Determinacy of Refinements to the Difference Hierarchy of Co-analytic Sets

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Abstract

In this paper we develop a technique for proving determinacy of classes of the form $\omega^2\text{-}\Pi^1_1 + \Gamma$ (a refinement of the difference hierarchy on Π^1_1 lying between $\omega^2\text{-}\Pi^1_1$ and $(\omega^2+1)\text{-}\Pi^1_1$) from weak principles, establishing upper bounds for the determinacy-strength of the classes $\omega^2\text{-}\Pi^1_1 + \Sigma^0_\alpha$ for all computable α and of $\omega^2\text{-}\Pi^1_1 + \Delta^1_1$. This bridges the gap between previously known hypotheses implying determinacy in this region.

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1. Introduction

We work towards proving the following theorem (the relevant definitions can be found in the next section):

Theorem 1.1. *If there exists a non-trivial mouse $\mathcal{M}$ with measurable cardinal κ satisfying the theory T, then $\mathrm{Det}(\omega^2\text{-}\Pi^1_1 + \Gamma)$ for the following combinations of T and Γ:*

1. $T =$ *"cleverness + there exists a clever mouse,"* $\Gamma = \Sigma^0_1$;
2. $T = \mathsf{KP} + \Sigma_1\text{-}$, $\Gamma = \Sigma^0_2$;
3. $T = \Sigma_2\text{-}\mathsf{KP} + \Sigma_2\text{-}$, $\Gamma = \Sigma^0_3$;
4. $T = \Sigma_{n+1}\text{-}\mathsf{KP} + \Sigma_{n+1}\text{-}$, $\Gamma = n\text{-}\Pi^0_3$;
5. $T = \mathsf{ZFC}^- + \mathcal{P}^\alpha(\kappa)$ *exists,* $\Gamma = \Sigma^0_{1+\alpha+3}$ *for computable* α;
6. $T = \mathsf{ZFC}$, $\Gamma = \Delta^1_1$.

This is thus an extension of $\omega^2\text{-}\Pi^1_1$ determinacy, without requiring all the strength of $0^\dagger$ required to prove determinacy of $(\omega^2+1)\text{-}\Pi^1_1$. The story of $\mathrm{Det}(\omega^2\text{-}\Pi^1_1)$ starts with Martin proving in [10] that the existence of a measurable cardinal implies the determinacy of $\boldsymbol{\Pi}^1_1$. The proof uses the measure to "integrate" many strategies together in a technique that proved very fruitful

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and which makes the core of the present paper. Analysis of the proof allows the result to be broken down to the following:

Theorem (Martin)**.**

$$\forall x \in \omega^\omega (x^\sharp \textit{ exists} \to \mathrm{Det}(\Pi^1_1(x))).$$

Hence $0^\sharp$ implies the determinacy of the lightface co-analytic sets. Here the role of the measure is seen through the lens of indiscernibility, and this is the form in which our determinacy proof will be.

It was then shown in [5] that actually $0^\sharp$ implies $\mathrm{Det}(3\text{-}\Pi^1_1)$, the third level of the difference hierarchy. Eventually the full strength of $0^\sharp$ came out in a proof by Martin (available in Martin's unpublished book manuscript [9] or an account by DuBose [4]) that

Theorem (Martin)**.**

$$0^\sharp \textit{ exists} \leftrightarrow \mathrm{Det}\left(\bigcup_{\alpha<\omega^2} \alpha\text{-}\Pi^1_1\right).$$

The much stronger principle of $0^\dagger$ was found to imply $\mathrm{Det}((\omega^2+1)\text{-}\Pi^1_1)$, and indeed this is an exact equivalence. Intermediate results were sought; in [11] Martin proves $\mathrm{Det}(\omega^2\text{-}\boldsymbol{\Pi^1_1})$ from a measurable cardinal and, providing inspiration for our current results, proves $\mathrm{Det}(\Delta(\omega^2+1)\text{-}\boldsymbol{\Pi^1_1})$ and hence $\mathrm{Det}(\omega^2\text{-}\boldsymbol{\Pi^1_1}+\boldsymbol{\Delta}^1_1)$ from the same hypothesis. Nonetheless a weaker hypothesis was sought and found in [15], the paper on which this is based:

Theorem (Welch)**.** *There exists a* clever mouse *iff* $\mathrm{Det}(\omega^2\text{-}\Pi^1_1)$.

A clever mouse is a certain type of iterable model. Principles such as $0^\sharp$ and $0^\dagger$ can be viewed in terms of mice, as well, so this result in some sense exists on the same continuum.

