On Asymptotic Stability of Stochastic Differential Equations with Delay in Infinite Dimensional Spaces

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by

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Abstract

In most stochastic dynamical systems which describe process in engineering, physics and economics, stochastic components and random noise are often involved. Stochastic effects of these models are often used to capture the uncertainty about the operating systems. Motivated by the development of analysis and theory of stochastic processes, as well as the studies of natural sciences, the theory of stochastic differential equations in infinite dimensional spaces evolves gradually into a branch of modern analysis. In the analysis of such systems, we want to investigate their stabilities.

This thesis is mainly concerned about the studies of the stability property of stochastic differential equations in infinite dimensional spaces, mainly in Hilbert spaces. Chapter 1 is an overview of the studies. In Chapter 2, we recall basic notations, definitions and preliminaries, especially those on stochastic integration and stochastic differential equations in infinite dimensional spaces. In this way, such notions as Q-Wiener processes, stochastic integrals, mild solutions will be reviewed. We also introduce the concepts of several types of stability. In Chapter 3, we are mainly concerned about the moment exponential stability of neutral impulsive stochastic delay partial differential equations with Poisson jumps. By employing the fixed point theorem, the p-th moment exponential stability of mild solutions to system is obtained. In Chapter 4, we firstly attempt to recall an impulsive-integral inequality by considering impulsive effects in stochastic systems. Then we define an attracting set and study the exponential stability of mild solutions to impulsive neutral stochastic delay partial differential equations with

Poisson jumps by employing impulsive-integral inequality. Chapter 5 investigates p-th moment exponential stability and almost sure asymptotic behaviours of mild solutions to stochastic delay integro-differential equations. Finally in Chapter 6, we study the exponential stability of neutral impulsive stochastic delay partial differential equations driven by a fractional Brownian motion.

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Notations

$\square$	end of proof
$:=$	equality of definition
$\mathcal{X}_{\{\cdot\}}$	the indictor function
$\mathbb{R}$	real field of real numbers
$\mathbb{C}$	complex field of complex numbers
$\mathbb{R}^+$	nonnegative real numbers
$\mathrm{Re}\lambda$	real part of $\lambda \in \mathbb{C}$
A	linear operator
$\mathcal{D}(A)$	domain of A
$\mathcal{R}(A)$	range of A
$\mathcal{B}(X)$	Borel σ-filed of X
$\mathcal{L}(X)$	the set of all bounded linear operators on X
$\mathcal{L}(X, Y)$	the set of all bounded linear operators from X into Y
$\mathcal{L}_1(X, Y)$	the set of all nuclear operators from X into Y
$\mathcal{L}_2(X, Y)$	the set of all Hilbert-Schmidt operators from X into Y
$C(X, Y)$	the set of all continuous functions from X to Y
$D(X, Y)$	the set of all càdlàg functions from X to Y

Contents

Chapter 1

Introduction

This thesis mainly works on the stability of stochastic differential equations in infinite dimensional spaces. We attempt to investigate stability properties such as asymptotic stability, mean square exponential stability, p-th moment exponential stability and almost sure exponential stability. We concentrate on various types of stochastic differential equations such as neutral stochastic functional partial equations with Poisson point processes and delays, stochastic integro-differential equations and impulsive delay neutral stochastic partial differential equations driven by a fractional Brownian motion.

In Chapter 2, we recall some standard concepts of the theory of stochastic differential equations in infinite dimensional spaces. In this chapter, we firstly introduce some basic definitions and preliminaries in functional analysis and Hilbert spaces valued stochastic differential equations, such as Q-Wiener processes, stochastic integral with respect to Wiener processes, jump processes, stochastic integral with respect to Poisson random measures. We also introduce mild solutions of stochastic differential equations and various types of stabilities. The required knowledge of this chapter is necessarily presented in order to help readers to understand the following chapters. Moreover, some of important mathematical tools are given in this chapter. The main source of reference of this thesis are based on the books Da Prato and Zabczyk [42], Kreyszig [67], Liu [80] and

Pazy [109]. As for applications of semigroup approaches to infinite dimensional stochastic systems, the variational method can be found in many literatures such as Chow [33], Métivier [100].

Stochastic partial differential equations driven by Wiener processes have been studied by many researchers. To the best of my knowledge, there have not been much studies of stochastic delayed partial differential equations with Jump processes and impulsive effects. The classical technique applied in the studies of stability of stochastic differential equations is based on a stochastic version of Lyapunov's method. However, it may be difficult to apply Lyapunov's direct method to specific issues on exponential stability of mild solutions of delayed stochastic differential equations. It is worth pointing out that Luo [?] employed the fixed point theory to study the exponential stability of mild solutions in stochastic systems, where the conditions do not require the boundedness of delays. Cui, Yan and Sun [36] proved the existence and exponential stability in mean square of mild solutions for a class of neutral partial differential equations with delays and Poisson jump. In Chapter 3, we are concerned about the stability of mild solutions to impulsive neutral stochastic partial delay differential equations driven by Poisson point process. In this class of equations, we do not only consider delay effects, but also the impulsive effects will be investigated.

When one talks about stability, or stability in the sense of Lyapunov, it is enough to investigate the stability problem for the null solution of some relevant systems. In Chapter 4, we firstly recall an impulsive-integral inequality which takes impulsive effects into account in our system. By using the impulsive-integral inequality, we obtain an attracting set of neutral stochastic partial differential equations with delays driven by Poisson point process. Moreover, we investigate the sufficient conditions for the p-th moment exponential stability of mild solutions of systems under investigation.

Caraballo and Liu [26] established the exponential stability of mild solutions of stochastic partial differential equations with delays by using the Gronwall inequality. Mao [96] discussed the stability of solutions of finite dimensional spaces valued stochastic differential equations by employing the same method in his book. We refer the reader to [96] for more details. In many applications, due to the complex random nature of situation, the stochastic problems could be formulated as some integro-differential systems. Recently, the existence, uniqueness and stability of integro-differential equations have been considered by some investigators, such as Diop and his cooperators [48], [49], [47], [50]. In Chapter 5, we are interested in the moment and almost sure stability properties of stochastic integro-differential equations with delays. We assume that the linear part of the system under consideration has a resolvent operator which has been given by Grimmer[54]. For more details on resolvent operators, we refer readers to [54] and [55]. In order to obtain sufficient conditions for the exponential stability of solutions of stochastic differential equations with delays, we shall employ a technique which has been developed by Caraballo [20].

We would like to mention that the theory for stochastic differential equations driven by a fractional Brownian motion (fBm) has recently been discussed intensively. The case of finite-dimensional equations driven by a fBm has been studied by many researchers such as Neuenkirch (2008), Boufoussi and Hajji (2011), Leon and Tindel (2012) and many others. The case of Hilbert spaces valued stochastic equalitions driven by a fBm has been studied by Caraballo and his cooperators [21]. They investigated the existence and uniqueness of mild solutions to stochastic differential equations driven by a fBm by using Lyapunov's method. Shortly, Boufoussi and Hajji [12] studied neutral stochastic functional differential equations with finite delay driven by a fBm in a Hilbert space. In Chapter 6, we shall study the stability of neutral stochastic functional differential equations driven by a fBm. In most of the work, finite delay is considered. This work is based on the one of Boufoussi and Hajji [12]. The difficulty in our work is the inclusion of impulsive effects and infinite delay in our system.

Finally, a conclusion chapter of this thesis is presented in Chapter 7.

Chapter 2

Stochastic Differential Equations in Infinite Dimensions

This thesis deals with stochastic differential equations with delay in infinite dimensional spaces. More specifically, we study solutions of stochastic differential equations, in which we are especially interested in stability of stochastic systems.

This chapter is devoted to the background knowledge in regard to the concepts of stochastic integrals and stochastic differential equations. We shall study stochastic differential equations driven by Wiener processes or Poisson processes. Firstly, we introduce some basic definitions and preliminaries from stochastic analysis in Section 2.1. In Section 2.2, we define Hilbert space valued Wiener processes and stochastic integrals with respect to them. The aim of Section 2.3 is to introduce jump processes and stochastic integrals with respect to compensated Poisson random measures. We shall deal with these equations which are driven by jump processes. In Section 2.4, we start from the definitions of strong and mild solutions of stochastic differential equations and establish their properties. At the end, we give some concepts of different stabilities of stochastic systems in Section 2.5. The material of this chapter is standard. The book, Liu [80] contributes to the development of this thesis as the main source of reference and we refer the reader to the books: Da Prato and Zabzcyk [42], Mao [96], Pazy [109] for more details. Proofs of the results presented in this chapter will not be given as they are available in the existing literature (c.f. Section 2.6).

2.1 Notations, Definitions and Preliminaries

Let H and K be two real separable Hilbert spaces with norms and their inner products denoted by $(H, \|\cdot\|_H)$, $(K, \|\cdot\|_K)$ and $\langle\cdot,\cdot\rangle_K$, $\langle\cdot,\cdot\rangle_H$ respectively. We denote by $\mathcal{L}(K, H)$ the set of all linear bounded operators from $K \to H$, equipped with the usual operator norm $\|\cdot\|$. In this thesis, we use the symbol $\|\cdot\|$ to denote norms of operators regardless of the spaces potentially involved when no confusion possibly arises. The set $\mathcal{L}(K, H)$ is a linear space and equipped with the operator norm. It becomes a Banach space. Unfortunately, if both the spaces K and H are infinite dimensional, the space $\mathcal{L}(K, H)$ is not generally separable. A direct consequence of this inseparability is that the usual Bochner's integral definition cannot be applied to $\mathcal{L}(K, H)$-valued random variables. We denote by K^* and H^* the dual space of K and H respectively.

Definition 2.1.1. (Nuclear operator or compact operator) An element $A \in \mathcal{L}(K, H)$ is said to be a *nuclear operator (or compact operator)* if there exist two sequences $\{a_j\} \subset H$, $\{b_j\} \subset K^*$ such that

$$\sum_{j=1}^{\infty} \|a_j\| \cdot \|b_j\| < +\infty, \tag{2.1.1}$$

and A has the representation

$$Ax = \sum_{j=1}^{\infty} a_j b_j(x), \qquad x \in K. \tag{2.1.2}$$

The space of all nuclear operators from K into H, endowed with the norm

$$\|A\|_{\mathcal{L}_1(K,H)} = \inf\left\{\sum_{j=1}^{\infty} \|a_j\| \cdot \|b_j\| : Ax = \sum_{j=1}^{\infty} a_j b_j(x)\right\}, \tag{2.1.3}$$

is a Banach space, and will be denoted by $\mathcal{L}_1(K, H)$.

Let H be a separable Hilbert space and $\{e_k\}$ be a complete orthonormal system in H. If $A \in \mathcal{L}_1(H, H)$, then we define trace of A by

$$TrA = \sum_{j=1}^{\infty} \langle Ae_j, e_j \rangle \tag{2.1.4}$$

(c.f. Da Prato and Zabczyk (1992), Appendix C).

Definition 2.1.2. (Hilbert-Schmidt operator) Let K and H be two separable Hilbert spaces with complete orhonormal bases $\{e_k\} \subset K$, $\{f_i\} \subset H$. A linear bounded operator $A : K \to H$ is called *Hilbert-Schmidt* if

$$\sum_{k=1}^{\infty} \|Ae_k\|^2 < \infty. \tag{2.1.5}$$

It may be shown the sum (2.1.5) is independent of the basis $\{e_j\}$. We define a Hilbert-Schmidt operator norm by

$$\|A\|_{\mathcal{L}_2} = \left(\sum_{k=1}^{\infty} \|Ae_k\|^2 \right)^{1/2}. \tag{2.1.6}$$

Since

$$\sum_{k=1}^{\infty} \|Ae_k\|^2 = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \|\langle Ae_k, f_j \rangle\|^2 = \sum_{k=1}^{\infty} \|A^* f_j\|^2, \tag{2.1.7}$$

thus, $\|A\|_{\mathcal{L}_2} = \|A^*\|_{\mathcal{L}_2}$

(c.f. Da Prato and Zabczyk (1992), Appendix C).

One checks easily that the set $\mathcal{L}_2(K, H)$ of all Hilbert-Schmidt operators from K into H, equipped with the norm (2.1.6), is a separable Hilbert space, with the inner product

$$\langle S, T \rangle_{\mathcal{L}_2} = \sum_{k=1}^{\infty} \langle Se_k, Ae_k \rangle, \qquad S, T \in \mathcal{L}_2(K, H),\ \{e_i\} \subset K. \tag{2.1.8}$$

Let $A : \mathcal{D}(A) \subseteq X \to X$ be a linear operator on a Banach space X. The *resolvent set* $\rho(A)$ of A is a set of all complex numbers $\lambda \in \mathbb{C}$ such that $(\lambda I - A)^{-1}$ exists and $(\lambda I - A)^{-1} \in \mathcal{L}(X)$ where I is the identity operator on X. For $\lambda \in \rho(A)$, we write $R(\lambda, A) = (\lambda I - A)^{-1}$ and it is called the *resolvent operator* of A. The *spectrum* of A is defined to be $\sigma(A) = \mathbb{C} \setminus \rho(A)$. It may be shown that the resolvent set $\rho(A)$ is open in $\mathbb{C}$.

Definition 2.1.3. Let A be a linear operator on Banach space X. Define

(i) $\sigma_p(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not injective}\}$, and $\sigma_p(A)$ is called the *point spectrum* of A. Moreover, each $\lambda \in \sigma_p(A)$ is called the *eigenvalue*, and each nonzero $x \in \mathcal{D}(A)$ satisfying $(\lambda I - A)x = 0$ is called the *eigenvector* of A corresponding to λ.

(ii) $\sigma_c(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is injective, } \mathcal{R}(\lambda I - A) \neq X \text{ and } \overline{\mathcal{R}(\lambda I - A)} = X\}$, and $\sigma_c(A)$ is called the *continuous spectrum* of A.

(iii) $\sigma_r(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is injective and } \mathcal{R}(\lambda I - A) \neq X\}$, and $\sigma_r(A)$ is called the *residual spectrum* of A.

From this definition, it is immediate that $\sigma_p(A)$, $\sigma_c(A)$ and $\sigma_r(A)$ are mutually exclusive and their union is $\sigma(A)$. If A is self-adjoint, we have $\sigma_r(A) = \emptyset$. Note that if dim $X < \infty$, all the linear operators A on X are compact and in this case $\sigma(A) = \sigma_p(A)$, a fact which is extendable to any compact operators in infinite dimensional spaces.

Theorem 2.1.1. *Let X be a Banach space. If $A \in \mathcal{L}(X)$ is compact, then*

(i) $0 \in \sigma(A)$;

(ii) $\sigma(A)\backslash\{0\} = \{\lambda : \lambda \neq 0,\ \lambda \text{ is eigenvalue of } A\}$;

(iii) one of the following cases holds:

(a) $\sigma(A) = \{0\}$,

(b) $\sigma(A)\backslash\{0\}$ *is a finite set,*

(c) $\sigma(A)\backslash\{0\}$ *is a sequence with the only possible point of accumulation* 0.

In this thesis, we shall employ the theory of linear semigroups which usually allows a uniform treatment of many systems such as some parabolic, hyperbolic and delay equations.

Definition 2.1.4. A *strongly continuous* or *C_0-semigroup* $S(t) \in \mathcal{L}(X),\ t \geq 0$, on a Banach space X is a family of bounded linear operators $S(t) : X \to X,\ t \geq 0$, satisfying:

(i) $S(0)x = x$ for all $x \in X$;

(ii) $S(t+s) = S(t)S(s)$ for all $t, s \geq 0$;

(iii) $S(t)$ is strongly continuous, i.e., for any $x \in X$, $S(\cdot)x : [0, \infty) \to X$ is continuous.

For any C_0-semigroup $S(t)$ on X, there exist constants $M \geq 1$ and $\mu \in \mathbb{R}$ such that

$$\|S(t)\| \leq Me^{\mu t}, \qquad t \geq 0. \tag{2.1.9}$$

In particular, the semigroup $S(t)$ is called *(uniformly) bounded* if $\|S(t)\| \leq M$ for all $t \geq 0$. The semigroup $S(t),\ t \geq 0$, is called *norm continuous* if the map $t \to S(t)$ is continuous from $(0, \infty)$ to $\mathcal{L}(X)$. If $M = 1$ in (2.1.9), the semigroup $S(t),\ t \geq 0$, is called a *pseudo-contraction* C_0-semigroup, and if further $\mu = 0$, it is called a *contraction* C_0-semigroup.

In association with the C_0-semigroup $S(t)$, we may define a linear operator $A : \mathcal{D}(A) \subseteq X \to X$ by

$$\mathcal{D}(A) = \left\{ x \in X : \lim_{t \downarrow 0} \frac{S(t)x - S(0)x}{t} \text{ exists in } X \right\},$$

$$Ax = \lim_{t \downarrow 0} \frac{S(t)x - S(0)x}{t}, \ x \in \mathcal{D}(A).$$

The operator A is called the *infinitesimal generator*, or simply *generator*, of the semigroup $\{S(t)\}_{t \geq 0}$ which is frequently written as $e^{tA},\ t \geq 0$, in this thesis. It may be shown that A is densely defined and closed.

Theorem 2.1.2. (Hille-Yosida Theorem) *Let X be a Banach space and $A : \mathcal{D}(A) \subseteq X \to X$ be a linear operator. Then the following are equivalent:*

(i) A generates a C_0-semigroup e^{tA}, $t \geq 0$, on X such that (2.1.9) holds for some $M \geq 1$ and $\mu \in \mathbb{R}$.

(ii) A is densely defined, closed and there exist constants $\mu \in \mathbb{R}$, $M \geq 1$ such that $\rho(A) \supset \{\lambda \in \mathbb{C} : Re\lambda > \mu\}$ and

$$\|R(\lambda, A)^n\| \leq \frac{M}{(Re\lambda - \mu)^n} \tag{2.1.10}$$

for any $n \in \mathbb{N}_+$, $Re\lambda > \mu$.

Furthermore, we review some specific types of C_0-semigroups with delicate properties.

Definition 2.1.5. Let e^{tA}, $t \geq 0$, be a C_0-semigroup on a Banach space X with the generator $A : \mathcal{D}(A) \subset X \to X$.

(i) The semigroup e^{tA}, $t \geq 0$, is called *(eventually) compact* if there exists $r \geq 0$ such that $e^{tA} \in \mathcal{L}(X)$ is compact for any $t \in (r, \infty)$.

(ii) The semigroup e^{tA}, $t \geq 0$, is called *analytic* if it admits an extension e^{zA} on $z \in \Delta_\theta := \{z \in \mathbb{C} : |\text{arg}z| < \theta\}$ for some $\theta \in (0, \pi]$, such that $z \to e^{zA}$ is analytic on Δ_θ and satisfies:

(a) $e^{(z_1+z_2)A} = e^{z_1 A} e^{z_2 A}$ for any $z_1, z_2 \in \Delta_\theta$;

(b) $\lim_{\Delta_{\bar{\theta}} \ni z \to 0} \|e^{zA}x - x\|_X = 0$ for all $x \in X$ and $0 < \bar{\theta} < \theta$.

Moreover, we define fractional powers of certain unbounded linear operators and study some of their properties which will play an important role in this thesis. We concentrate mainly on fractional powers if operators A for which $-A$ is the infinitesimal generator of an exponentially stable analytic semigroup. The results of this section will be used on the study of solutions of semilinear initial value problems.

Theorem 2.1.3. *Let $S(t) = e^{tA}$, $t \geq 0$, be a C_0-semigroup with generator A on X. The semigroup e^{tA}, $t \geq 0$, is analytic if there exists $M > 0$ and $\mu \in \mathbb{R}$ such that*

$$\rho(A) \supset \{\lambda : Re\lambda \geq \mu\} \quad \text{and} \quad \|R(\lambda, A)\| \leq \frac{M}{1 + |\lambda|} \quad \text{for all } Re\lambda \geq \mu.$$

Assume that A generates an exponentially stable analytic semigroup and the spectrum of A lies entirely in the (open) left half-plane. For any $\beta \in (0, 1)$, we define

$$(-A)^{-\beta} = -\frac{1}{2\pi i} \int_\Gamma \lambda^{-\beta}(\lambda + A)^{-1} d\lambda, \tag{2.1.11}$$

where Γ is a curve from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$, $\theta \in (\pi/2, \pi/2 + \delta)$ for some $\delta > 0$, such that the spectrum of $-A$ lies to the right and the origin lies to the left of Γ. It can be shown that $(-A)^{-\beta}$ is bounded and one-to-one. The inverse $(-A)^{\beta}$ of $(-A)^{-\beta}$ is called *fractional power* of $-A$ with domain $\mathcal{D}((-A)^{\beta})$.

Lemma 2.1.1. *There exists a constant C such that*

$$\|(-A)^{-\beta}\| \leq C \qquad \text{for } 0 \leq \beta \leq 1. \tag{2.1.12}$$

Proof. The proof can be found in Lemma 6.3, Pazy [109]. $\square$

Theorem 2.1.4. *Let A be the infinitesimal generator of an exponentially stable analytic semigroup $S(t)$. For any $0 < \beta < 1$, the following equality holds:*

(a) $S(t) : X \to \mathcal{D}((-A)^{\beta})$ for every $t > 0$ and $\alpha \geq 0$.

(b) For every $x \in \mathcal{D}((-A)^{\beta})$ we have

$$S(t)(-A)^{\beta}x = (-A)^{\beta}S(t)x. \tag{2.1.13}$$

(c) For every $t > 0$ the operator $(-A)^{\beta}S(t)$ is bounded. There exist numbers $M_\beta > 0$ such that

$$\|(-A)^{\beta}S(t)\| \leq M_\beta t^{-\beta} e^{-\gamma t} \tag{2.1.14}$$

(d) Let $0 < \beta \leq 1$, and $x \in \mathcal{D}((-A)^\beta)$ then

$$\|S(t)x - x\| \leq C_\beta t^n \|(-A)^\beta x\|, \quad t > 0, \tag{2.1.15}$$

where $C_\beta > 0$ is a constant dependent on β.

Proof. The proof can be found in Theorem 6.13, Pazy [109]. □

A *fixed point* of a mapping $T : X \to X$ of a complete space X into itself is an $x \in X$ which is mapped onto itself, that is

$$Tx = x.$$

The Banach fixed point theorem plays an important role as a source of existence and uniqueness theorems in different branches of analysis. In this way the theorem provides an impressive illustration of the unifying power of functional analytic methods and usefulness of fixed point theorems in analysis.

The Banach fixed point theorem, sometimes, called contraction theorem, concerns certain contraction mappings from a complete metric space into itself. It gives sufficient conditions for the existence and uniqueness of a fixed point. The theorem also gives an iterative process by which we can obtain approximations to the fixed point.

Definition 2.1.6. (Contraction) Let $X = (X, d)$ be a complete metric space. A mapping $T : X \to X$ is called a contraction on X if there is a positive real number $\alpha < 1$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq \alpha d(x, y), \qquad \alpha < 1. \tag{2.1.16}$$

Geometrically, this means that any points x and y have images that are closer together that those points x and y. More precisely, the ratio $d(Tx, Ty)/d(x, y)$ does not exceed a constant α which is strictly less than 1.

Theorem 2.1.5. (Banach Fixed Point Theorem) Consider a metric space $X = (X, d)$, where $X \neq \varnothing$. Suppose that X is complete and let $T : X \to X$ be a contraction on X. Then T has a unique fixed point.

Let $(X, \mathcal{S}, m)$ be a measurable space and $1 < p < \infty$. The collection of all measurable functions $f(\cdot)$ for which $\|f(\cdot)\|^p$ is integrable will be denoted by $L^p(X)$, that is

$$L^p(X) := \Big\{ f : \int_X \|f(x)\|^p m(dx) < \infty \Big\}.$$

Then $L^1(X)$ is the space of all Lebesgue integrable functions on X. The space $L^p(X)$ is a Banach space. If $1 \leq p < \infty$, $f,\ g \in L^p(X)$ and $\alpha, \beta \in \mathbb{R}$ then

(a). $\alpha f + \beta g \in L^p(X)$;

(b). $\|f\|_p \geq 0$;

(c). $\|\alpha f\|_p = |\alpha| \|f\|_p$.

Theorem 2.1.6. (Hölder's Inequality) *Let $1 < p < \infty$ and $1 < q < \infty$ be real values, such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f(\cdot) \in L^p(X)$ and $g(\cdot) \in L^q(X)$ then $f(\cdot)g(\cdot) \in L^1(X)$ and*

$$\begin{aligned} \int_X \|f(x)g(x)\| m(dx) &\leq \Big(\int_X \|f(x)\|^p m(dx) \Big)^{1/p} \cdot \Big(\int_X \|g(x)\|^q m(dx) \Big)^{1/q} \\ &= \|f(x)\|_p \|g(x)\|_q. \end{aligned} \tag{2.1.17}$$

In particular, if $p = q = 2$, Hölder's inequality is the so-called Schwarz's inequality.

Theorem 2.1.7. (Minikowski's Inequality) *Let $1 < p < \infty$. Then for every pair $f, g \in L^p$,*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p. \tag{2.1.18}$$

A *measurable space* is a pair $(\Omega, \mathcal{F})$ where Ω is a set and $\mathcal{F}$ is a σ-field, also called a σ-algebra, of subsets of Ω. This means that the family $\mathcal{F}$ contains the set Ω and is closed under the operation of taking complements and countable unions of its elements. If $(\Omega, \mathcal{F})$ and $(S, \mathcal{S})$ are two measurable spaces, then a mapping ξ from Ω into S such that the set $\{\omega \in \Omega : \xi(\omega) \in A\} = \{\xi \in A\}$ belongs to $\mathcal{F}$ for arbitrary $A \in \mathcal{S}$ is called a *random variable* from $(\Omega, \mathcal{F})$ into $(S, \mathcal{S})$. A random variable is called *simple* if it takes on only a finite number of values. In this thesis, we shall only be concerned with the case where S is a complete, separable metric space. Then we always take $\mathcal{S} = \mathcal{B}(S)$, the Borel σ-field of S which is the smallest σ-field containing all closed (or open) subsets of S. If S is a separable Banach, we shall denote its norm by $\|\cdot\|_S$ and its topological dual by S^*.

A *probability measure* on a measurable space $(\Omega, \mathcal{F})$ is a σ-additive function $\mathbb{P}$ from $\mathcal{F}$ into $[0, 1]$ such that $\mathbb{P}(\Omega) = 1$. The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space.* If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, we set

$$\bar{\mathcal{F}} = \{A \subset \Omega : \exists B, C \in \mathcal{F},\ B \subset A \subset C,\ \mathbb{P}(B) = \mathbb{P}(C)\}.$$

Then $\bar{\mathcal{F}}$ is a σ-field, called the *completion* of $\mathcal{F}$. If $\mathcal{F} = \bar{\mathcal{F}}$, the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be *complete.*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a complete probability space. A family $\{\mathcal{F}_t\}, t \geq 0$, for which all the $\mathcal{F}_t$ are sub-σ-fields of $\mathcal{F}$ and form an increasing family of σ-fields, is called a *filtration* if $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for $s \leq t$. With $\{\mathcal{F}_t\}_{t\geq 0}$, one can associate two other filtration by setting: $\mathcal{F}_{t-} = \bigvee_{s<t} \mathcal{F}_s$ if $t > 0$, $\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$ if $t \geq 0$, where $\bigvee_{s<t} \mathcal{F}_s$ is the smallest σ-filed containing $\bigcup_{s<t} \mathcal{F}_s$. The σ-field $\mathcal{F}_{0-}$ is not defined and, by convention, we put $\mathcal{F}_{0-} = \mathcal{F}_0$, and also $\mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t$. An increasing family $\{\mathcal{F}_t\}_{t\geq 0}$ is *right-continuous* if for each $t \geq 0$, $\mathcal{F}_{t+} = \mathcal{F}_t$.

For many purposes we need to assume that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$. Unless otherwise stated, completeness of $(\Omega, \mathcal{F}, \mathbb{P})$ and the above assumptions will always be assumed to hold in this thesis. Sometimes, we also call a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying *usual conditions* if

(i) for each $t \geq 0$, $\{\mathcal{F}_t\}_{t\geq 0}$ is a right-continuous and increasing family.

(ii) $\mathcal{F}_0$ contains all $\mathbb{P}-$null sets in $\mathcal{F}$.

If ξ is a random variable from $(\Omega, \mathcal{F})$ into $(S, \mathcal{S})$ and $\mathbb{P}$ a probability measure on Ω, then by $\mathcal{Q}(\xi)(\cdot)$ we will denote the image of $\mathbb{P}$ under the mapping ξ:

$$\mathcal{Q}(\xi)(A) = \mathbb{P}\{\omega \in \Omega : \xi(\omega) \in A\}, \quad \forall A \in \mathcal{S}.$$

We say that the measure $\mathcal{Q}$ is the *distribution* or the *law* of ξ.

A mapping $\Phi(\cdot)$ from Ω into $\mathcal{L}(K, H)$ is said to be *strongly measurable* if for arbitrary $k \in K$, $\Phi(\cdot)k$ is measurable as a mapping from $(\Omega, \mathcal{F})$ into $(H, \mathcal{B}(H))$. Let $\mathcal{F}(\mathcal{L}(K, H))$ be the smallest σ-field of subsets of $\mathcal{L}(K, H)$ containing all sets of the form

$$\{\Phi \in \mathcal{L}(K, H) : \Phi k \in A\}, \quad k \in K, \quad A \in \mathcal{B}(H).$$

Elements of $(\mathcal{F}(\mathcal{L}(K, H)))$ are called *strongly measurable.* Then $\Phi : \Omega \to \mathcal{L}(K, H)$ is a strongly measurable mapping from $(\Omega, \mathcal{F})$ into the space $(\mathcal{L}(K, H), \mathcal{F}(\mathcal{L}(K, H)))$. Mapping Φ is said to be *Bochner integrable* with respect to the measure $\mathbb{P}$ if for arbitrary k, the mapping $\Phi(\cdot)k$ is Bochner integrable and there exists a bounded linear operator $\Psi \in \mathcal{L}(K, H)$ such that

$$\int_\Omega \Phi(\omega)k\mathbb{P}(d\omega) = \Psi k, \quad k \in K.$$

The operator Ψ is then denoted as

$$\Psi = \int_\Omega \Phi(\omega)\mathbb{P}(d\omega)$$

and called the *strong Bochner integral* of Φ. This integral has many of the properties of the Lebesgue integral. For instance, it is easy to show that if K and H are both separable, then $\|\Phi(\cdot)\|$ is a measurable function and

$$\|\Psi\| \leq \int_\Omega \|\Phi(\omega)\|\mathbb{P}(d\omega).$$

Assume that E is a Banach space with norm $\|\cdot\|_E$ and let $\mathcal{B}(E)$ be the σ-field of its Borel subsets. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A family $X = \{X(t)\}_{t\geq 0}$ of E-valued random variables $X(t)$, $t \geq 0$, defined on Ω is called a E-valued *stochastic process.* Sometimes, we also write $X(t,\omega) = X(t)$ for all $t \geq 0$ and $\omega \in \Omega$. The functions $X(\cdot,\omega)$ are called the *trajectories* of X. We now introduce several definitions of regularity for a process X on $I = [0,T)$, where T could be finite or infinite.

(a). X is *measurable* if the mapping $X(\cdot,\cdot) : I \times \Omega \to E$ is $\mathcal{B}(I) \times \mathcal{F}$-measurable;

(b). Let $\{\mathcal{F}_t\}$, $t \in I$, be an increasing family of σ-fields. The process X is $\{\mathcal{F}_t\}_{t\in I}$-*adapted* if each $X(t)$ is measurable with respect to $\mathcal{F}_t$ for every $t \in I$;

(c). X is *stochastically continuous* at $t_0 \in I$ if $\forall \varepsilon > 0$, $\forall \delta > 0$ $\exists \rho > 0$ such that

$$P\{\|X(t) - X(t_0)\|_E \geq \varepsilon\} \leq \delta, \qquad \forall t \in [t_0 - \rho, t_0 + \rho] \cap [0,T);$$

(d). X is *stochastically continuous in* I if it is stochastically continuous at every point of I;

(e). X is *continuous with probability one* if its trajectories $X(\cdot,\omega)$ are continuous almost surely;

(f). X is *càdlàg* (*right-continuous and left limit*) if it is right-continuous and for almost all $\omega \in \Omega$ the left limit $X(t-,\omega) = \lim_{s\uparrow t} X(s,\omega)$ exists for all $t > 0$.

Let E be a separable Banach space with norm $\|\cdot\|_E$ and $M = M(t),\ t \in [0,T]$, an E-valued stochastic process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$. If $\mathbb{E}\|M(t)\|_E < \infty$ for all $t \in [0,T]$, then the process is called *integrable.* An integrable and adapted E-valued process $M(t)$, $t \in [0,T)$, is said to be a *martingale* with respect to $\{\mathcal{F}_t\}_{t\in[0,T]}$ if

$$\mathbb{E}(M(t)\,|\,\mathcal{F}_s) = M(s) \qquad \mathbb{P} - a.s. \tag{2.1.19}$$

for arbitrary $t \geq s$, $t,s \in [0,T]$. If $\mathbb{E}\|M(t)\|_E^2 < \infty$, for all $t \in [0,T]$ then M_t is called *square integrable.* By the definition of conditional expectations, the equality (2.1.19) is equivalent to the following statement

$$\int_F M(t)d\mathbb{P} = \int_F M(s)d\mathbb{P}, \qquad \forall F \in \mathcal{F}_s, \quad s \le t, \ \ s, \ t \in [0,T]. \tag{2.1.20}$$

A real-valued integrable and adapted process $M(t), \ t \in [0,T]$, is said to be a *submartingale* with respect to $\{\mathcal{F}_t\}_{t\in[0,T]}$ if

$$\mathbb{E}(M(t) \,|\, \mathcal{F}_s) \ge M(s), \qquad \mathbb{P} - a.s. \tag{2.1.21}$$

for any $s \le t, \ s, \ t \in [0,T]$. The process $M(t), \ t \in [0,T]$, is said to be a *supermartingale* with respect to $\{\mathcal{F}_t\}_{t\in[0,T]}$ if

$$\mathbb{E}(M(t) \,|\, \mathcal{F}_s) \le M(s), \qquad \mathbb{P} - a.s. \tag{2.1.22}$$

for any $s \le t, \ s, \ t \in [0,T]$.

Let $[0,T], \ 0 \le T < \infty$, be a subinterval of $[0,\infty)$. A continuous E-valued stochastic process $M(t), \ t \in [0,T]$, defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, is a *continuous square integrable* with respect to $\{\mathcal{F}_t\}_{t\in[0,T]}$ if it is a martingale with almost surely continuous trajectories and satisfies, in addition, $\sup_{t\in[0,T]} \mathbb{E}\|M(t)\|_E^2 < \infty$. Let us denote by $\mathcal{M}_T^2(E)$ the space of all E-valued continuous, square integrable martingales M.

Theorem 2.1.8. *The space $\mathcal{M}_T^2(E)$, equipped with the norm*

$$\|M\|_{\mathcal{M}_T^2(E)} = \left(\mathbb{E} \sup_{t\in[0,T]} \|M(t)\|_E^2\right)^{1/2}, \tag{2.1.23}$$

is a Banach space.

Proof. The proof can be found in Theorem 1.1.8 Liu [80] or Proposition 3.9 Da Prato and Zabczyk [42]. □

An $\mathcal{L}_1$-valued process $V(\cdot)$ is said to be *increasing* if the operators $V(t), \ t \in [0,T]$, are nonnegative, denoted by $V_t \ge 0$, i.e., for any $k \in K, \ \langle V(t)k, k\rangle_K \ge 0, \ t \in [0,T]$, and $0 \le V(s) - V(t)$ if $0 \le t \le s \le T$. An $\mathcal{L}_1$-valued continuous, adapted and increasing process $V(t)$ such that $V_0 = 0$ is said to be a *tensor* *quadratic variation process* of the martingale $M(t) \in \mathcal{M}_T^2(K)$ if and only if for arbitrary $a, b \in K$, the process

$$\langle M(t), a\rangle_K \langle M(t), b\rangle_K - \langle V(t)a, b\rangle_K, \qquad t \in [0, T],$$

is a continuous $\mathcal{F}_t$-martingale, or equivalently, if and only if the process

$$M(t) \otimes M(t) - V(t), \qquad t \in [0, T],$$

is a continuous $\mathcal{F}_t$-martingale, where $(a \otimes b)k := a\langle b, k\rangle_K$ for any $k \in K$ and $a, b \in K$. One can show that the process V_t is uniquely determined and can be denoted therefore by $\ll M(t) \gg,\ t \in [0, T]$.

On the other hand, one can also show that there exists a real-valued, increasing, continuous process which is uniquely determined up to probability one, denoted by $[M(t)]$ with $[M_0] = 0$, called the *quadratic variation* of $M(t)$, such that

$$\|M(t)\|_K^2 - [M(t)]$$

is an $\mathcal{F}_t$-martingale.

