}_9$, and there exists a maximal subgroup A of S that is abelian. Also, by Theorem 5.1 and Lemma 4.23 we have $T < \mathrm{Z}(S) < E$. From $E = T\Omega_1(E)$ (Lemma 4.18) we get $\mathrm{Z}(S) \cong \mathrm{C}_3 \times \mathrm{C}_9$. Note that $[E, \mathrm{Z}_2(S)] \leq \mathrm{Z}(S) \leq E$ so $\mathrm{Z}_2(S) \leq \mathrm{N}_S(E)$ and $S = \mathrm{N}_S(\mathrm{N}_S(E))$ has nilpotency class 3.

We enter this information in *Magma*, recalling that S has sectional rank 3 and $S/\Phi(E)$ is isomorphic to either SmallGroup(3^5, 52) or SmallGroup(3^5, 53) (see Table 1). As output we get $[52, 277]$ and $[53, 278]$.

Finally, if A is not $\mathcal{F}$-essential then by [Oli14, Theorem 2.8] we have $|\mathrm{Z}(S)| = 3$, that is false. Thus A is an $\mathcal{F}$-essential subgroup of S and $\mathcal{F}$ is among the fusion systems described in [COS16].

```
for j in [52,53] do
for i in [1..NumberOfSmallGroups(3^6)] do S:=SmallGroup(3^6,i);
  if IsIsomorphic(Center(S), DirectProduct(CyclicGroup(3), CyclicGroup(9)))
  eq true and NilpotencyClass(S) eq 3 and
    ♯[M : M in MaximalSubgroups(S)| IsAbelian(M‘subgroup) eq true] ne 0 and
    ♯[M : M in Subgroups(S)|
      ♯(M‘subgroup/ FrattiniSubgroup(M‘subgroup)) ge 81] eq 0 and
    ♯[C : C in Subgroups(Center(S))| ♯C‘subgroup eq 3 and
      IsIsomorphic(S/C‘subgroup, SmallGroup(3^5,j)) eq true] ne 0 then
    [j,i];
end if; end for; end for;
```

Table 1

Case 1b: suppose $[E\colon T] = 3^3$ and $Z_S < Z_E$. Then by Theorem B we have $\mathrm{Z}(S) \cong \mathrm{C}_3 \times \mathrm{C}_3$ and $E \cong \mathrm{C}_9 \times \mathrm{C}_3 \times \mathrm{C}_3$. Also note that $\mathrm{Z}_2(S) \leq \mathrm{N}_S(E)$, so S has nilpotency class at least 3. Moreover, the group $S/\Phi(E)$ is isomorphic to SmallGroup$(3^5, j)$, for $j \in \{56, 57, 58\}$. We enter this information in *Magma* to find the isomorphism type of S.

```
for j in [56,57, 58] do
for i in [1..NumberOfSmallGroups(3^6)] do S:=SmallGroup(3^6,i);
if ♯Center(S) eq 9 and NilpotencyClass(S) ge 3 and Exponent(Center(S)) eq 3
  and ♯[E: E in Subgroups(S)| ♯E‘subgroup eq 81 and
    IsAbelian(E‘subgroup) eq true and
    FrattiniSubgroup(E‘subgroup) subset Center(S) and
    IsIsomorphic(S/FrattiniSubgroup(E‘subgroup), SmallGroup(3^5,j)) eq true
    and Centralizer(S,E‘subgroup) subset E‘subgroup and
    ♯Normalizer(S,E‘subgroup) eq 3*♯E‘subgroup] ne 0
  and ♯[H : H in Subgroups(S)|
    ♯(H‘subgroup/FrattiniSubgroup(H‘subgroup)) ge 81] eq 0
  then [j,i];
end if; end for; end for;
```

As output we get:

- $[56,183],[56,210]$;
- $[57,184],[57,212],[57,281],[57,356]$;
- $[58,182],[58,214],[58,282],[58,355]$.

Case 1c: assume $[E\colon T]=3^3$ and $Z_S=Z_E$. Then $Z_S=T\cong \mathrm{C}_3$. We use *Magma* to identify the group S.

```
for j in [56,57,58] do
for i in [1..NumberOfSmallGroups(3^6)] do S:=SmallGroup(3^6,i);
if ♯Omega(Center(S),1) eq 3 and NilpotencyClass(S) ge 3 and
  IsIsomorphic(S/Omega1(Center(S),1),SmallGroup(3^5,j)) eq true and
  ♯[E: E in Subgroups(S)| (♯E‘subgroup eq 81) and
    Centralizer(S,E‘subgroup) subset E‘subgroup and
    ♯Normalizer(S,E‘subgroup) eq 3*♯E‘subgroup] ne 0 and
  ♯[H : H in Subgroups(S)|
    ♯(H‘subgroup/FrattiniSubgroup(H‘subgroup)) ge 81] eq 0 then
  [j,i];
end if; end for;
```

As output we get

$$[58,411],[58,412],[58,413],[58,414].$$

Also, all the groups listed have center of order 3, so we have $T=\mathrm{Z}(S)$.