The difference hierarchy is already a refinement of the projective hierarchy, here, so we consider further refinements; if $\Gamma \subseteq \Pi^1_1$ is a pointclass, $\omega^2\text{-}\Pi^1_1+\Gamma$ is viewed as $(\omega^2+1)\text{-}\Pi^1_1$ with the final set in the ω^2+1-sequence being constrained to being Γ. Using the methods in [15], we find that such results are closely connected to the determinacy of a class related to Γ which we call $\widetilde{\Gamma}$. This is the analogue of Γ in a larger space, arising from the auxiliary game used in Section 4 , and in Section 2.2 we will spend some time developing the theory of this space.

The way we will prove Theorem 1.1 is via the following:

Theorem 1.2. *Let Γ be an arithmetic pointclass. Suppose there exists a class C of Σ_n generating indiscernibles for a theory T such that for any $\vec{c} \in [C]^\omega$, any $\widetilde{\Gamma}$ game has a Σ_n-definable winning strategy in the smallest transitive model of T with $\vec{c} \in T$. Then* $\mathrm{Det}(\omega^2\text{-}\Pi^1_1+\Gamma)$.

We will define exactly what generating indiscernibles are in Section 3, where we will also show their existence, and Theorem 1.2 will be proved in Section

4, where $\widetilde{\Gamma}$ will be defined implicitly. Finally in Section 5 we will prove the determinacy of $\widetilde{\Gamma}$ for each Γ mentioned in 1.1 in the relevant model.

Section 2.1 contains definitions of the difference hierarchy and our refinements to it, as well as some small lemmas that help in establishing the determinacy of dual classes. Readers familiar with the difference hierarchy may wish to skip that section except for the main Definition 2.2. Section 2.2 contains essential definitions required to complete the determinacy proofs, but the material closely mirrors the usual effective descriptive set theory. Sections 3 and 4 are independent of one another and can be read in either order, with the exception of Definition 3.1.

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2. Preliminaries

Our set theoretic notation is standard, and we follow [9] for our determinacy notation. Thus a game is specified by a *game-tree* T, usually $\omega^{<\omega}$, and the payoff set $A \subseteq \lceil T \rceil$. $\lceil T \rceil$ is the set of all possible plays, and is equal to $[T]$, the set of branches, if T has no terminal-nodes, which for us will always be the case. Determinacy for a class of sets Γ is denoted $\mathrm{Det}(\Gamma)$. The Borel hierarchy and co-analytic sets in the space $[T]$ are defined as usual as in [13], but since we will be dealing with small models without all of the reals we will need to use the lightface counterparts of these classes. In Section 2.2 this will be generalised to uncountable T; once the basic definitions are in order, the results here work the same as in the classical case.

We require familiarity with admissible structures and L, where we follow the notation of [2]. We will also use the Jensen hierarchy J_α, but since we mostly deal with admissible structures this can be thought of as L_α. KP_n denotes the axioms of Kripke-Platek set theory augmented by the schemes of Σ_n Collection and Separation. We make extensive use of structures of the form $\langle L_\alpha[A], \in, A\rangle$ for some predicate A (usually a countable sequence of ordinals.) If we say a structure of this form is admissible, or a model of KP_n, we mean a model in said theory in the language with a predicate for A, adjusting the Lévy hierarchy appropriately. Most results from the usual L_α hierarchy generalise in the obvious way; in particular we will need Σ_n Skolem functions for $n > 1$. Normally this would require the full fine structure theory, but we will always be in a model of at least Σ_{n-1} admissibility. Thus we will freely make use of the fact that "the M-least set x such that $\varphi(x)$" for φ a Σ_n formula is Σ_n over M.

While the statement of Theorem 1.1 is in terms of mice, we won't need much mouse theory at all, since the mice we use will be models of enough set theory to obviate the need for much of the fine structure theory. The important part of being a mouse is the property of iterability, i.e. having wellfounded iterated ultrapowers. Thus when we talk of M being a mouse the important features that we will use are its transitivity and wellfounded ultrapowers, and indeed the hypotheses of the main theorem could just as well be stated in terms of the existence of any transitive, iterable model of enough set theory. The full theory can be found in [18], but we will only need the theory for mice with a single measure, which can be found (albeit with older notation) in [3].

Also touched upon in section 3 is the canonical mouse order, $<_*$, the details of which are not important for this paper.

For the first result of the main theorem we refer to the cleverness property defined in [15] that was the key to getting that paper's main result. Essentially this is a version of Rowbottom's theorem for classes Σ_1-definable over a model, a principle which we get for free in models of Σ_1 (or more) separation. The definition uses the concept of a Q-structure, defined at 1.8 in [15]. Let M be a mouse in the old sense of [3], then:

Definition 2.1. Let $M \vDash$ "F is a normal measure on κ", where F is, in V, the closed-unbounded filter on κ. Then the *Q-structure of M at κ* is:

$$Q^M_\kappa = \langle J^F_\theta, \in, F \cap J^F_\theta \rangle$$

where θ is largest such that:

$$J^F_\theta \vDash \text{“}F \text{ is normal, and the club filter on } \kappa.\text{”}$$

We will not make extensive use of this material and haven't changed much from [15], so we omit the majority of the details, commenting rather on the changes where they exist.