With regard to the relation between $\ll M_t \gg$ and $[M_t]$ of M_t, we have the following theorem:

Theorem 2.1.9. *For arbitrary $M(t) \in \mathcal{M}_T^2(K)$, there exists a unique predictable, positive symmetric element $Q_M(\omega, t)$, or simply $Q(\omega, t)$ of $\mathcal{L}_1(K)$ such that*

$$\ll M(t) \gg = \int_0^t Q_M(\omega, s)d[M(s)], \tag{2.1.24}$$

for all $t \in [0, T]$. In particular, we also call the K-valued stochastic process $M(t),\ t \geq 0,$ a $Q_M(\omega, t)$-martingale process.

Proof. The proof can be found in Theorem 21.6 Mètivier [99]. □

2.2 Hilbert Space Valued Wiener Processes and Stochastic Integration

Let K be a real separable Hilbert space with norm $\|\cdot\|_K$ and inner product $\langle\cdot,\cdot\rangle_K$, respectively. A probability measure $\mathcal{N}$ on $(K, \mathcal{B}(K))$ is called *Gaussian* if for arbitrary $u \in K$, there exist numbers $\mu \in \mathbb{R}$, $\sigma \geq 0$, such that

$$\mathcal{N}\{x \in K : \langle u, x\rangle_K \in A\} = N(\mu, \sigma^2)(A), \qquad A \in \mathcal{B}(\mathbb{R}^1),$$

where $N(\mu, \sigma^2)$ is the usual one dimensional normal distribution with mean μ and variance σ^2. It can be proved that if $\mathcal{N}$ is Gaussian, then there exist an element $m \in K$ and a symmetric nonnegative trace class operator $Q \in \mathcal{L}_1(K)$ such that

$$\int_K \langle k, x\rangle_K \mathcal{N}(dx) = \langle m, k\rangle_K, \qquad \forall k \in K, \tag{2.2.1}$$

$$\begin{aligned}\int_K \langle k_1, x\rangle_K \langle k_2, x\rangle_K \mathcal{N}(dx) - \langle m, k_1\rangle_K \langle m, k_2\rangle_K \\ = \langle Qk_1, k_2\rangle_K, \qquad \forall k_1, k_2 \in K,\end{aligned} \tag{2.2.2}$$

and the characteristic function of $\mathcal{N}$ takes the form:

$$\hat{\mathcal{N}}(\lambda) = \int_K e^{i\langle\lambda,x\rangle_K} \mathcal{N}(dx) = e^{i\langle\lambda,m\rangle_K - \frac{1}{2}\langle Q\lambda,\lambda\rangle_K}, \qquad \lambda \in K. \tag{2.2.3}$$

Therefore, the measure $\mathcal{N}$ is uniquely determined by m and Q and denoted also by $\mathcal{N}(m, Q)$. In particular, in this case we call m the *mean* and Q the *covariance operator* of $\mathcal{N}$.

We assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is equipped with a right continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that $\mathcal{F}_0$ contains all sets of $\mathbb{P}$-measure zero. We consider two Hilbert spaces K and H, and a symmetric nonnegative operator $Q \in \mathcal{L}_1(K)$. We will firstly consider the case that $TrQ < +\infty$. Then there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in space K, and a bounded sequence of positive real numbers $\{\lambda_k\}_{k\geq 1}$ such that

$$Qe_k = \lambda_k e_k, \qquad k = 1, 2, \cdots$$

Definition 2.2.1. (K**-valued** Q**-Wiener process**) A K-valued stochastic process $W(t),\ t \geq 0$, is called a Q-Wiener process if

(a). $W(0) = 0$;

(b). $W(t)$ has continuous trajectories;

(c). $W(t)$ has independent increments;

(d). $\mathbb{E}(W(t)) = 0$ and $Cov(W(t) - W(s)) = (t - s)Q$ for all $t \geq s \geq 0$, where $Cov(X)$ denotes the *covariance operator* of $X \in H$ (cf. Da Prato and Zabczyk [42]).

If the covariance Q is the identity operator I, then the Wiener process $W(t)$ is called a *cylindrical Wiener process* in K.

Proposition 2.2.1. *Assume that $W(t)$ is a Q-Wiener process with $TrQ < +\infty$. Then the following statements hold:*

(a) $W(t)$ is a Gaussian process on K and

$$\mathbb{E}(W(t)) = 0, \ \ Cov(W(t)) = tQ, \qquad t \geq 0. \tag{2.2.4}$$

(b) For arbitrary $t \geq 0$, $W(t)$ has the expansion

$$W(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\beta_j(t)e_j \tag{2.2.5}$$

where

$$\beta_j(t) = \frac{1}{\sqrt{\lambda_j}}\langle W(t), e_j\rangle, \qquad j = 1, 2, \cdots, \tag{2.2.6}$$

are real valued Brownian motions mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$ and the series in (2.2.5) is convergent in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Proposition 2.2.2. *For an arbitrary trace class symmetric nonnegative operator Q on the separable Hilbert space K, there exists a Q-Wiener process $W(t),\ t \geq 0$.*

Proof. The proof can be found in Proposition 4.1 Da Prato and Zabczyk [42].

$\square$

Theorem 2.2.1. *Let $W(t)$ be a Q-Wiener process such that (2.2.4) holds. Then the series (2.2.5) is uniformly convergent on $[0,T]$ $\mathbb{P}$-a.s., for arbitrary $T > 0$.*

Proof. The proof can be found in Theorem 4.3 Da Prato and Zabczyk [42].

$\square$

We may also derive the following direct generalization of Lévy's celebrated characterization result.

Theorem 2.2.2. *A continuous martingale $M \in \mathcal{M}_T^2(K),\ M(0) = 0$, is a Q-Wiener process on $[0,T]$ adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and with increments $M(t) - M(s),\ 0 \leq t \leq s \leq T$, independent of $\mathcal{F}_s$, for $s \in [0,T]$, if and only if*

$$\ll M(t) \gg = tQ,\ t \in [0,T].$$

Proof. The proof can be found in Theorem 4.4 Da Prato and Zabczyk (1992).

$\square$

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space, with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $W(t), t \geq 0$ denote a K-valued Wiener process defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, with covariance operator Q, that is

$$\mathbb{E}\langle W(t), x\rangle_K \langle W(s), y\rangle_K = (t \wedge s)\langle Qx, y\rangle_K, \quad \text{for all } \ x, y \in K,$$

where $t \wedge s = \min\{t, s\}$ and Q is a positive, self-adjoint, trace class operator on K. To define stochastic integrals with respect to the Q-Wiener process $W(t)$, we introduce the subspace $K_0 = Q^{\frac{1}{2}}K$ of K endowed with the inner product $\langle u, v\rangle_{K_0} = \langle Q^{-\frac{1}{2}}u, Q^{-\frac{1}{2}}v\rangle_K$ which is a Hilbert space. We assume that there exist a complete orthonormal system $\{e_i\}_{i\geq 1}$ in K, a bounded sequence of positive numbers λ_i such that $Qe_i = \lambda_i e_i, i = 1, 2, \ldots,$ and sequence $\{\beta_i(t)\}_{i\geq 1}$ of independent standard real Brownian motions such that

$$W(t) = \sum_{i=1}^{+\infty} \sqrt{\lambda_i}\beta_i(t)e_i \qquad \text{for} \quad t \geq 0$$

and

$$\mathcal{F}_t = \mathcal{F}_t^W,$$

where $\mathcal{F}_t^W$ is the σ-algebra generated by $\{W(t) : t \in [0, \infty)\}$.

Roughly speaking, the stochastic integral $\int_0^t \Phi(s, \omega)dW(s)$ may be defined in the following way. Let $\mathcal{L}_2^0(K_0, H)$ denote the sapce of all Hilbert-Schmidt operators from K_0 into H. Then $\mathcal{L}_2^0(K_0, H)$ turns out to be a separable Hilbert space under the inner product

$$\langle L, P\rangle_{\mathcal{L}_2^0(K_0,H)} = Tr\left[LQ^{\frac{1}{2}}\left(PQ^{\frac{1}{2}}\right)^*\right],$$

for any $L,\ P \in \mathcal{L}_2^0(K_0, H)$. For arbitrarily given $T \geq 0$, let $\Phi(t, \omega),\ t \in [0, T]$, be an $\mathcal{F}_t$-adapted, $\mathcal{L}_2^0(K_0, H)$-valued process. We define the following norm for arbitrary $t \in [0, T]$,

$$\begin{aligned} |\Phi|_t &:= \left\{\mathbb{E}\int_0^t \|\Phi(s, \omega)\|^2_{\mathcal{L}_2^0} ds\right\}^{\frac{1}{2}} \\ &= \left\{\mathbb{E}\int_0^t Tr\left[\left(\Phi(s, \omega)\cdot Q^{\frac{1}{2}}\right)\left(\Phi(s, \omega)\cdot Q^{\frac{1}{2}}\right)^*\right]ds\right\}^{\frac{1}{2}}. \end{aligned} \tag{2.2.7}$$

In general, we denote all $\mathcal{L}_2^0(K_0, H)$-valued predictable process Φ such that $|\Phi|_T < \infty$ by $\mathcal{W}^2([0, T]; \mathcal{L}_2^0)$. In particular, if $\Phi(t, \omega) \in \mathcal{L}_2^0(K_0, H),\ t \in [0, T]$, is an $\mathcal{F}_t$-adapted, $\mathcal{L}(K, H)$-valued process, (2.2.7) turns out to be

$$|\Phi|_t = \left\{\mathbb{E}\int_0^t Tr\left(\Phi(s, \omega)Q\Phi(s, \omega)^*\right)ds\right\}^{\frac{1}{2}}, \qquad t \in [0, T] \tag{2.2.8}$$

Proposition 2.2.3. *For arbitrary $T > 0$ and $\Phi \in \mathcal{W}^2([0, T]; \mathcal{L}_2^0)$, the stochastic integral $\int_0^t \Phi(s, \omega)dW(s)$ is a continuous, square integrable H-valued martingale* *on* $[0,T]$ *and*

$$\mathbb{E}\left\|\int_0^t \Phi(s,\omega)dW(s)\right\|_H^2 = |\Phi|_t^2, \qquad t \in [0,T]. \tag{2.2.9}$$

As a matter of fact, the stochastic integral

$$\int_0^t \Phi(s,\omega)dW(s), \quad t \geq 0, \tag{2.2.10}$$

may be generalized for any $\mathcal{L}_2^0(K_0,H)$-valued adapted process $\Phi(\cdot,\omega)$ satisfying

$$\mathbb{P}\left\{\int_0^t \|\Phi(s,\omega)\|_{\mathcal{L}_2^0}^2 ds < \infty, \ \ 0 \leq t \leq T\right\} = 1. \tag{2.2.11}$$

Moreover, we may deduce the following generalized relation of (2.2.9)

$$\mathbb{E}\left\|\int_0^t \Phi(s,\omega)dW(s)\right\|_H^2 \leq \mathbb{E}\int_0^t \|\Phi(s,\omega)\|_{\mathcal{L}_2^0}^2 ds, \quad 0 \leq t \leq T. \tag{2.2.12}$$

with the equality holding in (2.2.12) if the right hand side is finite.

Proposition 2.2.4. *Let* $\Phi \in \mathcal{W}^2([0,T];\mathcal{L}_2^0)$, *then* $\int_0^t \Phi(s,\omega)dW(s)$ *is a continuous square integrable martingale, and its tensor quadratic variance is of the form*

$$\ll \int_0^t \Phi(s,\omega)dW(s) \gg = \int_0^t Q_\Phi(s,\omega)ds, \tag{2.2.13}$$

where

$$Q_\Phi(t,\omega) = \left(\Phi(t,\omega)Q^{\frac{1}{2}}\right)\left(\Phi(t,\omega)Q^{\frac{1}{2}}\right)^*, \quad t \in [0,T]. \tag{2.2.14}$$

Theorem 2.2.3. (Fubini Theorem) *Let* (Z,Ω,m) *be a measurable space and* $(\Phi(t,z)_{(t,z)\in[0,T]\times Z})$ *be a* $\mathcal{L}_2^0$*-valued stochastic process. Assume that*

$$\int_Z \int_0^T \|\Phi(s,z)\|_{\mathcal{L}_2^0}^2 ds m(dz) < +\infty, \tag{2.2.15}$$

then with probability one

$$\int_Z \left(\int_0^T \Phi(s,z)dW(s)\right)m(dz) = \int_0^T \left(\int_Z \Phi(s,z)\mu(dz)\right)dW(s). \tag{2.2.16}$$

Theorem 2.2.4. (Doob's inequalities) *Assume $T \geq 0$ and*

$$\mathbb{E}\int_0^T \|\Phi(s,\omega)\|^p_{\mathcal{L}_2^0} ds < \infty.$$

(i) For arbitrary $p > 1$ and $\lambda > 0$,

$$P\bigg\{\sup_{0\leq t\leq T}\bigg\|\int_0^t \Phi(s,\omega)dW(s)\bigg\|^p_H \geq \lambda\bigg\} \leq \frac{1}{\lambda^p}\mathbb{E}\bigg\|\int_0^T \Phi(s,\omega)dW(s)\bigg\|^p_H.$$

(ii) For arbitrary $p > 1$,

$$\mathbb{E}\bigg(\sup_{0\leq t\leq T}\bigg\|\int_0^t \Phi(s,\omega)dW(s)\bigg\|^p_H\bigg) \leq \frac{p}{p-1}\mathbb{E}\bigg\|\int_0^T \Phi(s,\omega)dW(s)\bigg\|^p_H.$$

Proof. The proof can be found in Theorem 7.1 Da Prato and Zabczyk [42]. □

Theorem 2.2.5. (Burkholder-Davis-Gundy) *For arbitrary $p > 0$, and let Φ be a $\mathcal{L}_2^0-$valued process such that*

$$\mathbb{E}\bigg(\int_0^T \|\Phi(s)\|^p_{\mathcal{L}_2^0} ds\bigg) < +\infty.$$

Then there exists a constant $C_p > 0$, dependent only on p, such that for any $T \geq 0$,

$$\mathbb{E}\bigg\{\sup_{0\leq t\leq T}\bigg\|\int_0^t \Phi(s,\omega)dW(s)\bigg\|^p_H\bigg\} \leq C_p\mathbb{E}\bigg\{\int_0^T \|\Phi(s,\omega)\|^2_{\mathcal{L}_2^0} ds\bigg\}^{p/2}. \quad (2.2.17)$$

Proof. The proof can be found in Theorem 7.2 Da Prato and Zabczyk [42]. □

Assume A is a linear operator, generally unbounded, on H and $S(t),\ t \geq 0$, is a strongly continuous semigroup of bounded linear operator with infinitesimal generator A. Suppose $\Phi(t,\omega) \in \mathcal{W}^2([0,T];\mathcal{L}_2^0),\ t \in [0,T]$, is an $\mathcal{L}_2^0(K_0,H)$-valued process such that the stochastic integral

$$\int_0^t S(t-s)\Phi(s,\omega)dW(s) = W_A^\Phi(t,\omega), \qquad t \in [0,T],$$

is well defined. Then the process $W_A^\Phi(t,\omega)$ is called the *stochastic convolution* of Φ. In general, the stochastic convolution is no longer a martingale. However, we have the following result which could be regarded as an infinite dimensional version of Burkholder-Davis-Gundy type of inequality for stochastic convolutions.

Proposition 2.2.5. [42] *Let $p > 2$, $T > 0$ and Φ be a $\mathcal{L}_2^0$-valued process such that*

$$\mathbb{E}\left(\int_0^T \|\Phi(s)\|^p_{\mathcal{L}_2^0} ds\right) < +\infty.$$

There exists a value $C_T > 0$ such that

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|S(t-s)\Phi(s)dW(s)\|_H^p\right) \leq C_T \mathbb{E}\left(\int_0^T \|\Phi(s)\|^p_{\mathcal{L}_2^0} ds\right). \tag{2.2.18}$$

Moreover, if $TrQ < \infty$, then

$$\lim_{n\to\infty} \mathbb{E} \sup_{t\in[0,T]} \|W_A^\Phi(t) - W_{A,n}^\Phi(t)\|^p = 0, \tag{2.2.19}$$

where $W_A^\Phi(t)$ and $W_{A,n}^\Phi(t)$ are defined as

$$W_A^\Phi(t) = \int_0^t S(t-s)\Phi(s)dW(s), \tag{2.2.20}$$

$$W_{A,n}^\Phi(t) = \int_0^t S_n(t-s)\Phi(s)dW(s) \quad t \in [0,T], \tag{2.2.21}$$

where $S_n(t-s)$ is the C_0-semigroup generated by A_n which are the Yosida approximation of A.

Proof. The proof can be found in Proposition 7.3 Da Prato and Zabczyk [42].

□

Theorem 2.2.6. *Let $p \geq 2$ and assume that A generates a contraction semigroup $S(t)$, $t \geq 0$, and $\Phi(t,\omega) \in \mathcal{W}^2([0,T];\mathcal{L}_2^0)$, $t \in [0,T]$, is an $\mathcal{L}_2^0(K_0,H)$-valued process. Then the stochastic convolution $W_A^\Phi(t,\omega)$ has a continuous modification and there exists a constant $C_{p,T} > 0$, dependent of p and T, such that*

$$\mathbb{E}\left\{\sup_{0\leq t\leq T}\left\|\int_0^t S(t-s)\Phi(s,\omega)dW(s)\right\|_H^p\right\} \leq C_{T,p}\cdot \mathbb{E}\left\{\int_0^T \|\Phi(s,\omega)\|^2_{\mathcal{L}_2^0} ds\right\}^{p/2}. \tag{2.2.22}$$

Proof. The proof can be found in Theorem 7.4 Da Prato and Zabczyk [42]. □

Moreover, if A generates a contraction C_0-semigroup the number $C_{T,p} > 0$ may be chosen to depend on $p \geq 2$ only.

Note that in Proposition 2.2.5, there is a restriction on the condition $p > 2$ to secure the validness of (2.2.18) for any C_0-semigroup S(t), $t \geq 0$, on H. A version of Theorem 2.2.6 is possible to cover the case $p = 2$. However, we have to restrict in this case the C_0-semigroup S(t), $t \geq 0$, to a pseudo-contraction one.

Lemma 2.2.1. *For any $p \geq 2$, let $\Phi \in \mathcal{W}([0,T], \mathcal{L}_2^0(K_0, H)$, $t \in [0,T]$, is an $\mathcal{L}_2^0(K_0, H)$-valued process, then*

$$\sup_{s\in[0,T]} \mathbb{E}\left\| \int_0^t \Phi(s,\omega)dW(s) \right\|_H^p \leq C_p \left(\int_0^T (E\|\Phi(s,\omega)\|_{\mathcal{L}_2^0}^p)^{2/p} ds \right)^{p/2}, \quad (2.2.23)$$

where $C_p = \left(\frac{p(p-1)}{2}\right)^p$, $t \in [0,T]$.

Proof. The proof can be found in Lemma 7.7 Da Prato and Zabczyk [42]. □

As another important tool, we mention the following infinite dimensional version of the classic Itô's formula which plays an essential role in stochastic processes studies. Suppose that $V(t,x) : I \times H \to \mathbb{R}$ is a continuous function with properties:

(i). $V(t,x)$ is differentiable in t and $V_t^{'}(t,x)$ is continuous on $I \times H$;

(ii). $V(t,x)$ is twice Fréchet differentiable in x, $V_t^{'}(t,x) \in H$ and $V_t^{''}(t,x) \in \mathcal{L}(H)$ are continuous on $I \times H$, where $I = [0,T]$, $T > 0$.

Assume that $\Phi(t,\omega) \in \mathcal{W}^2([0,T]; \mathcal{L}_2^0)$ is an $\mathcal{L}_2^0(K_0, H)$-valued process, $\phi(t,\omega)$ is an H-valued continuous, Bochner integrable process on $[0,T]$, and x_0 is an $\mathcal{F}_0$-measurable, H-valued random variable. Then the following H-valued process

$$X(t) = x_0 + \int_0^t \phi(s,\omega)ds + \int_0^t \Phi(s,\omega)dW(s), \qquad t \in [0,T], \quad (2.2.24)$$

can be well defined.

Theorem 2.2.7. (Itô's formula) *Suppose the above condition (i) and (ii) hold,* *then for all $t \in [0,T]$, $Z(t) = V(t, X(t))$ has the stochastic differential*

$$\begin{aligned} dZ(t) =& \Big\{ V_t^{'}(t,x) + \langle V_x^{'}(t,X(t)), \phi(t) \rangle_H \\ &+ \frac{1}{2} Tr\Big[V_{xx}^{''}(t,X(t)) \big(\Phi(t)Q^{1/2}\big) \big(\Phi(t)Q^{1/2}\big)^* \Big] \Big\} dt \\ &+ \langle V_x^{'}(t,X(t)), \Phi(t) dW(t) \rangle_H. \end{aligned} \tag{2.2.25}$$

2.3 Jump Process

Let U be a Hilbert space with its norm $\|\cdot\|_U$ and inner product $\langle \cdot, \cdot \rangle_U$. Suppose that $Z = \{Z(t)\},\ t \geq 0$, is an U-valued Lévy process so that Z has stationary and independent increments, is stochastically continuous and satisfies $Z(0) = 0$ almost surely. Let p_t be the law of $Z(t)$ for each $t \geq 0$, then $(p_t, t \geq 0)$ is a weakly continuous convolution semigroup of probability measures on U. Associated with the Lévy process Z, we have the following Lévy-Khintchine formula or infinitely divisible distribution: for any $t \geq 0$ and $h \in U$,

$$\mathbb{E}\bigg(e^{i\langle h, Z(t) \rangle_U} \bigg) = e^{t\eta_{b,Q,\nu}(h)}, \tag{2.3.1}$$

with the exponent

$$\begin{aligned} \eta_{b,Q,\nu}(h) =& i\langle b, h\rangle_U - \frac{1}{2}\langle h, Qh\rangle_U \\ &+ \int_U \Big[e^{i\langle h,x\rangle_U} - 1 - i\langle h, x\rangle_U \cdot \mathcal{X}_{\{\|x\|_U \leq 1\}}(x) \Big] \nu(dx), \end{aligned} \tag{2.3.2}$$

where $b \in U,\ Q$ is a positive, self-adjoint and trace class operator on U. And ν is the so-called Lévy measure on U, satisfying the relations that

$$\nu(\{0\}) = 0 \qquad \text{and} \qquad \int_U (\|x\|_U^2 \wedge 1)\nu(dx) < \infty. \tag{2.3.3}$$

Here we use the symbol $\mathcal{X}_E(x)$ to denote the characteristic function on set $E \subset U$, i.e., $\mathcal{X}_E(x) = 1$ if $x \in E$ and $\mathcal{X}_E(x) = 0$ if $x \notin E$. We call the triple (b, Q, ν) the characteristics of the process Z, and mapping $\eta_{b,Q,\nu}$ the characteristic exponent of Z. It can be shown that Lévy process has a càdlàg version which, unless otherwise specified, will be always assumed to be the case in this thesis. We also strengthen the independent increment requirement on Z by assuming that $Z(t) - Z(s)$ is independent of $\mathcal{F}_s$ for all $0 \leq s < t < \infty$.

If Z is a Lévy process on U, we write $\Delta Z(t) = Z(t) - Z(t-)$ for all $t \geq 0$ where $Z(t-) := \lim_{s \uparrow t} Z(s)$. We obtain then a counting Poisson random measure N on $U \backslash \{0\}$ by

$$N(t, E) = \#\{0 \leq s \leq t : \Delta Z(s) \in E\} < \infty, \quad t \geq 0, \tag{2.3.4}$$

almost surely for any $E \in \mathcal{B}(U \backslash \{0\})$ with $0 \notin \bar{E}$, the closure of E in U. Here the symbol $\#$ means the counting and $\mathcal{B}(U \backslash \{0\})$ is the Borel σ-field on $U \backslash \{0\}$. The associated compensated Poisson random measure $\tilde{N}$ is defined by

$$\tilde{N}(t, dx) = N(t, dx) - t\nu(dx). \tag{2.3.5}$$

Let $\mathcal{O} \in \mathcal{B}(U \backslash \{0\})$ with $0 \notin \bar{\mathcal{O}}$ and $\nu_{\mathcal{O}}$ denote the restriction of measure ν to $\mathcal{O}$, still denote it by ν, so that ν is finite on $\mathcal{O}$. Let $\mathcal{V}([0, T] \times \mathcal{O}; H)$ denote the spaces of all predictable mappings $L : [0, T] \times \mathcal{O} \times \Omega \to H$ with

$$\int_0^T \int_{\mathcal{O}} \mathbb{E}\|L(t, x)\|_H^2 \nu(dx) dt < \infty.$$

We may then define the following stochastic integral

$$\int_0^T \int_{\mathcal{O}} L(t, x) N(dt, dx) = \sum_{0 \leq t \leq T} L(t, \Delta Z(t)) 1_{\mathcal{O}}(\Delta Z(t)),$$

which enables us to define further the stochastic integral

$$\int_0^T \int_{\mathcal{O}} L(t, x) \tilde{N}(dt, dx) := \int_0^T \int_{\mathcal{O}} L(t, x) N(dt, dx) - \int_0^T \int_{\mathcal{O}} L(t, x) \nu(dx) dt.$$

By standard arguments, it is known that

$$M(t) = \int_0^t \int_{\mathcal{O}} L(s, x) \tilde{N}(ds, dx), \ t \geq 0,$$

is actually an H-valued square-integrable martingale, for each $T \geq 0$, satisfying

$$\mathbb{E}\left(\left\| \int_0^T \int_{\mathcal{O}} L(t,x)\tilde{N}(dt,dx)\right\|_H^2\right) \leq \int_0^T \int_{\mathcal{O}} \mathbb{E}\|L(t,x)\|_H^2 \nu(dx)dt.$$

Theorem 2.3.1. (Lévy-Itô decomposition) *Suppose that $Z(t)$, $t \geq 0$, is a càdlàg U-valued Lévy process with characteristic exponent given by (2.3.2), then for each $t \geq 0$,*

$$Z(t) = bt + W_Q(t) + \int_{\|x\|_U < 1} x\tilde{N}(t,dx) + \int_{\|x\|_U \geq 1} xN(t,dx), \tag{2.3.6}$$

where $W_Q(t)$ is a Q-Wiener process, independent of N.

Proof. The proof can be found in Theorem 4.1 Albeverio and Rüdiger (2005).

□

In many situations, the term in equation (2.3.6) involving large jumps maybe handled by using an interlacing technique (c.f. Applebaum (2004)). In the rest of this thesis, for the sake of simplicity, we proceed by omitting this term and concentrate on the study of the equation with small jumps.

2.4 Semigroup Approach and Mild Solutions of Stochastic Differential Equations

In this section, we consider the following semilinear stochastic differential equation on $I = [0,T]$, $T \geq 0$,

$$\begin{cases} dX(t) = (AX(t) + F(t,X(t)))dt + G(t,X(t))dW(t), \\ X_0 = x_0 \in H, \end{cases} \tag{2.4.1}$$

where A is the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$, of bounded linear operators on the Hilbert space H. The coefficients F and G are two non-linear measurable mappings from $[0,T] \times H \to H$ and $[0,T] \times H \to \mathcal{L}(K,H)$, respectively, satisfying the following Lipschitz continuity conditions:

$$\begin{aligned} &\|F(t,y)-F(t,z)\|_H \leq \alpha(T)\|y-z\|_H, \quad \alpha(T)>0,\ y,z\in H,\ t\in[0,T], \\ &\|G(t,y)-G(t,z)\|_{\mathcal{L}_2^0} \leq \beta(T)\|y-z\|_H, \quad \beta(T)>0,\ y,z\in H,\ t\in[0,T]. \end{aligned} \tag{2.4.2}$$

Definition 2.4.1. A stochastic process $X(t),\ t\in I$, defined on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$, is called a *strong solution* of equation (2.4.1) if

(i) $X(t)\in\mathcal{D}(A),\ 0\leq t\leq T$, almost surely and is adapted to $\mathcal{F}_t,\ t\in I$;

(ii) $X(t)$ is continuous in $t\in I$ almost surely. For arbitrary $0\leq t\leq T$,

$$\mathbb{P}\left\{\omega : \int_0^t \|X(s,\omega)\|_H^2 ds < \infty\right\} = 1$$

and

$$X(t) = x_0 + \int_0^t (AX(s)+F(s,X(s))) + \int_0^t G(s,X(s))dW(s), \tag{2.4.3}$$

for any $x_0\in\mathcal{D}(A)$ almost surely.

In most situations, one finds that the concept of strong solution is too limited to include important examples. There is a weaker concept, mild solution, which is found to be more appropriate for practical purposes.

Definition 2.4.2. A stochastic process $X(t),\ t\in I$, define on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ is called a *mild solution* of equation (2.4.1) if

(i) $X(t)$ is adapted to $\mathcal{F}_t,\ t\geq 0$;

(ii) For arbitrary $0\leq t\leq T$,

$$\mathbb{P}\left\{\omega : \int_0^t \|X(s,\omega)\|_H^2 ds < \infty\right\} = 1,$$

and

$$X(t) = S(t)x_0 + \int_0^t S(t-s)F(s,X(s))ds + \int_0^t S(t-s)G(s,X(s))dW(s), \tag{2.4.4}$$

for any $x_0\in H$ almost surely.

As a direct application of the properties of semigroup theory, it may be proved that:

Proposition 2.4.1. *For arbitrary $x_0 \in \mathcal{D}(A)$, the domain of A, assume $X(t) \in \mathcal{D}(A)$, $t \in I$, is a solution of equation (2.4.1) in the sense of satisfying*

$$X(t) = x_0 + \int_0^t (AX(s) + F(s, X(s)))ds + \int_0^t G(s, X(s))dW(s), \qquad (2.4.5)$$

then it is also a mild solution.

By a straightforward argument, it is possible to establish the following result.

Proposition 2.4.2. *Assume that the Lipschitz condition (2.4.2) holds, then there exists at most one mild solution of equation (2.4.1). In other words, under the condition (2.4.2) the mild solution of (2.4.1) is unique.*

The following stochastic version of the classic Fubini theorem will be frequently used in the thesis and its proof can be found in Da Prato and Zabczyk [42].

Proposition 2.4.3. *Let $I = [0, T]$, $T \geq 0$, and*

$$G : I \times I \times \Omega \to (\mathcal{L}(K, H), \mathcal{F}(\mathcal{L}(K, H))),$$

be strongly measurable in the sense of Section 2.1. such that $G(s, t)$ is $\{\mathcal{F}_t\}$-measurable for each $s \geq 0$ with

$$\int_0^T \int_0^T \|G(s, t)\|^2_{\mathcal{L}_2^0} dsdt < \infty \qquad a.s. \qquad (2.4.6)$$

Then

$$\int_0^T \int_0^T G(s, t)dW(t)ds = \int_0^T \int_0^T G(s, t)dsdW(t) \qquad a.s. \qquad (2.4.7)$$

The following result gives sufficient conditions for a mild solution to be also a strong solution.

Proposition 2.4.4. *Suppose that the following conditions hold:*

(a) $x_0 \in \mathcal{D}(A),\ S(t-s)F(s,t) \in \mathcal{D}(A),\ S(t-s)G(s,t)k \in \mathcal{D}(A)$ *for each* $x \in H,\ k \in K,$ *and* $t \geq s$;

(b) $\|AS(t-s)F(s,x)\|_H \leq f(t-s)\|x\|_H, \quad f(\cdot) \in L^1(0,T;\mathbb{R}_+)$;

(c) $\|AS(t-s)G(s,x)\|_{\mathcal{L}_2^0} \leq g(t-s)\|x\|_H, \quad g(\cdot) \in L^2(0,T;\mathbb{R}_+)$.

Then a mild solution $X(t),\ t \in I$, *of equation (2.4.1) is also a strong solution with* $X(t) \in \mathcal{D}(A),\ t \in I$, *in the sense of Definition 2.2.3.*

Proof. The proof can be in Proposition 1.3.5 Liu [80]. □

Theorem 2.4.1. *[80] Assume that the conditions (2.4.2) hold. Suppose that* $x_0 \in H$ *is an arbitrarily given* $\mathcal{F}_0$*-measurable random variable with* $\mathbb{E}\|x_0\|_H^p < \infty$ *for some integer* $p \geq 2$. *Then there exists a unique mild solution of (2.4.1) in the space* $C(0,T;L^p(\Omega,\mathcal{F},\mathbb{P};H))$.

As we pointed out in Section 2.2, the stochastic convolution in (2.4.4) is no longer a martingale. A remarkable consequence of this fact is that we cannot employ Itô's formula for mild solutions directly in most of our arguments. We can deal with this problem, however, by introducing approximating systems of strong solutions to which Itô's formula can be well applied. In particular, by virtue of Proportion 2.4.4, we may obtain an approximation result of mild solutions. To this end, we introduce an approximating system of (2.4.1) as follows:

$$\begin{cases} dX(t) = AX(t)dt + R(l)F(t,X(t))dt + R(l)G(t,X(t))dW(t), \\ X_0 = R(l)x_0, \quad x_0 \in H, \end{cases} \tag{2.4.8}$$

where $l \in \rho(A)$, the resolvent set of A and $R(l) := lR(l,A),\ R(l,A)$ is the resolvent of A.

Proposition 2.4.5. *Let* x_0 *be an arbitrarily given random variable in* H *with* $\mathbb{E}\|x_0\|_H^p < \infty$ *for some integer* $p > 2$. *Suppose the nonlinear terms* $F(\cdot,\cdot)$, $G(\cdot,\cdot)$ *in (2.4.1) satisfy the Lipschitz condition (2.4.2). Then, for each* $l \in$

$\rho(A)$, the stochastic differential equation (2.4.8) has a unique strong solution $X(t,l) \in \mathcal{D}(A)$, which lies in $L^p(\Omega, \mathcal{F}, \mathbb{P}; C(0,T;H))$ for all $T > 0$ and $p > 2$. Moreover, there exist a subsequence, denoted by $X^n(t)$, such that for arbitrary $T > 0$, $X^n(t) \to X(t)$ almost surely as $n \to \infty$, uniformly with respect to $[0,T]$.

Proof. The proof can be in Proposition 1.3.6 Liu [80]. □

2.5 Definitions and Methods of Stability

In 1892, A.M. Lyapunov introduced the concept of stability of a dynamic system. Roughly speaking, stability means insensitivity of the state of the system to small changes in the initial state or the parameters of the system. Indeed, an individual predictable process can be physically realized only when it is stable in a natural sense. If a system can be solved explicitly, it would be rather easy to determine whether the trivial solution is stable. But only in very special cases, the equations can be solved explicitly. However, Lyapunov introduced a method to determine stability without necessarily solving the equation. This method is now well-known as the Lyapunov direct method.

Let us study the following system to motivate the stability ideas. We now consider the solutions of $Y(t, y_0)$, $t \geq 0$, of a deterministic differential equation on the Hilbert space H,

$$\begin{cases} dY(t) = g(t, Y(t))dt, \qquad t \geq 0, \\ Y_0 = y_0 \in H, \end{cases} \tag{2.5.1}$$

where $g(\cdot,\cdot)$ is some given function. Let $\tilde{Y}(t)$, $t \geq 0$, be a particular solution of system (2.5.1). The system associated with other solutions $Y(t, y_0)$ are regarded as perturbed ones. When one talks about stability of the solution $\tilde{Y}(t)$, $t \geq 0$, it means that the norm $\|Y(t) - \tilde{Y}(t)\|_H$ could be made small enough under some suitable assumptions, for instance, that the initial perturbation scale $\|Y_0 - \tilde{Y}_0\|_H$

is small enough or t is very large. Let $X(t) = Y(t) - \tilde{Y}(t)$, then the equation (2.5.1) can be written as:

$$\begin{aligned} dX(t) = dY(t) - d\tilde{Y}(t) &= (g(t, Y(t)) - g(t, \tilde{Y}(t)))dt \\ &= (g(t, X(t) + \tilde{Y}(t)) - g(t, \tilde{Y}(t)))dt := G(t, X(t))dt, \end{aligned} \tag{2.5.2}$$

where $G(t, 0) = 0,\ t \geq 0$. Therefore, we could content ourselves with defining and studying stability for the null solution of (2.5.2).

Definition 2.5.1. (Stability) The null solution of (2.5.2) is said to be stable, if for arbitrarily given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\|x_0\|_H < \delta$, then

$$\|X(t, x_0)\|_H < \varepsilon$$

for all $t \geq 0$.

Definition 2.5.2. (Asymptotic Stability) The null solution of (2.5.2) is said to be asymptotically stable if it is stable and there exists $\delta > 0$ such that if $\|x_0\|_H < \delta$, then

$$\lim_{t \to \infty} \|X(t, x_0)\|_H = 0.$$

Definition 2.5.3. (Exponential Stability) The null solution of (2.5.2) is said to be exponentially stable if it is asymptotically stable and there exists some numbers $\alpha > 0$ and $\beta > 0$ such that

$$\|X(t, x_0)\|_H < \beta \|x_0\|_H e^{-\alpha t},$$

for all $t \geq 0$.