Case 1d: assume $[E\colon T]=3^2$ and $Z_S=Z_E$. Then by Theorem 5.1 we have $Z_S\leq T$. Also $T\leq \mathrm{Z}(S)$ by Lemma 4.23 and $E=\Omega_1(E)T$ by Lemma 4.18, so we have $T=\mathrm{Z}(S)\cong \mathrm{C}_9$.

Since $O_3(\mathcal{F})=1$, there exists an $\mathcal{F}$-essential subgroup $P\leq S$ such that $Z_S<Z_P$. If P is not normal in S then $[S\colon P]=3^2$ by assumption and so we are in one of the situations

described above (with P in place of E). So we may assume that P is normal in S (and the $\mathcal{F}$-automorphism group of all non-normal essential subgroups of S normalizes Z_S). Hence by Theorem B we can assume that P is $\mathcal{F}$-characteristic in S. Since the group $\mathrm{N}_S(E)/\mathrm{Z}(S)$ is not abelian by Lemma 4.16, we have $\mathrm{Z}_2(S) \leq E$. Thus $\mathrm{Z}_3(S) \leq \mathrm{N}_S(E)$ and S has nilpotency class 4.

We enter this information in *Magma*, recalling that S has sectional rank 3 and $S/\Phi(E)$ is isomorphic to either SmallGroup$(3^5, 52)$ or SmallGroup$(3^5, 53)$.

```
for j in [52,53] do
for i in [1..504] do S:=SmallGroup(3^6,i);
if ♯Center(S) eq 9 and
  NilpotencyClass(S) eq 4 and
  ♯[C : C in Subgroups(Center(S))| ♯C‘subgroup eq 3 and
    IsIsomorphic(S/C‘subgroup, SmallGroup(3^5,j)) eq true] ne 0 and
  ♯[M : M in Subgroups(S))|
    ♯(M‘subgroup/FrattiniSubgroup(M‘subgroup)) ge 81] eq 0 then
  [j,i];
end if; end for; end for;
```

As output we get $j = 53$ and $i \in \{394, 395, 396, 397, 402, 403, 404, 405\}$.

Recall that $\Phi(P) = \Phi(\mathrm{N}_S(E))$ by Lemma A. So $T \cap \Phi(P) = \Phi(E)$. We can check with *Magma* that if M is a maximal subgroup of S containing $\mathrm{Z}_3(S)$ then

- either $|\mathrm{Z}(M)| = 9$ (and so $\mathrm{Z}(M) = \mathrm{Z}(S)$);
- or $i \in \{394, 395, 396, 397\}$ and M is abelian;
- or $i \in \{402, 403, 404, 405\}$ and $|\mathrm{Z}(M)| = 27$.

Since $Z_S < Z_P$, we deduce that either P is abelian or $[P\colon \mathrm{Z}(P)] = 3^2$. Suppose for a contradiction that P is not abelian. In the second case, by Lemma 1.6 we have

$|[P,P]| = 3$. So $[P,P] \leq \mathrm{Z}(S) \cap \Phi(P) = \Phi(E)$.

If there exists an $\mathcal{F}$-essential subgroup $Q \leq S$ such that $[S\colon Q] = 3^2$ and $[Q\colon \mathrm{core}_{\mathcal{F}}(Q)] = 3^3$, then we are in the situation described at the previous point (with Q in place of E). Thus we may assume that every $\mathcal{F}$-essential subgroup of S not normal in S has $\mathcal{F}$-core of index 3^2 in it. Thus $[P,P]$ is $\mathcal{F}$-characteristic in every $\mathcal{F}$-essential subgroup of S not normal in S. If $Q \trianglelefteq S$ is $\mathcal{F}$-essential then $\Phi(Q) = \Phi(P)$ by Lemma 5.4 and so $[P,P] = \Phi(Q) \cap \mathrm{Z}(S)$ is $\mathcal{F}$-characteristic in Q. Hence we conclude that $[P,P] \trianglelefteq \mathcal{F}$, contradicting the fact that $O_3(\mathcal{F}) = 1$.