The next two subsections introduce more notation as well as the required generalisations.

2.1. The Difference Hierarchy and Refinements

The difference hierarchy was developed by Hausdorff, and a discussion can be found in [6]. We first cover the basic case, then develop the refinement we are considering and prove some basic lemmas.

Following the nomenclature of [13], let us fix Γ to be an *adequate* pointclass closed under countable intersection in some ambient Polish space $\mathcal{X}$, for instance in the intended case, the class of Π^1_1 sets in Baire space, ω^ω. Then, if α is a countable ordinal we denote the αth level of the *difference hierarchy* on Γ by α-Γ: a pointset A is in α-Γ iff there is a sequence $\langle A_\xi \mid \xi \leq \alpha \rangle$, with each $A_\xi \in \Gamma$ and $A_\alpha = \varnothing$ such that:

$$x \in A \leftrightarrow \text{the least } \xi \text{ such that } x \notin A_\xi \text{ is odd},$$

in which case we say the sequence $\langle A_\xi \mid \xi \leq \alpha \rangle$ *witnesses* that $A \in \alpha$-Γ. Since Γ is assumed to be closed under countable intersections we can take the sequence to be downwards-closed, that is $A_\alpha \supseteq A_\beta$ if $\alpha < \beta$, by rewriting A_β as $\bigcap_{\alpha \leq \beta} A_\alpha$

if necessary. The definition of the difference hierarchy ensures that intersections with previous members of the sequence do not affect the resulting set A.

These pointclasses serve as a further stratification in the Borel hierarchy; each $\alpha\text{-}\mathbf{\Pi}^0_n$ is strictly between $\mathbf{\Pi}^0_n$ and $\mathbf{\Delta}^0_{n+1}$ as long as α is countable, by a result of Hausdorff. Note while we can consider pointclasses Γ that are not closed under countable intersections, say $\mathbf{\Sigma}^0_n$, then already $\omega\text{-}\mathbf{\Sigma}^0_n$ captures all of the $\mathbf{\Pi}^0_{n+1}$ sets, so this is not as interesting, and the results below become trivial.

We can make the hierarchy even finer by restricting the final set in the witnessing sequence. We adopt the following notation for this:

Definition 2.2. If $\Lambda \subseteq \Gamma$ is a pointclass, we say $A \in (\alpha\text{-}\Gamma) + \Lambda$ iff there is a sequence $\langle A_\xi \mid \xi \leq \alpha + 1\rangle$ witnessing that $A \in (\alpha + 1)\text{-}\Gamma$, with $A_\alpha \in \Lambda$. Note that $A_{\alpha+1}$ is still $\varnothing$.

With this notation it is clear that $\alpha\text{-}\Gamma + \Gamma$ is just $(\alpha + 1)\text{-}\Gamma$, but for general Λ we end up with a different pointclass.

We now present a couple of elementary results about these refinements to the difference hierarchy. Let $\neg\Gamma$ denote the dual class of the pointclass Γ.

Lemma 2.3. *If λ is a countable limit ordinal, then the dual class, $\neg((\lambda\text{-}\Gamma) + \Lambda) = (\lambda\text{-}\Gamma) + \neg\Lambda$.*

Proof. Let $\langle A_\alpha \mid \alpha \leq \lambda+1\rangle$ witness that $A^c \in (\lambda\text{-}\Gamma)+\Lambda$, and we want to find a sequence witnessing that $A \in (\lambda\text{-}\Gamma)+\neg\Lambda$. Define the sequence $\langle B_\alpha \mid \alpha \leq \lambda+1\rangle$ as follows:

$$\begin{aligned} B_\eta &= \mathcal{X}, && \text{when } \eta < \lambda \text{ is zero or a limit ordinal} \\ B_{\alpha+1} &= A_\alpha, && \text{when } \alpha \leq \lambda \\ B_\lambda &= (A_\lambda)^c, && \\ B_{\lambda+1} &= \varnothing. && \end{aligned}$$

So assuming $\lambda > \omega$, the sequence looks like:

$$\langle \mathcal{X}, A_1, A_2, \ldots, \mathcal{X}, A_\omega, \ldots, (A_\lambda)^c, \varnothing\rangle,$$

with the whole space $\mathcal{X}$ inserted at zero and each limit position below λ. Now, this sequence is a witness that some set is in $(\lambda\text{-}\Gamma) + \neg\Lambda$, since it is of the correct length and $B_\lambda \in \neg\Lambda$. Denote this set B and we show $A^c = B$.