When we try to carry over the principles of the Lyapunov stability theory from deterministic systems to stochastic ones, we may need to consider what is the proper definition of stochastic stability. There are at least three basic types of stochastic stabilities: stability in probability, moment stability and almost sure stability. Stochastic stability has been one of the most active areas in stochastic analysis and many researchers have contributed a lot in this field. We here mention Arnold [4], Chow [33], Curtain [39], Da Prato and Zabczyk [42], Khas'minskii [62], Liu [80], Mao [96], Mohammed [101], Truman [117] among others.

Consider the following semilinear stochastic differential equation in H:

$$\begin{cases} dX(t) = (AX(t) + F(t, X(t)))dt + G(t, X(t))dW(t), \\ X_0 = x_0 \in H, \end{cases} \tag{2.5.3}$$

where A is the infinitesimal generator of a C_0-semigroup $S(t),\ t \geq 0$, of bounded linear operators on the Hilbert space H. The coefficients F, G are two nonlinear measurable mappings from $[0, T] \times H \to H$ and $[0, T] \times H \to \mathcal{L}_2^0(K_0, H)$, respectively, satisfying the following Lipschitz condition:

(**H1**) $\|F(t, x) - F(t, y)\|_H \leq \alpha(T)\|x - y\|_H, \quad \alpha(T) > 0,\ \ x, y \in H,$

(**H2**) $\|G(t, x) - G(t, y)\|_{\mathcal{L}_2^0} \leq \beta(T)\|x - y\|_H, \quad \beta(T) > 0,\ \ x, y \in H,$

for all $t \in [0, T]$.

Definition 2.5.4. A stochastic process $X(t),\ t \in [0, T]$, defined on $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$ is called a mild solution of (2.5.3) if (a) X_t is adapted to $\mathcal{F}_t,\ t \geq 0$;

(b) For arbitrary $t \in [0, T]$,

$$\mathbb{P}\left\{\omega : \int_0^t \|X(s, \omega)\|_H^2 ds < \infty\right\} = 1$$

and

$$X(t) = S(t)x_0 + \int_0^t S(t-s)F(s, X(s))ds + \int_0^t S(t-s)G(s, X(s))dW(s), \tag{2.5.4}$$

for any $x_0 \in H$ almost surely.

Definition 2.5.5. (Stable in Probability) The null solution of (2.5.4) is said to be stable in probability, if for arbitrarily given $\varepsilon,\ \varepsilon^{'} > 0$, there exists $\delta(\varepsilon, \varepsilon^{'}) > 0$

such that if $\|x_0\|_H < \delta$, then

$$P\Big\{\|X(t, x_0)\|_H > \varepsilon'\Big\} < \varepsilon$$

for all $t \geq 0$.

Definition 2.5.6. (Stability in p-th Moment) The null solution of (2.5.4) is said to be stable in p-th moment, $p > 0$, if for arbitrarily given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if $\|x_0\|_H < \delta$, then

$$\mathbb{E}\|X(t, x_0)\|_H^p < \varepsilon$$

for all $t \geq 0$.

Definition 2.5.7. (Almost Sure Stability) The null solution of (2.5.4) is said to be almost sure stable, if for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\|x_0\|_H < \delta$, then

$$\mathbb{P}\Big\{\|X(t, x_0)\|_H^p < \varepsilon\Big\} = 1$$

for all $t \geq 0$.

We now define the asymptotic stability and exponential stability.

Definition 2.5.8. (Asymptotic Stability in Probability) The null solution of (2.5.4) is said to be asymptotically stable in probability, if it is stable in probability and for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\|x_0\|_H < \delta$ guarantees

$$\lim_{t\to\infty} \mathbb{P}\big\{\|X(t, x_0)\|_H > \varepsilon\big\} = 0.$$

Definition 2.5.9. (Asymptotic Stability in p-th Moment) The null solution of (2.5.4) is said to be asymptotically stable in p-th moment, $p > 0$, if it is stable in p-th moment and there exists $\delta > 0$ such that $\|x_0\|_H < \delta$ guarantees

$$\lim_{t\to\infty} \mathbb{E}\|X(t, x_0)\|_H^p = 0.$$

Definition 2.5.10. (Asymptotic Almost Sure Stability) The null solution of (2.5.4) is said to be asymptotic almost sure stable if it is stable in probability and there exists $\delta > 0$ such that $\|x_0\|_H < \delta$ guarantees

$$\mathbb{P}\left\{ \lim_{t\to\infty} \|X(t,x_0)\|_H^p = 0 \right\} = 1.$$

Definition 2.5.11. (p-th Moment Exponential Stability) The null solution of (2.5.4) is said to be p-th moment exponentially stable, $p > 0$, if there exist positive numbers $\alpha > 0$ and $\beta > 0$ such that

$$\mathbb{E}\|X(t,x_0)\|_H^p \leq \beta \|x_0\|_H^p e^{-\alpha t}.$$

for all $t \geq 0$.

Definition 2.5.12. (Almost Sure Exponential Stability) The null solution of (2.5.4) is said to be almost sure exponentially stable, if there exist positive numbers $\alpha > 0$ and $\beta > 0$ such that

$$\mathbb{P}\left\{ \|X(t,x_0)\|_H \leq \beta \|x_0\|_H e^{-\alpha t} \right\} = 1.$$

for all $t \geq 0$.

2.6 Notes and Remarks

The material of this chapter is classical and standard. The concepts in Section 2.1 are adapted from Kreszig [67] and Pazy [109]. The result in Section 2.2, 2.4 and 2.5 are taken mainly from Liu [79]. For the material of Section 2.3, we refer the reader for Da Prato and Zabczyk [42]. For more details of this chapter, see also Kallianpur [62], Kozin [66], Rozovskii [63], Teman [117], Wu [120], Yosida [125]. For finite dimensional stochastic differential equations, see also Arnold [5], Mao [96].

Chapter 3

Moment Exponential Stability of Neutral Impulsive Stochastic Delay Partial Differential Equations with Poisson Jumps

3.1 Introduction

In recent years, studies on stochastic partial differential equations (SPDEs) have been widely noticed in the literature. Stochastic partial differential equations have received much attention, since many real world issues can be modelled by SPDEs. SPDEs can be used in many applications, such as finance, engineering, and science.

Mao [96] has given some results for the stability of solutions to stochastic differential equations in finite dimensional spaces, as well as among others. Caraballo and Liu [26] have studied the exponential stability of mild solutions of SPDEs with delays by stochastic analysis technique. As we known, the Lyapunov's direct method is a classic and powerful tool to investigate the existence, uniqueness and asymptotic behaviour of solutions of the stochastic systems. But it is not useful to discuss those problems such as delay systems. However, many dynamical systems not only depend on present and past states, but also involve derivatives with delays. The neutral SPDEs with delays are more often used to describe such systems. Luo [87] has applied the Banach fixed point theorem to deal with the asymptotical stability in mean square of SPDEs with delays. This valuable method has also been employed by Burton and his co-authors ([13], [14], [15], [16]) to investigate the stability of both deterministic and stochastic differential equations.

In practical applications, impulsive differential equations have been used to model interesting problems. Sakthivel and Luo [110] [111] have discussed asymptotic stability of nonlinear impulsive stochastic differential equations and asymptotic stability of impulsive SPDEs with infinite delays. Zhang and his co-authors [129] have investigated moment exponential stability of neutral impulsive nonlinear SPDEs with delays. Maheswari and Karunanithi [92] have studied asymptotic stability of stochastic impulsive neutral partial functional differential equations. The study of SPDEs with impulsive effects is a new area of research, which can be found in the literature.

Recently, SPDEs driven by jump processes have received attentions. Cui [35] has discussed exponential stability for neutral SPDEs with delays and Poisson jumps. Some results on SDEs or SPDEs with Poisson jumps have been noticed in the literature. To the best of my knowledge, there are only a few papers on stability analysis for impulsive neutral SPDEs with delays and Poisson jumps. We study in this chapter the moment exponential stability of mild solution of neutral impulsive stochastic delay partial differential equations with Poisson jumps under natural conditions. Here we shall apply the so-called fixed point theorem to investigate the existence and uniqueness moment exponential stability of mild solutions of this class of systems.

3.2 Problem Formulation and Assumptions

Let H and K be two real separable Hilbert spaces with norms $(H, \|\cdot\|_H)$ and $(K, \|\cdot\|_K)$, and their inner products denoted by $\langle\cdot,\cdot\rangle_K$, $\langle\cdot,\cdot\rangle_H$. We denote by $\mathcal{L}(K, H)$ the set of all linear bounded operators from $K \to H$, equipped with the usual operator norm $\|\cdot\|$. In this chapter, we use the symbol $\|\cdot\|$ to denote norms of operators regardless of the spaces potentially involved when no confusion possibly arises.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered complete probability space, with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. The filtration is a right continuous increasing family and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $r > 0$ and $D := D([-r, 0]; H)$ be the space of all bounded, càdlàg (i.e, is right continuous and has left limits) functions from $[-r, 0]$ into H, equipped with the norm $\|\phi\|_D = \sup_{t\in[-r,0]} \|\phi(t)\|_H,\ \phi \in D$. It can be shown that the space D, under the norm

$$\|\phi\|_D = \sup_{t\in[-r,0]} \|\phi(t)\|_H,\ \phi \in D,$$

is a Banach space.

Let $W(t), t \geq 0$ denote a K-valued Wiener process defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, with covariance operator Q, that is

$$\mathbb{E}\langle W(t), x\rangle_K \langle W(s), y\rangle_K = (t \wedge s)\langle Qx, y\rangle_K, \qquad \forall x, y \in K,$$

where $t \wedge s = min\{t, s\}$ and Q is a positive, self-adjoint, trace class operator on K. We assume that there exists a complete orthonormal system $\{e_i\}_{i\geq 1}$ in K, a bounded sequence of positive numbers λ_i such that $Qe_i = \lambda_i e_i, i = 1, 2, \ldots$, and sequence $\{\beta_i(t)\}_{i\geq 1}$ of independent standard real Brownian motions such that

$$W(t) = \sum_{i=1}^{+\infty} \sqrt{\lambda_i}\beta_i(t)e_i, \qquad t \geq 0$$

and

$$\mathcal{F}_t = \mathcal{F}_t^W,$$

where $\mathcal{F}_t^W$ is the σ-algebra generated by $\{W(t) : t \in [0,\infty)\}$. We introduce the subspace $K_0 = Q^{\frac{1}{2}}K$ of K endowed with the inner product $\langle u, v\rangle_{K_0} = \langle Q^{1\frac{1}{2}}u, Q^{-\frac{1}{2}}v\rangle$ which is a Hilbert space. Furthermore, let $\mathcal{L}_2^0(K_0, H)$ denotes the space of all Hilbert-Schmidt operators from K_0 to H with the norm

$$|\xi|^2_{\mathcal{L}_2^0} := tr(\xi Q^{1/2}(\xi Q^{1/2})^*) < \infty, \quad \xi \in \mathcal{L}_2^0(K_0, H).$$

For the construction of stochastic integral in Hilbert space, see Da. Prato and Zabczyk [42].

Let A be a linear operator from H to H. Assume that $\{S(t), t \geq 0\}$ is an exponentially stable analytic semigroup with its infinitesimal generator A. Then it is possible (see[109]), under some circumstances, to define the fractional power $(-A)^\alpha$ for any $\alpha \in (0,1]$ which is a closed linear operator with its domain $\mathcal{D}((-A)^\alpha)$, furthermore, the subspace $\mathcal{D}((-A)^\alpha)$ is dense in H and the expression

$$\|x\|_\alpha = |(-A)^\alpha x|_H, \quad x \in \mathcal{D}((-A)^\alpha),$$

defines a norm on $\mathcal{D}((-A)^\alpha)$.

Let U be a Hilbert space with norm $\|\cdot\|_U$ and inner product $\langle\cdot,\cdot\rangle_U$. For a Borel set $\mathcal{O} \in \mathcal{B}(U - \{0\})$, we consider the following impulsive neutral stochastic delay partial differential equation with both Poisson point process and Brownian motions in the form:

$$\begin{cases} d[x(t) + u(t, x(t-r))] = [Ax(t) + f(t, x(t-r))]dt + g(t, x(t-r))dW(t) \\ \qquad\qquad\qquad\qquad + \int_{\mathcal{O}} h(t, x(t-r), z)\tilde{N}(dt, dz), \quad t \geq 0, \quad t \neq t_k \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad t = t_k, k = 1, 2, \dots, \\ x_0(t) = \phi(t) \in D([-r, 0]; H). \end{cases} \tag{3.2.1}$$

where $D := D([-r,0]; H)$ is the space of all càdlàg functions from $[-r, 0]$ into H, equipped with the supremum norm $\|\phi\|_D = \sup_{t\in[-r,0]} \|\phi(t)\|_H$. Here u, f :

$[0,+\infty) \times H \to H$, $g : [0,+\infty) \times H \to \mathcal{L}_2^0(K_0, H)$, $h : [0,+\infty) \times H \times U \to H$ are measurable functions and $I_k : H \to H$ is measurable and $0 < t_1 < \ldots < t_k < \ldots, \lim_{k\to\infty} t_k = \infty$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k, k = 1, 2, \ldots$, respectively. The mapping I_k represents the size of the jump at $t_k, k = 1, 2, \ldots$.

Definition 3.2.1. A stochastic process $\{x(t), t \in [0,T]\}, 0 \le T$, is called a mild solution of equation (3.2.1) if:

(i). $x(t)$ is adapted to $\mathcal{F}_t$, for each $t \ge 0$, and

$$\mathbb{P}\left\{\omega : \int_0^T \|x(t,\omega)\|_H^p dt < \infty\right\} = 1, \quad T \ge 0;$$

(ii). $x(t) \in H$ has càdlàg path on $t \in [0,T]$ a.s and for each $t \in [0,T]$, $x(t)$ satisfies the following integral equation.

$$\begin{aligned} x(t) =& S(t)[\phi(0) + u(0,\phi)] - u(t, x(t-r)) - \int_0^t AS(t-s)u(s, x(s-r))ds \\ &+ \int_0^t S(t-s)f(s, x(s-r))ds + \int_0^t S(t-s)g(s, x(s-r))dW(s) \\ &+ \int_0^t \int_{\mathcal{O}} S(t-s)h(t, x(s-r), z)\tilde{N}(ds, dz) \\ &+ \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)), \\ x(0) =& \phi \in D([-r,0]; H). \end{aligned} \tag{3.2.2}$$

We are mainly concerned about the p-th moment exponentially stable of mild solutions to stochastic systems in this chapter. We shall recall the following stability notation which is the definition of exponential stability in p-th moment.

Definition 3.2.2. The mild solution of equation (3.2.1) is said to be *exponentially stable* in the p-th moment, if for any initial $\phi \in D$, there exist a pair of numbers

$M_0 > 0$ and $\gamma > 0$ such that

$$\mathbb{E}\|x(t)\|_H^p < M_0(\|\phi\|_D^p)e^{-\gamma t}, \quad t \geq 0.$$

In order to obtain the main result, we impose the following reasonable assumptions:

(**H1**) A is the infinitesimal generater of an exponentially stable analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$ in H such that the following inequality holds

$$\|S(t)\| \leq Me^{-\gamma t}, \quad t \geq 0,$$

for $\gamma > 0$.

(**H2**) The coefficients f, g, h satisfy the Lipschitz conditions, i.e. there exist some positive constants $p > 2$, $K_1 > 0$, $K_2 > 0$ and $K_3 > 0$ such that for any $x, y \in H$ and $t \geq 0$,

$$\|f(t,x) - f(t,y)\|_H \leq K_1\|x - y\|_H,$$

$$\|g(t,x) - g(t,y)\|_{\mathcal{L}_2^0} \leq K_2\|x - y\|_H,$$

$$\int_{\mathcal{O}} \|h(t,x,z) - h(t,y,z)\|_H^p \nu(dz) \leq K_3^p\|x - y\|_H^p,$$

Moreover, we assume that $f(t,0) = g(t,0) = h(t,0,z) = 0$.

(**H3**) There exist constants $\alpha \in (0,1]$ and $K_4 > 0$ such that for any $x \in H$ and $t \geq 0$, $u(t,x) \in \mathcal{D}((-A)^\alpha)$

$$\|(-A)^\alpha u(t,x) - (-A)^\alpha u(t,y)\|_H \leq K_4\|x - y\|_H, \quad x, y \in H,$$

with $u(t,0) = 0$ for $t \geq 0$.

(**H4**) There exist a series $\{q_k\}$, $q_k > 0$, $k \in \{1,2,3,\ldots\}$, such that $\sum_{k=1}^{\infty} q_k = \kappa < \infty$ and

$$\|I_k(x) - I_k(y)\| \leq q_k\|x - y\|, \quad x, y \in H,$$

and $I(0) = 0,\ k = 1, 2, \cdots$.

Theorem 3.2.1. *Suppose that the assumptions (**H2**)-(**H4**) are satisfied. Let* $\phi(t, \omega) : [-r, 0] \times \Omega \to H$ *and* $r > 0$ *is some given initial datum such that* $\phi(t)$ *is* $\mathcal{F}_0$*-measurable for any* $t \in [-r, 0]$ *and* $\sup_{-r \le s \le 0} \mathbb{E}\|\phi(s)\|_H^p < \infty$*. Then, there exists a unique mild solution to equation (3.2.1) on* $[0, T]$ *for all* $T \geq 0$.

Proof. As $t - r$ for all $t \geq 0$, so we have that $t - r \leq 0$ for $t \in [0, r]$ and therefore the problem on $[0, r]$ can be rewritten as

$$\begin{aligned} x(t) = & S(t)[\phi(0) + u(0, \phi)] - u(t, \phi(t-r)) - \int_0^t AS(t-s)u(s, \phi(s-r))ds \\ & + \int_0^t S(t-s)f(s, \phi(s-r))ds + \int_0^t S(t-s)g(s, \phi(s-r))dW(s) \\ & + \int_0^t \int_{\mathcal{O}} S(t-s)h(t, \phi(s-r), z)\tilde{N}(ds, dz) \\ & + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)), \end{aligned} \tag{3.2.3}$$

which is a nondelay problem. We then obtain the existence of a unique mild solution on $[0, r]$. By induction, the problem can be solved on $[r, 2r]$, $[2r, 3r]$, $\cdots$, $[nr, (n+1)r]$ for all natural numbers $n \geq 0$ and therefore on $[0, \infty)$. Then there exist a unique mild solution to equation (3.2.1). $\square$

In the deterministic framework, there exist a large literature on the existence of solutions to stochastic differential equations (see [37], [112], [116], [108], [108]). Caraballo, Liu and Truman [29] established conditions to ensure existence and uniqueness of solutions of general stochastic functional differential equations. The theorem of an infinite-dimensional Bichteler-Jacod inequality for stochastic integral with respect to Poisson random measures has been established in Marineli, Prévôt, Röckner [97]. We shall recall the following theorem.

Lemma 3.2.1. (Marineli [97]) *Let* $p \geq 2$*. Assume that* $h : [0, T] \to H$ *is a mea-*

surable process such that the expectation on the right-hand side of (3.2.4) below is finite. Then for all $p \in [2, \infty)$ *there exists a constant* $C_p > 0$ *such that*

$$
\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}\left\| \int_0^t \int_{\mathcal{O}} h(s,z)\tilde{N}(ds,dz)\right\|_H^p \\
&\leq C_p\mathbb{E}\int_0^T \left[\int_{\mathcal{O}} \|h(s,z)\|_H^p \nu(dz) + \left(\int_{\mathcal{O}} \|h(s,z)\|_H^2 \nu(dz)\right)^{p/2}\right] ds,
\end{aligned} \tag{3.2.4}
$$

where $C_p = \left(\frac{p(p-1)}{2}\right)^{p/2}$.

Proof. The proof can be found in Lemma 3.1. Marineli et al. [97].

□

The inequality (3.2.4) can be extended also to stochastic convolutions, even though in general that the stochastic convolutions are not martingales. Recalled that an operator A on H is called *dissipative* if $Re\langle Ax, x\rangle_H \leq 0$ for all $x \in \mathcal{D}(A) \subset H$. An operator A on H is called *m-dissipative* if A is dissipative and $(I - A)$ is surjective.

Proposition 3.2.1. (Marineli [97]) *Let A be m-dissipative on H and h satisfies the hypotheses of Lemma 3.2.1. Then for all* $p \in [2, \infty)$, *there exists a constant* $C_p > 0$ *such that*

$$
\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}\left\| \int_0^t \int_{\mathcal{O}} e^{(t-s)A}h(s,z)\tilde{N}(ds,dz)\right\|_H^p \\
&\leq C_p\mathbb{E}\int_0^T \left[\int_{\mathcal{O}} \|h(s,z)\|_H^p \nu(dz) + \left(\int_{\mathcal{O}} \|h(s,z)\|_H^2 \nu(dz)\right)^{p/2}\right] ds,
\end{aligned} \tag{3.2.5}
$$

where $C_p = \left(\frac{p(p-1)}{2}\right)^{p/2}$.

Proof. The proof can be found in Proposition 3.3 Marineli et al. [97]. □

Lemma 3.2.2. (Pazy [109]) *Suppose that the assumption (**H1**) holds. Then for any* $0 < \beta \leq 1$, *the following equality holds:*

$$
S(t)(-A)^\beta x = (-A)^\beta S(t)x, \quad x \in \mathcal{D}((-A)^\beta), \tag{3.2.6}
$$

and there exists a positive value M_β such that

$$\|(-A)^\beta S(t)\| \le M_\beta t^{-\beta} e^{-\gamma t}, \quad t > 0. \tag{3.2.7}$$

Proof. The proof can be found in Theorem 6.13 Pazy [109]. $\square$

3.3 Exponential Stability in p-th Moment of Mild Solutions

In this section, the exponential stability in p-th moment of mild solutions of equation (3.2.1) will be considered by employing the contraction mapping theorem.

Theorem 3.3.1. *Suppose that the assumptions **(H1)**-**(H4)** hold for some $\alpha \in (0,1)$, $p > 2$. Assume further that*

$$\begin{aligned} 6^{(p-1)}\Bigg[&{K_4}^p\|(-A)^{-\alpha}\|^p + M_{1-\alpha}^p {K_4}^p \gamma^{-p\alpha}\big(\Gamma(1+p(\alpha-1)/(p-1))\big)^{p-1} \\ &+ M^p K_1^p \gamma^{-p} + C_p M^p K_2^p \left(\frac{p-2}{p-1}\right)^{p/2-1} (2\gamma)^{1-p/2}\gamma^{-1} \\ &+ C_p M^p K_3^p \left(\gamma^{-p} + \gamma^{p/2}\left(\frac{p-2}{2(p-1)}\right)^{(p-2)/2}\right) + M^p \kappa^p \gamma^{-p}\Bigg] < 1, \end{aligned} \tag{3.3.1}$$

where $\Gamma(\cdot)$ is the standard Gamma function, $M_{1-\alpha}$ is the corresponding number in Lemma 3.2.2, and $C_p = (\frac{p(p-1)}{2})^{\frac{p}{2}}$, $\frac{1}{p} + \frac{1}{q} = 1$. Then the mild solution of equation (3.2.1) is exponentially stable in the p-th moment. In other words, there exist some numbers $M_0(\phi) > 0$ and $\mu > 0$ such that

$$\mathbb{E}\|y(t)\|_H^p \le M_0(\phi) e^{-\mu t}, \quad t \ge 0.$$

Proof. Firstly, we define a space $\mathcal{S}$ as the family of all stochastic process $x(t)$, $t \in [-r, \infty)$, such that

$$\mathbb{E}\|x(t)\|_H^p \le \tilde{M}\mathbb{E}\|\phi\|_D^p e^{-\eta t}, \quad t \ge 0, \tag{3.3.2}$$

for some constants $\tilde{M} > 0$ and $\gamma > \eta > 0$, where γ is the constant in condition (**H1**). It can be shown that $\mathcal{S}$, under this norm

$$\|x\|_{\mathcal{S}} := \sup_{t\in[0,\infty)} \mathbb{E}\|x(t)\|_H^p, \qquad x \in \mathcal{S},$$

is a Banach space.

Now, we define a nonlinear operator π on $\mathcal{S}$ by $\pi(x)(t) = \phi(t)$ for $t \in [-r, 0]$ and for $t \geq 0$,

$$\begin{aligned}\pi(x)(t) =& S(t)[\phi(0) + u(0,\phi)] - u(t, x(t-r)) \\ & - \int_0^t AS(t-s)u(s, x(s-r))ds \\ & + \int_0^t S(t-s)f(s, x(s-r))ds \\ & + \int_0^t S(t-s)g(s, x(s-r))dW(s) \\ & + \int_0^t \int_{\mathcal{O}} S(t-s)h(t, x(s-r), z)\tilde{N}(ds, dz) \\ & + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)).\end{aligned} \tag{3.3.3}$$

To prove the p-th moment stability of mild solutions, it is enough to prove that the operator π has a fixed point in space $\mathcal{S}$. In order to show this result, we are going to use the usual contraction mapping theorem.

We first show that π is a mapping from $\mathcal{S}$ into $\mathcal{S}$. Let $x(t) \in \mathcal{S}$, then from the definition of π, we have for $t \geq r$ that

$$\begin{aligned}
\mathbb{E}\|\pi(x)(t)\|_H^p \leq\ & 7^{p-1}\mathbb{E}\|S(t)[\phi(0)+u(0,\phi)]\|_H^p + 7^{p-1}\mathbb{E}\|u(t,x(t-r))\|_H^p \\
& + 7^{p-1}\mathbb{E}\left\|\int_0^t AS(t-s)u(s,x(s-r))ds\right\|_H^p \\
& + 7^{p-1}\mathbb{E}\left\|\int_0^t S(t-s)f(s,x(s-r))ds\right\|_H^p \\
& + 7^{p-1}\mathbb{E}\left\|\int_0^t S(t-s)g(s,x(s-r))dW(s)\right\|_H^p \\
& + 7^{p-1}\mathbb{E}\left\|\int_0^t \int_{\mathcal{O}} S(t-s)h(s,x(s-r),z)\tilde{N}(ds,dz)\right\|_H^p \\
& + 7^{p-1}\mathbb{E}\left\|\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-))\right\|_H^p := 7^{p-1}\sum_{i=1}^{7} I_i(t).
\end{aligned} \tag{3.3.4}$$

By the definition of $\mathcal{S}$ and assumption (**H3**) it follows that for each $t \geq r$,

$$\begin{aligned}
I_1(t) &= \mathbb{E}\|S(t)[\phi(0)+u(0,-r)]\|_H^p \\
&\leq \|S(t)\|^p\mathbb{E}\|\phi(0)+u(0,-r)\|_H^p \\
&\leq M^p e^{-p\gamma t}\mathbb{E}\|\phi(0)+u(0,-r)\|_H^p.
\end{aligned} \tag{3.3.5}$$

Moreover, by taking (3.3.2), assumptions (**H2**) and (**H3**) into account, for each $t \geq r$, we have that

$$\begin{aligned}
I_2(t) &= \mathbb{E}\|u(t,x(t-r))\|_H^p \\
&= \mathbb{E}\|(-A)^{-\alpha}(-A)^{\alpha}u(t,x(t-r))\|_H^p \\
&\leq \|(-A)^{-\alpha}\|^p\mathbb{E}\|(-A)^{\alpha}u(t,x(t-r))\|_H^p \\
&\leq K_4^p\|(-A)^{-\alpha}\|^p\mathbb{E}\|x(t-r)\|_H^p.
\end{aligned} \tag{3.3.6}$$

Since $x \in \mathcal{S}$, x satisfies the relation

$$\mathbb{E}\|x(t-r)\|_H^p \leq \tilde{M}e^{\eta r}\mathbb{E}\|\phi\|_D^p e^{-\eta t}, \qquad t \geq r. \tag{3.3.7}$$

Substituting (3.3.7) into (3.3.6), we get that

$$I_2(t) \leq K_4^p\|(-A)^{-\alpha}\|^p\tilde{M}e^{\eta r}\mathbb{E}\|\phi\|_D^p e^{-\eta t}, \qquad t \geq r. \tag{3.3.8}$$

Under assumption (**H3**), and $\alpha \in (0,1)$, we have that, for each $t \geq r$,

$$\begin{aligned} I_3(t) &= \mathbb{E}\left\| \int_0^t (-A)S(t-s)u(s,x(s-r))ds \right\|_H^p \\ &\leq \mathbb{E}\left\| \int_0^t (-A)^{1-\alpha}(-A)^{\alpha}S(t-s)u(s,x(s-r))ds \right\|_H^p \\ &\leq \mathbb{E}\bigg(\int_0^t \|(-A)^{1-\alpha}(-A)^{\alpha}S(t-s)u(s,x(s-r))\|_H ds \bigg)^p \\ &\leq \mathbb{E}\bigg(\int_0^t \|(-A)^{(1-\alpha)}S(t-s)\| \|(-A)^{\alpha}u(s,x(s,-r))\|_H ds \bigg)^p. \end{aligned} \tag{3.3.9}$$

Thus, in view of Lemma 3.2.2 and Hölder inequality, for each $t \geq r$, we get

$$\begin{aligned} I_3(t) &\leq \mathbb{E}\bigg(\int_0^t \|M_{1-\alpha}(t-s)^{-(1-\alpha)}e^{-\gamma(t-s)}\| \|(-A)^{\alpha}u(s,x(s-r))\|_H ds \bigg)^p \\ &\leq M_{1-\alpha}^p \mathbb{E}\bigg(\int_0^t (t-s)^{\alpha-1}e^{-\gamma(t-s)}\|(-A)^{\alpha}u(s,x(s-r))\|_H ds \bigg)^p. \end{aligned} \tag{3.3.10}$$

On the other hand, by using Lipschitz condition (**H3**), we have the following inequality for $t \geq r$

$$\begin{aligned} I_3(t) &\leq M_{1-\alpha}^p \bigg(\int_0^t (t-s)^{\frac{p(\alpha-1)}{p-1}} e^{-\gamma(t-s)} ds \bigg)^{p-1} {K_4}^p \int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-r)\|_H^p ds \\ &\leq M_{1-\alpha}^p {K_4}^p \gamma^{1-p\alpha} \bigg(\Gamma\bigg(1 + \frac{p(\alpha-1)}{p-1}\bigg)\bigg)^{p-1} \int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-r)\|_H^p ds. \end{aligned} \tag{3.3.11}$$

Since $x \in \mathcal{S}$, x satisfies that

$$\mathbb{E}\|x(t-r)\|_H^p \leq \tilde{M}e^{\eta r}\mathbb{E}\|\phi\|_D^p e^{-\eta t}, \qquad t \geq r. \tag{3.3.12}$$

Substituting inequality (3.3.12) into (3.3.11), for each $t \geq r$, we get that

$$\begin{aligned} I_3(t) &\leq M_{1-\alpha}^p {K_4}^p \gamma^{1-p\alpha}\big(\Gamma(1+p(\alpha-1)/(p-1))\big)^{p-1} \\ &\quad \cdot \int_0^t e^{-\gamma(t-s)}\tilde{M}e^{\eta r}\mathbb{E}\|\phi\|_D^p e^{-\eta s} ds \\ &= M_{1-\alpha}^p {K_4}^p \gamma^{1-p\alpha}\big(\Gamma(1+p(\alpha-1)/(p-1))\big)^{p-1} \\ &\quad \cdot \tilde{M}\mathbb{E}\|\phi\|_D^p e^{\eta\gamma} e^{-\gamma t}\int_0^t e^{(\gamma-\eta)s} ds. \end{aligned} \tag{3.3.13}$$

The inequality (3.3.13) turns to be

$$\begin{aligned}
I_3(t) &= M_{1-\alpha}^p {K_4}^p \gamma^{1-p\alpha}\big(\Gamma(1+p(\alpha-1)/(p-1))\big)^{p-1} \\
&\quad \cdot \tilde{M}\mathbb{E}\|\phi\|_D^p e^{\eta\gamma} e^{-\gamma t}\frac{1}{\gamma-\eta}\big(e^{(\gamma-\eta)t}-1\big) \\
&\leq M_{1-\alpha}^p {K_4}^p \gamma^{1-p\alpha}\big(\Gamma(1+p(\alpha-1)/(p-1))\big)^{p-1} \\
&\quad \cdot \tilde{M}\mathbb{E}\|\phi\|_D^p e^{\eta\gamma} e^{-\gamma t}\frac{1}{\gamma-\eta} e^{\gamma t} e^{-\eta t} \\
&= M_{1-\alpha}^p {K_4}^p \gamma^{1-p\alpha}\big(\Gamma(1+p(\alpha-1)/(p-1))\big)^{p-1}\frac{\tilde{M}\mathbb{E}\|\phi\|_D^p e^{\eta\gamma}}{\gamma-\eta} e^{-\eta t}.
\end{aligned} \tag{3.3.14}$$

Now employing assumption (**H1**), similarly we have that for each $t \geq r$,

$$\begin{aligned}
I_4(t) &= \mathbb{E}\bigg\|\int_0^t S(t-s)f(s,x(s-r))ds\bigg\|_H^p \\
&\leq \mathbb{E}\bigg(\int_0^t \|S(t-s)\| \cdot \|f(s,x(s-r))\|ds\bigg)^p \\
&\leq \mathbb{E}\bigg(\int_0^t Me^{-\gamma(t-s)}K_1\|x(s-r)\|_H ds\bigg)^p.
\end{aligned} \tag{3.3.15}$$

On the other hand, by using Lipschitz condition (**H2**) and Hölder inequality, for each $t \geq r$, we obtain

$$\begin{aligned}
I_4(t) &\leq M^p K_1^p \bigg(\int_0^t e^{-\gamma(t-s)}ds\bigg)^{p-1}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-r)\|_H^p ds \\
&\leq M^p K_1^p \gamma^{1-p}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-r)\|_H^p ds \\
&\leq M^p K_1^p \gamma^{1-p}\frac{\tilde{M}\mathbb{E}\|\phi\|_D^p e^{\eta r}}{\gamma-\eta} e^{-\eta t}.
\end{aligned} \tag{3.3.16}$$

On the other hand, by taking (**H1**) into account, we have for each $t \geq r$,

$$\begin{aligned}
I_5(t) &= \mathbb{E}\bigg\|\int_0^t S(t-s)g(s,x(s-r))dW(s)\bigg\|_H^p \\
&\leq \mathbb{E}C_p\bigg(\int_0^t \big(\|S(t-s)g(s,x(s-r))\|_H^p\big)^{2/p}ds\bigg)^{p/2} \\
&\leq \mathbb{E}C_p\bigg(\int_0^t \|S(t-s)\|^2\big(\|g(s,x(s-r))\|_{\mathcal{L}_2^0}^p\big)^{2/p}ds\bigg)^{p/2} \\
&\leq M^p C_p\bigg(\int_0^t \big(e^{-2\gamma(t-s)}\mathbb{E}\|g(s,x(s-r))\|_{\mathcal{L}_2^0}^p\big)^{2/p}ds\bigg)^{p/2}.
\end{aligned} \tag{3.3.17}$$

Under the Lipschitz condition (**H2**), Lemma 2.2.1 and Hölder inequality, for

$t \geq r$, the inequality (3.3.17) turns to be

$$
\begin{aligned}
I_5(t) &\leq C_p M^p K_2^p \bigg(\int_0^t \left(e^{-2\gamma(t-s)} \mathbb{E}\|x(s-r)\|_H^p\right)^{2/p} ds \bigg)^{p/2} \\
&\leq C_p M^p K_2^p \bigg(\int_0^t e^{-\frac{2(p-1)}{p-2}\gamma(t-s)} ds \bigg)^{p-1} \int_0^t e^{-\gamma(t-s)} \mathbb{E}\|x(s-r)\|_H^p ds \\
&\leq C_p M^p K_2^p \bigg(\frac{p-2}{2\gamma(p-1)} \bigg)^{p/2-1} \int_0^t e^{-\gamma(t-s)} \mathbb{E}\|x(s-r)\|_H^p ds \\
&\leq C_p M^p K_2^p \bigg(\frac{p-2}{2\gamma(p-1)} \bigg)^{p/2-1} \frac{\tilde{M}\mathbb{E}\|\phi\|_D^p e^{\eta r}}{\gamma - \eta} e^{-\eta t}.
\end{aligned}
\tag{3.3.18}
$$

Further, taking into account Lemma 3.2.1, for $t \geq r$, we obtain that

$$
\begin{aligned}
I_6(t) &= \mathbb{E}\bigg\| \int_0^t \int_{\mathcal{O}} S(t-s)h(s, x(s-r), z)\tilde{N}(ds, dz) \bigg\|_H^p \\
&\leq \mathbb{E}C_p \bigg[\int_0^t \int_{\mathcal{O}} \|S(t-s)h(s, x(s-r), z)\|^p \nu(dz) ds \\
&\quad + \bigg(\int_0^t \int_{\mathcal{O}} \|S(t-s)h(s, x(s-r), z)\|^2 \nu(dz) ds \bigg)^{p/2} \bigg].
\end{aligned}
\tag{3.3.19}
$$

Under the assumption (**H1**) and Hölder inequality, for $t \geq r$, one can have

$$
\begin{aligned}
I_6(t) &\leq \mathbb{E}C_p \bigg[\int_0^t \int_{\mathcal{O}} \|S(t-s)\|^p \|h(s, x(s-r), z)\|^p \nu(dz) ds \\
&\quad + \bigg(\int_0^t \int_{\mathcal{O}} \|S(t-s)\|^2 \|h(s, x(s-r), z)\|^2 \nu(dz) ds \bigg)^{p/2} \bigg] \\
&\leq M^p C_p \bigg[\int_0^t \int_{\mathcal{O}} e^{-p\gamma(t-s)} \mathbb{E}\|h(s, x(s-r), z)\|^p \nu(dz) ds \\
&\quad + \bigg(\int_0^t \int_{\mathcal{O}} e^{-2\gamma(t-s)} \mathbb{E}\|h(s, x(s-r), z)\|^2 \nu(dz) ds \bigg)^{p/2} \bigg].
\end{aligned}
\tag{3.3.20}
$$

By using Lipschitz condition (**H2**), for $t \geq r$, we further have that,

$$
\begin{aligned}
I_6(t) &\leq M^p C_p \bigg[K_3^p \gamma^{(1-p)} \int_0^t e^{-\gamma(t-s)} \mathbb{E}\|x(s-r)\|^p ds \\
&\quad + K_3^p \bigg(\frac{p-2}{2(p-1)\gamma} \bigg)^{(p-2)/2} \int_0^t e^{-\gamma(t-s)} \mathbb{E}\|x(s-r)\|^p ds \bigg] \\
&\leq M^p C_p K_3^p \bigg(\gamma^{(1-p)} + \bigg(\frac{p-2}{2(p-1)\gamma} \bigg)^{(p-2)/2} \bigg) \int_0^t e^{-\gamma(t-s)} \mathbb{E}\|x(s-r)\|^p ds \\
&\leq M^p C_p K_3^p \bigg(\gamma^{-p} + \bigg(\frac{p-2}{2(p-1)\gamma} \bigg)^{(p-2)/2} \bigg) \frac{\tilde{M}\mathbb{E}\|\phi\|_D^p e^{\eta r}}{\gamma - \eta} e^{-\eta t}.
\end{aligned}
\tag{3.3.21}
$$

Now, we estimate the impulsive term, from assumption (**H1**), we get for $t \geq r$ that,

$$\begin{aligned} I_7(t) &= \mathbb{E}\Big\| \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-))\Big\|_H^p \\ &\leq \mathbb{E}\bigg(\sum_{0<t_k<t} Me^{-\gamma(t-t_k)} q_k \|x(t_k^-)\|_H \bigg)^p. \end{aligned} \tag{3.3.22}$$

On the other hand, by employing assumption (**H4**) for each $t \geq r$, we obtain

$$\begin{aligned} I_7(t) &\leq \mathbb{E}\bigg(\kappa \int_0^t Me^{-\gamma(t-s)} \|x(s)\|_H ds \bigg)^p \\ &\leq M^p \kappa^p \bigg(\int_0^t e^{-\gamma(t-s)} ds \bigg)^{p-1} \int_0^t e^{-\gamma(t-s)} \mathbb{E}\|x(s)\|_H^p ds \\ &\leq M^p \kappa^p \gamma^{(1-p)} \int_0^t e^{-\gamma(t-s)} \mathbb{E}\|x(s)\|_H^p ds. \end{aligned} \tag{3.3.23}$$

Since $x \in \mathcal{S}$, x satisfies that for $s \geq 0$,

$$\mathbb{E}\|x(s)\|_H^p \leq \tilde{M} \mathbb{E}\|\phi\|_D^p e^{-\eta s}. \tag{3.3.24}$$

Substituting (3.3.24) into (3.3.23), we get that for $t \geq r$,

$$I_7(t) \leq M^p \kappa^p \gamma^{(1-p)} \frac{\tilde{M} \mathbb{E}\|\phi\|_D^p e^{\eta r}}{\gamma - \eta} e^{-\eta t}. \tag{3.3.25}$$

Recalling inequality (3.3.4), from inequalities (3.3.5) to (3.3.25), one can see that there exists numbers $M_1 > 0$ and $\eta_1 > 0$ such that,

$$\mathbb{E}\|(\pi x)(t)\|_H^p \leq M_1 \mathbb{E}\|\phi\|_D^p e^{-\eta_1 t}. \tag{3.3.26}$$

Thus, we conclude that $\pi(\mathcal{S}) \subset \mathcal{S}$.