Therefore the group P has to be abelian and $i \in \{394, 395, 396, 397\}$.

Also, since $[E\colon (E \cap P)] = 3$, $S = EP$ and $[E\colon \mathrm{Z}(S)] = 3^2$, we deduce that $E \cap P \neq \mathrm{Z}(S)$ and so E is non abelian. Thus $E \cong 3_+^{1+2} \times \mathrm{C}_3$ if $\mathrm{Z}(S) \cong \mathrm{C}_3 \times \mathrm{C}_3$ and $E \cong 3_+^{1+2} \circ \mathrm{C}_9$ if $\mathrm{Z}(S) \cong \mathrm{C}_9$. Using *Magma* we can check the exponent of the center of S and the order of the group $\Omega_1(M)$, for every maximal subgroup M of S containing the group $\mathrm{Z}_3(S)$.

```
for i in [394, 395, 396, 397] do S:=SmallGroup(3^6,i);
  Exponent(Center(S));
  [♯Omega(M‘subgroup,1) :  M in MaximalSubgroups(S)|
    UpperCentralSeries(S)[4] subset M‘subgroup];
end for;
Output:
3, [243, 27, 27, 27]
3, [243, 243, 243, 27]
3, [27, 27, 27, 27]
9, [243, 243, 243, 27]
```

Note that if the center of S has exponent 3 then $E = \Omega_1(E)T = \Omega_1(E)\mathrm{Z}(S) = \Omega_1(E)$ and by the maximality of T we deduce $\Omega_1(\mathrm{N}_S(E)) = \mathrm{N}_S(E)$. Thus $\mathrm{N}_S(E)$ is a maximal subgroup of S containing $\mathrm{Z}_3(S)$ and such that $|\Omega_1(\mathrm{N}_S(E))| = 3^5$. In particular $i \neq 396$.

Case 2: suppose $|S| = 3^7$. Then by Theorem C there exists an $\mathcal{F}$-characteristic $\mathcal{F}$-essential subgroup P of S such that $Z_S < Z_P$ and $\Phi(\Phi(P)) = 1$ and $Z_S = Z_Q$ for every $\mathcal{F}$-essential subgroup Q distinct from P. In particular $Z_S = Z_E$. Also, T is not cyclic (look at the proof of Theorem C).

Case 2a: suppose $[E\colon T] = 3^3$. Then $\mathrm{Z}(S)T \le \Phi(\mathrm{N}^1)$ and since $\Phi(\mathrm{N}^1)$ is elementary abelian we deduce that T and $\mathrm{Z}(S)$ are elementary abelian and so $\mathrm{Z}(S) = Z_S \le T$. So $T \cong \mathrm{C}_3 \times \mathrm{C}_3$ and $\mathrm{Z}(S) \le T$. Also, if $V = [E, O^{p'}(\mathrm{Aut}_{\mathcal{F}}(E))]T$, then by Lemma 4.18 we get $V^3 = T^3 = 1$. So V has exponent 3 and $|\Omega 1(S)| \ge |V| = 3^4$.

Recall that the quotient $S/\Phi(E)$ is isomorphic to the group SmallGroup$(3^5, j)$, for $j \in \{56, 57, 58\}$. Using *Magma* we can prove that indeed $\mathrm{Z}(S) = T$, that the quotient $S/\mathrm{Z}(S)$ is isomorphic to the group SmallGroup$(3^5, 58)$ and that there are 154 possibilities for the isomorphism type of S.

```
for j in [56,57,58] do
for i in [1..NumberOfSmallGroups(3^7)] do S:=SmallGroup(3^7,i);
  if ♯Center(S) le 9 and
    Exponent(Center(S)) eq 3 and
    NilpotencyClass(S) ge 4 and
    ♯Omega(S,1) ge 3^4 and
    ♯[M : M in Subgroups(S))|
      ♯(M‘subgroup/FrattiniSubgroup(M‘subgroup) ) ge 81] eq 0 and;
    ♯[C : C in NormalSubgroups(S)| C‘subgroup subset FrattiniSubgroup(S)
      and ♯C‘subgroup eq 9 and
      IsIsomorphic(S/C‘subgroup, SmallGroup(3^5,j)) eq true] ne 0 and
    ♯[P : P in MaximalSubgroups(S)|
      ♯FrattiniSubgroup(FrattiniSubgroup(P‘subgroup)) eq 1] ne 0 then
    [j, i];
end if; end for; end for;
```