Let $x \in \mathcal{X}$, then let α be the least ordinal such that $x \notin A_\alpha$. By construction, the least β such that $x \notin B_\beta$ is $\alpha + 1$, unless $\alpha = \lambda + 1$ in which case $\beta = \alpha - 1$. In either case $x \in A^c \leftrightarrow x \in B$.

We've shown that $\neg((\lambda\text{-}\Gamma) + \Lambda) \subseteq (\lambda\text{-}\Gamma) + \neg\Lambda$, but note that we can apply the same argument to get that $\neg((\lambda\text{-}\Gamma) + \neg\Lambda) \subseteq (\lambda\text{-}\Gamma) + \Lambda$, since $\neg\neg A = A$. Hence since taking dual classes preserves subsets, $((\lambda\text{-}\Gamma) + \neg\Lambda) \subseteq \neg((\lambda\text{-}\Gamma) + \Lambda)$. □

For a pointclass Γ, $\Delta(\Gamma)$ denotes the self-dual pointclass, $\Gamma \cap \neg\Gamma$.

Corollary 2.4. *If λ is a countable limit ordinal, then $\lambda\text{-}\Gamma + \Delta(\Gamma) \subseteq \Delta((\lambda+1)\text{-}\Gamma)$.*

Proof. This follows immediately from the previous lemma:

$$\lambda\text{-}\Gamma + \Delta(\Gamma) = \lambda\text{-}\Gamma + \neg\Delta(\Gamma) = \neg(\lambda\text{-}\Gamma + \Delta(\Gamma)) \subseteq \neg((\lambda+1)\text{-}\Gamma).$$

□

Together these results will help extend the pattern down in the Borel hierarchy, that if we know $\mathrm{Det}(\Sigma^0_n)$, say, we get the dual class, $\mathrm{Det}(\Pi^0_n)$, for free. Hence in the cases we are considering, having proved $\mathrm{Det}(\omega^2\text{-}\Pi^1_1 + \Sigma^0_2)$, we will know immediately that $\mathrm{Det}(\omega^2\text{-}\Pi^1_1 + \Pi^0_2)$, and indeed $\mathrm{Det}(\omega^2\text{-}\Pi^1_1 + \Delta^0_2)$ hold.

2.2. Effective Descriptive Set Theory with Uncountable Spaces

The auxiliary game we construct to prove $\mathrm{Det}(\omega^2\text{-}\Pi^1_1 + \Gamma)$ has a payoff set in the space $(\omega \times \aleph_\omega)^\omega$. While it's not hard to see that such a set will be in the boldface counterpart of Γ in this space, we will need a closer analysis than this.

Firstly, we need to know that the payoff set is at least definable over the weak models for which we have indiscernibles, and secondly proving determinacy in these models requires tools from effective descriptive set theory. Effective descriptive set theory is defined in terms of computable functions on ω, which will not suffice when working with an uncountable space. We therefore need to generalise the theory and reprove several basic results in the uncountable context.

Let T^* be the tree $F^{<\omega}$ for $F = \omega \times \aleph_\omega$. This will be the tree for the auxiliary game.

Definition 2.5 (Generalised Recursive Pointclasses)**.** Define the following structure, the analogue of the hereditarily finite sets for classical recursion theory:

$$\mathcal{H} = \langle L_{\aleph_\omega}[\langle \aleph_i \mid i < \omega \rangle], \in, \langle \aleph_i \mid i < \omega \rangle \rangle.$$

Fix a predicate $R(x^*, a) \subseteq [T^*] \times F$ (although the below definitions generalise trivially to predicates on $[T^*]^n \times F^m \times \omega^l$). Then R is *generalised-semi-recursive* if there is a $\Sigma_1(\mathcal{H})$-definable set $X \subseteq T^* \times F$ such that

$$R(x^*, a) \iff \exists m \langle x^* \upharpoonright m, a \rangle \in X.$$

In other words, if R is the $\Sigma_1(\mathcal{H})$-union of basic open subsets of $[T^*] \times F$. R is called *generalised-recursive* if its complement is also of this form.

Remark. Compare this with HF, over which the computably enumerable relations on ω are Σ_1, or to α-recursion theory which considers subsets of α. This can be seen as extending α-recursion theory to subsets of α^ω, where we in particular take $\alpha = \aleph_\omega$.