Next, we show the mapping π is contractive. For any $x, y \in \mathcal{S}$, we have,

$$
\begin{aligned}
&\mathbb{E}\|(\pi x)(t) - (\pi y)(t)\|_H^p \\
&\leq 6^{p-1}\mathbb{E}\|u(t, x(t-r)) - u(t, y(t-r))\|_H^p \\
&\quad + 6^{p-1}\mathbb{E}\left\| \int_0^t AS(t-s)\big(u(s, x(s-r)) - u(s, y(s-r))\big)ds \right\|_H^p \\
&\quad + 6^{p-1}\mathbb{E}\left\| \int_0^t S(t-s)\big(f(s, x(s-r)) - f(s, y(s-r))\big)ds \right\|_H^p \\
&\quad + 6^{p-1}\mathbb{E}\left\| \int_0^t S(t-s)\big(g(s, x(s-r)) - g(s, y(s-r))\big)dWs \right\|_H^p \\
&\quad + 6^{p-1}\mathbb{E}\left\| \int_0^t \int_{\mathcal{O}} S(t-s)\big(h(s, x(s-r), z) - h(s, y(s-r), z)\big)\tilde{N}(ds, dz) \right\|_H^p \\
&\quad + 6^{p-1}\mathbb{E}\left\| \sum_{0<t_k<t} S(t-t_k)(I_k)(x(t_k^-) - y(t_k^-)) \right\|_H^p \\
&:= 6^{p-1}\sum_{i=1}^{6} J_i(t).
\end{aligned}
\tag{3.3.27}
$$

Noting that $x(s) = y(s) = \phi(s)$ for $s \in [-r, 0]$, then from assumption (**H3**), we have

$$
\begin{aligned}
J_1(t) &= \mathbb{E}\|u(t, x(t-r)) - u(t, y(t-r))\|_H^p \\
&\leq K_4^p\|(-A)^{-\alpha}\|^p\mathbb{E}\|x(t-r) - y(t-r)\|_H^p.
\end{aligned}
\tag{3.3.28}
$$

By using Lemma 3.2.2 and assumption (**H4**), for $t \geq r$, one can be

$$
\begin{aligned}
J_2(t) &= \mathbb{E}\left\| \int_0^t (-A)S(t-s)\big(u(s, x(s-r)) - u(s, y(s-r))\big)ds \right\|_H^p \\
&= \mathbb{E}\left\| \int_0^t (-A)^{1-\alpha}(-A)^{\alpha}S(t-s)\big(u(s, x(s-r)) - u(s, y(s-r))\big)ds \right\|_H^p \\
&\leq \mathbb{E}\bigg(\int_0^t \|(-A)^{1-\alpha}S(t-s)\|\|(-A)^{\alpha}u(s, x(s-r)) - u(s, y(s-r))\|_H ds \bigg)^p \\
&\leq \mathbb{E}\bigg(\int_0^t M_{1-\alpha}(t-s)^{-(1-\alpha)}e^{-\gamma(t-s)} \\
&\qquad \cdot \big\|(-A)^{\alpha}\big(u(s, x(s-r)) - u(s, y(s-r))\big)\big\|_H ds \bigg)^p.
\end{aligned}
\tag{3.3.29}
$$

Under assumption (**H3**) and by using Hölder's inequality, for $t \geq r$, inequality (3.3.29) turns out to be

$$
\begin{aligned}
J_2(t) \leq M_{1-\alpha}^p \bigg(\int_0^t (t-s)^{p/(p-1)(\alpha-1)} e^{-\gamma(t-s)} ds \bigg)^{p-1} \\
\cdot \int_0^t e^{-\gamma(t-s)} \mathbb{E}\|u(s, x(s-r)) - u(s, y(s-r))\|_H^p ds.
\end{aligned}
\tag{3.3.30}
$$

By the definition of Gamma function $\Gamma(x) = \int_0^\infty x^{t-1} e^{-x} dx$, for $t \geq r$, we get

$$
\begin{aligned}
J_2(t) &\leq M_{1-\alpha}^p K_4{}^p \gamma^{1-p\alpha} \big(\Gamma(1 + p(\alpha-1)/(p-1)))\big)^{p-1} \\
&\quad \cdot \int_0^t e^{-\gamma(t-s)} \mathbb{E}\|x(s-r)) - y(s-r)\|_H^p ds \\
&\leq M_{1-\alpha}^p K_4{}^p \gamma^{1-p\alpha} \big(\Gamma(1 + p(\alpha-1)/(p-1)))\big)^{p-1} \\
&\quad \cdot \int_0^t e^{-\gamma(t-s)} \sup_{-r\leq s<\infty} \mathbb{E}\|x(s-r)) - y(s-r)\|_H^p ds \\
&= M_{1-\alpha}^p K_4{}^p \gamma^{1-p\alpha} \big(\Gamma(1 + p(\alpha-1)/(p-1)))\big)^{p-1} \\
&\quad \cdot \sup_{-r\leq s<\infty} \mathbb{E}\|x(s-r)) - y(s-r)\|_H^p \int_0^t e^{-\gamma(t-s)} ds \\
&\leq M_{1-\alpha}^p K_4{}^p \gamma^{-p\alpha} \big(\Gamma(1 + p(\alpha-1)/(p-1)))\big)^{p-1} \\
&\quad \cdot \sup_{-r\leq s<\infty} \mathbb{E}\|x(s-r) - y(s-r)\|_H^p ds.
\end{aligned}
\tag{3.3.31}
$$

By employing assumption (**H1**), for $t \geq r$, we have that

$$
\begin{aligned}
J_3(t) &= \mathbb{E}\bigg\| \int_0^t S(t-s)\big(f(s, x(s-r)) - f(s, y(s-r))\big) ds \bigg\|_H^p \\
&\leq \mathbb{E}\bigg(\int_0^t \|S(t-s)\| \|f(s, x(s-r)) - f(s, y(s-r))\|_H ds \bigg)^p \\
&\leq \mathbb{E}\bigg(\int_0^t M e^{-\gamma(t-s)} \|f(s, x(s-r)) - f(s, y(s-r))\|_H ds \bigg)^p.
\end{aligned}
\tag{3.3.32}
$$

On the other hand, using the Lipschitz condition (**H2**) and Hölder's inequality, for $t \geq r$, we get

$$
\begin{aligned}
J_3(t) &\leq M^p K_1^p \bigg(\int_0^t e^{-\gamma(t-s)} ds \bigg)^{p-1} \int_0^t e^{-\gamma(t-s)} \mathbb{E}\|x(s-r) - y(s-r)\|_H^p ds \\
&\leq M^p K_1^p \gamma^{1-p} \int_0^t e^{-\gamma(t-s)} \sup_{-r \leq s < \infty} \mathbb{E}\|x(s-r) - y(s-r)\|_H^p ds \\
&= M^p K_1^p \gamma^{1-p} \sup_{-r \leq s < \infty} \mathbb{E}\|x(s-r) - y(s-r)\|_H^p \int_0^t e^{-\gamma(t-s)} ds \\
&\leq M^p K_1^p \gamma^{-p} \sup_{-r \leq s < \infty} \mathbb{E}\|x(s-r) - y(s-r)\|_H^p .
\end{aligned} \tag{3.3.33}
$$

By taking Lemma 2.2.1 and condition (**H2**) into account, for $t > r$, we have that

$$
\begin{aligned}
J_4(t) &= \mathbb{E}\bigg\| \int_0^t S(t-s)\big(g(s,x(s-r)) - g(s,y(s-r))\big) dW(s) \bigg\|_H^p \\
&\leq \mathbb{E} C_p \bigg(\int_0^t \big(\|S(t-s)\big(g(s,x(s-r)) - g(s,y(s-r))\big)\|_H^p\big)^{2/p} ds \bigg)^{p/2} \\
&\leq \mathbb{E} C_p \bigg(\int_0^t \|S(t-s)\|^2 \big(\|g(s,x(s-r)) - g(s,y(s-r))\|_{\mathcal{L}_2^0}^p\big)^{2/p} ds \bigg)^{p/2} .
\end{aligned} \tag{3.3.34}
$$

Under condition (**H1**), for $t > r$, we get that

$$
\begin{aligned}
J_4(t) &\leq M^p C_p \bigg(\int_0^t \big(e^{-2\gamma(t-s)} \mathbb{E}\|g(s,x(s-r)) - g(s,y(s-r))\|_{\mathcal{L}_2^0}^p\big)^{2/p} ds \bigg)^{p/2} \\
&\leq C_p M^p K_2^p \bigg(\int_0^t \big(e^{-2\gamma(t-s)} \mathbb{E}\|x(s-r) - y(s-r)\|_H^p\big)^{2/p} ds \bigg)^{p/2} .
\end{aligned} \tag{3.3.35}
$$

Now employing Hölder inequality, we have that

$$
\begin{aligned}
J_4(t) \leq{}& C_p M^p K_2^p \bigg(\int_0^t e^{-\frac{2(p-1)}{p-2}\gamma(t-s)} ds \bigg)^{p-1} \\
&\cdot \int_0^t e^{-\gamma(t-s)} \sup_{-r \leq s < \infty} \mathbb{E}\|x(s-r) - y(s-r)\|_H^p ds \\
={}& C_p M^p K_2^p \bigg(\frac{p-2}{2\gamma(p-1)} \bigg)^{\frac{p}{2}-1} \sup_{-r \leq s < \infty} \mathbb{E}\|x(s-r) - y(s-r)\|_H^p \int_0^t e^{-\gamma(t-s)} ds \\
\leq{}& C_p M^p K_2^p \bigg(\frac{p-2}{p-1} \bigg)^{p/2-1} (2\gamma)^{1-p/2} \gamma^{-1} \sup_{-r \leq s < \infty} \mathbb{E}\|x(s-r) - y(s-r)\|_H^p .
\end{aligned}
$$

$$(3.3.36)$$

Further, by employing the Lemma 3.2.1, for $t \geq r$, we get

$$\begin{aligned} J_5(t) &= \mathbb{E}\bigg\| \int_0^t \int_{\mathcal{O}} S(t-s)(h(s,x(s-r),z) - h(s,y(s-r),z))\tilde{N}(ds,dz) \bigg\|_H^p \\ &\leq \mathbb{E}C_p \bigg[\int_0^t \int_{\mathcal{O}} \|S(t-s)(h(s,x(s-r),z) - h(s,y(s-r),z))\|^p \nu(dz)ds \\ &\quad + \bigg(\int_0^t \int_{\mathcal{O}} \|S(t-s)(h(s,x(s-r),z) - h(s,y(s-r),z))\|^2 \nu(dz)ds \bigg)^{p/2} \bigg]. \end{aligned} \tag{3.3.37}$$

Thus, under the assumption (**H1**) and using Hölder inequality, for $t \geq r$, we have that

$$\begin{aligned} J_5(t) &\leq \mathbb{E}C_p \bigg[\int_0^t \int_{\mathcal{O}} \|S(t-s)\|^p \|h(s,x(s-r),z) - h(s,y(s-r),z)\|^p \nu(dz)ds \\ &\quad + \bigg(\int_0^t \int_{\mathcal{O}} \|S(t-s)\|^2 \|h(s,x(s-r),z) - h(s,y(s-r),z)\|^2 \nu(dz)ds \bigg)^{p/2} \bigg] \\ &\leq M^p C_p \bigg[\int_0^t \int_{\mathcal{O}} e^{-p\gamma(t-s)} \mathbb{E}\|h(s,x(s-r),z) - h(s,y(s-r),z)\|^p \nu(dz)ds \\ &\quad + \bigg(\int_0^t \int_{\mathcal{O}} e^{-2\gamma(t-s)} \mathbb{E}\|h(s,x(s-r),z) - h(s,y(s-r),z)\|^2 \nu(dz)ds \bigg)^{p/2} \bigg]. \end{aligned} \tag{3.3.38}$$

On the other hand, by using the Lipschitz condition (**H2**), for $t \geq r$, we also have

$$\begin{aligned} J_5(t) &\leq M^p C_p \bigg[K_3^p \gamma^{(1-p)} \int_0^t e^{-\gamma(t-s)} \mathbb{E}\|x(s-r) - y(s-r)\|^p ds \\ &\quad + K_3^p \bigg(\frac{p-2}{2(p-1)\gamma} \bigg)^{(p-2)/2} \int_0^t e^{-\gamma(t-s)} \mathbb{E}\|x(s-r) - y(s-r)\|^p ds \bigg] \\ &\leq M^p C_p \bigg[K_3^p \gamma^{(1-p)} \int_0^t e^{-\gamma(t-s)} \sup_{-r \leq s < \infty} \mathbb{E}\|x(s-r) - y(s-r)\|^p ds \\ &\quad + K_3^p \bigg(\frac{p-2}{2(p-1)\gamma} \bigg)^{(p-2)/2} \int_0^t e^{-\gamma(t-s)} \sup_{-r \leq s < \infty} \mathbb{E}\|x(s-r) - y(s-r)\|^p ds \bigg]. \end{aligned} \tag{3.3.39}$$

By combining the coefficient, for $t \geq r$, inequality (3.3.39) turns to be

$$\begin{aligned} J_5(t) &= M^p C_p K_3^p \Big(\gamma^{(1-p)} + \Big(\frac{p-2}{2(p-1)\gamma} \Big)^{(p-2)/2} \Big) \\ &\quad \cdot \int_0^t e^{-\gamma(t-s)} \sup_{-r \leq s < \infty} \mathbb{E} \| x(s-r) - y(s-r) \|^p ds \\ &= M^p C_p K_3^p \Big(\gamma^{(1-p)} + \Big(\frac{p-2}{2(p-1)\gamma} \Big)^{(p-2)/2} \Big) \\ &\quad \cdot \sup_{-r \leq s < \infty} \mathbb{E} \| x(s-r) - y(s-r) \|^p \int_0^t e^{-\gamma(t-s)} ds \\ &\leq M^p C_p K_3^p \Big(\gamma^{-p} + \Big(\frac{p-2}{2(p-1)\gamma} \Big)^{(p-2)/2} \Big) \sup_{-r \leq s < \infty} \mathbb{E} \| x(s-r) - y(s-r) \|^p. \end{aligned} \tag{3.3.40}$$

Now, from the assumption (**H1**) and (**H4**), for $t \geq r$, we have

$$\begin{aligned} J_6(t) &= \mathbb{E} \Big\| \sum_{0<t_k<t} S(t-t_k) I_k(x(t_k^-) - y(t_k^-)) \Big\|_H^p \\ &\leq \mathbb{E} \Big(\sum_{0<t_k<t} M e^{-\gamma(t-t_k)} q_k \| (x(t_k^-) - y(t_k^-)) \|_H \Big)^p \\ &\leq \mathbb{E} \Big(M\kappa \int_0^t e^{-\gamma(t-s)} \| x(s) - y(s) \|_H ds \Big)^p. \end{aligned} \tag{3.3.41}$$

By using Hölder inequality and inequality (3.3.2) for each $t \geq 0$, we get

$$\begin{aligned} J_6(t) &\leq M^p \kappa^p \Big(\int_0^t e^{-\gamma(t-s)} ds \Big)^{p-1} \int_0^t e^{-\gamma(t-s)} \mathbb{E} \| x(s) - y(s) \|_H^p ds \\ &\leq M^p \kappa^p \Big(\int_0^t e^{-\gamma(t-s)} ds \Big)^{p-1} \int_0^t e^{-\gamma(t-s)} \sup_{-r \leq s < \infty} \mathbb{E} \| x(s) - y(s) \|_H^p ds \\ &= M^p \kappa^p \Big(\int_0^t e^{-\gamma(t-s)} ds \Big)^{p-1} \sup_{-r \leq s < \infty} \mathbb{E} \| x(s) - y(s) \|_H^p \int_0^t e^{-\gamma(t-s)} ds \\ &\leq M^p \kappa^p \gamma^{(-p)} \sup_{-r \leq s < \infty} \mathbb{E} \| x(s) - y(s) \|_H^p ds. \end{aligned} \tag{3.3.42}$$

Now we are in a position to show that π is a contraction mapping.

$$\begin{aligned}
&\|(\pi x) - (\pi y)\|_{\mathcal{S}}^p \\
&\leq 6^{(p-1)}\Bigg[K_4{}^p\|(-A)^{-\alpha}\|^p + M_{1-\alpha}^p K_4{}^p\gamma^{-p\alpha}\big(\Gamma(1+p(\alpha-1)/(p-1))\big)\big)^{p-1} \\
&\quad + M^p K_1^p\gamma^{-p} + M^p C_p K_2^p\left(\frac{p-2}{(p-1)}\right)^{p/2-1}(2\gamma)^{1-p/2}\gamma^{-1} \\
&\quad + M^p C_p K_3^p\left(\gamma^{-p} + \gamma^{p/2}\left(\frac{p-2}{2(p-1)}\right)^{(p-2)/2}\right) + M^p\kappa^p\gamma^{(-p)}\Bigg]\|x-y\|_{\mathcal{S}}^p,
\end{aligned} \tag{3.3.43}$$

where $C_p = (\frac{p(p-1)}{2})^{\frac{p}{2}}$, for $p > 2$, is a constant. Therefore, π is a contraction mapping and hence there exists a unique fixed point $x(\cdot)$ in $\mathcal{S}$ which is the mild solution of the equation (3.2.1) with $x(0) = 0$ and $E\|x(t)\|_H^p \to 0$, as $t \to \infty$. This completes the proof. □

3.4 Illustrative Example

Let us consider the following neutral stochastic impulsive partial functional differential equation with delays and Poisson jumps of the form:

$$\begin{cases}
d\left[x(t) + \alpha_0\left(-\frac{\partial^2}{\partial x^2}\right)^{-\alpha}x(t-r)\right] = \left[\frac{\partial^2}{\partial x^2}x(t) + \alpha_1 x(t-r)\right]dt + \alpha_2 x(t-r)d\beta(t) \\
\qquad\qquad\qquad\qquad + \displaystyle\int_{\mathcal{O}} \alpha_3 z x(t-r)\tilde{N}(dz,dt), \quad t \geq 0,\ t \neq t_k; \\
\Delta x(t_k) = x(t_k^+) - x(t_k^-) = \alpha_4 x(t_k^-), \quad t = t_k, k = 1,2,\ldots,m \\
x_0(s) = \phi(s) \in D([-r,0];H),
\end{cases} \tag{3.4.1}$$

where $\alpha_i > 0,\ i = 0,1,2,3,4,\ \alpha \in (0,1)$, $\beta(t)$ denotes the one-dimensional Brownian motion and $\mathcal{O} = \{z \in \mathbb{R}, 0 < |z| \leq c,\ c > 0\}$

We rewrite equation (3.4.1) into the abstract form of (3.2.1). Let $H =$

$L^2(0,\pi)$. We shall define $A : H \longrightarrow H$ by $A = \frac{\partial^2}{\partial x^2}$ with domain

$$\mathcal{D}(A) = \{x \in H : x, x^{'} \text{ are absolutely continous, } x^{''} \in H \text{ and } x(0) = x(\pi) = 0\}, \tag{3.4.2}$$

then

$$Ax = \sum_{n=1}^{\infty} -n^2 \langle x, e_n \rangle_H e_n, \quad x \in \mathcal{D}(A), \tag{3.4.3}$$

where $e_n(\xi) = \sqrt{\frac{2}{n}} \sin n\xi$, $n = 1, 2, \ldots$ is the set of eigenvector of $-A$. It is well known that A is the infinitesimal generator of an analytic semigroup $S(t), t \geq 0$, in H and

$$S(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} \langle x, e_n \rangle_H e_n, \quad x \in H.$$

Moreover, $\|S(t)\| \leq M e^{-\gamma t}, t \geq 0$.

For $t \geq r$, let

$$\begin{aligned} &u(t, x(t-r)) = \alpha_0 (-A)^{-\alpha} x(t-r), \quad f(t, x(t-r)) = \alpha_1 x(t-r), \\ &g(t, x(t-r)) = \alpha_2 x(t-r), \quad h(t, x(t-r), y) = \alpha_3 y x(t-r), \\ &x(t_k) = \alpha_4 x(t_k^-). \end{aligned} \tag{3.4.4}$$

It is obvious that all the assumptions (**H1**)-(**H6**) are satisfied with

$$M = 1,\ \gamma = 1,\ K_1 = \alpha_1,\ K_2 = \alpha_2,\ K_3 = \alpha_3 \int_{\mathcal{O}} z^2 \nu(dz),$$

$$K_4 = \alpha_0 \|(-A)^{-\alpha}\|,\ \kappa = \alpha_4.$$

Thus, by Theorem 3.3.1, for $t \in [-r, 0]$, if

$$\mathbb{E}\|y(t)\|_H^p \leq M_0(\phi) e^{-\mu t},$$

where $M_0(\phi) > 0, 1 > \mu > 0$ are some constants. If $p > 2$, then

$$\begin{aligned}\Bigg[&\alpha_0^p \|(-A)^{-\alpha}\|^{2p} + M_{1-\alpha}^p \alpha_0^p \|(-A)^{-\alpha}\|^p \big(\Gamma(1 + p(\alpha - 1)/(p-1))\big)\big)^{p-1} \\ &+ \alpha_1^p + \left(\frac{p(p-1)}{2}\right)^p \alpha_2^p \left(\frac{p-2}{(p-1)}\right)^{p/2-1} 2^{1-p/2} \\ &+ \left(\frac{p(p-1)}{2}\right)^p \alpha_3^p \left(1 + \left(\frac{p-2}{2(p-1)}\right)^{(p-2)/2}\right) + \alpha_3^p\Bigg] < \frac{1}{6^{(p-1)}},\end{aligned} \tag{3.4.5}$$

so the mild solution of equation (3.4.1) is exponentially stable in p-th moment.

Chapter 4

Attracting Set of Neutral Impulsive Stochastic Delay Partial Differential Equations with Poisson Jumps

4.1 Introduction

A stochastic dynamical system can be frequently described by some stochastic differential equations. The equation of a given system specifies its behaviour over any given short period of time. To determine the system's behaviour for a long period, we often study the integration of the equations. When we investigate real world systems, we are interested in settling the system into its typical behaviour. The subset of the phase space of the dynamical system corresponding to the typical behaviour is the attractor, also known as the attracting set. In the mathematical field of dynamical systems, an attractor is a set of numerical values toward which a system tends to evolve, for a wide variety of starting conditions of the system. System values that get close enough to the attractor values remain close even if slightly disturbed. The attracting set of a stochastic dynamical system has been extensively studied over the past several years. Although, some results have been given in finite dimensional space valued systems in Li and Xu [69], Liao, Luo and Zeng [72], Xu [121] and among others . However, not many investigations of attracting set of stochastic systems in infinite dimensional spaces. Chen [30] has studied the exponential stability for stochastic partial differential equations with delays by establishing an impulsive-integral inequality in the following form

$$\begin{cases} dx(t) = [Ax(t) + f(t, x(t - r(s)))]dt + g(t, x(t - r(s)))dW(t), & t \geq 0, \quad t \neq t_k, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), & t = t_k, \ k = 1, 2, \dots, \\ x_0(t) = \phi(t), \qquad t \in [-r, 0]. \end{cases} \tag{4.1.1}$$

Recently, Long and his co-authors [85] studied a class of stochastic neural partial differential equations with impulsive. The impulsive effects also have been considered in this chapter. Motivated by the above discussions and based on the result in Chen [30] and Long [85]. We firstly recall an impulsive-integral inequality to deal with impulsive effects. Next, by employing the impulsive-integral inequality, we shall study the existence, uniqueness and stability of mild solutions of neutral stochastic impulsive partial differential equations with delay and jumps.

4.2 Problem Formulation and Assumptions

Let $P(I,\ H) = \{\psi : I \to H$ is continuous for all but at accountable number of points $t \in I$ and at these points $t \in I,\ \psi(t^+)$ and $\psi(t^-)$ exist, $\psi(t^+) = \psi(t)\}$, where $I \subset H$ is a bounded interval, $\psi(t^+)$ and $\psi(t^-)$ denote the right-hand and left-hand limits of the function $\psi(t)$, respectively.

Let $D := D^b_{\mathcal{F}_0}([-r, 0], H)$ denotes the space of all bounded $\mathcal{F}_0$-measurable càdlàg functions from $[-r, 0]$ into H, equipped with the supremum norm

$$\|\phi\|_D = \sup_{t\in[-r,0]} \mathbb{E}\|\phi(t)\|_H < \infty.$$

Let $P := P^b_{\mathcal{F}_0}([-r, 0], H)$ denotes the family of all bounded $\mathcal{F}_0$-measurable, $P([-r, 0], H)$-valued random variables ϕ, equipped with the supremum norm

$$\|\phi\|^p_{L^p} = \sup_{\theta\in[-r,0]} \mathbb{E}\|\phi(\theta)\|^p_H < \infty, \quad \text{for } p > 0.$$

We consider a class of neutral stochastic partial differential delay equation with impulsive and Poisson jumps in following form:

$$\begin{cases} d[x(t) + u(t, x(t-r))] = [Ax(t) + f(t, x(t-r))]dt + g(t, x(t-r))dW(t) \\ \qquad\qquad\qquad\qquad + \displaystyle\int_{\mathcal{O}} h(t, x(t-r), z)\tilde{N}(dt, dz), \quad t \geq 0, \quad t \neq t_k, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad t = t_k, \; k = 1, 2, \ldots, \\ x_0(t) = \phi(t) \in D^b_{\mathcal{F}_0}([-r, 0]; H). \end{cases} \tag{4.2.1}$$

where $A : \mathcal{D}(A) \subset H \to H$ is the infinitesimal generator of an analytic semigroup of linear operator $(S(t))_{t\geq 0}$ on a Hilbert space H. Here $u,\ f : [0, +\infty) \times H \to H,\ g : [0, +\infty) \times H \to \mathcal{L}_2^0(K_0, H),\ h : [0, +\infty) \times H \times U \to H$ are measurable functions and $I_k : H \to H$ is measurable and $0 < t_1 < \ldots < t_k < \ldots,\ \lim_{k\to\infty} t_k = \infty$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, $k = 1, 2, \ldots,$ respectively. The mapping I_k represents the size of the jump at t_k, $k = 1, 2, \ldots$.

Definition 4.2.1. A stochastic process $\{x(t), t \in [0, T]\}, 0 \leq T$, is called a mild solution of the equation (4.2.1) if:

(i) $x(t)$ is adapted to $\mathcal{F}_t$, for each $t \geq 0$, and

$$\mathbb{P}\left\{\omega : \int_0^T \|x(t, \omega)\|^p_H dt < \infty\right\} = 1, \quad T \geq 0;$$

(ii) $x(t) \in H$ has càdlàg path on $t \in [0, T]$ almost surely.

(iii) For each $t \in [0,T], x(t)$ satisfies the following integral equation,

$$
\begin{aligned}
x(t) =& S(t)[\phi(0) + u(0,\phi)] - u(t, x(t-r)) \\
&- \int_0^t AS(t-s)u(s, x(s-r))ds \\
&+ \int_0^t S(t-s)f(s, x(s-r))ds \\
&+ \int_0^t S(t-s)g(s, x(s-r))dW(s) \qquad (4.2.2) \\
&+ \int_0^t \int_{\mathcal{O}} S(t-s)h(t, x(s-r), z)\tilde{N}(ds, dz) \\
&+ \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)), \\
x(0) =& \phi \in D^b_{\mathcal{F}_0}([-r,0], H).
\end{aligned}
$$

We shall introduce two classes of notations: one is the definition of the attracting set and other is the definition of exponentially stable in p-th moment.

Definition 4.2.2. A set $\mathcal{A} \subset H$ is called an attracting set of equation (4.2.1), if for any initial value $\phi \in D^b_{\mathcal{F}_0}([-r,0], H)$, the solution $x(t,\phi)$ converges to the attracting set $\mathcal{A}$ as t is large enough. That is,

$$d(x(t,\phi), \mathcal{A}) \to 0, \quad \text{as} \quad t \to +\infty,$$

where

$$d(x, \mathcal{A}) = \inf_{y \in \mathcal{A}} \mathbb{E}\big(\|x(t,\phi) - y\|_H^p\big).$$

Definition 4.2.3. For any $p \geq 2$, the mild solution of equation (4.2.1) is said to be exponentially stable in p-th moment if there exists a pair of positive constants $\gamma > 0$ and $M_0 > 1$ such that for any solution $x(t,\phi)$ with the initial condition $\phi \in D^b_{\mathcal{F}_0}([-r,0], H)$,

$$\mathbb{E}\big(\|x(t,\phi)\|_H^p\big) \leq M_0(\|\phi\|_D^p)e^{-\gamma t}, \quad t \geq 0. \qquad (4.2.3)$$

In particular, when $p = 2$, the equation (4.2.3) is said to be exponentially stable in mean square.

In order to obtain the main result, for system (4.2.1), we impose the following assumptions:

(**H1**) A is the infinitesimal generator of an analytic semigroup of bounded linear operator $\{S(t)\ t \geq 0\}$ in H such that the following inequality holds

$$\|S(t)\| \leq Me^{-\gamma t}, \quad t \geq 0,$$

for some constants $M > 0$ and $\gamma > 0$.

(**H2**) The coefficients $f,\ g,\ h$ satisfy Lipschitz conditions, i.e. there exist some positive constants $p > 2$, $K_1 > 0$, $K_2 > 0$, $K_3 > 0$ such that for any $x, y \in H$ and $t \geq 0$,

$$\|f(t,x) - f(t,y)\|_H \leq K_1 \|x - y\|_H, \quad \|f(t,0)\| \leq b_f,$$

$$\|g(t,x) - g(t,y)\|_{\mathcal{L}_2^0} \leq K_2 \|x - y\|_H, \quad \|g(t,0)\|_{\mathcal{L}_2^0} \leq b_g,$$

$$\int_{\mathcal{O}} \|h(t,x,z) - h(t,y,z)\|_H^p \nu(dz) \leq K_3^p \|x - y\|_H^p, \quad \|h(t,0,z)\| \leq b_h,$$

where $b_f > 0$, $b_g > 0$ and $b_h > 0$ are some constants.

(**H3**) There exist a positive series $\{q_k\}$ for each $k \in \{1, 2, 3, \ldots\}$, such that $\sum_{k=1}^{\infty} q_k = \kappa < \infty$ and

$$\|I_k(x) - I_k(y)\| \leq q_k \|x - y\|, \quad x, y \in H,$$

and $I_k(0) = 0$, $k = 1, 2, \ldots$.

(**H4**) There exist some constants $\alpha \in (0, 1]$ and $K_4 > 0$ such that for any $x \in H$, $t \geq 0$, $u(t,x) \in \mathcal{D}((-A)^\alpha)$ and

$$\|(-A)^\alpha u(t,x) - (-A)^\alpha u(t,y)\|_H \leq K_4 \|x - y\|_H, \quad x, y \in H,$$

with $u(t,0)=0$ for $t \geq 0$.

4.3 Impulsive-integral Inequality

In this section, in order to get an attracting set and exponential stability in p-th moment for mild solutions of system (4.2.1). We shall recall the following impulsive-integral inequality [85] to overcome the difficulty when the impulsive effects are presented.