Since $T = \mathrm{Z}(S)$, we can prove that $\mathrm{Z}(E)$ is the preimage in E of $\mathrm{C}_{E/T}(O^{3'}(\mathrm{Out}_{\mathcal{F}}(E))$ and so $|\mathrm{Z}(E)| = 3^3$. We now determine the isomorphism type of E.

```
for i in [1..NumberOfSmallGroups(3^5)] do E:=SmallGroup(3^5,i);
  if ♯Center(E) eq 27 and
    Exponent(FrattiniSubgroup(E)) eq 3 and
    ♯(FrattiniSubgroup(E)) eq 9 and
    ♯Omega(E,1) ge 3^4 then
  i; end for; end for;

Output:  [32, 35]
```

We now impose that S has a subgroup E that contains its centralizer in S, is isomorphic to the group $\mathrm{SmallGroup}(3^5, k)$, for $k \in \{32, 35\}$, and such that $S/\Phi(E)$ is isomorphic to $\mathrm{SmallGroup}(3^5, 58)$. Out of the 154 possibilities for S calculated before, 66 of them satisfy this condition, and only for $k = 35$. Thus $E \cong \mathrm{SmallGroup}(3^5, 35) \cong 3^{1+2}_{+} \times \mathrm{C}_9$.

Case 2b: suppose that $[E \colon T] = 3^2$. Then $Z_S \leq T \leq \mathrm{Z}(S)$ by Lemmas 4.23 and 5.1. Also, $E = \Omega_1(E)T$ by Lemma 4.18. Therefore we conclude that $T = \mathrm{Z}(S)$. finally, since $|S| = 3^7$ and $\Phi(\Phi(T)) = 1$ we conclude $T = \mathrm{Z}(S) \cong \mathrm{C}_3 \times \mathrm{C}_9$. Thus by Theorem 13 we have $E \cong 3^{1+2}_{3} \times \mathrm{C}_9$. Also, by what we proved above, all the $\mathcal{F}$-essential subgroups of S distinct from P have $\mathcal{F}$-core of index 3^2 in them. Since $E^3 = T^3$ and $\Omega_1(\mathrm{N}_S(E)) \nleq E$ by Lemma 4.18, we deduce that $|\Omega_1(S)| \geq \Omega_1(\mathrm{N}_S(E))| > |\Omega_1(E)| > 3^4$.

If P is not abelian, then $[\mathrm{Z}(P) \colon \mathrm{Z}(S)] = 3$ and so $\mathrm{Z}(P) \leq \mathrm{Z}_2(S) < E$. Thus $\mathrm{Z}(P) = \mathrm{Z}_2(S)$ and $\Phi(\mathrm{Z}(P)) = \Phi(\mathrm{Z}(S))$. Since $O_3(\mathcal{F}) = 1$ and $\mathrm{Z}(S) = \mathrm{core}_{\mathcal{F}}(Q)$ for every essential subgroup Q of S, we get $\Phi(\mathrm{Z}(S)) = 1$, that is a contradiction. Thus if there exists an essential subgroup E of S such that $[E \colon \mathrm{core}_{\mathcal{F}}(E)] = 3^2$ then the group P is abelian.

Recall that the quotient $S/\Phi(E)$ is isomorphic to the group $\mathrm{SmallGroup}(3^5, j)$, for $j \in \{52, 53\}$. We enter the information in *Magma* to determine the isomorphism type of the 3-group S.

```
for j in [52, 53] do
for i in [1..NumberOfSmallGroups(3^7)] do S:=SmallGroup(3^7,i);
  if ♯Center(S) eq 27 and
    Exponent(Center(S)) eq 9 and
    NilpotencyClass(S) eq 4 and
    ♯Omega(S,1) ge 3^4 and
    ♯[M : M in Subgroups(S)|
      ♯(M‘subgroup/FrattiniSubgroup(M‘subgroup)) ge 81]eq 0 and
    ♯[C : C in Subgroups(Center(S))| ♯C‘subgroup eq 9 and
    IsIsomorphic(S/C‘subgroup, SmallGroup(3^5,j)) eq true] ne 0 and
    ♯[P : P in MaximalSubgroups(S))| IsAbelian(P‘subgroup) eq true
    and ♯FrattiniSubgroup(FrattiniSubgroup(P‘subgroup)) eq 1] ne 0 then
    [j, i];
  end if;
end for; end for;
```

As output we get $[53, 5402], [53, 5403]$. In particular the quotient $S/\Phi(E)$ is isomorphic to the group SmallGroup$(3^5, 53)$. $\square$

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