A coding of $(\aleph_\omega)^{<\omega}$ into $\aleph_\omega$ will allow us to talk about generalised-recursive relations on T^* itself. Since $\mathcal{H}$ is closed under Gödel pairing, the Gödel pairing function G on $\aleph_\omega$ is total and hence $\Delta^{\mathcal{H}}_1$, allowing consideration of such relations.

Definition 2.6 (Generalised Kleene Pointclasses)**.** Then for a predicate P (which may be a subset of $[T^*] \times F^m \times \omega^l$ in general, but which we write below as a subset of $[T^*] \times F$ for clarity):

1. P is $\widetilde{\Sigma}^0_1$ iff P is generalised-semi-recursive;
2. P is $\widetilde{\Sigma}^0_{n+1}$ iff there is a $\widetilde{\Pi}^0_n$ predicate $R \subseteq [T^*] \times F \times \omega$ such that $P(x^*, a) \iff \exists b \in \omega(R(x^*, a, b))$;
3. P is $\widetilde{\Pi}^0_n$ iff $\neg P$ is $\widetilde{\Sigma}^0_n$;
4. P is $\widetilde{\Delta}^0_n$ iff it is $\widetilde{\Sigma}^0_n$ and $\widetilde{\Pi}^0_n$.

We then define the generalised analytic sets for $X \subseteq [T^*] \times F$:

$$X \in \widetilde{\Sigma}^1_1 \iff \exists Y \in \widetilde{\Pi}^0_1(x \in X \leftrightarrow \exists y \in [T^*](\langle x, y\rangle \in Y)).$$

The $\widetilde{\Sigma}^1_n$, $\widetilde{\Pi}^1_n$ and $\widetilde{\Delta}^1_n$ sets follow as usual.

Remark. The notion of computability is still the only difference with the usual lightface hierarchy; the subsequent levels are still built by (generalised) recursive unions, and complementation. Note that we take countable unions so as to align this hierarchy with the usual Borel hierarchy.

Lemma 2.7. *Suppose P is $\widetilde{\Sigma}^0_n$ or $\widetilde{\Pi}^0_n$ and $f : [T^*] \to [T^*]$ is such that the relation $G \subseteq [T^*] \times T^*$ given by $G(a, p) \iff p \subseteq f(a)$. Then the set f^{-1}“P is also $\widetilde{\Sigma}^0_n$ or $\widetilde{\Pi}^0_n$, respectively.*

Proof. First suppose P is $\widetilde{\Sigma}^0_1$ and consider the set f^{-1}“$P = \{a \mid f(a) \in P\}$. By our assumption on f there is a $\Sigma_1(\mathcal{H})$ set C such that

$$p \subseteq f(a) \iff \exists m(\langle a \restriction m, p\rangle \in C).$$

Let $\bar{C}$ be the $\Sigma_1(\mathcal{H})$ set $\{q \mid \exists p \in P(\langle q, p\rangle \in C)\}$, and note that:

$$\begin{aligned} \exists m(a \restriction m \in \bar{C}) &\iff \exists m(\exists p \in P(p \subseteq f(a) \restriction m) \\ &\iff \exists m(f(a) \restriction m \in P) \\ &\iff a \in f^{-1}\text{“}P, \end{aligned}$$

which is $\widetilde{\Sigma}^0_1$.

If the result holds for $\widetilde{\Sigma}^0_n$, and $\neg Q = P \in \widetilde{\Sigma}^0_n$ then:

$$a \in f^{-1}\text{“}p \iff f(a) \in P \iff f(a) \notin Q \iff a \notin f^{-1}\text{“}Q.$$

So the result holds for $\widetilde{\Pi}^0_n$.

Now assuming inductively that the result holds for $\widetilde{\Pi}^0_n$, let P be $\widetilde{\Sigma}^0_{n+1}$. There is thus a $\widetilde{\Pi}^0_n$ set Q such that:

$$\begin{aligned} f^{-1}\text{“}P &= \{a \mid \exists m(\langle f(a), m\rangle \in Q)\} \\ &= \{a \mid \exists m(\langle a, m\rangle \in \bar{f}^{-1}\text{“}Q)\}, \end{aligned}$$

Where $\bar{f}$ is the function given by $\bar{f}(\langle a, m\rangle) = \langle f(a), m\rangle$. ☐

The following two propositions are obvious relationships between these pointclasses and the usual pointclasses.

Proposition 2.8. $\Sigma^0_n \subseteq \widetilde{\Sigma}^0_n \subseteq \boldsymbol{\Sigma}^0_n$*, where the pointclasses are taken to be in the spaces of* ω^ω*,* $[T^*]$ *and* $[T^*]$*, respectively; the boldface Borel hierarchy on* $[T^*]$ *being the usual topological definition.*

Proof. The first inclusion is seen by an easy induction starting with the observation that Σ^0_1 relations on ω^ω are a union of a Σ^{HF}_1 set of basic open sets, which is thus also a $\Sigma_1(\mathcal{H})$ set. The second inclusion follows directly from the definition. $\square$

In the following proposition, $\vee$ and $\wedge$ denote the pointclasses formed by sets being the union (respectively intersection) of a set in the first class with one in the second.