Lemma 4.3.1. *[85] Suppose $y : [-r, +\infty) \to [0, +\infty)$ is a measurable function satisfying the following impulsive-integral inequality:*

$$y(t) \leq \begin{cases} \|\psi\|_r e^{-\mu(t-t_0)} + a_1\|y_t\|_r + a_2 \displaystyle\int_{t_0}^{t} e^{-\mu(t-s)}\|y_s\|_r ds \\ \qquad + \displaystyle\sum_{t_k \in (0,t)} c_k e^{-\mu(t-t_k)} y(t_k^-) + J, \quad t \geq t_0, \\ \psi(t), \quad t \in [t_0 - r, t_0], \end{cases} \tag{4.3.1}$$

where $\psi(t) \in P([t_0-r,t_0], \mathbb{R}^+)$ and some nonnegative constants $a_1,\ a_2,\ c_k,\ (k = 1,2,\ldots),\ \mu$ and J satisfying,

$$\sigma \triangleq a_1 + \frac{a_2}{\mu} + \sum_{k=1}^{+\infty} c_k < 1. \tag{4.3.2}$$

Then there exist some positive constants $\lambda \in (0,\mu)$ and $K \leq N$ such that

$$y(t) \leq N e^{-\lambda(t-t_0)} + \frac{J}{1-\sigma}, \quad t \geq t_0, \tag{4.3.3}$$

where λ and N satisfy that

$$\sigma_\lambda \triangleq a_1 e^{\lambda r} + \frac{a_2 e^{\lambda r}}{\mu - \lambda} + \sum_{k=1}^{+\infty} c_k < 1 \quad \text{and} \quad N \geq \frac{K}{1-\sigma_\lambda}, \tag{4.3.4}$$

or $a_2 \neq 0$ and

$$\sigma_\lambda \leq 1 \quad \text{and} \quad N \geq \frac{(\mu-\lambda)\left[K - \frac{a_2 J}{\mu(1-\sigma)}\right]}{a_2 e^{\lambda r}}. \tag{4.3.5}$$

Proof. From (4.3.2) to (4.3.5), for any $\hat{N} > N$, we shall prove that,

$$y(t) \leq \hat{N}e^{-\lambda(t-t_0)} + \frac{J}{1-\sigma} \equiv v(t), \quad t \geq t_0. \tag{4.3.6}$$

Now we prove the above inequality (4.3.6) by contradiction. If the inequality (4.3.16) is not true, from our condition $N \geq K \geq \|\psi\|_r$, there must exist a $\hat{t} > t_0$ such that

$$y(\hat{t}) \leq v(\hat{t}), \quad y(t) \geq v(t), \quad \forall t \in [t_0 - r, \hat{t}). \tag{4.3.7}$$

From inequality (4.3.1), we obtain the following inequality,

(1) For $y(\hat{t}) < \|y_{\hat{t}}\|_r := \sup_{-r\leq t\leq 0} \|y(\hat{t}+t)\|$, by using (4.3.2), (4.3.1), (4.3.3), (4.3.4) and (4.3.5), we have that,

$$\begin{aligned} y(\hat{t}) \leq\ & Ke^{-\mu(t-t_0)} + a_1\left[\hat{N}e^{-\lambda(\hat{t}-t_0)}e^{\lambda r} + \frac{J}{1-\sigma}\right] \\ & + a_2\int_{t_0}^{\hat{t}} e^{-\mu(t-s)}\left[\hat{N}e^{-\lambda(s-t_0)}e^{\lambda r} + \frac{J}{1-\sigma}\right]ds \\ & + \sum_{t\in(t_0,\hat{t})} c_k e^{-\mu(\hat{t}-t_k)}\left[\hat{N}e^{-\lambda(t_k-t_0)} + \frac{J}{1-\sigma}\right] + J \\ \leq & \left(K - \frac{a_2 J}{\mu(1-\sigma)} - \frac{a_2 e^{\lambda r}}{\mu-\lambda}\hat{N}\right)e^{-\lambda(\hat{t}-t_0)} \\ & + \left(a_1 e^{\lambda r} + \frac{a_2 e^{\lambda r}}{\mu-\lambda} + \sum_{k=1}^{+\infty} c_k\right)\hat{N}e^{-\lambda(\hat{t}-t_0)} \\ & + \left(a_1 + \frac{a_2}{\mu} + \sum_{k=1}^{+\infty} c_k\right)\frac{J}{1-\sigma} + J. \end{aligned} \tag{4.3.8}$$

Under the condition (4.3.4), by substituting σ_λ and σ into the inequality (4.3.8), we have that,

$$\begin{aligned} y(\hat{t}) < & \left(K - \frac{a_2 J}{\mu(1-\sigma)} - \frac{a_2 e^{\lambda r}}{\mu-\lambda}\hat{N}\right)e^{-\lambda(\hat{t}-t_0)} + \sigma_\lambda \hat{N}e^{-\lambda(\hat{t}-t_0)} + \frac{\sigma J}{1-\sigma} + J \\ & \leq Ke^{-\lambda(\hat{t}-t_0)} + \sigma_\lambda \hat{N}e^{-\lambda(\hat{t}-t_0)} + \frac{\sigma J}{1-\sigma} + \frac{J - J\sigma}{1-\sigma} \\ & \leq Ke^{-\lambda(\hat{t}-t_0)} + \sigma_\lambda \hat{N}e^{-\lambda(\hat{t}-t_0)} + \frac{J}{1-\sigma}. \end{aligned} \tag{4.3.9}$$

Supposing that the condition (4.3.5) is satisfied. Similarly, we obtain that

$$\begin{aligned} y(\hat{t}) &< \left(\frac{a_2 e^{\lambda r}}{\mu-\lambda}N - \frac{a_2 e^{\lambda r}}{\mu-\lambda}\hat{N}\right)e^{-\lambda(\hat{t}-t_0)} + \sigma_\lambda \hat{N} e^{-\lambda(\hat{t}-t_0)} + \frac{\sigma J}{1-\sigma} + J \\ &\leq \sigma_\lambda \hat{N} e^{-\lambda(\hat{t}-t_0)} + \frac{J}{1-\sigma}. \end{aligned} \tag{4.3.10}$$

Thus, under the conditions (4.3.4) and (4.3.5), one can have,

$$y(\hat{t}) \leq \hat{N} e^{-\lambda(\hat{t}-t_0)} + \frac{J}{1-\sigma} = v(\hat{t}), \tag{4.3.11}$$

which contradicts with assumption (4.3.7).

(2) For $y(\hat{t}) = \|y_{\hat{t}}\|_r$, by using (4.3.2), (4.3.1), (4.3.3), (4.3.4) and (4.3.5), we have that,

$$\begin{aligned} y(\hat{t}) \leq\ & Ke^{-\mu(t-t_0)} + a_1 y(\hat{t}) + a_2 \int_{t_0}^{\hat{t}} e^{-\mu(t-s)}\left[\hat{N} e^{-\lambda(s-t_0)} e^{\lambda r} + \frac{J}{1-\sigma}\right] ds \\ & + \sum_{t_k \in (t_0,\hat{t})} c_k e^{-\mu(\hat{t}-t_k)}\left[\hat{N} e^{-\lambda(t_k-t_0)} + \frac{J}{1-\sigma}\right] + J \\ \leq\ & a_1 y(\hat{t}) + \left(K - \frac{a_2 J}{\mu(1-\sigma)} - \frac{a_2 e^{\lambda r}}{\mu-\lambda}\hat{N}\right)e^{-\lambda(\hat{t}-t_0)} \\ & + \left(\frac{a_2 e^{\lambda r}}{\mu-\lambda} + \sum_{k=1}^{+\infty} c_k\right)\hat{N} e^{-\lambda(\hat{t}-t_0)} + \left(\frac{a_2}{\mu} + \sum_{k=1}^{+\infty} c_k\right)\frac{J}{1-\sigma} + J \end{aligned} \tag{4.3.12}$$

Under condition (4.3.4), by substituting σ_λ and σ into the inequality (4.3.8), we have that,

$$\begin{aligned} y(\hat{t}) &< a_1 y(\hat{t}) + Ke^{-\lambda(\hat{t}-t_0)} + (\sigma_\lambda - a_1 e^{\lambda r})\hat{N} e^{-\lambda(\hat{t}-t_0)} + (\sigma - a_1)\frac{J}{1-\sigma} + J \\ &\leq a_1 y(\hat{t}) + \big(K + (\sigma_\lambda - a_1 e^{\lambda r})\big)\hat{N} e^{-\lambda(\hat{t}-t_0)} + (\sigma - a_1)\frac{J}{1-\sigma} + J. \end{aligned} \tag{4.3.13}$$

Next, supposing that condition (4.3.5) is satisfied. Similarly, we have

$$y(\hat{t}) < a_1 y(\hat{t}) + \big(1 - a_1 e^{\lambda r}\big)\hat{N} e^{-\lambda(\hat{t}-t_0)} + (\sigma - a_1)\frac{J}{1-\sigma} + J. \tag{4.3.14}$$

Since under condition (4.3.4), we have that $K < (1-\sigma_\lambda)\hat{N}$. And for any $\lambda > 0$, we have that $1 - a_1 \geq 1 - a_1 e^{\lambda r}$, then inequality (4.3.12) implies that

$$y(\hat{t}) \leq \hat{N} e^{-\lambda(\hat{t}-t_0)} + \frac{J}{1-\sigma} = v(\hat{t}), \quad \hat{t} \geq t_0, \tag{4.3.15}$$

which contradicts with inequality (4.3.7). Therefore, the following inequality

$$y(t) \leq \hat{N} e^{-\lambda(t-t_0)} + \frac{J}{1-\sigma} \equiv v(\hat{t}), \quad t \geq t_0, \tag{4.3.16}$$

is true. Letting $\hat{N} \to N$, we complete the proof to get inequality (4.3.3). $\square$

4.4 Attracting Set of the System

Theorem 4.4.1. *Assume that (**H1**) – (**H4**) hold and the following inequality*

$$\begin{aligned} \hat{\sigma} \triangleq & 7^{p-1} K_4^p \|(-A)^{-\alpha}\|^p + 7^{p-1} M^p \kappa^p \gamma^{(-p)} + 7^{p-1} \gamma^{-1} (C_1 + C_3 + C_5) \\ & + 7^{p-1} M_{1-\alpha}^p K_4{}^p \gamma^{-p\alpha} \big(\Gamma(1 + p(\alpha-1)/(p-1))\big)^{p-1} < 1, \end{aligned} \tag{4.4.1}$$

holds for $\alpha \in (0,1]$ and $p > 2$, then the attracting set of system (4.2.1) is

$$S = \left\{ x \in H : \ \|x\| \leq \sqrt[p]{(1-\hat{\sigma})^{-1} \hat{J}} \right\}, \tag{4.4.2}$$

where $\hat{J} \triangleq 7^{p-1}(C_2 + C_4 + C_6)$, $C_p = (\frac{p(p-1)}{2})^{\frac{p}{2}}$ and when $b_f > 0$, $b_g > 0$, $b_h > 0$,

$$\begin{aligned} & C_1 = 2^{p-1} M^p K_1^p \gamma^{1-p}, \ C_4 = 2^{p-1} M^p C_p \left(\frac{p-2}{p-1}\right)^{p/2-1} (2\gamma)^{1-p/2} \gamma^{-1} b_g^p, \\ & C_2 = 2^{p-1} M^p \gamma^{-p} b_f^p, \ C_3 = 2^{p-1} M^p K_2^p C_p \left(\frac{p-2}{p-1}\right)^{p/2-1} (2\gamma)^{1-p/2}, \\ & C_5 = 2^{p-1} M^p K_3^p C_p \left(\gamma^{(1-p)} + \left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}\right), \\ & C_6 = 2^{p-1} M^p C_p \left(\gamma^{(1-p)} + \left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}\right) \gamma^{-1} b_h^p. \end{aligned} \tag{4.4.3}$$

Proof. From the definition (4.2.1), we have the following inequality,

$$\begin{aligned}\mathbb{E}\|\pi(x)(t)\|_H^p \leq & 7^{p-1}\mathbb{E}\|S(t)[\phi(0)+u(0,\phi)]\|_H^p + 7^{p-1}\mathbb{E}\|u(t,x(t-r))\|_H^p \\ & + 7^{p-1}\mathbb{E}\left\|\int_0^t AS(t-s)u(s,x(s-r))ds\right\|_H^p \\ & + 7^{p-1}\mathbb{E}\left\|\int_0^t S(t-s)f(s,x(s-r))ds\right\|_H^p \\ & + 7^{p-1}\mathbb{E}\left\|\int_0^t S(t-s)g(s,x(s-r))dW(s)\right\|_H^p \\ & + 7^{p-1}\mathbb{E}\left\|\int_0^t\int_{\mathcal{O}} S(t-s)h(s,x(s-r),z)\tilde{N}(ds,dz)\right\|_H^p \\ & + 7^{p-1}\mathbb{E}\left\|\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-))\right\|_H^p := 7^{p-1}\sum_{i=1}^{7} I_i(t).\end{aligned} \tag{4.4.4}$$

By assumption (**H3**) it follows that for each $t \geq r$,

$$\begin{aligned} I_1(t) &= \mathbb{E}\|S(t)[\phi(0)+u(0,-r)]\|_H^p \\ &\leq \|S(t)\|^p\mathbb{E}\|\phi(0)+u(0,-r)\|_H^p \\ &\leq M^p e^{-p\gamma t}\mathbb{E}\|\phi(0)+u(0,-r)\|_H^p. \end{aligned} \tag{4.4.5}$$

Moreover, under assumption (**H3**), for each $t \geq r$, we have

$$\begin{aligned} I_2(t) &= \mathbb{E}\|u(t,x(t-r))\|_H^p \\ &= \mathbb{E}\|(-A)^{-\alpha}(-A)^{\alpha}u(t,x(t-r))\|_H^p \\ &\leq \|(-A)^{-\alpha}\|^p\mathbb{E}\|(-A)^{\alpha}u(t,x(t-r))\|_H^p \\ &\leq K_4^p\|(-A)^{-\alpha}\|^p \sup_{-r\leq s\leq\infty}\mathbb{E}\|x(t-r)\|_H^p. \end{aligned} \tag{4.4.6}$$

Under assumption (**H3**), and $\alpha \in (0,1)$, we have that for each $t \geq r$,

$$\begin{aligned} I_3(t) &= \mathbb{E}\left\|\int_0^t (-A)S(t-s)u(s,x(s-r))ds\right\|_H^p \\ &\leq \mathbb{E}\left\|\int_0^t (-A)^{1-\alpha}(-A)^{\alpha}S(t-s)u(s,x(s-r))ds\right\|_H^p \\ &\leq \mathbb{E}\left(\int_0^t \|(-A)^{1-\alpha}(-A)^{\alpha}S(t-s)u(s,x(s-r))\|_H ds\right)^p \\ &\leq \mathbb{E}\left(\int_0^t \|(-A)^{(1-\alpha)}S(t-s)\|\|(-A)^{\alpha}u(s,x(s,-r))\|_H ds\right)^p. \end{aligned} \tag{4.4.7}$$

On the other hand, by using Lemma (2.1.4), we get that, for each $t \geq r$

$$\begin{aligned} I_3(t) &\leq \mathbb{E}\bigg(\int_0^t \|M_{1-\alpha}(t-s)^{-(1-\alpha)} e^{-\gamma(t-s)}\| \|(-A)^\alpha u(s, x(s-r))\|_H ds \bigg)^p \\ &\leq M_{1-\alpha}^p \mathbb{E}\bigg(\int_0^t (t-s)^{\alpha-1} e^{-\gamma(t-s)} \|(-A)^\alpha u(s, x(s-r))\|_H ds \bigg)^p. \end{aligned} \tag{4.4.8}$$

Moreover, by employing Hölder inequality, for each $t \geq r$, inequality (4.4.8) turns to be

$$\begin{aligned} I_3(t) &\leq M_{1-\alpha}^p \bigg(\int_0^t (t-s)^{p/(p-1)(\alpha-1)} e^{-\gamma(t-s)} ds \bigg)^{p-1} K_4^p \int_0^t e^{-\gamma(t-s)} \mathbb{E}\|x(s-r)\|_H^p ds \\ &\leq M_{1-\alpha}^p K_4^p \gamma^{1-p\alpha} \big(\Gamma(1 + p(\alpha-1)/(p-1)) \big)^{p-1} \int_0^t e^{-\gamma(t-s)} \sup_{-r \leq s \leq \infty} \mathbb{E}\|x(s-r)\|_H^p ds. \end{aligned} \tag{4.4.9}$$

Now employing assumption (**H2**) and Hölder inequality, we similarly have that for each $t \geq r$,

$$\begin{aligned} I_4(t) &= \mathbb{E}\bigg\| \int_0^t S(t-s) f(s, x(s-r)) ds \bigg\|_H^p \\ &\leq \mathbb{E}\bigg(\int_0^t \|S(t-s)\| \cdot \|f(s, x(s-r))\| ds \bigg)^p \\ &\leq \mathbb{E}\bigg(\int_0^t M e^{-\gamma(t-s)} (K_1 \|x(s-r)\|_H + \|f(s,0)\|) ds \bigg)^p. \end{aligned} \tag{4.4.10}$$

Since $\|f(t,0)\| \leq b_f$ by the condition (**H2**), for $b_f > 0$ and $t \geq r$, inequality (4.4.10) implies that

$$\begin{aligned} I_4(t) &\leq 2^{p-1} M^p K_1^p \gamma^{1-p} \int_0^t e^{-\gamma(t-s)} \mathbb{E}\|x(s-r)\|_H^p ds + 2^{p-1} M^p \gamma^{-p} b_f^p \\ &\leq C_1 \int_0^t e^{-\gamma(t-s)} \sup_{-r \leq s \leq \infty} \mathbb{E}\|x(s-r)\|_H^p ds + C_2, \end{aligned} \tag{4.4.11}$$

where

$$C_1 = 2^{(p-1)} M^p K_1^p \gamma^{1-p}, \quad C_2 = 2^{p-1} M^p \gamma^{-p} b_f^p. \tag{4.4.12}$$

On the other hand, by taking into account (**H1**), Lemma (2.2.1), for $t \geq r$, we have that

$$
\begin{aligned}
I_5(t) &= \mathbb{E}\bigg\| \int_0^t S(t-s)g(s,x(s-r))dW(s)\bigg\|_H^p \\
&\le C_p\bigg(\mathbb{E}\int_0^t \big(\|S(t-s)g(s,x(s-r))\|_H^p\big)^{2/p} ds\bigg)^{p/2} \\
&\le C_p\bigg(\mathbb{E}\int_0^t \|S(t-s)\|^2\big(\|g(s,x(s-r))\|_{\mathcal{L}_2^0}^p\big)^{2/p} ds\bigg)^{p/2}.
\end{aligned} \tag{4.4.13}
$$

Moreover, by the definition of stability and Hölder inequality, for $t \ge r$, we obtain

$$
\begin{aligned}
I_5(t) &\le C_p\bigg(\int_0^t \big(M^p e^{-p\gamma(t-s)}\mathbb{E}\|g(s,x(s-r))\|_{\mathcal{L}_2^0}^p\big)^{2/p} ds\bigg)^{p/2} \\
&\le C_p M^p\bigg(\int_0^t \big(e^{-p\gamma(t-s)}\mathbb{E}\|g(s,x(s-r))\|_{\mathcal{L}_2^0}^p\big)^{2/p} ds\bigg)^{p/2} \\
&\le C_p M^p\bigg(\int_0^t e^{-\frac{2(p-1)}{p-2}\gamma(t-s)} ds\bigg)^{p-1} \int_0^t e^{-\gamma(t-s)}\mathbb{E}\|g(s,x(s-r))\|_{\mathcal{L}_2^0}^p ds.
\end{aligned} \tag{4.4.14}
$$

Since $\|g(t,0)\| \le b_g$ by the condition (**H2**), for $b_g > 0$ and $t \ge r$, then inequality (4.4.14) implies that

$$
\begin{aligned}
I_5(t) &\le 2^{p-1}M^p K_2^p C_p\bigg(\frac{p-2}{2\gamma(p-1)}\bigg)^{p/2-1} \int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-r)\|_H^p ds \\
&\quad + 2^{p-1}M^p C_p\bigg(\frac{p-2}{p-1}\bigg)^{p/2-1}(2\gamma)^{1-p/2}\gamma^{-1}b_g^p \\
&\le C_3\int_0^t e^{-\gamma(t-s)} \sup_{-r\le s\le\infty}\mathbb{E}\|x(s-r)\|_H^p ds + C_4,
\end{aligned} \tag{4.4.15}
$$

where

$$
\begin{aligned}
C_3 &= 2^{p-1}M^p K_2^p C_p\bigg(\frac{p-2}{p-1}\bigg)^{p/2-1}(2\gamma)^{1-p/2}, \\
C_4 &= 2^{p-1}M^p C_p\bigg(\frac{p-2}{p-1}\bigg)^{p/2-1}(2\gamma)^{1-p/2}\gamma^{-1}b_g^p.
\end{aligned} \tag{4.4.16}
$$

By employing Lemma (3.2.1), for $t \geq r$, we obtain

$$
\begin{aligned}
I_6(t) &= \mathbb{E}\bigg\| \int_0^t \int_{\mathcal{O}} S(t-s)h(s,x(s-r),z)\tilde{N}(ds,dz)\bigg\|_H^p \\
&\leq \mathbb{E}C_p\bigg[\int_0^t \int_{\mathcal{O}} \|S(t-s)h(s,x(s-r),z)\|^p \nu(dz)ds \\
&\quad + \bigg(\int_0^t \int_{\mathcal{O}} \|S(t-s)h(s,x(s-r),z)\|^2 \nu(dz)ds\bigg)^{p/2}\bigg].
\end{aligned}
\tag{4.4.17}
$$

Further, by Hölder inequality, for $t \geq r$, we have that

$$
\begin{aligned}
I_6(t) &\leq \mathbb{E}C_p\bigg[\int_0^t \int_{\mathcal{O}} \|S(t-s)\|^p \|h(s,x(s-r),z)\|^p \nu(dz)ds \\
&\quad + \bigg(\int_0^t \int_{\mathcal{O}} \|S(t-s)\|^2 \|h(s,x(s-r),z)\|^2 \nu(dz)ds\bigg)^{p/2}\bigg] \\
&\leq C_p\bigg[\int_0^t \int_{\mathcal{O}} M^p e^{-p\gamma(t-s)} \mathbb{E}\|h(s,x(s-r),z)\|^p \nu(dz)ds \\
&\quad + \bigg(\int_0^t \int_{\mathcal{O}} M^p e^{-2\gamma(t-s)} \mathbb{E}\|h(s,x(s-r),z)\|^2 \nu(dz)ds\bigg)^{p/2}\bigg].
\end{aligned}
\tag{4.4.18}
$$

By using assumption (**H2**) and $\|h(t,0,z)\| \leq b_h$, for $b_h > 0$ and $t \geq r$, we have

$$
\begin{aligned}
I_6(t) &\leq 2^{p-1} M^p K_3^p C_p \bigg(\gamma^{(1-p)} + \bigg(\frac{p-2}{2(p-1)\gamma}\bigg)^{\frac{p-2}{2}}\bigg) \int_0^t e^{-\gamma(t-s)} \mathbb{E}\|x(s-r)\|^p ds \\
&\quad + 2^{p-1} M^p C_p \bigg(\gamma^{(1-p)} + \bigg(\frac{p-2}{2(p-1)\gamma}\bigg)^{\frac{p-2}{2}}\bigg) \gamma^{-1} b_h^p \\
&\leq C_5 \int_0^t e^{-\gamma(t-s)} \sup_{-r \leq s \leq \infty} \mathbb{E}\|x(s-r)\|_H^p ds + C_6,
\end{aligned}
\tag{4.4.19}
$$

where

$$
\begin{aligned}
C_5 &= 2^{(p-1)} M^p K_3^p C_p \bigg(\gamma^{(1-p)} + \bigg(\frac{p-2}{2(p-1)\gamma}\bigg)^{(p-2)/2}\bigg), \\
C_6 &= 2^{p-1} M^p C_p \bigg(\gamma^{(1-p)} + \bigg(\frac{p-2}{2(p-1)\gamma}\bigg)^{(p-2)/2}\bigg) \gamma^{-1} b_h^p.
\end{aligned}
\tag{4.4.20}
$$

Now, we estimate the impulsive term, for each $t \geq r$, we obtain

$$
\begin{aligned}
I_7(t) &= \mathbb{E}\bigg\| \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-))\bigg\|_H^p \\
&\leq \mathbb{E}\bigg(\sum_{0<t_k<t} e^{-\gamma(t-t_k)} q_k \|x(t_k^-)\|_H \bigg)^p.
\end{aligned}
\tag{4.4.21}
$$

Thus, by using condition(**H2**) and Hölder inequality, for $t \geq r$, we get

$$
\begin{aligned}
I_7(t) &\leq \mathbb{E}\bigg(\kappa \int_0^t e^{-\gamma(t-s)} \|x(s)\|_H ds \bigg)^p \\
&\leq \kappa^p \bigg(\int_0^t e^{-\gamma(t-s)} ds \bigg)^{p-1} \int_0^t e^{-\gamma(t-s)} \mathbb{E}\|x(s)\|_H^p ds \\
&\leq \kappa^p \gamma^{(1-p)} \int_0^t e^{-\gamma(t-s)} \sup_{-r\leq s\leq\infty} \mathbb{E}\|x(s)\|_H^p ds.
\end{aligned}
\tag{4.4.22}
$$

Finally, by substituting (4.4.5)-(4.4.22) into (4.4.4), we have that,

$$
\begin{aligned}
\mathbb{E}\|x(t)\|^p \leq\ & 7^{p-1}\tilde{M}\|\phi\|^p e^{-\gamma t} + 7^{p-1}K_4^p\|(-A)^{-\alpha}\|^p \sup_{-r\leq s\leq\infty} \mathbb{E}\|x(t-r)\|_H^p \\
&+ 7^{p-1}\kappa^p\gamma^{(1-p)} \int_0^t e^{-\gamma(t-s)} \sup_{-r\leq s\leq\infty} \mathbb{E}\|x(s)\|_H^p ds \\
&+ 7^{p-1}M_{1-\alpha}^p {K_4}^p \gamma^{1-p\alpha} \bigg(\Gamma\bigg(1+\frac{p(\alpha-1)}{p-1}\bigg)\bigg)^{p-1} \\
&\cdot \int_0^t e^{-\gamma(t-s)} \sup_{-r\leq s\leq\infty} \mathbb{E}\|x(s-r)\|_H^p ds \\
&+ 7^{p-1}(C_1+C_3+C_5) \int_0^t e^{-\gamma(t-s)} \sup_{-r\leq s\leq\infty} \mathbb{E}\|x(s-r)\|_H^p ds + \hat{J},
\end{aligned}
\tag{4.4.23}
$$

where $\hat{J} = 7^{p-1}(C_2+C_4+C_6)$. Let

$$
\begin{aligned}
& a_0 \triangleq 7^{p-1}\tilde{M},\ \ \hat{a}_1 \triangleq 7^{p-1}K_4^p\|(-A)^{-\alpha}\|^p,\ \ \hat{c}_k \triangleq 7^{p-1}\kappa^p\gamma^{(1-p)}, \\
& \hat{a}_2 \triangleq 7^{p-1}M_{1-\alpha}^p {K_4}^p \gamma^{1-p\alpha}\big(\Gamma(1+p(\alpha-1)/(p-1))\big)^{p-1} \\
& \qquad + 7^{p-1}(C_1+C_3+C_5) + \hat{J}.
\end{aligned}
\tag{4.4.24}
$$

From (4.4.1), we know

$$
\hat{\sigma} \triangleq \hat{a}_1 + \frac{\hat{a}_2}{\gamma} + \sum_{k=1}^{+\infty} \hat{c}_k < 1. \tag{4.4.25}
$$

We can choose such $\phi \in D^b_{\mathcal{F}_0}([-r,0],H)$, so that there exist $\hat{K} \geq 0,\ \ \hat{N} > 0,\ \ \lambda \in (0,\mu)$ such that

$$a_0\|\phi\|^p_{L^p} \leq \hat{K}, \quad \hat{\sigma_\lambda} \triangleq \hat{a}_1 e^{\lambda r} + \frac{\hat{a}_2 e^{\lambda r}}{\mu - \lambda} + \sum_{k=1}^{+\infty} \hat{c}_k \leq 1 \tag{4.4.26}$$

and

$$\frac{(\mu-\lambda)\left[\hat{K} - \frac{\hat{a}_2 \hat{J}}{\mu(1-\hat{\sigma})}\right]}{\hat{a}_2 e^{\lambda r}} \leq \hat{N}. \tag{4.4.27}$$

We are combining above results with Lemma (4.3.1), from inequality (4.3.3), there exist constant $\lambda \in (0,\mu)$ such that for any $t \geq t_0$,

$$\mathbb{E}\|x(t)\|^p \leq \hat{N} e^{-\lambda(t-t_0)} + \frac{\hat{K}}{1-\hat{\sigma}},$$

as $t \to \infty$, $\mathbb{E}\|x(t)\|^p \leq \frac{\hat{K}}{1-\hat{\sigma}}$, then we have that the set

$$\mathcal{A} = \left\{ x(t) \in H,\ \mathbb{E}\|x(t)\|^p \leq (1-\hat{\sigma})^{-1}\hat{J} \right\}$$

is an attracting set of system (4.2.1). Now, we know the conclusion of Theorem (4.4.1) is true. □

4.5 Illustrative Example

Let us consider a class of neutral stochastic impulsive partial functional differential equation with delays and Poisson jumps in the following form:

$$
\begin{cases}
d\Big[x(t)+\alpha_0\Big(-\frac{\partial^2}{\partial x^2}\Big)^{-\alpha}x(t-r))\Big] = \Big[\frac{\partial^2}{\partial x^2}x(t)+\alpha_1 x(t-r))+\beta_1\Big]dt \\
\qquad\qquad\qquad\qquad\qquad\qquad +(\alpha_2 x(t-r)+\beta_2)d\beta(t) \\
\qquad\qquad\qquad\qquad\qquad\qquad +\int_{\mathcal{O}}(\alpha_3 z x(t-r)+\beta_3)\tilde{N}(dz,dt), \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t\geq 0, t\neq t_k, \\
\Delta x(t_k)=x(t_k^+)-x(t_k^-)=\alpha_4 x(t_k^-), \quad t=t_k, k=1,2,\ldots,m \\
x_0(t)=\phi(t)\in D^b_{\mathcal{F}_0}([-r,0],L^2[0,\pi]),\ x(t,0)=x(t,\pi)=0, \quad -r\leq t\leq 0,
\end{cases}
\tag{4.5.1}
$$

where $\alpha_i>0,\ i=0,1,2,3,4,\ \beta_j\geq 0,\ j=1,2,3,$ are constants, $\beta(t)$ denotes the one-dimensional Brownian motion and $\mathcal{O}=\{z\in\mathbb{R}:0<|z|\leq c, c>0\}$.

We rewrite (4.5.1) into the abstract form of (4.2.1). Let $H=L^2(0,\pi)$. Define bounded linear operator $A:\mathcal{D}(A)\subset H\to H$ by $Ax=\frac{\partial^2}{\partial x^2}\quad\forall x\in\mathcal{D}(A)$. Then we get

$$
Ax=\sum_{n=1}^{+\infty}-n^2\langle x,e_n\rangle_H e_n, \quad x\in\mathcal{D}(A),
$$

where $e_n(\pi)=\sqrt{\frac{2}{\pi}}\sin nz, n=1,2,\ldots$ is the set of eigenvector of $-A$. It is well known that A is the infinitesimal generator of an analytic semigroup $S(t), t\geq 0$, in H and

$$
S(t)x=\sum_{n=1}^{\infty}e^{-n^2t}\langle x,e_n\rangle_H e_n, \quad x\in H.
$$

Moreover, $\|S(t)\|\leq e^{-t},\ t\geq 0$, the unbounded linear operator $(-A)^{\frac{3}{4}}$ is given by

$$
(-A)^{\frac{3}{4}}x=\sum_{n=1}^{+\infty}n^{\frac{3}{2}}\langle x,e_n\rangle_H e_n,
$$

with domain

$$
\mathcal{D}((-A)^{\frac{3}{4}})=\big\{x\in H,\sum_{n=1}^{+\infty}n^{\frac{3}{2}}\langle x,e_n\rangle_H e_n\in H\big\}.
$$

Let

$$u(t,x(t-r)) = \alpha_0(-A)^{-\alpha}x(t-r), \ \ f(t,x(t-r)) = \alpha_1 x(t-r) + \beta_1,$$
$$g(t,x(t-r)) = \alpha_2 x(t-r) + \beta_2, \ \ h(t,x(t-r),y) = \alpha_3 y x(t-r) + \beta_3. \tag{4.5.2}$$

It is obvious that all the assumptions are satisfied with

$$M = \gamma = 1, \ K_4 = \alpha_0\|(-A)^{-\alpha}\|, \ K_1 = \alpha_1, \ K_2 = \alpha_2, \ K_3 = \alpha_3\int_{\mathcal{O}} z^2\nu(dz),$$

$$\kappa = \alpha_4, \ b_f = \beta_1, \ \ b_g = \beta_2, \ \ b_h = \beta_3.$$

Thus, let $p > 2$,

$$\begin{aligned}\hat{\sigma} \triangleq & 7^{p-1}\alpha_0^p\|(-A)^{-\alpha}\|^{2p} + 7^{p-1}\alpha_4^p + 14^{p-1}\alpha_1^p \\ & + 14^{p-1}\alpha_2^p\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\frac{p-2}{2(p-1)}\right)^{\frac{p}{2}-1} \\ & + 14^{p-1}\alpha_3^p\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(1+\left(\frac{p-2}{2(p-1)}\right)^{\frac{p}{2}-1}\right) \\ & + 7^{p-1}M_{1-\alpha}^p\alpha_0^p\|(-A)^{-\alpha}\|^p\big(\Gamma(1+p(\alpha-1)/(p-1))\big)^{p-1} < 1, \\ \hat{J} \triangleq & 14^{p-1}\beta_1^p + 14^{p-1}\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\frac{p-2}{2(p-1)}\right)^{\frac{p}{2}-1}\beta_2^p \\ & + 14^{p-1}\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(1+\left(\frac{p-2}{2(p-1)}\right)^{\frac{p}{2}-1}\right)\beta_3^p \end{aligned} \tag{4.5.3}$$

From employing Theorem 4.4.1, for $\alpha \in (\frac{1}{2}, 1]$, the attracting set of system (4.5.1) is

$$S = \left\{x \in H: \ \|x\| \le \sqrt[p]{(1-\hat{\sigma})^{-1}\hat{J}}\right\}. \tag{4.5.4}$$

In addition, let $b_f, \ b_g, \ b_h \to 0$, by Theorem 4.4.1, we know that the mild solution of system (4.5.1) is exponential stability in p-th moment provided that:

$$\begin{aligned}
\hat{\sigma} \triangleq & 7^{p-1}\alpha_0^p\|(-A)^{-\alpha}\|^{2p} + 7^{p-1}\alpha_4^p + 14^{p-1}\alpha_1^p \\
& + 14^{p-1}\alpha_2^p\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\frac{p-2}{2(p-1)}\right)^{\frac{p}{2}-1} \\
& + 14^{p-1}\alpha_3^p\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(1+\left(\frac{p-2}{2(p-1)}\right)^{\frac{p}{2}-1}\right) \\
& + 7^{p-1}M_{1-\alpha}^p\alpha_0^p\|(-A)^{-\alpha}\|^p\big(\Gamma(1+p(\alpha-1)/(p-1))\big)^{p-1} < 1, \\
\hat{J} = & 0
\end{aligned} \tag{4.5.5}$$

Chapter 5

Exponential Stability of Stochastic Partial Integro-differential Equations with Delays

5.1 Introduction

Stochastic partial differential equations find their applications in a various areas. In the past several years, existence, uniqueness and stability of solutions of stochastic differential equations with delays have been investigated by many researchers. Taniguchi [116] has investigated the almost sure exponential stability of mild solutions of a class of stochastic partial differential equations. Shortly, Liu and Truman [83] have improved their result by using some analytic techniques.

Stochastic integro-differential equations are more general. Stochastic delay integro-differential equations can model many real world problems, such as population dynamics, optional control, biotechnology, biological and many others in science and engineering. More recently, existence, uniqueness and stability of mild solutions of stochastic integro-differential equations with delays seem to be receiving more attention by many investigators. In particular, Diop, Ezzinbi and Lo [47] studied the existence and exponential stability for some stochastic partial functional integro-differential equations. The exponential and asymptotic stability of mild solutions of stochastic integro-differential equations with delays have been considered by Diop [48]. Diop, Ezzinbi and Lo [49] have employed the Banach fixed point approach for the existence of mild solutions of stochastic integro-differential equations with delays to achieve the required result.

In this chapter, we consider the following stochastic integro-differential equation with delays

$$\begin{cases} dx(t) = \left[Ax(t) + \displaystyle\int_0^t B(t-s)x(s) + F(x(\sigma_1(t)))\right]dt + G(x(\sigma_2(t)))dW(t), \\ \hfill t \geq 0, \\ x_0(t) = \phi(t), \quad t \in [-r, 0]. \end{cases} \tag{5.1.1}$$

where $r > 0, A : \mathcal{D}(A) \subset H \to H$ is a generator of some C_0-semigroup e^{tA}, $B(t)$ is a closed linear operator with domain $\mathcal{D}(B(t)) \supset \mathcal{D}(A)$, for each $t \geq 0$. The process $W(t)$ is a Wiener process on the separable Hilbert space K with covariance operator $Q \in \mathcal{L}_1(K)$. The mappings $F : [0, +\infty) \times H \to H$, $G : [0, +\infty) \times H \to \mathcal{L}_2^0(K_0, H)$ are measurable, $\sigma_1, \sigma_2 : [0, +\infty] \to [-r, T)$ are suitable delay functions, and $\phi : [-r, +\infty) \times \Omega \to H$ is the initial value.

The theory of integro-differential equations with resolvent operators is an important branch of differential equations, which has an extensive physical background. The resolvent operator is similar to the semigroup operator for abstract differential equations in Banach spaces. Caraballo and Liu [26] studied the exponential stability of mild solution of the following stochastic partial differential equation with delays:

$$\begin{cases} dx(t) = Ax(t)dt + F(x(\sigma_1(t)))dt + G(x(\sigma_2(t)))dW(t), \quad t \geq 0, \\ x_0(t) = \phi(t), \quad t \in [-r, 0], \end{cases} \tag{5.1.2}$$

by employing Gronwall inequality. The main purpose of this chapter is to obtain the sufficient condition for p-th moment exponential stability of mild solutions to integro-differential delay equations.

5.2 Stochastic Integro-differential Equations in Banach Spaces

Throughout this chapter, let K and H be two real separable Hilbert spaces. And $\mathcal{L}(K, H)$ denotes the space of all bounded linear operators from K into H. We denote by $\langle \cdot, \cdot \rangle_K, \langle \cdot, \cdot \rangle_H$ their inner products and by $\| \cdot \|_K, \| \cdot \|_H$ their norms respectively. We shall assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ is a complete probability space with a normal filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Let $\{W(t), t \geq 0\}$ denote a K-valued Wiener process defined on the probability spaces $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, and $Q \in \mathcal{L}_1(K)$ is the incremental covariance operator of $W(t)$ which is a positive, self-adjoint, trace class operator on K.