Proposition 2.9. $(\Sigma^0_n \vee \widetilde{\Sigma}^0_1), (\Sigma^0_n \wedge \widetilde{\Pi}^0_1) \subseteq \widetilde{\Sigma}^0_n$ *for* $n > 1$*.*

Proof. For the first we need to observe that $[T^*]$ is a metrizable space, hence $\widetilde{\Sigma}^0_1$ sets are $\widetilde{\Sigma}^0_n$. (The proof works as in the classical setting.) By the above, for the second part we just need to see that $\widetilde{\Pi}^0_1 \subseteq \widetilde{\Sigma}^0_n$ for $n > 1$ as for the usual pointclasses. $\square$

We note at this point that ultimately we will be interested in the analogue of $\mathcal{H}$ built on some sequence $\vec{c} \in [\aleph_\omega]^\omega$ instead of $\langle \aleph_i \mid i \in \omega \rangle$. For concreteness we present this section in terms of the $\aleph_i$s, since this doesn't change anything. For the rest of this section, by "admissible" we mean "admissible in the language of set theory augmented with constants $\vec{c}$;" for now they will be interpreted as the $\aleph_i$s but this need not be the case in general.

Lemma 2.10. *If* R *is a generalised-recursive well-founded relation and* M *is admissible with* $\mathcal{H} \in M$*, then* $\operatorname{rk} R \in M$*.*

Proof. If R is $\widetilde{\Delta}^0_1$ and wellfounded, then let

$$S = \{\langle p_0, \ldots, p_k \rangle \mid \forall i < k (p_{i+1} R p_i)\}.$$

For $p, q \in S$ let $p <_R q$ if either $p \supsetneq q$ or, for i the least point where $p_i \neq q_i$, $p_i < q_i$. Then $<_R$ wellorders S, is generalised-recursive and ot $<_R$ is at least $\operatorname{rk} R$.

Now we need to show that the order-type of a generalised-recursive wellorder—a *generalised-recursive ordinal*—is an element of M. Let $\alpha = On \cap M$, hence $\alpha > \aleph_\omega$ is an admissible ordinal and

$$N = \langle L_\alpha[\langle \aleph_i \mid i \in \omega \rangle], \in, \langle \aleph_i \mid i \in \omega \rangle \rangle$$

is admissible. Now, since $<_R$ is definable over $\mathcal{H}$, it is an element of

$$\langle L_{(\aleph_\omega)+1}[\langle \aleph_i \mid i \in \omega \rangle], \in, \langle \aleph_i \mid i \in \omega \rangle \rangle$$

which is a subset of N. By admissibility (i.e. recursion) let f be a Σ^N_1 order-preserving bijection between $<_R$ and $\operatorname{ot}(<_R)$. Hence $\operatorname{ot}(<_R) \in N$, and $On \cap M = On \cap N$. $\square$

Lemma 2.11. *Let $x^* \in [T^*]$ be $\widetilde{\Sigma}^1_1$. Then there is a $\widetilde{\Pi}^0_1$ tree $R \subseteq \omega \times F \times T^*$ such that*

$$x^*(n) = a \iff \exists y \in [T^*](\langle\langle n, a\rangle, y\rangle \in [R]).$$

Proof. A $\widetilde{\Pi}^0_1$ relation on $\omega^n \times F^m \times T^*$ is a tree if its T^* component is closed under initial segments, that is:

$$\langle i_1, \dots i_n, a_1, \dots, a_m, q\rangle \in R \wedge p \subseteq q \implies \langle i_1, \dots i_n, a_1, \dots, a_m, p\rangle \in R.$$

Let R be such a tree with $n = m = 1$ for simplicity, then the set of branches of R is:

$$[R] = \{\langle n, a, x^*\rangle \in \omega \times F \times [T^*] \mid \forall m(\langle n, a, x^* \restriction m \in R)\}.$$

Now, being $\widetilde{\Sigma}^1_1$ means that there is a $\widetilde{\Pi}^0_1$ predicate Y such that:

$$\begin{aligned} x^*(n) = a &\iff \exists y \in [T^*](Y(\langle\langle n, a\rangle, y\rangle)) \\ &\iff \exists y \in [T^*]\forall m(\langle\langle n, a\rangle, y \restriction m\rangle \notin A_Y), \end{aligned}$$

where A_Y is the $\Sigma_1(\mathcal{H})$ set witnessing that Y is $\widetilde{\Pi}^0_1$. So let R consist of all elements $\langle\langle n, a\rangle, p\rangle$ such that $\forall m \leq |p|\langle\langle n, a\rangle, p \restriction m\rangle \notin A_Y$, so that R is a tree, is $\widetilde{\Pi}^0_1$ and $x^*(n) = a \iff \exists y\langle\langle n, a\rangle, y\rangle \in [R]$. □