Now for the question of existence and uniqueness of mild solution of the integro-differential equation (5.1.1), we recall some fundamental results needed. Regarding the theory of resolvent operators, we refer the reader to Grimmer [54]. At the moment, let X be a Banach space, A and $B(t)$, $t \geq 0$, are closed linear operators on X. Let $\mathcal{D}(A)$ denotes the domain of A, equipped with the graph norm defined by

$$\|y\|_{\mathcal{D}(A)} := \|Ay\|_X + \|y\|_X, \quad \text{for } y \in \mathcal{D}(A).$$

The notations $C([0, T];\ \mathcal{D}(A))$ and $\mathcal{L}(\mathcal{D}(A),\ X)$ stand for the spaces of all continuous functions from $[0, T]$ into $\mathcal{D}(A)$, and the set of all bounded linear operators from $\mathcal{D}(A)$ into X.

We consider the following Cauchy problem,

$$\begin{cases} v'(t) = Av(t) + \displaystyle\int_0^t B(t-s)v(s)ds, \quad t \geq 0, \\ v(0) = v_0 \in H. \end{cases} \tag{5.2.1}$$

Definition 5.2.1. [55] A resolvent operator for equation (5.2.1) is a bounded linear operator valued function $R(t) \in \mathcal{L}(X)$ for $t \geq 0$ with the following properties:

(a) $R(0) = I$ and $\|R(t)\| \leq Ne^{\mu t}$ for some constants $N > 0$, $\mu \in \mathbb{R}$ and all $t \geq 0$.

(b) For each $x \in X$, $R(t)x$ is strongly continuous for $t \geq 0$.

(c) For $x \in \mathcal{D}(A)$, $R(\cdot)x \in C^1([0, T]; X) \cap C([0, T]; \mathcal{D}(A))$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t B(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)B(s)xds. \end{aligned} \tag{5.2.2}$$

The resolvent operator $R(\cdot)$ plays an important role in the study of the existence and uniqueness of solutions and establishes a variation of constants formula for many systems. For additional details on resolvent operators, we refer the read to [54] and [55]. We need to know when the linear system (5.1.1) has a resolvent operator. Theorem 5.2.1 below provides a satisfactory answer to this problem.

In what follows, we impose the following assumptions:

(**H1**) A is the infinitesimal generater of a C_0-semigroup $S(t)_{t\geq 0}$ on X.

(**H2**) For each $0 \leq t \leq T, B(t)$ is a continuous linear operator from $(\mathcal{D}(A), \|\cdot\|_{\mathcal{D}(A)})$ into $(X, \|\cdot\|_X)$. Moreover, there is an integrable function $c : [0, T] \to \mathbb{R}^+$ such that for any $y \in \mathcal{D}(A)$, $t \mapsto B(t)y$ belongs to $W^{1,1}([0, T], X)$ and

$$\left\| \frac{d}{dt} B(t)y \right\|_X \leq c(t)\|y\|_{\mathcal{D}(A)}, \quad t \in [0, T]. \tag{5.2.3}$$

Theorem 5.2.1. *Assume that the assumptions (**H1**) and (**H2**) hold. Then (5.2.1) admits a unique resolvent operator $R(t)_{t\geq 0}$.*

Proof. The proof can be found in Theorem 2.5. Grimmer [55].

□

In the following, we give some results on the existence of solutions for the following integro-differential equation:

$$\begin{cases} v'(t) = Av(t) + \displaystyle\int_0^t B(t-s)v(s)ds + f(t), \quad t \geq 0, \\ v(0) = v_0 \in X, \end{cases} \tag{5.2.4}$$

where $f : [0, T) \to X$ is a continuous function.

Definition 5.2.2. [55] A continuous function $v : [0, T) \to X$ is said to be a strict solution of (5.2.4) if

(a) $v \in C^1([0, T]; X) \cap C([0, T]; \mathcal{D}(A))$,

(b) v satisfies (5.2.4) for $t \geq 0$.

Theorem 5.2.2. *Assume that (**H1**) and (**H2**) hold. If $v(t)$ is a strict solution of equation (5.2.4), then the following variation of constants formula holds*

$$v(t) = R(t)v_0 + \int_0^t R(t-s)f(s)ds, \quad t \geq 0. \tag{5.2.5}$$

Proof. The proof can be found in Theorem 2.7. Grimmer [55].

□

Accordingly, we introduce the following definitions.

Definition 5.2.3. [55] For $v_0 \in X$. A function $v : [0, +\infty) \to X$ is called a mild solution of equation (5.2.4), if v satisfies the variation of constants formula (5.2.5).

The next theorem provides sufficient conditions for the regularity of solutions of equation (5.2.4).

Theorem 5.2.3. *Let $f \in C^1([0,T];X)$ and v be defined by (5.2.4). If $v_0 \in \mathcal{D}(A)$, then v is a strict solution of equation (5.2.4).*

Proof. The proof can be found in Theorem 5.4. Grimmer [55]. $\square$

In the sequel, we suppose that X is a Hilbert space H and (**H1**)-(**H2**) hold. Moreover, we suppose the following assumptions:

(**H3**) There exist some constants $M > 0$ and $\gamma > 0$, such that for $t \geq 0$,

$$\|R(t)\| \leq Me^{-\gamma t}.$$

(**H4**) There exist some constants $K_1,\ K_2,\ C > 0$ such that for any $t \geq 0$, the coefficients F, G satisfy the following conditions:

$$\|F(t,x) - F(t,y)\|_H \leq K_1\|x-y\|_H, \quad x, y \in H,$$

$$\|G(t,x) - G(t,y)\|_{\mathcal{L}_2^0} \leq K_2\|x-y\|_H, \quad x, y \in H,$$

$$\|F(t,x)\|_H^2 + \|G(t,x)\|_{\mathcal{L}_2^0}^2 \leq C(1+\|x\|_H^2), \quad x, y \in H.$$

(**H5**) The initial value $\phi : [-r,0] \times \Omega \to H$ satisfies that $\phi(t)$ is $\mathcal{F}_0$-measurable for all $t \in [-r,0]$ and

$$\sup_{-r\leq t\leq 0} \mathbb{E}\|\phi(t)\|_H^2 < +\infty.$$

(**H6**) The delays functions $\sigma_1(t) = t - \rho_1(t)$ and $\sigma_2(t) = t - \rho_2(t)$: $[0,+\infty) \to [-r,T), r > 0$ are continuously differentiable and satisfy that for any $t \geq 0$

$$\rho_1^{'}(t) \leq 0,\ \rho_2^{'}(t) \leq 0 \text{ and } -r \leq \sigma_1(t) \leq t\ \ -r \leq \sigma_2(t) \leq t.$$

Note that the functions $\sigma_1 = t - r_1$, $\sigma_2 = t - r_2$ with $r_1, r_2 > 0$ satisfy the precedent hypotheses by setting $r = \max\{r_1, r_2\}$.

Remark 5.2.1. *From (**H6**), we observe that there exist a constant $k \geq 0$ for any*

$t \geq -r$ such that

$$\sigma_1'(t) \geq 1,\ \sigma_2'(t) \geq 1,\ \textit{and}\ \sigma_1^{-1}(t) \leq t + k,\ \sigma_2^{-1}(t) \leq t + k \tag{5.2.6}$$

5.3 Existence Uniqueness and Exponential Stability in p-th Mean of Mild Solutions

Definition 5.3.1. Let $(R(t))_{t\geq 0}$ be a resolvent operator for equation (5.2.1). An H-valued stochastic process $\{x(t),\ t \geq 0\}$ is called a mild solution of stochastic integro-differential equation (5.1.1) for $T > 0$ such that

(i) $x(t)$ is adapted to $\mathcal{F}_t$ and

$$\mathbb{P}\left\{\omega : \int_0^T \|x(t,\omega)\|_H^2 dt < +\infty\right\} = 1,\ \ T \geq 0,$$

(ii) $x(t) \in H$ and for each $t \in [0,T]$, $x(t)$ satisfies the following integral equation

$$\begin{cases} x(t) = R(t)\phi(0) + \displaystyle\int_0^t R(t-s)F(x(\sigma_1(s)))ds + \int_0^t R(t-s)f(x(\sigma_2(s)))dW(s), \\ x_0(t) = \phi(t)\ \ t \in [-r,0]. \end{cases} \tag{5.3.1}$$

Theorem 5.3.1. *Suppose that the assumptions (**H1**)-(**H4**) are satisfied. Then there exist a unique mild solution to stochastic partial integro-differential equation (5.1.1).*

Proof. The proof can be find in Theorem 3.3 Diop et al. [48]. □

Caraballo and Liu [26] established the exponentially stable of mild solutions of stochastic partial differential equations with delays by using the Gronwall inequality. In this section, we shall discuss the exponential stability in p-th moment of mild solutions of stochastic partial integro-differential equation (5.1.1).

Definition 5.3.2. Let $p \geq 2$, the mild solution $x_\phi(t)$ of equation (5.1.1) is said to be *exponentially stable* in the p-th moment if there exist $\eta > 0$ and $M_0 \geq 1$ such that, for any mild solution of equation (5.1.1) $x_\varphi(t)$ corresponding to an initial value φ with $\mathbb{E}\|\varphi(0)\|_H^p + \int_{-r}^0 \mathbb{E}\|\varphi(s)\|_H^p ds < \infty$, the following inequality holds:

$$\mathbb{E}\|x_\phi(t) - x_\varphi(t)\|_H^p \leq M_0\|\phi - \varphi\|_1^p e^{-\eta t}, \quad t \geq 0, \tag{5.3.2}$$

where

$$\|\phi - \varphi\|_1^p = \max\left\{\mathbb{E}\|\phi(0) - \varphi(0)\|_H^p,\ \int_{-r}^0 \mathbb{E}\|\phi(s) - \varphi(s)\|_H^p ds\right\}. \tag{5.3.3}$$

Theorem 5.3.2. *Let $p \geq 2$ be an integer and $x(t) \equiv x_\phi(t)$ and $y(t) \equiv y_\varphi(t)$ be solutions of equation (5.3.1) with initial values ϕ and φ respectively. Assume that conditions (**H3**)-(**H6**) are satisfied. Then, the following inequality holds:*

$$\mathbb{E}\|x(t) - y(t)\|_H^p \leq \alpha\|\phi - \varphi\|_1^p e^{-(\gamma-\beta)t}, \quad t \geq 0, \tag{5.3.4}$$

where

$$C_p = \left(\frac{p(p-1)}{p}\right)^{p/2}, \quad \alpha = 3^{p-1}M^p\left(1 + \gamma^{1-p}K_1^p e^{\gamma k} + C_p K_2^p\left(\frac{p-2}{2\gamma(p-1)}\right)^{\frac{2-p}{2}}\right),$$

$$\beta = 3^{p-1}M^p\left(\gamma^{1-p}K_1^p e^{\gamma k} + C_p K_2^p\left(\frac{p-2}{2\gamma(p-1)}\right)^{\frac{2-p}{2}}\right).$$

Proof. Since $x(t)$ and $y(t)$ are two solutions of equation (5.3.1). We have that for $t \geq 0$,

$$x(t) = R(t)\phi(0) + \int_0^t R(t-s)F(x(\sigma_1(s)))ds + \int_0^t R(t-s)G(x(\sigma_2(s)))dW(s), \tag{5.3.5}$$

$$y(t) = R(t)\varphi(0) + \int_0^t R(t-s)F(y(\sigma_1(s)))ds + \int_0^t R(t-s)G(y(\sigma_2(s)))dW(s). \tag{5.3.6}$$

Thus, it follows that for any $x, y \in \mathcal{S}$,

$$\begin{aligned}
\mathbb{E}\|x(t)-y(t)\|_H^p \leq\; & 3^{p-1}\mathbb{E}\|R(t)(\phi(0)-\varphi(0))\|_H^p \\
& + 3^{p-1}\mathbb{E}\bigg\| \int_0^t R(t-s)\big[F(x(\sigma_1(s))) - F(y(\sigma_1(s)))\big] ds \bigg\|_H^p \\
& + 3^{p-1}\mathbb{E}\bigg\| \int_0^t R(t-s)\big[G(x(\sigma_2(s))) - G(y(\sigma_2(s)))\big] dW(s) \bigg\|_H^p \\
=\; & 3^{p-1}\sum_{i=1}^{3} I_i.
\end{aligned} \tag{5.3.7}$$

By the assumption (**H3**), it follows that for each $t \geq r$,

$$\begin{aligned}
I_1(t) &= \mathbb{E}\|R(t)(\phi(0)-\varphi(0))\|_H^p \\
&\leq M^p e^{-p\gamma t}\|\phi(0)-\varphi(0)\|_H^p.
\end{aligned} \tag{5.3.8}$$

By employing the assumption (**H3**), we have that, for $t \geq r$,

$$\begin{aligned}
I_2(t) &= \mathbb{E}\bigg\| \int_0^t R(t-s)\big[F(x(\sigma_1(s))) - F(y(\sigma_1(s)))\big] ds \bigg\|_H^p \\
&\leq \mathbb{E}\bigg(\int_0^t \|R(t-s)\| \|F(x(s-\sigma_1(s))) - F(y(s-\sigma_1(s)))\|_H ds \bigg)^p \\
&\leq \mathbb{E}\bigg(\int_0^t M e^{-\gamma(t-s)} \|F(x(\sigma_1(s))) - F(y(\sigma_1(s)))\|_H ds \bigg)^p.
\end{aligned} \tag{5.3.9}$$

One the other hand, by using Hölder's inequality, we have for $t \geq r$,

$$\begin{aligned}
I_2(t) &\leq M^p\mathbb{E}\bigg(\int_0^t e^{-\gamma(p-1)(t-s)/p} e^{-\gamma(t-s)/p} \|F(x(\sigma_1(s))) - F(y(\sigma_1(s)))\|_H ds \bigg)^p \\
&\leq M^p\mathbb{E}\Bigg(\bigg[\int_0^t \big(e^{-\gamma(p-1)(t-s)/p}\big)^{\frac{p}{p-1}} ds \bigg]^{\frac{p-1}{p}} \\
&\quad \cdot \bigg[\int_0^t \Big(e^{-\gamma(t-s)/p} \|F(x(\sigma_1(s))) - F(y(\sigma_1(s)))\|_H \Big)^p ds \bigg]^{\frac{1}{p}} \Bigg)^p.
\end{aligned} \tag{5.3.10}$$

Moreover, under the Lipschitz condition (**H4**), we get for $t \geq r$,

$$\begin{aligned} I_2(t) &\leq K_1^p M^p \left(\int_0^t e^{-\gamma(t-s)} ds \right)^{p-1} \mathbb{E} \int_0^t e^{-\gamma(t-s)} \|x(\sigma_1(s)) - y(\sigma_1(s))\|_H^p ds \\ &\leq K_1^p M^p \gamma^{1-p} \mathbb{E} \int_0^t e^{-\gamma(t-s)} \|x(\sigma_1(s)) - y(\sigma_1(s))\|_H^p ds. \end{aligned} \tag{5.3.11}$$

Let $u = \sigma_1(s)$ in (5.3.9), from Remark 5.2.1, for $t \geq r$, we have that

$$s = \sigma_1^{-1}(u) \leq u + k \quad \text{and} \quad \sigma_1^{-1}(u) = \frac{du}{\sigma_1'(\sigma_1^{-1}(u))}. \tag{5.3.12}$$

By substituting (5.3.12) into (5.3.11) and using assumption (**H5**), for $t \geq r$, we have that

$$\begin{aligned} I_2(t) &\leq K_1^p M^p \gamma^{1-p} \mathbb{E} \int_{\sigma_1(0)}^{\sigma_1(t)} e^{-\gamma(t-\sigma_1^{-1}(u))} \|x(u) - y(u)\|_H^p \frac{du}{\sigma_1'(\sigma_1^{-1}(u))} \\ &\leq K_1^p M^p \gamma^{1-p} \mathbb{E} \int_{\sigma_1(0)}^{\sigma_1(t)} e^{-\gamma(t-u-k)} \|x(u) - y(u)\|_H^p du \\ &\leq K_1^p M^p \gamma^{1-p} e^{-\gamma k} \mathbb{E} \int_{-r}^t e^{-\gamma(t-u)} \|x(u) - y(u)\|_H^p du. \end{aligned} \tag{5.3.13}$$

The equation (5.3.13) can be re-written in the following form

$$\begin{aligned} I_2(t) \leq\ & K_1^p M^p \gamma^{1-p} e^{\gamma k} \mathbb{E} \int_0^t e^{-\gamma(t-u)} \|x(u) - y(u)\|_H^p du \\ &+ K_1^p M^p \gamma^{1-p} e^{\gamma k} \mathbb{E} \int_{-r}^0 e^{-\gamma(t-u)} \|x(u) - y(u)\|_H^p du. \end{aligned} \tag{5.3.14}$$

Since $x(u) = \phi(u)$, $y(u) = \varphi(u)$ for any $u \in [-r, 0]$, we have that,

$$\begin{aligned} I_2(t) \leq\ & K_1^p M^p \gamma^{1-p} e^{\gamma k} \mathbb{E} \int_0^t e^{-\gamma(t-u)} \|x(u) - y(u)\|_H^p du \\ &+ K_1^p M^p \gamma^{1-p} e^{\gamma k} \mathbb{E} \int_{-r}^0 e^{-\gamma(t-u)} \|\phi(u) - \varphi(u)\|_H^p du. \end{aligned} \tag{5.3.15}$$

As $e^{-\gamma u} \leq 1$ for any $u \in [-r, 0]$, then we get,

$$
\begin{aligned}
I_2(t) &\leq K_1^p M^p \gamma^{1-p} e^{\gamma k} \mathbb{E} \int_0^t e^{-\gamma(t-u)} \|x(u) - y(u)\|_H^p du \\
&\quad + K_1^p M^p \gamma^{1-p} e^{\gamma k} e^{-\gamma t} \mathbb{E} \int_{-r}^0 \|\phi(u) - \varphi(u)\|_H^p du \\
&\leq K_1^p M^p \gamma^{1-p} e^{\gamma k} \mathbb{E} \int_0^t e^{-\gamma(t-u)} \|x(u) - y(u)\|_H^p du \\
&\quad + K_1^p M^p \gamma^{1-p} e^{\gamma k} e^{-\gamma t} \|\phi - \varphi\|_1^p.
\end{aligned}
\tag{5.3.16}
$$

By taking Lemma 2.2.1 and assumption (**H3**) into account, we have for $t \geq r$ that

$$
\begin{aligned}
I_3(t) &= \mathbb{E} \Big\| \int_0^t R(t-s) \big[G(x(\sigma_2(s))) - G(y(\sigma_2(s))) \big] dW(s) \Big\|_H^p \\
&\leq C_p \mathbb{E} \bigg(\int_0^t \big(\big\| R(t-s) \big[G(x(\sigma_2(s))) - G(y(\sigma_2(s))) \big] \big\|_H^p \big)^{2/p} ds \bigg)^{p/2} \\
&\leq C_p \mathbb{E} \bigg(\int_0^t \big(\|R(t-s)\|^p \|G(x(\sigma_2(s))) - G(y(\sigma_2(s)))\|_{\mathcal{L}_2^0}^p \big)^{2/p} ds \bigg)^{p/2}.
\end{aligned}
\tag{5.3.17}
$$

On the other hand, by using assumptions (**H3**) and (**H4**), for $t \geq r$, we get

$$
\begin{aligned}
I_3(t) &\leq C_p M^p \bigg(\int_0^t \big(e^{-p\gamma(t-s)} \mathbb{E} \|G(x(\sigma_2(s))) - G(y(\sigma_2(s)))\|_{\mathcal{L}_2^0}^p \big)^{2/p} ds \bigg)^{p/2} \\
&\leq C_p K_2^p M^p \bigg(\int_0^t \big(e^{-p\gamma(t-s)} \mathbb{E} \|x(\sigma_2(s)) - y(-\sigma_2(s))\|_H^p \big)^{2/p} ds \bigg)^{p/2}.
\end{aligned}
\tag{5.3.18}
$$

Now employing Hölder inequality, one can have that, for $t \geq r$,

$$
\begin{aligned}
I_3(t) &\leq C_p K_2^p M^p \Bigg(\bigg[\int_0^t e^{-(p-1)\gamma(t-s)} e^{-\gamma(t-s)} \mathbb{E} \|x(\sigma_2(s)) - y(\sigma_2(s))\|_H^p \bigg]^{2/p} ds \Bigg)^{p/2} \\
&\leq C_p K_2^p M^p \Bigg(\bigg[\int_0^t e^{-(p-1)\frac{2}{p}\frac{p}{p-2}\gamma(t-s)} ds \bigg]^{\frac{p-2}{p}} \bigg[\int_0^t \bigg(\Big[e^{-\gamma(t-s)} \mathbb{E} \|x(\sigma_2(s)) \\
&\quad - y(\sigma_2(s))\|_H^p \Big]^{2/p} \bigg)^{p/2} ds \bigg]^{2/p} \Bigg)^{p/2} \\
&\leq C_p K_2^p \bigg(\frac{p-2}{2\gamma(p-1)} \bigg)^{p/2-1} \int_0^t e^{-\gamma(t-s)} \mathbb{E} \|x(\sigma_2(s)) - y(\sigma_2(s))\|_H^p ds.
\end{aligned}
\tag{5.3.19}
$$

Let $u=\sigma_2(s)$ in (5.3.19), from Remak 5.2.1, for $t\geq r$, we have that

$$s=\sigma_2^{-1}(u)\leq u+k \quad\text{and}\quad \sigma_2^{-1}(u)=\frac{du}{\sigma_1'(\sigma_2^{-1}(u))}. \tag{5.3.20}$$

Similarly by using the method which we employ in (5.3.11), we have that, for $t\geq r$,

$$\begin{aligned} I_3(t) &\leq C_pK_2^p\left(\frac{p-2}{2\gamma(p-1)}\right)^{p/2-1}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(\sigma_2(s))-y(\sigma_2(s))\|_H^p ds \\ &\leq C_pK_2^p\left(\frac{p-2}{2\gamma(p-1)}\right)^{p/2-1} e^{\gamma k}\int_{-r}^t e^{-\gamma(t-u)}\mathbb{E}\|x(u)-y(u)\|_H^p du. \end{aligned} \tag{5.3.21}$$

As $e^{-\gamma u}\leq 1$, for all $u\in[-r,0]$, we get,

$$\begin{aligned} I_3(t) &\leq C_pK_2^p\left(\frac{p-2}{2\gamma(p-1)}\right)^{p/2-1}\mathbb{E}\int_0^t e^{-\gamma(t-u)}\|x(u)-y(u)\|_H^p du \\ &\quad + C_pK_2^p\left(\frac{p-2}{2\gamma(p-1)}\right)^{p/2-1} e^{-\gamma t}\mathbb{E}\int_{-r}^0 \|\phi(u)-\varphi(u)\|_H^p du \\ &\leq C_pK_2^p\left(\frac{p-2}{2\gamma(p-1)}\right)^{p/2-1}\mathbb{E}\int_0^t e^{-\gamma(t-u)}\|x(u)-y(u)\|_H^p du \\ &\quad + C_pK_2^p\left(\frac{p-2}{2\gamma(p-1)}\right)^{p/2-1} e^{-\gamma t}\|\phi-\varphi\|_1^p, \end{aligned} \tag{5.3.22}$$

where $\|\phi-\varphi\|_1^p$ is given in (5.3.3).

Recalling inequality (5.3.7) and (5.3.8)-(5.3.22) we can have that

$$\begin{aligned} &\mathbb{E}\|x(t)-y(t)\|_H^p \\ &\leq 3^{p-1}\left(M^pe^{-p\gamma t}+K_1^pM^p\gamma^{1-p}e^{\gamma k}e^{-\gamma t}+C_pK_2^p\left(\frac{p-2}{2\gamma(p-1)}\right)^{p/2-1}\right)e^{-\gamma t}\|\phi-\varphi\|_1^p \\ &\quad + 3^{p-1}\left(K_1^pM^p\gamma^{1-p}e^{\gamma k}+C_pK_2^p\left(\frac{p-2}{2\gamma(p-1)}\right)^{p/2-1}\right) \\ &\quad \cdot\mathbb{E}\int_0^t e^{-\gamma(t-u)}\|x(u)-y(u)\|_H^p du \\ &\leq 3^{p-1}M^p\left(1+K_1^p\gamma^{1-p}e^{\gamma k}e^{-\gamma t}+C_pK_2^p\left(\frac{p-2}{2\gamma(p-1)}\right)^{p/2-1}\right)e^{-\gamma t}\|\phi-\varphi\|_1^p \\ &\quad + 3^{p-1}M^p\left(K_1^p\gamma^{1-p}e^{\gamma k}+C_pK_2^p\left(\frac{p-2}{2\gamma(p-1)}\right)^{p/2-1}\right)e^{-\gamma t} \\ &\quad \cdot\mathbb{E}\int_0^t e^{\gamma u}\|x(u)-y(u)\|_H^p du. \end{aligned}$$

$$(5.3.23)$$

Then for any $t \geq 0$, we have that,

$$\begin{aligned} & e^{\gamma t}\mathbb{E}\|x(t)-y(t)\|_H^p \\ & \leq 3^{p-1}M^p\Bigg(1+K_1^p\gamma^{1-p}e^{\gamma k}e^{-\gamma t}+C_pK_2^p\bigg(\frac{p-2}{2\gamma(p-1)}\bigg)^{p/2-1}\Bigg)\|\phi-\varphi\|_1^p \\ & \quad + 3^{p-1}M^p\Bigg(K_1^p\gamma^{1-p}e^{\gamma k}+C_pK_2^p\bigg(\frac{p-2}{2\gamma(p-1)}\bigg)^{p/2-1}\Bigg) \\ & \quad \cdot \mathbb{E}\int_0^t e^{\gamma u}\|x(u)-y(u)\|_H^p du. \end{aligned} \tag{5.3.24}$$

Now employing Gronwall's inequality, one can have

$$\begin{aligned} & e^{\gamma t}\mathbb{E}\|x(t)-y(t)\|_H^p \\ & \leq 3^{p-1}M^p\Bigg(1+K_1^p\gamma^{1-p}e^{\gamma k}e^{-\gamma t}+C_pK_2^p\bigg(\frac{p-2}{2\gamma(p-1)}\bigg)^{p/2-1}\Bigg)\|\phi-\varphi\|_1^p \\ & \quad \cdot \exp\Bigg\{3^{p-1}M^p\Bigg(K_1^p\gamma^{1-p}e^{\gamma k}+C_pK_2^p\bigg(\frac{p-2}{2\gamma(p-1)}\bigg)^{p/2-1}\Bigg)t\Bigg\} \end{aligned} \tag{5.3.25}$$

Hence we have that

$$\mathbb{E}\|x(t)-y(t)\|_H^p \leq \alpha\|\phi-\varphi\|_1^p e^{-(\gamma-\beta)t}, \qquad t \geq 0. \tag{5.3.26}$$

where

$$\alpha = 3^{p-1}M^p\Bigg(1+K_1^p\gamma^{1-p}e^{\gamma k}e^{-\gamma t}+C_pK_2^p\bigg(\frac{p-2}{2\gamma(p-1)}\bigg)^{p/2-1}\Bigg),$$

and

$$\beta = 3^{p-1}M^p\Bigg(K_1^p\gamma^{1-p}e^{\gamma k}+C_pK_2^p\Bigg).$$

This completes the proof. □

Remark 5.3.1. *Observe that the constant β in Theorem 5.3.2 depends on values $M,\ \gamma,\ K_1,\ K_2,$ and k. Therefore, if the problem we are dealing with is such that $\gamma > \beta$, then we can assure that all mild solutions to this problem are exponentially stable in p-th mean.*

5.4 Almost Sure Asymptotic Stability

In this section, we study the almost sure asymptotic stability for the mild solution of equation (5.1.1) by using the technique close to Haussmann's [58]. Firstly, we recall two useful lemmas which were proved by Da Prato and Zabczyk[42].

Lemma 5.4.1. *[42] Let $\|S(t)\| \leq M$ for all $t \geq 0$, $p > 2$ be an integer and $\Phi : [0, +\infty) \to \mathcal{L}_2^0$ be an $\mathcal{F}_t$-adapted process with $\int_0^t \mathbb{E}\|\Phi(s)\|_{\mathcal{L}_2^0}^p ds < +\infty$, for all $t \geq 0$. There exists a constant $c_1 > 0$ such that for any natural number n, we have*

$$\mathbb{E}\bigg(\sup_{t\in[n,n+1]} \bigg\| \int_n^t S(t-s)\Phi(s)dW(s) \bigg\|_H^p \bigg) \leq c_1 \int_n^{n+1} \mathbb{E}\|\Phi(s)\|_{\mathcal{L}_2^0}^p ds. \tag{5.4.1}$$

Proof. The proof can be found in Da Prato and Zabczyk [42], p.144. $\square$

Lemma 5.4.2. *[42] Assume that operator A generates a strongly continuous constraction semigroup. Let $\Psi : [0, +\infty) \to \mathcal{L}_2^0$ be an $\mathcal{F}_t$-adapted process with $\int_0^t \mathbb{E}\|\Phi(s)\|_{\mathcal{L}_2^0}^2 ds < +\infty$, for any $t \geq 0$. There exists a constant $c_2 > 0$ such that for any natural number n, we have*

$$\mathbb{E}\bigg(\sup_{t\in[n,n+1]} \bigg\| \int_n^t S(t-s)\Phi(s)dW(s) \bigg\|_H^2 \bigg) \leq c_2 \int_n^{n+1} \mathbb{E}\|\Phi(s)\|_{\mathcal{L}_2^0}^2 ds. \tag{5.4.2}$$

Proof. The proof can be found in Da Prato and Zabczyk [42], p.160. $\square$

Now, we prove the following new version of Lemma 5.4.1 by using the resolvent operator $R(t)$ instead of the semigroup $S(t)$. Let $p > 2$, $\Psi : [0, +\infty) \to \mathcal{L}_2^0$ be a $\mathcal{F}_t$-adapted process. The stochastic convolution is defined by

$$W_{A,B}^{\Psi}(t) = \int_0^t R(t-s)\Psi(s)dW(s), \quad t \geq 0. \tag{5.4.3}$$

Lemma 5.4.3. *Let $\|R(t)\| \leq M$ for all $t \geq 0$ and let $\Psi : [0, +\infty) \to \mathcal{L}_2^0$ be an $\mathcal{F}_t$-adapted process with $\int_0^t \mathbb{E}\|\Psi(s)\|_{\mathcal{L}_2^0}^p ds < +\infty$, for some integer $p > 2$. Then there exists a constant $C_T > 0$ such that for any natural number n, we have*

$$\mathbb{E}\bigg(\sup_{t\in[n,n+1]} \bigg\| \int_n^t R(t-s)\Psi(s)dW(s) \bigg\|_H^p \bigg) \leq C_T \int_n^{n+1} \mathbb{E}\|\Psi(s)\|_{\mathcal{L}_2^0}^p ds. \tag{5.4.4}$$

Proof. We shall use the factorization method. Let $\theta \in (\frac{1}{p}, \frac{1}{2})$ and $\frac{1}{p} + \frac{1}{q} = 1$, we suppose that

$$\mathbb{E}\left(\int_n^{n+1} \|\Psi(s)\|^p_{\mathcal{L}_2^0} ds \right) < +\infty.$$

And for any $n \leq h < t \leq n+1$, we have

$$\int_h^t (t-s)^{\theta-1}(s-h)^{-\theta} ds = \frac{\pi}{\sin \pi\theta}. \tag{5.4.5}$$

By substituting (5.4.5) into (5.4.3), we give the following stochastic convolution $W^{\Psi}_{A,B}$

$$W^{\Psi}_{A,B}(t) = \frac{\sin \pi\theta}{\pi} \int_n^t R(t-h)\Psi(h) \left[\int_h^t (t-s)^{\theta-1}(s-h)^{-\theta} ds \right] dW(h) \quad t \geq 0. \tag{5.4.6}$$

Taking stochastic Fubini Theorem 2.2.3 into account, we have that

$$W^{\Psi}_{A,B}(t) = \frac{\sin \pi\theta}{\pi} \int_h^t (t-s)^{\theta-1} \mathcal{Z}(s) ds \quad t \geq 0, \tag{5.4.7}$$

where

$$\mathcal{Z}(s) = \int_n^s R(s-h)(s-h)^{-\theta}\Psi(h) dW(h) \quad 0 \leq s \leq T. \tag{5.4.8}$$

Since $\theta \in (\frac{1}{p}, \frac{1}{2})$, applying Hölder's inequality, we can obtain that there exists a constant $C_{T_1} > 0$ such that

$$\begin{aligned} \|W^{\Psi}_{A,B}(t)\| &\leq \frac{\sin \pi\theta}{\pi} \int_h^t (t-s)^{\theta-1} \|\mathcal{Z}(s)\|^p_H ds \\ &\leq \frac{\sin \pi\theta}{\pi} \left[\int_h^t (t-s)^{q(\theta-1)} ds \right]^{1/q} \left[\int_n^t \|\mathcal{Z}(s)\|^p_H ds \right]^{1/p} \\ &\leq \frac{\sin \pi\theta}{\pi} \frac{t^{\theta-\frac{1}{p}}}{(q(\theta-1)+1)^{\frac{1}{q}}} \left[\int_n^t \|\mathcal{Z}(s)\|^p_H ds \right]^{1/p}. \end{aligned} \tag{5.4.9}$$

Thus,

$$\sup_{t\in[n,n+1]} \|W^{\Psi}_{A,B}(t)\|^p_H \leq C_{T_1} \int_n^{n+1} \|\mathcal{Z}(s)\|^p_H ds. \tag{5.4.10}$$

Moreover, by Lemma 2.2.1, there exist a constant $C_{T_2} > 0$ such that

$$\mathbb{E}\|\mathcal{Z}(s)\|^p \leq C_{T_2}\mathbb{E}\bigg(\int_n^s (s-h)^{-2\theta}\|\Psi(h)\|^2_{\mathcal{L}_2^0}dh\bigg)^{p/2}. \tag{5.4.11}$$

Substituting (5.4.11) into (5.4.10), we obtain the following inequality

$$\begin{aligned}&\mathbb{E}\bigg(\sup_{t\in[n,n+1]}\|W^{\Psi}_{A,B}(t)\|^p_H\bigg)\\ &\leq C_{T_2}\bigg(\frac{\sin\pi\theta}{\pi}\bigg)^p\frac{T^{p\theta-1}}{(q(\theta-1)+1)^{\frac{p}{q}}}\mathbb{E}\int_n^{n+1}\bigg(\int_n^s (s-h)^{-2\theta}\|\Psi(h)\|^2_{\mathcal{L}_2^0}dh\bigg)^{p/2}ds.\end{aligned} \tag{5.4.12}$$

From which, using the classic Young inequality, there exist a constant $C_{T_3} > 0$ such that

$$\begin{aligned}&\int_n^{n+1}\bigg(\int_n^s (s-h)^{-2\theta}\|\Psi(h)\|^2_{\mathcal{L}_2^0}dh\bigg)^{p/2}ds\\ &\leq \bigg(\int_n^{n+1}(s-h)^{-2\theta}ds\bigg)^{p/2}\int_n^{n+1}\|\Psi(s)\|^{p/2}_{\mathcal{L}_2^0}ds\\ &\leq C_{T_3}\int_n^{n+1}\|\Psi(s)\|^p_{\mathcal{L}_2^0}ds.\end{aligned} \tag{5.4.13}$$

Replacing the above expression in (5.4.12), there exists a constant $C_T > 0$, we have that

$$\begin{aligned}\mathbb{E}\bigg(\sup_{t\in[0,T]}\|W^{\Psi}_{A,B}(t)\|^p_H\bigg) &\leq C_{T_2}C_{T_3}\bigg(\frac{\sin\pi\theta}{\pi}\bigg)^p\frac{T^{p\theta-1}}{(q(\theta-1)+1)^{\frac{p}{q}}}\mathbb{E}\int_n^{n+1}\|\Psi(s)\|^p_{\mathcal{L}_2^0}ds\\ &\leq C_T\mathbb{E}\int_n^{n+1}\|\Psi(s)\|^p_{\mathcal{L}_2^0}ds.\end{aligned} \tag{5.4.14}$$

This finishes the proof of Lemma 5.4.3. □

We shall recall the Borel-Cantelli's Lemma which plays important role in our argument.

Lemma 5.4.4. (**Borel-Cantelli's Lemma**) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $A_1,\ A_2, \cdots$, are a group of events. Let $A = \cap_n \cup_{m=n}^{\infty} A_m$ be the event that infinitely many of the A_n occur. Then:*

(a). $\mathbb{P}(A) = 0$ *if* $\sum_n \mathbb{P}(A_n) < \infty$,

(b). $\mathbb{P}(A) = 1$ *if* $\sum_n \mathbb{P}(A_n) = \infty$ *and* $A_1, A_2, \cdots$ *are independent events.*

If the assumption of independence is dropped, the statement (b) could be false. For example, consider some event E with $0 < \mathbb{P}(E) < 1$ and let $A_n = E$ for all n. Then $A = E$ and $\mathbb{P}(A) = \mathbb{P}(E)$.

Theorem 5.4.1. *Suppose that all assumptions (**H1**)-(**H6**) of Theorem 5.3.2 are hold with $p > 2$. Let $x(t)$ and $y(t)$ are solutions of equation (5.1.1) with initial values ϕ and φ respectively. If $\gamma > \beta$, then there exists a random variable $\tau(\omega) \geq 0$ such that for all $t \geq \tau(\omega)$*

$$\|x(t) - y(t)\|_H^p \leq \|\phi - \varphi\|_1^p e^{-(\gamma-\beta)t/2} \quad \mathbb{P} \;\; a.s. \tag{5.4.15}$$

Proof. Let n_0 be an integer such that $\sigma_1(n_0) \geq 0$, $\sigma_2(n_0) \geq 0$. Since the assumption (**H5**) is satisfied, we have that $\sigma_1(n) > 0$ and $\sigma_2(n) > 0$. Let $n > n_0$, we denote I_n the interval $[n, n+1]$. For any $t \in [n, n+1]$, we have

$$x(t) = R(t-n)x(n) + \int_n^t R(t-s)F(x(\sigma_1(s)))ds + \int_n^t R(t-s)G(x(\sigma_2(s)))dW(s), \tag{5.4.16}$$

$$y(t) = R(t-n)y(n) + \int_n^t R(t-s)F(y(\sigma_1(s)))ds + \int_n^t R(t-s)G(y(\sigma_2(s)))dW(s). \tag{5.4.17}$$

It follows that

$$\begin{aligned}
&\|x(t) - y(t)\|_H^p \\
&\leq \|R(t-n)[x(n) - y(n)]\|_H^p \\
&\quad + \left\| \int_n^t R(t-n)\big[F(x(\sigma_1(s))) - F(y(\sigma_1(s)))\big]ds \right\|_H^p \\
&\quad + \left\| \int_n^t R(t-n)\big[G(x(\sigma_2(s))) - G(y(\sigma_2(s)))\big]dW(s) \right\|_H^p
\end{aligned} \tag{5.4.18}$$

For any constant $\varepsilon > 0$, we have

$$\begin{aligned}
&\mathbb{P}\bigg(\sup_{n\le t\le n+1}\|x(t)-y(t)\|_H^p > \varepsilon\bigg)\\
&\le \mathbb{P}\bigg[\sup_{n\le t\le n+1}\|R(t-n)[x(n)-y(n)]\|_H^p > \frac{\varepsilon}{3}\bigg]\\
&\quad+\mathbb{P}\bigg[\sup_{n\le t\le n+1}\bigg\|\int_n^t R(t-n)\big[F(x(\sigma_1(s)))-F(y(\sigma_1(s)))\big]ds\bigg\|_H^p > \frac{\varepsilon}{3}\bigg]\\
&\quad+\mathbb{P}\bigg[\sup_{n\le t\le n+1}\bigg\|\int_n^t R(t-n)\big[G(x(\sigma_2(s)))-G(y(\sigma_2(s)))\big]dW(s)\bigg\|_H^p > \frac{\varepsilon}{3}\bigg]
\end{aligned} \tag{5.4.19}$$

By using Markov inequality in the above inequality (5.4.19), one follows that

$$\begin{aligned}
&\mathbb{P}\bigg(\sup_{n\le t\le n+1}\|x(t)-y(t)\|_H^p > \varepsilon\bigg)\\
&\le \left(\frac{3}{\varepsilon}\right)^p\mathbb{E}\bigg[\sup_{n\le t\le n+1}\|R(t-n)[x(n)-y(n)]\|_H^p\bigg]\\
&\quad+\left(\frac{3}{\varepsilon}\right)^p\mathbb{E}\bigg[\sup_{n\le t\le n+1}\bigg\|\int_n^t R(t-n)\big[F(x(\sigma_1(s)))-F(y(\sigma_1(s)))\big]ds\bigg\|_H^p\bigg]\\
&\quad+\left(\frac{3}{\varepsilon}\right)^p\mathbb{E}\bigg[\sup_{n\le t\le n+1}\bigg\|\int_n^t R(t-n)\big[G(x(\sigma_2(s)))-G(y(\sigma_2(s)))\big]dW(s)\bigg\|_H^p\bigg]\\
&= \left(\frac{3}{\varepsilon}\right)^p(\Pi_1+\Pi_2+\Pi_3).
\end{aligned} \tag{5.4.20}$$

By employing Theorem 5.4.1, assumption (**H3**) and Hölder inequality, we have that

$$\begin{aligned}
\Pi_1 &= \mathbb{E}\bigg[\sup_{n\le t\le n+1}\|R(t-n)[x(n)-y(n)]\|_H^p\bigg]\\
&\le \mathbb{E}\bigg[\sup_{n\le t\le n+1}M^p e^{-p\gamma(t-n)}\|x(n)-y(n)\|_H^p\bigg]\\
&\le M^p\mathbb{E}\|x(n)-y(n)\|_H^p\\
&\le M^p\alpha\|\phi-\varphi\|_1^p e^{-(\gamma-\beta)n}.
\end{aligned} \tag{5.4.21}$$

Applying assumptions (**H3**), we can see that

$$\begin{aligned} \Pi_2 &= \mathbb{E}\Bigg[\sup_{n\le t\le n+1}\Bigg\|\int_n^t R(t-n)\big[F(x(\sigma_1(s))) - F(y(\sigma_1(s)))\big]ds\Bigg\|_H^p\Bigg] \\ &\le \mathbb{E}\Bigg[\sup_{n\le t\le n+1}\Bigg(\int_n^t \|R(t-n)\big[F(x(\sigma_1(s))) - F(y(\sigma_1(s)))\big]\|_H ds\Bigg)^p\Bigg]. \end{aligned} \tag{5.4.22}$$

Under the Lipschitz condition (**H4**) and by using Hölder inequality, we have,

$$\begin{aligned} \Pi_2 &\le (MK_1)^p \mathbb{E}\Bigg[\sup_{n\le t\le n+1}\Bigg(\int_0^t e^{-p\gamma(t-n)}\|x(\sigma_1(s)) - y(\sigma_1(s))\|_H ds\Bigg)^p\Bigg] \\ &\le (MK_1)^p \mathbb{E}\Bigg[\sup_{n\le t\le n+1}\Bigg(\int_n^t 1\times\|x(\sigma_1(s)) - y(\sigma_1(s))\|_H ds\Bigg)^p\Bigg] \\ &\le (MK_1)^p \mathbb{E}\Bigg[\sup_{n\le t\le n+1}(t-n)^{(p-1)/p}\int_n^t \|x(\sigma_1(s)) - y(\sigma_1(s))\|_H^p ds\Bigg] \\ &\le (MK_1)^p \int_n^{n+1} \mathbb{E}\|x(\sigma_1(s)) - y(\sigma_1(s))\|_H^p ds. \end{aligned} \tag{5.4.23}$$

Let $u = \sigma_1(s)$, then the above expression (5.4.23) implies

$$\begin{aligned} \Pi_2 &\le (MK_1)^p \int_{\sigma_1(n)}^{\sigma_1(n+1)} \mathbb{E}\|x(u) - y(u)\|_H^p d(\sigma_1^{-1}(u)) \\ &\le (MK_1)^p \int_{\sigma_1(n)}^{\sigma_1(n+1)} \mathbb{E}\|x(u) - y(u)\|_H^p du \\ &\le (MK_1)^p \int_{\sigma_1(n)}^{\sigma_1(n+1)} \alpha\|\phi - \varphi\|_1^p e^{-(\gamma-\beta)u} du \\ &\le (MK_1)^p \alpha\|\phi - \varphi\|_1^p \int_{\sigma_1(n)}^{\sigma_1(n+1)} e^{-(\gamma-\beta)u} du. \end{aligned} \tag{5.4.24}$$

Since $\gamma > \beta$, we have

$$\begin{aligned} \Pi_2 &\le (MK_1)^p \left(\frac{\alpha}{\gamma-\beta}\right)\|\phi - \varphi\|_1^p \big(e^{-(\gamma-\beta)\sigma(n)} - e^{-(\gamma-\beta)\sigma(n+1)}\big) \\ &\le (MK_1)^p \left(\frac{\alpha}{\gamma-\beta}\right)\|\phi - \varphi\|_1^p e^{-(\gamma-\beta)\sigma(n)} \\ &\le (MK_1)^p \left(\frac{\alpha}{\gamma-\beta}\right)\|\phi - \varphi\|_1^p e^{-(\gamma-\beta)(n-r)} \\ &\le (MK_1)^p \left(\frac{\alpha}{\gamma-\beta}\right) e^{(\gamma-\beta)r}\|\phi - \varphi\|_1^p e^{-(\gamma-\beta)n}. \end{aligned} \tag{5.4.25}$$

On the other hand, by taking Lemma 5.5.1 into account, we have that

$$\begin{aligned}\Pi_3 &= \mathbb{E}\Big[\sup_{n\le t\le n+1}\Big\|\int_n^t R(t-n)\big[G(x(\sigma_2(s)))-G(y(\sigma_2(s)))\big]dW(s)\Big\|_H^p\Big]\\ &\le (MK_2)^p C_T\int_n^{n+1}\mathbb{E}\|x(\sigma_2(s))-y(\sigma_2(s))\|_H^p ds.\end{aligned} \tag{5.4.26}$$

By using the change of variable $u=\sigma_2(s)$ in (5.4.26), we get

$$\begin{aligned}\Pi_3 &= (MK_1)^p\int_{\sigma_1(n)}^{\sigma_1(n+1)}\mathbb{E}\|x(u)-y(u)\|_H^p du\\ &\le (MK_1)^p\Big(\frac{\alpha C_T}{\gamma-\beta}\Big)e^{(\gamma-\beta)r}\|\phi-\varphi\|_1^p e^{-(\gamma-\beta)n}.\end{aligned} \tag{5.4.27}$$

Recalling (5.4.19), from (5.4.21) to (5.4.27), one can see that there exist a $M_1>0$ such that

$$\mathbb{P}\Big(\sup_{t\in[n,n+1]}\|x(t)-y(t)\|_H^p>\varepsilon\Big)\le\frac{M_1}{\varepsilon}\|\phi-\varphi\|_1^p e^{-(\gamma-\beta)n} \tag{5.4.28}$$

where $M_1=(3M)^p\alpha\Big(1+K_1^p\frac{1}{\gamma-\beta}e^{(\gamma-\beta)r}+K_2^p\frac{C_p}{\gamma-\beta}e^{(\gamma-\beta)r}\Big)$.

For each integer $n\ge n_0$, we set $\varepsilon_n=\|\phi-\varphi\|_1 e^{-(\gamma-\beta)n/2p}$. Then one can have that

$$\mathbb{P}\Big(\sup_{t\in[n,n+1]}\|x(t)-y(t)\|_H>\|\phi-\varphi\|_1 e^{-(\gamma-\beta)n/2p}\Big)\le M_1 e^{-(\gamma-\beta)n/2}. \tag{5.4.29}$$

Hence, the Lemma 5.5.2 implies that there exist a random variable $\tau(\omega),\ t\ge 0$ such that for all $t\ge\tau(\omega)$

$$\|x(t)-y(t)\|_H^p\le\|\phi-\varphi\|_1^p e^{-(\gamma-\beta)t/2}\quad \mathbb{P}\ \ a.s. \tag{5.4.30}$$

$\square$

5.5 Illustrative Example

We consider the following system

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u(t,x)}{\partial x^2} + \displaystyle\int_0^t \alpha(t-s)\frac{\partial^2}{\partial x^2}u(s,x)ds - \frac{u(\sigma_1(t),x)}{1+|u(\sigma_1(t),x)|} \\ \qquad\qquad\qquad\qquad\qquad\qquad +\sigma\dfrac{u(\sigma_2(t),x)}{1+|u(\sigma_2(t),x)|}d\beta(t), \quad t \geq 0, \\ u(t,0) = u(t,\pi) = 0, \quad t \geq 0, \\ u(t,x) = u_0(t,x), \quad t \in [-r,0], \;\; x \in [0,\pi]. \end{cases} \tag{5.5.1}$$

where $r > 0$, $\beta(t)$ is a real standard Wiener process and $\alpha : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function. Let $H = L^2([0,\pi])$ with the norm $\|\cdot\|$ and $e_n := \sqrt{\frac{2}{\pi}}\sin(nx)$, $n = 1,2,3,\cdots$ denote the completed orthonormal basis in H. Let $W(t) = \sum_{n=1}^{\infty}\sqrt{\lambda_n}\delta_n(t)e_n$, where $\delta_n(t)$ are one dimensional standard Brownian motion mutually independent on a complete probability space $(\Omega,\mathcal{F},\mathcal{F}_{t\geq 0},\mathbb{P})$.

Define $A : \mathcal{D}(A) \subset H \to H$ by $A = \frac{\partial^2}{\partial x^2}$, with domain $\mathcal{D}(A) = H^2[0,\pi] \cap H_0^1[0,\pi]$.

Then

$$Ah = \sum_{n=1}^{\infty} -n^2\langle h, e_n\rangle e_n, \quad h \in \mathcal{D}(A),$$

where e_n, $n = 1,2,3,\cdots$, is also the orthonormal set of eigenvectors of A. It is well-known that A is the infinitesimal generator of a strongly continuous semigroup $S(t)$ on H, given by

$$S(t)h = \sum_{n=1}^{\infty} e^{-n^2 t}\langle h, e_n\rangle e_n, \quad h \in H,$$

which is compact.

Let $B(t) : \mathcal{D}(A) \subset H \to H$, $t \geq 0$, be the operator defined by

$$B(t)(z) = \alpha(t)Az \quad t \geq 0, \;\; z \in \mathcal{D}(A).$$

If we put

$$\begin{cases} x(t) = u(t,\cdot), \quad & t \geq 0, \\ \phi(t) = u_0(t,\cdot), \quad & t \in [-r, 0]. \end{cases} \tag{5.5.2}$$

Then the equation (5.5.1) takes the following abstract from

$$\begin{cases} dx(t) = \left[Ax(t) + \int_0^t B(t-s)x(s) + F(x(\sigma_1(t)))\right]dt + G(x(\sigma_2(t)))dW(t), \ t \geq 0, \\ x_0(t) = \phi(t) \quad t \in [-r, 0]. \end{cases} \tag{5.5.3}$$

We suppose b is bounded and b' is bounded and uniformly continuous, which implies that the operator $B(t)$ satisfies conditions (**H1**) and (**H2**) and hence, by Theorem (5.2.1), equation (5.2.1) has a resolvent operator $(R(t))_{t\geq 0}$ on H. By Lipchiz condition (**H4**), we suppose that for all $t \geq 0$,

$$\|R(t)\|_{\mathcal{L}(H)} \leq Me^{-\gamma t}, \quad M \geq 1, \ \gamma > 0,$$

then all the assumptions of Theorem 5.3.2 are fulfilled. Therefore, the stochastic integro-differential equation (5.5.1) has a unique mild solution which is exponentially stable in p-th moment provided

$$M^p\left(1 + \gamma^{1-p}e^{\gamma r} + C_p\left(\frac{p-2}{2(p-1)\gamma}\right)^{\frac{p-2}{2}}\right) < \frac{1}{3^{p-1}}, \tag{5.5.4}$$

for any $p > 2$ and there exists a

$$C_p = \left(\frac{p(p-1)}{2}\right)^{p/2}. \tag{5.5.5}$$

Chapter 6

Exponential Stability of Neutral Impulsive Stochastic Delay Partial Differential Equations Driven by a Fractional Brownian Motion

6.1 Introduction

The fractional Brownian motion is a special stochastic process. It differs significantly from the standard Brownian motion and semi-martingales in the theory of stochastic processes. As a family of centered Gaussian processes, it is characterized by the stationarity of its increments and a medium or long-memory property. It also exhibits power scaling with exponent H. Its paths are Hölder continuous of any order $H \in (0, 1)$. When $H = \frac{1}{2}$, the fractional Brownian motion becomes the standard Brownian motion. However, when $H \neq \frac{1}{2}$, fBm B^H behaves in a completely different way from the standard Brownian motion. In particular, it is neither a semi-martingale nor a Markov process.

We shall define the stochastic integral with respect to fractional Brownian motion in infinite dimensional spaces in the same way as in Caraballo et al. [21]. They have also discussed the existence and exponential behaviour of mild solutions. Although, stochastic functional differential equations driven by a fractional Brownian motion have recently been studied intensively, as far as we know, there are only a few satisfactory results. The following stochastic functional differential equation driven by a fractional Brownian motion in a Hilbert space with finite delay has been studied by Boufoussi and Hajji [12]

$$\begin{cases} d[x(t)+u(t,x(t-r(t)))] = [Ax(t)+f(t,x(t-r(t)))]dt + \sigma(t)dB^H(t), \\ \hfill t \geq 0, \\ x_0(t) = \phi(t), \quad t \in (-r,0]. \end{cases} \tag{6.1.1}$$

R. Maheswari and S. Karunanithi [93] have discussed the existence, uniqueness and asymptotic behaviors of mild solutions for a neutral stochastic differential equation with finite delays driven by fractional Bownian motion in the following form

$$\begin{cases} d[x(t)+u(t,x(t-r(t)))] = [Ax(t)+f(t,x(t-r(t)))]dt \\ \qquad\qquad +g(t,x(t-r(t)))dW(t) + \sigma(t)dB^H(t), \quad t \geq 0, \\ x_0(t) = \phi(t), \quad t \in (-r,0]. \end{cases} \tag{6.1.2}$$

Moreover, in addition to stochastic effects, it is known that the impulsive effects exist in many different areas of real world such as mechanics, medicine and biology, economics and finance. Impulsive effects often make systems under investigation unstable. Therefore, impulsive effects should be taken into account in the research of stochastic delay differential equations driven by fractional Brownian motion. However, to our best knowledge, so far not many works have been reported on the corresponding problems for impulsive stochastic neutral stochastic differential equations driven by a fractional Brownian motion. Motivated by this consideration in this chapter, we are interested in existence, uniqueness and asymptotic behaviors of mild solutions for the following Hilbert space valued neutral impulsive stochastic differential equation driven by a fractional Brownian motion with finite ($r > 0$) or infinite delays ($r = +\infty$):

$$\begin{cases} d[x(t) + u(t, x(t-\tau(t)))] = [Ax(t) + f(t, x(t-\tau(t)))]dt + g(t, x(t-\tau(t)))dW(t) \\ \qquad\qquad\qquad\qquad +\sigma(t)dB^H(t), \qquad\qquad\qquad t \geq 0, \ \ t \neq t_k, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad t = t_k, k = 1, 2, \ldots, \\ x(t) = \phi(t), \quad t \in (-r, 0]. \end{cases} \tag{6.1.3}$$

where A is the infinitesimal generator of an analytic semigroup of bounded liner operators $(S(t))_{t\geq 0}$ in a Hilbert space Y with norm $\|\cdot\|$, $B^H(t)$ is a fractional Brownian motion with $H > \frac{1}{2}$ on a real and separable Hilbert space K, $\tau : [0, \infty) \to [0, r)$ is continuous. Here u, $f : [0, +\infty) \times Y \to Y$, $g : [0, +\infty) \times Y \to \mathcal{L}_2^0(K_0, Y)$, are measurable functions, mapping $\sigma : [0, +\infty) \to Y$ and $I_k : Y \to Y$ is measurable and $0 < t_1 < \ldots < t_k < \ldots$, $\lim_{k\to\infty} t_k = \infty$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, $k = 1, 2, \ldots$, respectively. The mapping I_k represents the size of the jump at t_k, $k = 1, 2, \ldots$. The initial value $\phi \in P((-r, 0], Y)$ the space of all continuous functions from $(-r, 0)$ to Y, but $\phi(t^+)$ and $\phi(t^-)$ exist at accountable number of points on $(-r, 0)$. $\phi(t^+)$ and $\phi(t^-)$ denote the right-hand and left-hand limits of the function $\psi(t)$, respectively.

6.2 Fractional Brownian Motion

In this section, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. We recall the definition of Wiener integral with respect to fractional Brownian motions with Hurst parameter $H > \frac{1}{2}$. We also establish some definitions and lemmas which play an important role throughout this chapter.

Definition 6.2.1. An one-dimensional fractional Brownian motion with Hurst parameter $H \in (0,1)$ is a continuous centered Gaussian process $\beta^H(t),\ t \in \mathbb{R}$, with the covariance function

$$R_H(t,s) = \mathbb{E}\big[\beta^H(t)\beta^H(s)\big] = \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big), \qquad t,s \in \mathbb{R}, \tag{6.2.1}$$

where H is the Hurst parameter. The process above is called a two-side one-dimensional fractional Brownian motion.

We shall define the stochastic integral with respect to fractional Brownian motion in infinite dimensional spaces in the same way as in Caraballo et al. [21]. Firstly, we introduce stochastic integral with respect to the one-dimensional fractional Brownian motion β^H. Let $T > 0$ and denote by Λ the linear space of $\mathbb{R}$-valued step functions on $[0,T]$, that is, $\varphi \in \Lambda$ if

$$\varphi(t) = \sum_{i=1}^{n-1} x_i \mathcal{X}_{[t_i, t_{i+1})}(t), \tag{6.2.2}$$

where $t \in [0,T],\ x_i \in \mathbb{R}$ and $0 = t_1 < t_2 < \cdots < t_n = T$. For $\varphi \in \Lambda$ we define its stochastic integral with respect to β^H as

$$\int_0^T \varphi(s) d\beta^H(s) = \sum_{i=1}^{n-1} x_i(\beta^H(t_{i+1}) - \beta^H(t_i)). \tag{6.2.3}$$

Let $\mathcal{H}$ be a Hilbert space defined as the closure of Λ with respect to the scalar product

$$\langle \mathcal{X}_{[0,t]}, \mathcal{X}_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s)$$

It can be shown that the mapping

$$\varphi = \sum_{i=1}^{n-1} x_i \mathcal{X}_{[t_i, t_{i+1})} \mapsto \int_0^T \varphi(s) d\beta^H(s) \tag{6.2.4}$$

is an isometry between Λ and the linear space span $\{\beta^H, t \in [0,T]\}$, which can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos of the fBm $\overline{span}^{L^2(\Omega)}\{\beta^H, t \in [0,T]\}$. The image of an element $\varphi \in \mathcal{H}$ by this ismetry is called the stochastic integral of φ with respect to β^H.

Next, we will give an explicit expression of this integral. To this end, we consider the kernel function

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_0^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, \qquad t \leq s, \tag{6.2.5}$$

where $c_H = \sqrt{\frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})}}$, with B denoting the Beta function. It is not difficult to see that

$$\frac{\partial K_H}{\partial t}(t,s) = c_H \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}}. \tag{6.2.6}$$

Consider the linear operator $K_H^* : \Lambda \to L^2([0,T])$ defined by

$$(K_H^* \varphi)(s) = \int_s^t \varphi(t) \frac{\partial K_H}{\partial t}(t,s) dt \quad \varphi \in \Lambda. \tag{6.2.7}$$

Then

$$(K_H^* \mathcal{X}_{[0,t]})(s) = K_H(t,s) \mathcal{X}_{[0,t]}(s). \tag{6.2.8}$$

It can be shown that K_H^* is an isometry between Λ and $L^2[0,T]$ that can be extended to $\mathcal{H}$. We define that

$$W(t) = \beta^H((K_H^*)^{-1} \mathcal{X}_{[0,t]}), \quad t \in [0,T]. \tag{6.2.9}$$

It turns out that $W(t)$ is a Wiener process and β^H has the following Wiener integral representation:

$$\beta^H(t) = \int_0^t K_H(t,s)dW(s), \quad t \geq 0.. \tag{6.2.10}$$

In addition, for any $\varphi \in \mathcal{H}$, it can be shown that

$$\int_0^T \varphi(s)d\beta^H(s) = \int_0^T (K_H^*\varphi)(t)dW(t), \tag{6.2.11}$$

if and only if $K_H^*\varphi \in L^2([0,T])$.

Also denoting $L^2_{\mathcal{H}}([0,T]) = \{\varphi \in \mathcal{H}, K_H^*\varphi \in L^2([0,T])\}$ and noticing $H > \frac{1}{2}$, we have

$$L^{1/H}([0,T]) \subset L^2_{\mathcal{H}}([0,T]), \tag{6.2.12}$$

for more details, we refer reader to [98]. Moreover, the following result can be shown.

Lemma 6.2.1. *For $\varphi \in L^{1/H}([0,T])$,*

$$H(2H-1)\int_0^T \int_0^T |\varphi(s)||\varphi(u)||s-u|^{2H-2}dsdu \leq c_H\|\varphi\|^2_{L^{1/H}([0,T])}, \tag{6.2.13}$$

where $c_H = \sqrt{\frac{H(2H-1)}{B(2-2H,H-\frac{1}{2})}}$, with B denoting the Beta function.

Proof. The proof can be found in Nualart, D. [107]. □

Further, we shall introduce the Hilbert space valued fractional Brownian motion and give the definition of the corresponding stochastic integral.

Let $(Y, \|\cdot\|_Y, \langle\cdot,\cdot\rangle_Y)$ and $(K, \|\cdot\|_K, \langle\cdot,\cdot\rangle_K)$ be separable Hilbert spaces. Let $\mathcal{L}_1(K,Y)$ denote the space of all bounded linear operators from K to Y. Let $Q \in \mathcal{L}_1(K,K)$ be a positive self-adjoint operator. Denote by $\mathcal{L}_2^0(K_0,Y)$ the space of all Hilbert-Schmidt operator from K_0 to Y. The norm is given by

$$\|\xi\|^2_{\mathcal{L}_2^0} = Tr(\xi Q^{1/2}(\xi Q^{1/2})^*) < \infty, \quad \xi \in \mathcal{L}_2^0(K_0,Y).$$

Let $\{\beta_n^H(t)\}_{n\in\mathbb{N}}$ be a sequence of two-sided one-dimensional standard fractional Brownian motion mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$. We consider a K-valued stochastic process $B^H(t)$ given formally by the following series:

$$B^H(t) = \sum_{n=1}^{\infty} \beta_n^H(t) Q^{\frac{1}{2}} e_n, \quad t \geq 0.$$

If Q is a non-negative self-adjoint trace class operator, then this series converges in the space Y, this is, it holds that $B^H(t) \in L^2(\Omega, K)$. Then, we say that the above $B^H(t)$ is a K-valued Q fractional Brownian motion with covariance operator Q. For example, if $\{\sigma_n\}_{n\in\mathbb{N}}$ is a bounded sequence of non-negative real numbers such that $Qe_n = \sigma_n e_n$ and assume that Q is a nuclear operator in K (that is, $\sum_{n=1}^{\infty} \sigma_n < \infty$), then the stochastic process

$$B^H(t) = \sum_{n=1}^{\infty} \beta_n^H(t) Q^{\frac{1}{2}} e_n = \sum_{n=1}^{\infty} \sqrt{\sigma_n} \beta_n^H(t) e_n, \quad t \geq 0, \tag{6.2.14}$$

is well-defined as a K-valued Q fractional Brownian motion. Let $\varphi : [0, T] \to \mathcal{L}_2^0(K_0, Y)$ be a measurable map such that

$$\sum_{n=1}^{\infty} \|K_H^*(\varphi Q^{\frac{1}{2}} e_n)\|_{L^2([0,T];Y)} < \infty. \tag{6.2.15}$$

Definition 6.2.2. Suppose that $\varphi : [0, T] \to \mathcal{L}_2^0(K_0, Y)$ satisfies (6.2.15). Its stochastic integral with respect to the fractional Brownian motion B^H is defined, for $t \geq 0$, as follows

$$\int_0^t \varphi(s) dB^H(s) := \sum_{n=1}^{\infty} \int_0^t \varphi(s) Q^{\frac{1}{2}} e_n d\beta_n^H = \sum_{n=1}^{\infty} \int_0^t (K_H^*(\varphi Q^{\frac{1}{2}} e_n))(s) d\beta(s), \tag{6.2.16}$$

where $\beta(s)$ is the standard Brownian motion used to present $B^H(s)$ as in 6.2.10. Notice that if

$$\|\varphi\|_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty} \|\varphi Q^{\frac{1}{2}} e_n\|_{L^{1/H}([0,T];Y)} < \infty, \tag{6.2.17}$$

then (6.2.15) holds, which follows immediately from (6.2.12).

Now we close this subsection by stating the following result which is fundamental to prove our result.

Lemma 6.2.2. *For any $\sigma : [0,T] \to \mathcal{L}_2^0(K_0, Y)$ satisfies*

$$\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0(K_0,Y)} ds < \infty$$

then the above sum in (6.2.17) is well defined as a Y-valued random variable and we have

$$\mathbb{E}\left\| \int_0^t \sigma(s) dB^H(s) \right\|_Y^2 \leq c_H H(2H-1) t^{2H-1} \int_0^t \|\sigma(s)\|^2_{\mathcal{L}_2^0(K_0,Y)} ds \qquad (6.2.18)$$

Proof. Lemma 6.2.2 is obtained as an application of Lemma 6.2.1. The proof can be found in Lemma 2 Caraballo et al. [17].

□

A similar lemma has also been proved in Lemma 2 by Boufoussi and Hajji [12].

6.3 The Existence of Mild Solutions for the System with Finite Delays

In this section, we establish the result for the system with finite delays by using contraction mapping principle.

Let $0 < t < \infty$, we have the following definition of mild solutions for system (6.1.3).

Definition 6.3.1. A stochastic process $\{x(t), t \in (-r, \infty)\}$ is called a mild solution of equation (6.1.3) if the following conditions hold:

(i) $x(t)$ is continuous on $(0, t_1]$ and each interval $(t_k, t_{k+1}]$, $k = 1, 2, 3, \ldots$,

(ii) For each t_k, $x(t_k^+) = \lim_{t \downarrow t_k^+} x(t)$ exists,

(iii) For each $t \in [0,T]$, $x(t)$ satisfies the following integral equation

$$
\begin{aligned}
x(t) = S(t)\big[\phi(0) + u(0,\phi(-\tau(0)))\big] &- u(t, x(t-\tau(t))) \\
&- \int_0^t AS(t-s)u(s, x(s-\tau(s)))ds \\
&+ \int_0^t S(t-s)f(s, x(s-\tau(s)))ds \\
&+ \int_0^t S(t-s)g(s, x(s-\tau(s)))dW(s) \\
&+ \int_0^t S(t-s)\sigma(s)dB^H(s) + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)),
\end{aligned}
\tag{6.3.1}
$$

and $x(0) = \phi \in P([-r,0];Y)$.

In order to prove the main results, we assume the following assumptions:

(**H1**) A is the infinitesimal generater of an exponentially stable analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$ in X, such that the following inequality holds

$$\|S(t)\| \leq Me^{-\gamma t}, \qquad t \geq 0,$$

for $M > 0$ and $\gamma > 0$.

(**H2**) The coefficients f, g satisfy Lipschitz conditions, i.e. there exist some positive constants C_1, C_2 and K_1, K_2 such that for any $x, y \in H$ and $t \geq 0$,

(i) $\|f(t,x) - f(t,y)\|_Y \leq K_1\|x-y\|_Y$,

(ii) $\|f(t,x)\|_Y \leq C_1(1 + \|x\|_Y)$,

(iii) $\|g(t,x) - g(t,y)\|_{\mathcal{L}_2^0} \leq K_2\|x-y\|_Y$,

(iv) $\|g(t,x)\|_{\mathcal{L}_2^0} \leq C_2(1 + \|x\|_Y)$.

Moreover, we assume that $f(t,0) = g(t,0) = 0$, for $x, y \in Y$.

(**H3**) There exist constants $\alpha \in (0,1]$ and $K_3 > 0$ such that for any $x \in Y$ and $t \geq 0, u(t,x) \in \mathcal{D}((-A)^\alpha)$ and

$$\|(-A)^\alpha u(t,x) - (-A)^\alpha u(t,y)\|_Y \leq K_3\|x-y\|_Y, \qquad x,\ y \in Y,$$

with $u(t,0)=0$.

(**H4**) There exist a positive constant series $\{q_k\}$ for each $k \in \{1,2,3,\ldots\}$, such that $\sum_{k=1}^{\infty} q_k = \kappa < \infty$ and

$$\|I_k(x) - I_k(y)\| \leq q_k \|x-y\|,$$

with $I_k(0)=0$ for each $x, y \in Y$.

(**H5**) The function $\sigma : [0,\infty) \to \mathcal{L}_2^0(K_0, Y)$ satisfies

$$\int_0^{\infty} e^{2\lambda s} \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds < \infty \ \text{ for some } \ \lambda > 0.$$

Theorem 6.3.1. *Suppose the assumptions (**H1**)-(**H5**) hold for some $\alpha \in (0,1)$, $p > 2$. We further assume that*

$$5\bigg[K_4{}^2\|(-A)^{-\alpha}\|^2 + M_{1-\alpha}^2 K_4{}^2\gamma^{-2\alpha}\Gamma(2\alpha-1) \\ + M^2K_1^2\gamma^{-2} + M^2K_2^2\gamma^{-2} + M^2\kappa^p\gamma^{-2}\bigg] < 1, \tag{6.3.2}$$

where $\Gamma(\cdot)$ is the Gamma function, $M_{1-\alpha}$ is the corresponding number in Lemma (2.1.4). Then the mild solution to (6.3.1) is exponential stability in mean square. In other words, there exists some constants $M_0(\phi) > 0, \mu > 0$ such that

$$\mathbb{E}\|y(t)\|_Y^2 \leq M_0(\phi)e^{-\mu t}, \quad t \geq 0.$$

Proof. First we define a space $\mathcal{S}$ as the family of all stochastic process $x(t)$, $t \in [-r, \infty)$, such that

$$\mathbb{E}\|x(t)\|_Y^2 \leq \tilde{M}\mathbb{E}\|\phi\|^2 e^{-\eta t}, \quad t \geq 0, \tag{6.3.3}$$

for some constants $\tilde{M} > 0$ and $\eta > 0$, where $\eta < \gamma$ with a norm

$$\|x\|_{\mathcal{S}} := \sup_{t\in[0,\infty)} \mathbb{E}\|x(t)\|_Y^p, \quad x \in \mathcal{S}.$$

It can be shown that $\mathcal{S}$, under this norm, is a Banach space.

Now we define a nonlinear map π on $\mathcal{S}$ by $\pi(x)(t) = \phi(t)$ for $t \in [-r, 0]$ and for $t \geq 0$,

$$\begin{aligned}\pi(x)(t) = & S(t)[\phi(0) + u(0, \phi(-\tau(0))] - u(t, x(t-\tau(t))) \\ & - \int_0^t AS(t-s)u(s, x(s-\tau(t)))ds + \int_0^t S(t-s)f(s, x(s-\tau(t)))ds \\ & + \int_0^t S(t-s)g(s, x(s-\tau(t)))dW(s) + \int_0^t S(t-s)\sigma(t)dB^H(s) \\ & + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-))\end{aligned} \tag{6.3.4}$$

Then it is clear that to prove stability of mild solutions to equation (6.1.3), it suffices to find a fixed point for the operator π in space $\mathcal{S}$. In order to show that π has a unique fixed point, we shall employ Banach fixed point theorem.