For $\langle n, a\rangle \in \omega \times F$ and a tree $R \subseteq \omega \times F \times T^*$, denote by $T_{\langle n,a\rangle}$ the $\langle n, a\rangle$'th part of R, that is, $\{p \in T^* \mid \langle\langle n, a\rangle, p\rangle \in R\}$. Then the above lemma says that x^* is $\widetilde{\Sigma}^1_1$ iff there is a $\widetilde{\Pi}^0_1$ tree R such that $x^*(n) = a \iff R_{\langle n,a\rangle}$ is illfounded (the reverse direction being obvious).

Of course, the lemma extends easily to relations $X \subseteq F^n \times (T^*)^m \times [T^*]^l$, in which case X is $\widetilde{\Sigma}^1_1$ if (and only if) there is a generalised-recursive tree $R \subseteq F^n \times F^m \times (T^*)^l \times T^*$ such that

$$\langle a, p, x\rangle \in X \iff \exists y \in [T^*](\langle a, e(p), x, y\rangle \in [R])$$

where e is a fixed generalised-recursive coding of F into T^*. Put more succinctly, $X \subseteq [T^*]$ is $\widetilde{\Sigma}^1_1$ iff there is a generalized-recursive tree $R \subseteq T^* \times T^*$ such that:

$$x \in X \iff R_x \text{ is illfounded}$$

where $R_x = \{y \in T^* \mid \langle x \restriction |y|, y\rangle \in R\}$.

Lemma 2.12. *If $x^* \in [T^*]$ is $\widetilde{\Sigma}^1_1$, then x^* is Π_1-definable over any admissible containing $\mathcal{H}$ as an element.*

Proof. This is essentially the Spector-Gandy theorem for elements of $[T^*]$ instead of 2^ω. Let M be an arbitrary admissible set with $\mathcal{H} \in M$.

By the above find $R \subseteq \omega \times F \times T^*$, a $\widetilde{\Pi}^0_1$ tree such that $x^*(n) = a \iff R_{\langle n,a\rangle}$ is illfounded. R is Π_1 definable over $\mathcal{H}$, hence Δ_1 definable over M, and so by Δ_1 separation in M, $R \in M$. Let $\varphi(\langle n, a\rangle)$ be the Σ_1 formula:

$$\exists \gamma, g(\gamma \in On \wedge g : R_{\langle n,a\rangle} \to \gamma \text{ is order-preserving}).$$

Suppose $x^*(n) \neq a$. Then $R_{\langle n,a\rangle}$ is wellfounded and an element of M, so by admissibility there is $\gamma < On \cap M$ and $g : R_{\langle n,a\rangle} \to \gamma$ witnessing the fact, hence $\varphi(\langle n, a\rangle)$ holds in M. On the other hand if $M \vDash \varphi(\langle n, a\rangle)$ then $R_{\langle n,a\rangle}$ is really wellfounded and so $x^*(n) \neq a$. Hence $x^*(n) = a \iff M \vDash \neg\varphi(\langle n, a\rangle)$. $\square$

Corollary 2.13. *Under the same hypotheses, x^* is an element of any admissible set which itself contains an admissible containing T^*.*

Proof. Let $M \in N$ be admissible and $T^* \in M$. Then x^* is definable over M and $M \in N$ is transitive. Hence, since the satisfaction relation is Δ_1^{KP}, by admissibility $(x^*)^M \in N$. Since $(x^*)^M$ is the "true" x^*, we have that $x^* \in N$. $\square$

Remark. This conclusion remains true if x^* is a $\widetilde{\Sigma}_1^1$ element of 2^{T^*}, since it will be a Π_1-definable class of M, hence an element of N.

The following is then a weak analogue of the Kleene Basis Theorem in our setting:

Theorem 2.14. *If $X^* \subseteq [T^*]$ is $\widetilde{\Sigma}_1^1$ and non-empty, then X^* has an element definable over any admissible set M with $T^* \in M$.*

Proof. In light of the above, it will suffice to show that X^* has an element which is recursive in some $\widetilde{\Sigma}_1^1$ element of $[T^*]$. This will then be definable over any M which is admissible and contains T^* as an element.