We first show that π is a mapping from $\mathcal{S}$ into $\mathcal{S}$. Let $x(t) \in \mathcal{S}$, from the definition of π we have for $t \geq 0$

$$\begin{aligned}\mathbb{E}\|\pi(x)(t)\|_Y^2 \leq & 7\mathbb{E}\|S(t)[\phi(0) + u(0, \phi)]\|_H^2 + 7\mathbb{E}\|u(t, x(t-\tau(s)))\|_Y^2 \\ & + 7\mathbb{E}\left\|\int_0^t AS(t-s)u(s, x(s-\tau(s)))ds\right\|_Y^2 \\ & + 7\mathbb{E}\left\|\int_0^t S(t-s)f(s, x(s-\tau(s)))ds\right\|_Y^2 \\ & + 7\mathbb{E}\left\|\int_0^t S(t-s)g(s, x(s-\tau(s)))dW(s)\right\|_Y^2 \\ & + 7\mathbb{E}\left\|\int_0^t S(t-s)\sigma(s)dB^H(s)\right\|_Y^2 \\ & + 7\mathbb{E}\left\|\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-))\right\|_Y^2 := 7\sum_{i=1}^{7} I_i(t).\end{aligned} \tag{6.3.5}$$

By the definition of $\mathcal{S}$ and assumption (**H3**) it follows that for each $t \geq 0$,

$$\begin{aligned}I_1(t) &= \mathbb{E}\|S(t)[\phi(0) + u(0, -\tau(0))]\|_Y^2 \\ &\leq \|S(t)\|^2\mathbb{E}\|\phi(0) + u(0, -\tau(0))\|_Y^2 \\ &\leq M^2e^{-2\gamma t}\mathbb{E}\|\phi(0) + u(0, -\tau(0))\|_Y^2.\end{aligned} \tag{6.3.6}$$

Moreover, by using (6.3.3), (**H2**) and (**H3**) for $t \geq r$, we have

$$\begin{aligned} I_2(t) &= \mathbb{E}\|u(t, x(t-\tau(s)))\|_Y^2 \\ &= \mathbb{E}\|(-A)^{-\alpha}(-A)^{\alpha}u(t, x(t-\tau(s)))\|_Y^2 \\ &\leq \|(-A)^{-\alpha}\|^2\mathbb{E}\|(-A)^{\alpha}u(t, x(t-\tau(s)))\|_Y^2 \\ &\leq K_3^2\|(-A)^{-\alpha}\|^2\mathbb{E}\|x(t-\tau(s))\|_Y^2. \end{aligned} \tag{6.3.7}$$

Since $x \in \mathcal{S}$, x satisfies the relation

$$\mathbb{E}\|x(t-\tau(s))\|_Y^2 \leq \tilde{M}e^{\eta r}\mathbb{E}\|\phi\|^2 e^{-\eta t}, \qquad t \geq r. \tag{6.3.8}$$

Substituting (6.3.8) into (6.3.7), we get that

$$I_2(t) \leq K_3^2\|(-A)^{-\alpha}\|^2\tilde{M}e^{\eta r}\mathbb{E}\|\phi\|^2 e^{-\eta t}, \qquad t \geq r. \tag{6.3.9}$$

By employing Theorem 2.1.4 and assumption (**H4**), for $t \geq r$, we have that

$$\begin{aligned} I_3(t) &= \mathbb{E}\left\|\int_0^t (-A)S(t-s)u(s, x(s-\tau(s)))ds\right\|_Y^2 \\ &\leq \mathbb{E}\left\|\int_0^t (-A)^{1-\alpha}(-A)^{\alpha}S(t-s)u(s, x(s-\tau(s)))ds\right\|_Y^2 \\ &\leq \mathbb{E}\left(\int_0^t \|(-A)^{(1-\alpha)}S(t-s)\|\|(-A)^{\alpha}u(s, x(s-\tau(s)))\|_Y ds\right)^2. \end{aligned} \tag{6.3.10}$$

Under the assumption (**H3**), we obtain for $t \geq r$ that

$$\begin{aligned} I_3(t) &\leq \mathbb{E}\left(\int_0^t \|M_{1-\alpha}(t-s)^{-(1-\alpha)}e^{-\gamma(t-s)}\|\|(-A)^{\alpha}u(s, x(s-\tau(s)))\|_Y ds\right)^2 \\ &\leq M_{1-\alpha}^2\mathbb{E}\left(\int_0^t (t-s)^{\alpha-1}e^{-\gamma(t-s)}\|(-A)^{\alpha}u(s, x(s-\tau(s)))\|_Y ds\right)^2. \end{aligned} \tag{6.3.11}$$

Moreover, by using Hölder inequality, we get that, for $t \geq r$

$$\begin{aligned} I_3(t) &\leq M_{1-\alpha}^2\left(\int_0^t (t-s)^{2(\alpha-1)}e^{-\gamma(t-s)}ds\right) \cdot K_4^2\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-r(s))\|_Y^2 ds \\ &\leq M_{1-\alpha}^2{K_3}^2\gamma^{1-2\alpha}\Gamma(2\alpha-1)\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-\tau(s))\|_Y^2 ds. \end{aligned} \tag{6.3.12}$$

Since $x \in \mathcal{S}$, x satisfies that

$$\mathbb{E}\|x(t-\tau(t))\|_Y^2 \le \tilde{M}e^{\eta r}\mathbb{E}\|\phi\|^2 e^{-\eta t}, \qquad t \ge r. \tag{6.3.13}$$

Substituting (6.3.13) into (6.3.12), for $t \ge r$, we get that

$$\begin{aligned} I_3(t) &\le M_{1-\alpha}^2 K_3^2 \gamma^{1-2\alpha}\Gamma(2\alpha-1)\int_0^t e^{-\gamma(t-s)}\tilde{M}e^{\eta r}\mathbb{E}\|\phi\|^2 e^{-\eta s}ds \\ &\le M_{1-\alpha}^2 K_3^2 \gamma^{1-2\alpha}\Gamma(2\alpha-1)\tilde{M}\mathbb{E}\|\phi\|^2 e^{\eta\gamma}e^{-\gamma t}\frac{1}{\gamma-\eta}e^{\gamma t}e^{-\eta t} \\ &= M_{1-\alpha}^2 K_3^2 \gamma^{1-2\alpha}\Gamma(2\alpha-1)\frac{\tilde{M}\mathbb{E}\|\phi\|^2 e^{\eta\gamma}}{\gamma-\eta}e^{-\eta t}. \end{aligned} \tag{6.3.14}$$

Now employing assumptions (**H1**), (**H2**), we similarly have that for each $t \ge r$,

$$\begin{aligned} I_4(t) &= \mathbb{E}\bigg\|\int_0^t S(t-s)f(s,x(s-\tau(s)))ds\bigg\|_Y^2 \\ &\le \mathbb{E}\bigg(\int_0^t \|S(t-s)\|\cdot\|f(s,x(s-\tau(s)))\|_Y ds\bigg)^2 \\ &\le \mathbb{E}\bigg(\int_0^t Me^{-\gamma(t-s)}K_1\|x(s-\tau(s))\|_Y ds\bigg)^2. \end{aligned} \tag{6.3.15}$$

Furthermore, by using Hölder inequality and substituting (6.3.13) into (6.3.15), for $t \ge r$, we get that

$$\begin{aligned} I_4(t) &\le M^2K_1^2\bigg(\int_0^t e^{-\gamma(t-s)}ds\bigg)\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-\tau(s))\|_Y^2 ds \\ &\le M^2K_1^2\gamma^{-1}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-\tau(s))\|_Y^2 ds \\ &\le M^2K_1^2\gamma^{-1}\frac{\tilde{M}\mathbb{E}\|\phi\|^2 e^{\eta r}}{\gamma-\eta}e^{-\eta t}. \end{aligned} \tag{6.3.16}$$

On the other hand, by taking (**H1**) into account, we have for each $t \ge r$,

$$\begin{aligned} I_5(t) &= \mathbb{E}\bigg\|\int_0^t S(t-s)g(s,x(s-\tau(s)))dW(s)\bigg\|_Y^2 \\ &\le \mathbb{E}\int_0^t \big(\|S(t-s)g(s,x(s-\tau(s)))\|_Y^2\big)ds \\ &\le \mathbb{E}\int_0^t \|S(t-s)\|^2\big(\|g(s,x(s-\tau(s)))\|_{\mathcal{L}_2^0}^2\big)ds \\ &\le \int_0^t M^2e^{-2\gamma(t-s)}\mathbb{E}\|g(s,x(s-\tau(s)))\|_{\mathcal{L}_2^0}^2 ds, \end{aligned} \tag{6.3.17}$$

Moreover, under Lipschitz condition (**H2**), Lemma (2.2.1) and Hölder inequality, for $t \geq r$, the above inequality (6.3.17) turns to be

$$\begin{aligned} I_5(t) &\leq M^2 K_2^2 \left(\int_0^t e^{-2\gamma(t-s)} \mathbb{E}\|x(s-\tau(s))\|_Y^2 ds \right) \\ &\leq M^2 K_2^2 \left(\int_0^t e^{-\gamma(t-s)} ds \right) \int_0^t e^{-\gamma(t-s)} \mathbb{E}\|x(s-\tau(s))\|_Y^2 ds \\ &\leq M^2 K_2^2 \int_0^t e^{-\gamma(t-s)} \mathbb{E}\|x(s-\tau(s))\|_Y^2 ds \\ &\leq M^2 K_2^2 \frac{\tilde{M}\mathbb{E}\|\phi\|^2 e^{\eta r}}{\gamma - \eta} e^{-\eta t}. \end{aligned} \tag{6.3.18}$$

Furthermore, by taking (**H1**), Lemma 6.2.2 and Hölder inequality into account, we have for each $t \geq r$,

$$\begin{aligned} I_6(t) &= \mathbb{E}\left\| \int_0^t S(t-s)\sigma(s) dB^H(s) \right\|_Y^2 \\ &\leq M^2 c_H H(2H-1) t^{2H-1} \int_0^t e^{-2\gamma(t-s)} \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds. \end{aligned} \tag{6.3.19}$$

From inequality (6.3.19), we can deduce that

$$\begin{aligned} I_6(t) &= M^2 c_H H(2H-1) t^{2H-1} \int_0^t e^{-2\gamma t} e^{2\gamma s} e^{-2\lambda s} e^{2\lambda s} \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds \\ &\leq \begin{cases} M^2 c_H H(2H-1) t^{2H-1} e^{-2\gamma t} \displaystyle\int_0^t e^{2\lambda s} \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds, & \gamma \leq \lambda, \\ M^2 c_H H(2H-1) t^{2H-1} \displaystyle\int_0^t e^{-2\gamma t} e^{(2\gamma - 2\lambda)t} e^{2\lambda s} \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds, & \gamma > \lambda, \end{cases} \\ &\leq \begin{cases} M^2 c_H H(2H-1) t^{2H-1} e^{-2\gamma t} \displaystyle\int_0^t e^{2\lambda s} \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds, & \gamma \leq \lambda, \\ M^2 c_H H(2H-1) t^{2H-1} e^{-2\lambda t} \displaystyle\int_0^t e^{2\lambda s} \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds, & \gamma > \lambda, \end{cases} \end{aligned} \tag{6.3.20}$$

Since $\sup_{t \geq 0} \left(t^{2H-1} e^{-(\gamma \wedge \lambda)t} \right) < \infty$, by taking (**H5**) into account, inequality (6.3.20) turns to be

$$I_6(t) \leq M_2 e^{-(\gamma \wedge \lambda)t}, \tag{6.3.21}$$

where $\gamma \wedge \lambda = \min\{\lambda, \gamma\}$.

Now, we estimate the impulsive term. From the condition (**H1**), for each $t \geq r$, we obtain

$$
\begin{aligned}
I_7(t) &= \mathbb{E}\Big\| \sum_{0<t_k<t} S(t-t_k) I_k(x(t_k^-)) \Big\|_Y^2 \\
&\leq \mathbb{E}\bigg(\sum_{0<t_k<t} M e^{-\gamma(t-t_k)} q_k \|(x(t_k^-)\|_Y \bigg)^2.
\end{aligned} \tag{6.3.22}
$$

Now by taking assumption (**H4**) and Hölder inequality into account, we have for each $t \geq r$,

$$
\begin{aligned}
I_7(t) &\leq M^2 \mathbb{E}\bigg(\kappa \int_0^t e^{-\gamma(t-s)} \|x(s)\|_Y ds \bigg)^2 \\
&\leq M^2 \kappa^2 \bigg(\int_0^t e^{-\gamma(t-s)} ds \bigg) \int_0^t e^{-\gamma(t-s)} \mathbb{E}\|x(s)\|_Y^2 ds \\
&\leq M^2 \kappa^2 \gamma^{-1} \int_0^t e^{-\gamma(t-s)} \mathbb{E}\|x(s)\|_Y^2 ds.
\end{aligned} \tag{6.3.23}
$$

Since $x \in \mathcal{S}$, x satisfies that for $t \geq r$,

$$
\mathbb{E}\|x(t)\|_Y^2 \leq \tilde{M} \mathbb{E}\|\phi\|^2 e^{-\eta s}. \tag{6.3.24}
$$

Substituting (6.3.24) into (6.3.22), we get that for $t \geq r$,

$$
I_7(t) \leq M^2 \kappa^2 \gamma^{-1} \frac{\tilde{M} \mathbb{E}\|\phi\|^2 e^{\eta r}}{\gamma - \eta} e^{-\eta t}. \tag{6.3.25}
$$

Recalling (6.3.5) and combining from (6.3.6) to (6.3.25), one can see that there exist some numbers $M_c > 0$ and $\eta_1 > 0$ such that,

$$
\mathbb{E}\|(\pi x)(t)\|_Y^2 \leq M_c \mathbb{E}\|\phi\|^2 e^{-\eta_1 t}, \tag{6.3.26}
$$

thus, we conclude that $\pi(\mathcal{S}) \subset \mathcal{S}$.

Next, we show that the mapping π is contractive. For any $x, y \in \mathcal{S}$, we have,

$$
\begin{aligned}
&\mathbb{E}\|(\pi x)(t)-(\pi y)(t)\|_Y^2 \\
\leq\ & 5\mathbb{E}\|u(t,x(t-\tau(s)))-u(t,y(t-\tau(s)))\|_Y^2 \\
&+5\mathbb{E}\Big\|\int_0^t AS(t-s)\big(u(s,x(s-\tau(s)))-u(s,y(s-\tau(s)))\big)ds\Big\|_Y^2 \\
&+5\mathbb{E}\Big\|\int_0^t S(t-s)\big(f(s,x(s-\tau(s)))-f(s,y(s-\tau(s)))\big)ds\Big\|_Y^2 \\
&+5\mathbb{E}\Big\|\int_0^t S(t-s)\big(g(s,x(s-\tau(s)))-g(s,y(s-\tau(s)))\big)dW(s)\Big\|_Y^2 \\
&+5\mathbb{E}\Big\|\sum_{0<t_k<t} S(t-t_k)(I_k)(x(t_k^-)-y(t_k^-))\Big\|_Y^2 = 5\sum_{i=1}^{5} J_i(t).
\end{aligned}
\tag{6.3.27}
$$

Noting that $x(s)=y(s)=\phi(s)$ for $s\in[-r,0]$, then from assumption (**H3**), we have for $t \geq r$ that

$$
\begin{aligned}
J_1(t) &= \mathbb{E}\|u(t,x(t-\tau(s)))-u(t,y(t-\tau(s)))\|_Y^2 \\
&\leq K_3^2\|(-A)^{-\alpha}\|^2\mathbb{E}\|x(t-\tau(s))-y(t-\tau(s))\|_Y^2 \\
&\leq K_3^2\|(-A)^{-\alpha}\|^2\sup_{t\geq 0}\mathbb{E}\|x(t)-y(t)\|_Y^2
\end{aligned}
\tag{6.3.28}
$$

By using Lemma (2.1.4) and Hölder's inequality, we have for $t \geq r$

$$
\begin{aligned}
J_2(t) &= \mathbb{E}\Big\|\int_0^t AS(t-s)\big[u(s,x(s-\tau(s)))-u(s,y(s-\tau(s)))\big]ds\Big\|_Y^2 \\
&\leq M_{1-\alpha}^2K_3^2\int_0^t (t-s)^{2(\alpha-1)}e^{-\gamma(t-s)}ds \\
&\quad\cdot\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-\tau(s))-y(s-\tau(s))\|_Y^2 ds \\
&\leq M_{1-\alpha}^2K_3^2\Gamma(2\alpha-1)\gamma^{1-2\alpha} \\
&\quad\cdot\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-\tau(s))-y(s-\tau(s))\|_Y^2 ds \\
&\leq M_{1-\alpha}^2K_3^2\Gamma(2\alpha-1)\gamma^{2\alpha}\sup_{t\geq 0}\mathbb{E}\|x(t)-y(t)\|_Y^2.
\end{aligned}
\tag{6.3.29}
$$

By employing the assumption (**H1**), we have for $t \geq r$ that

$$\begin{aligned} J_3(t) &= \mathbb{E}\bigg\| \int_0^t S(t-s)\big(f(s, x(s-\tau(s))) - f(s, y(s-\tau(s)))\big) ds \bigg\|_Y^2 \\ &\leq \mathbb{E}\bigg(\int_0^t \|S(t-s)\| \|f(s, x(s-\tau(s))) - f(s, y(s-\tau(s)))\|_Y ds \bigg)^2 \\ &\leq \mathbb{E}\bigg(\int_0^t M^2 e^{-\gamma(t-s)} \|f(s, x(s-\tau(s))) - f(s, y(s-\tau(s))\|_Y ds \bigg)^2. \end{aligned} \tag{6.3.30}$$

Furthermore, by taking assumption (**H2**) and Hölder's inequality into account, for $t \geq r$, one can get,

$$\begin{aligned} J_3(t) &\leq M^2 K_1^2 \gamma^{-1} \int_0^t e^{-\gamma(t-s)} \sup_{-r \leq s < \infty} \mathbb{E}\|x(s-\tau(s)) - y(s-\tau(s))\|_Y^2 ds \\ &= M^2 K_1^2 \gamma^{-1} \sup_{-r \leq s < \infty} \mathbb{E}\|x(s-\tau(s)) - y(s-\tau(s))\|_Y^2 \int_0^t e^{-\gamma(t-s)} ds \\ &\leq M^2 K_1^2 \gamma^{-2} \sup_{-r \leq s < \infty} \mathbb{E}\|x(s-\tau(s)) - y(s-\tau(s))\|_Y^2. \end{aligned} \tag{6.3.31}$$

By employing Lemma (2.2.1) and condition (**H2**), we have for $t \geq r$

$$\begin{aligned} J_4(t) &= \mathbb{E}\bigg\| \int_0^t S(t-s)\big(g(s, x(s-\tau(s))) - g(s, y(s-\tau(s)))\big) dW(s) \bigg\|_Y^2 \\ &\leq \mathbb{E} \int_0^t \big\| S(t-s)\big(g(s, x(s-\tau(s))) - g(s, y(s-\tau(s)))\big) \big\|_Y^2 ds \\ &\leq \mathbb{E} \int_0^t \|S(t-s)\|^2 \|g(s, x(s-\tau(s))) - g(s, y(s-\tau(s)))\|_{\mathcal{L}_2^0}^2 ds. \end{aligned} \tag{6.3.32}$$

By taking condition (**H1**) Hölder inequality into account, for $t \geq r$, we have that

$$\begin{aligned} J_4(t) &\leq \bigg(\int_0^t M^2 e^{-2\gamma(t-s)} \mathbb{E}\|g(s, x(s-\tau(s))) - g(s, y(s-\tau(s)))\|_{\mathcal{L}_2^0}^2 ds \bigg) \\ &\leq M^2 K_2^2 \bigg(\int_0^t e^{-2\gamma(t-s)} \mathbb{E}\|x(s-\tau(s)) - y(s-\tau(s))\|_Y^2 ds \bigg) \\ &\leq M^2 K_2^2 \int_0^t e^{-2\gamma(t-s)} \sup_{-r \leq s < \infty} \mathbb{E}\|x(s-\tau(s)) - y(s-\tau(s))\|_Y^2 ds \\ &\leq M^2 K_2^2 \sup_{-r \leq s < \infty} \mathbb{E}\|x(s-\tau(s)) - y(s-\tau(s))\|_Y^2 \int_0^t e^{-2\gamma(t-s)} ds \\ &\leq M^2 K_2^2 (2\gamma)^{-1} \sup_{-r \leq s < \infty} \mathbb{E}\|x(s-\tau(s)) - y(s-\tau(s))\|_Y^2. \end{aligned} \tag{6.3.33}$$

Now, from assumptions (**H1**)and (**H4**), for $t \geq r$, we get

$$
\begin{aligned}
J_5(t) &= \mathbb{E}\bigg\| \sum_{0<t_k<t} S(t-t_k) I_k(x(t_k^-) - y(t_k^-)) \bigg\|_Y^2 \\
&\leq M^2 \mathbb{E}\bigg(\sum_{0<t_k<t} e^{-\gamma(t-t_k)} q_k \|(x(t_k^-) - y(t_k^-))\|_Y \bigg)^2 \\
&\leq M^2 \mathbb{E}\bigg(\kappa \int_0^t e^{-\gamma(t-s)} \|x(s) - y(s)\|_Y ds \bigg)^2.
\end{aligned} \tag{6.3.34}
$$

By using Hölder inequality, for $t \geq 0$, we have

$$
\begin{aligned}
J_5(t) &\leq M^2\kappa^2 \bigg(\int_0^t e^{-\gamma(t-s)} ds \bigg) \int_0^t e^{-\gamma(t-s)} \mathbb{E}\|x(s) - y(s)\|_Y^2 ds \\
&\leq M^2\kappa^2 \bigg(\int_0^t e^{-\gamma(t-s)} ds \bigg) \int_0^t e^{-\gamma(t-s)} \sup_{-r \leq s < \infty} \mathbb{E}\|x(s) - y(s)\|_Y^2 ds \\
&\leq M^2\kappa^2\gamma^{-1} \sup_{-r \leq s < \infty} \mathbb{E}\|x(s) - y(s)\|_Y^2 \int_0^t e^{-\gamma(t-s)} ds \\
&\leq M^2\kappa^2\gamma^{(-2)} \sup_{-r \leq s < \infty} \mathbb{E}\|x(s) - y(s)\|_Y^2 ds.
\end{aligned} \tag{6.3.35}
$$

We proved that π is a contraction mapping.

$$
\begin{aligned}
&\|(\pi x) - (\pi y)\|_{\mathcal{S}}^2 \\
\leq 5 \bigg[&{K_3}^2 \|(-A)^{-\alpha}\|^2 + M_{1-\alpha}^2 K_3^2 \gamma^{-2\alpha} \Gamma(2\alpha - 1) \\
&+ M^2 K_1^2 \gamma^{-2} + M^2 K_2^2 (2\gamma)^{-1} + M^2 \kappa^2 \gamma^{(-2)} \bigg] \|x - y\|_{\mathcal{S}}^2.
\end{aligned} \tag{6.3.36}
$$

Since,

$$
\begin{aligned}
5 \bigg[&K_4^2 \|(-A)^{-\alpha}\|^2 + M_{1-\alpha}^2 K_4^2 \gamma^{-2\alpha} \Gamma(2\alpha - 1) \\
&\qquad + M^2 K_1^2 \gamma^{-2} + M^2 K_2^2 (2\gamma)^{-1} + M^2 \kappa^2 \gamma^{(-2)} \bigg] < 1,
\end{aligned} \tag{6.3.37}
$$

π is a contraction mapping and hence there exists a unique fixed point which is a mild solution of the equation (6.1.3) on $(-r, \infty)$. This completes the proof.

□

6.4 The Mild Solution of the System with Infinite Delays

Theorem 6.4.1. *Under the assumptions of (**H1**) to (**H5**), the mild solution to system (6.1.3) exists uniquely and converges to zero in mean square, i.e.,*

$$\lim_{t\to\infty} \mathbb{E}\|x(t)\|^2 = 0.$$

Proof. Denote by $\mathcal{S}'$ the space of all stochastic processes $x(t,\omega) : (-\infty,\infty)\times\Omega \to Y$ satisfying $x(t) = \phi(t),\ t \in (-\infty, 0]$ and the definition (6.3.1) and

$$\lim_{t\to\infty} \mathbb{E}\|x(t)\|^2 = 0. \tag{6.4.1}$$

We define a nonlinear the operator Ψ on $\mathcal{S}'$ by $(\Psi x)(t) = \phi(t)$, $-\infty < t \le 0$ and

$$\begin{aligned}(\Psi x)(t) = {} & S(t)[\phi(0) + u(0, \phi(-\tau(0))] - u(t, x(t-\tau(t))) \\ & - \int_0^t AS(t-s)u(s, x(s-\tau(s)))ds + \int_0^t S(t-s)f(s, x(s-\tau(s)))ds \\ & + \int_0^t S(t-s)g(s, x(s-\tau(s)))dW(s) + \int_0^t S(t-s)\sigma(t)dB^H(s) \\ & + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)) := \sum_{i=1}^{7} \Pi_i(t), \qquad t \ge 0.\end{aligned} \tag{6.4.2}$$

Since $(\Psi x)(t) = (\pi x)(t)$ on $[0,\infty)$, this implies that Ψ is contractive. Hence, it remains to check $\Psi x \subset \mathcal{S}'$. In order to obtain this result, we shall prove that for all $x \in \mathcal{S}'$,

$$\lim_{t\to\infty} \mathbb{E}\|(\Psi x)(t)\|^2 = 0.$$

By definition of $\mathcal{S}'$, assumption (**H5**) and the fact $t - \tau(t) \to \infty,\ t \to \infty$, we get

$$\lim_{t\to\infty} \mathbb{E}\|\Pi_1(t)\|^2 = \lim_{t\to\infty} \mathbb{E}\|\Pi_2(t)\|^2 = \lim_{t\to\infty} \mathbb{E}\|\Pi_6(t)\|^2 = 0. \tag{6.4.3}$$

We further have

$$\begin{aligned}
&\mathbb{E}\|\Pi_3(t)\|^2\\
&\leq \mathbb{E}\left\| \int_0^t AS(t-s)u(s,x(s-\tau(s)))ds \right\|_Y^2\\
&\leq M_{1-\alpha}^2 K_3^2 \int_0^t (t-s)^{\alpha-1} e^{-\gamma(t-s)} ds \int_0^t (t-s)^{\alpha-1} e^{-\gamma(t-s)} \mathbb{E}\|x(s-\tau(s))\|_Y^2 ds\\
&\leq M_{1-\alpha}^2 K_3^2 \Gamma(2\alpha-1)\gamma^{-\alpha} \int_0^t (t-s)^{\alpha-1} e^{-\gamma(t-s)} \mathbb{E}\|x(s-\tau(s))\|_Y^2 ds.
\end{aligned} \tag{6.4.4}$$

For any $x \in \mathcal{S}'$ and $\varepsilon > 0$, it follows from (6.4.1) that there exists $s_1 > 0$ such that $\mathbb{E}\|x(s-\tau(s))\|^2 < \varepsilon$ for all $s \geq s_1$. Thus we obtain

$$\begin{aligned}
\mathbb{E}\|\Pi_3(t)\|^2 \leq & M_{1-\alpha}^2 K_3^2 \Gamma(2\alpha-1)\gamma^{-\alpha} \int_0^{s_1} (t-s)^{\alpha-1} e^{-\gamma(t-s)} \mathbb{E}\|x(s-\tau(s))\|_Y^2 ds\\
& + M_{1-\alpha}^2 K_3^2 \Gamma^2(2\alpha-1)\gamma^{-2\alpha}\varepsilon,
\end{aligned} \tag{6.4.5}$$

which proves $\lim_{t\to\infty} \mathbb{E}\|\Pi_3(t)\|^2 \leq M_{1-\alpha}^2 K_3^2 \Gamma^2(2\alpha-1)\gamma^{-2\alpha}\varepsilon$.

Similarly, we also have

$$\begin{aligned}
\mathbb{E}\|\Pi_4(t)\|^2 &= \mathbb{E}\left\| \int_0^t S(t-s) f(s,x(s-\tau(s)))ds \right\|_Y^2\\
&\leq M^2 K_1^2 \int_0^t e^{-\gamma(t-s)} ds \int_0^t e^{-\gamma(t-s)} \mathbb{E}\|x(s-\tau(s)\|_Y^2 ds\\
&\leq M^2 K_1^2 \gamma^{-1} \int_0^t e^{-\gamma(t-s)} \mathbb{E}\|x(s-\tau(s)\|_Y^2 ds.
\end{aligned} \tag{6.4.6}$$

For any $x \in \mathcal{S}'$ and $\varepsilon > 0$, it follows from (6.4.1) that there exists $s_1 > 0$ such that $\mathbb{E}\|x(s-\tau(s))\|^2 < \varepsilon$ for all $s \geq s_1$. We have that

$$\mathbb{E}\|\Pi_4(t)\|^2 \leq M^2 K_1^2 \gamma^{-1} \int_0^{s_1} e^{-\gamma(t-s)} \mathbb{E}\|x(s-\tau(s)\|_Y^2 ds + M^2 K_1^2 \gamma^{-2}\varepsilon, \tag{6.4.7}$$

which proves that

$$\lim_{t\to\infty} \mathbb{E}\|\Pi_4(t)\|^2 \leq M^2 K_1^2 \gamma^{-2}\varepsilon, \tag{6.4.8}$$

Moreover, we have

$$\begin{aligned}\mathbb{E}\|\Pi_5(t)\|^2 &= \mathbb{E}\bigg\|\int_0^t S(t-s)g(s,x(s-\tau(s)))dW(s)\bigg\|_Y^2 \\ &\leq M^2K_2^2\int_0^t e^{-2\gamma(t-s)}\mathbb{E}\|x(s-\tau(s)\|_Y^2 ds,\end{aligned} \tag{6.4.9}$$

and

$$\mathbb{E}\|\Pi_5(t)\|^2 \leq M^2K_2^2\int_0^{s_1} e^{-2\gamma(t-s)}\mathbb{E}\|x(s-\tau(s)\|_Y^2 ds + M^2K_2^2(2\gamma)^{-1}\varepsilon, \tag{6.4.10}$$

which proves that

$$\lim_{t\to\infty}\mathbb{E}\|\Pi_5(t)\|^2 \leq M^2K_2^2(2\gamma)^{-1}\varepsilon, \tag{6.4.11}$$

Let $\varepsilon \to 0$, we have $\lim_{t\to\infty}\mathbb{E}\|\Pi_5(t)\|^2 = 0$. Furthermore, since

$$\begin{aligned}\mathbb{E}\|\Pi_7(t)\|^2 &= \mathbb{E}\|\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-))\|^2 \\ &\leq M^2\mathbb{E}\bigg(\sum_{0<t_k<t} e^{-\gamma(t-t_k)}q_k\|x(t_k^-)\|_Y\bigg)^2 \\ &\leq M^2\mathbb{E}\bigg(\kappa\int_0^t e^{-\gamma(t-s)}\|x(s)\|_Y ds\bigg)^2 \\ &\leq M^2\kappa^2\int_0^t e^{-\gamma(t-s)}ds\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s)\|_Y^2 ds \\ &\leq M^2\kappa^2\gamma^{-1}\int_0^{s_1} e^{-\gamma(t-s)}ds + M^2\kappa^2\gamma^{-2}\varepsilon,\end{aligned} \tag{6.4.12}$$

thus, $\lim_{t\to\infty}\mathbb{E}\|\Pi_7(t)\|^2 \leq M^2\kappa^2\gamma^{-2}\varepsilon$. Since ε is arbitrary, let $\varepsilon \to 0$, we have $\lim_{t\to\infty}\mathbb{E}\|\Pi_7(t)\|^2 = 0$.

Once again, We complete the proof of the theorem by employing the Banach fixed point theorem.

□

6.5 Illustrative Example

In this section, we consider the following neutral stochastic partial differential equation with delays and impulsive effects

$$\begin{cases} d[x(t) + \alpha_0\Big(-\dfrac{\partial^2}{\partial x^2}\Big)^{-\alpha} x(t-\tau(t))] = \dfrac{\partial^2}{\partial x^2} x(t) + \alpha_1 x(t-\tau(t))dt \\ \qquad\qquad\qquad\qquad\qquad\qquad +\alpha_2 x(t-\tau(t))d\beta(t) + e^{-t} d\beta^H(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \geq 0,\ t \neq t_k, \\ \Delta x(t_k) = x(t_k) - x(t_k^-) = \alpha_3 x(t_k), \qquad t = t_k,\ k \in \mathbb{N}, \\ x_0(t) = \phi(t), \quad t \in [-r, 0],\ \ x \in [0, \pi], \end{cases} \tag{6.5.1}$$

where $\alpha_i > 0,\ i = 0, 1, 2, 3,\ \alpha \in (0, 1)$ are constants and $\beta(t)$ denotes the one-dimetional Brownian motion.

Let $X = L^2([0, \pi])$ with the norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. Define $A : X \to X$ by $Ax = x^{''}$ with domain

$$\mathcal{D}(A) := \{x \in X : x, x^{'} \text{ are absolutely continuous } x^{''} \in X, x(0) = x(\pi) = 0\}.$$

Then the equation (6.5.1) can be written in the form of the system (6.1.3) with the coefficients

$$\begin{aligned} &u(t, x(t-\tau(t))) = \alpha_0(-A)^{-\alpha} x(t-\tau(t)), \\ &f(t, x(t-\tau(t))) = \alpha_1 x(t-\tau(t)), \\ &g(t, x(t-\tau(t))) = \alpha_2 x(t-\tau(t)), \\ &I_k(x(t_k)) = \alpha_3 x(t_k^-),\ \ \sigma(t) = e^{-t}. \end{aligned} \tag{6.5.2}$$

Thus, the assumptions (**H1**)-(**H6**) are satisfied with

$$M = \gamma = 1,\ K_3 = \alpha_0 \|(-A)^{-\alpha}\|,\ K_1 = \alpha_1,\ K_2 = \alpha_2,\ \kappa = \alpha_3,\ k \in \mathbb{N}$$

For the operator A, it is known from Pazy [109] that the following properties hold:

(1) $Ax = \sum_{n=1}^{\infty} -n^2 \langle x, e_n \rangle e_n,\ x \in \mathcal{D}(A)$, where $e_n(t) = \sqrt{\frac{2}{\pi}} \sin(nt),\ n = 1, 2, \cdots$ is the set of eigenvector of A.

(2) A is the infinitesimal generator of an analytic semigroup $S(t),\ t \geq 0$, in X:

$$S(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} \langle x, e_n \rangle e_n, \text{ for all } x \in X \text{ and every } t > 0.$$

(3) The unbounded linear operator $(-A)^{\frac{3}{4}}$ is well defined and given by

$$(-A)^{\frac{3}{4}} x = \sum_{n=1}^{\infty} n^{\frac{3}{2}} \langle x, e_n \rangle e_n, \qquad x \in X,$$

with domain

$$\mathcal{D}((-A)^{\frac{3}{4}}) := \left\{ x \in X : \sum_{n=1}^{\infty} n^{\frac{3}{2}} \langle x, e_n \rangle e_n \in X \right\}.$$

Consequently, we can conclude, by Theorem 6.3.2, that the stochastic partial equation (6.5.1) has a unique mild solution and that this solution converges to zero in mean square if the parameters $\alpha_1,\ \alpha_2,$ and α_3 satisfy the following relation:

$$\alpha_0^2 \|(-A)^{-\alpha}\|^4 + \alpha_0^2 \|(-A)^{-\alpha}\|^2 M_{1-\alpha}^2 \Gamma(2\alpha - 1) + \alpha_1^2 + 2\alpha_2^2 + \alpha_3^2 < \frac{1}{5}. \quad (6.5.3)$$

Chapter 7

Conclusion

In this chapter, we summarize the material presented in this thesis. We have studied and analysed some properties of stability of mild solutions to several different stochastic models in infinite dimensional spaces, mainly in Hilbert space. The stochastic models have been investigated such as neutral stochastic impulsive partial differential delay equations with Poisson jumps, stochastic partial integro-differential equations with delays and impulsive stochastic partial differential equations driven by a fractional Brownian motion with infinite delays.

In Chapter 3, we have discussed the stability of mild solutions to neutral stochastic impulsive partial differential delay equations with Poisson jumps. Under some natural conditions, by employing the Banach fixed point theorem, we have given the condition for p-th moment exponential stability of mild solutions to the system. The existence, uniqueness and asymptotic behaviour of solutions of the stochastic differential equations have been studied by using Lyapunov's direct method in Liu and Truman [82], Tanguchi [114] and among others. Comparing to their works, we applied fixed point theory to discuss the stability of mild solutions to stochastic delay systems, where the conditions do not require the boundedness of delays. In contrast with Cui and Yan [36] [35], we have studied more general type of equations which include both Poisson point processes and impulsive effects with delays. Moreover, we have also considered p-th moment exponential stability of mild solutions.

The main difficulty in studying the exponential stability of mild solutions to the case of impulsive stochastic differential delay equations comes from impulsive effects in the system. Those type of stochastic models have not been fully developed. Although the investigation of asymptotic stability of nonlinear impulsive stochastic impulsive stochastic differential equations has been given in Sakthivel and Lou [87]. Chen [30] established an impulsive-integral inequality, some sufficient conditions about the exponential stability of mild solutions for impulsive stochastic partial differential equations with delays are obtained. Comparing to Chen [30], Chapter 4 developed an impulsive-integral inequality for a more general type of system which contains impulsive effects, delays and Poisson jumps. We have also studied the p-th exponential stability of mild solutions to system with more general conditions.

Stochastic partial differential equations have applied in a various of application areas. Stochastic partial integro-differential equations are more general. The existence, uniqueness and asymptotic behaviours of mild solution to stochastic integro-differential equations has been investigated by using Banach fixed point theorem in Diop et al. [48], [49], [47], [50], [17]. Comparing to their studies, Chapter 5 mainly concerns the p-th moment and almost surely stability properties of the stochastic partial integro-differential system with delays. We have obtained the sufficient condition for p-th moment exponential stability of mild solutions to delay equations by using Theorem 5.5.2 which based on the properties of stochastic convolution. We have proved this theorem by using resolvent operator instead of the semigroup.

Finally, in Chapter 6, we have studied the exponential stability of mild solutions to neutral stochastic partial differential equations driven by a fractional Brownian motion with impulsive effects. Defining the stochastic integral with respect to fractional Brownian motion in infinite dimensional spaces in the same way as in Caraballo et al. [21], we have given the analysis of existence, uniqueness and exponential stability of mild solutions to a more general class of equations (6.1.3) in Theorem 6.3.1. In most of the works, the delays are finite. Comparing to Boufoussi and Hajji [12] and Maheswari and Karunanithi [93], we have considered infinite delays and impulsive effects in the stochastic system.

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