Since X^* is $\widetilde{\Sigma}_1^1$ there is a $\widetilde{\Pi}_1^0$ tree $T \subseteq T^* \times T^*$ such that $x^* \in X^* \iff T_{x^*}$ is illfounded. Hence any infinite branch of T determines an element of X^*, and we need to find such a branch in M. Define P to be the set of elements of T which can be extended to an infinite branch:

$$P = \{p = \langle (a_0, b_0), (a_1, b_1), \ldots, (a_t, b_t)\rangle \in T \mid \exists y \supseteq p \forall n (y \restriction n \in T)\}.$$

P is $\widetilde{\Sigma}_1^1$ by definition, so by the Spector-Gandy theorem for $T^* \times T^*$, we have that P is Π_1^M.

Then the leftmost path through P is defined by recursion by:

$$y(n) = \text{the least } a \in F \times F \text{ such that } (y \restriction n)^\frown a \in P$$

which is definable from P and hence over M, and the left part of y is an element of X^*. $\square$

Remark. It would be natural to consider the more direct analogue of Kleene's Basis Theorem by finding a $\widetilde{\Sigma}_1^1$ basis for the $\widetilde{\Sigma}_1^1$ sets, but for our purposes, the above suffices and is more expeditious.

We also note that Shoenfield Absoluteness holds for these models:

Theorem 2.15. *If $A \subseteq \omega^\omega$ is $\Sigma_2^1(a)$, then A is absolute for models M as considered above, i.e. which are admissible with $T^* \in M$, if in addition $a \in M$.*

Proof. Note we are now concerned with the usual Σ^1_2 sets, so the crucial things to know are that firstly $\Sigma^1_2(a)$ sets are ω_1-Suslin, and secondly that the Shoenfield tree can be ranked in our models. (See, for instance, the final remarks in [8], 13.15.) Thankfully, the Shoenfield tree is easily generalised-recursive; it is defined as:

$$\hat{T} = \{\langle p, u\rangle \in \omega^{<\omega} \times \omega_1^{<\omega} \mid \forall i, j < |p|(s_i \supseteq s_j \wedge \langle s \upharpoonright |s_i|, s_i\rangle \in T \to u(i) < u(j))\}$$

where $\langle s_i \mid i \in \omega\rangle$ is a standard recursive enumeration of elements of $\omega^{<\omega}$ and T is the $\Pi^1_1(a)$ tree witnessing that A is $\Sigma^1_2(a)$. Now this is a $\Delta_1(\{a\})$-definable element of $L_{\omega_1}[a]$, and is hence $\Delta_1^{\mathcal{H}[a]}$ where

$$\mathcal{H}[a] = \langle L_{\aleph_\omega}[\langle\aleph_i\rangle, a], \in, \langle\aleph_i\rangle, a\rangle$$

Hence $\hat{T}$ is also $\Delta_1^{\mathcal{H}[a]}$ and with minor modification, the method of Lemma 2.10 shows that wellfoundedness of such trees is absolute for M. But this is just what is needed to prove Shoenfield's absoluteness theorem. $\square$

3. Obtaining Indiscernibles

3.1. Introduction

We aim to define a notion of Prikry forcing which, when executed over a model of KP_n yields a generic extension which is also a model of KP_n, and which satisfies a form of *generating indiscernibility.* This is all a straightforward checking that the forcing can be defined over the model and preserves KP_n. First we define the indiscernibility principle we are interested in:

Definition 3.1. A closed-unbounded class of ordinals C is a class of Σ_n *generating indiscernibles* for the theory T if, for any $\vec{c}, \vec{d} \in [C]^\omega$, letting $\mathcal{A}_T[\vec{c}]$ be the least transitive model of T containing $\vec{c}$ as an element, we have $\mathcal{A}_T[\vec{c}] \equiv_{\Sigma_n} \mathcal{A}_T[\vec{d}]$.

First of all we seek to prove:

Theorem 3.2. *If there exists a non-trivial mouse which is a model of* KP_n*, then there exists a class of* Σ_n *generating indiscernibles for* KP_n.

Fix a mouse

$$M = \langle J^E_\theta, E, F\rangle \models \text{“}\mathsf{KP}_n + F \text{ is a normal measure on } \kappa\text{”}.$$

Let us fix some terminology for the remainder of the section:

Definition 3.3. Let $C = \{\kappa_0, \kappa_1, \ldots\}$ be the class of iteration points of M by F, and let $\vec{c} = \{c_0, c_1, \ldots\} \in [C]^\omega$ be any fixed ω-sequence of them indexed in increasing order. Then let λ be such that the measurable cardinal of M_λ, the λth iterate, is $\sup \vec{c}$, $\pi : M \to M_\lambda$ the iteration map. With E_λ, F_λ the filter sequence and top measure of M_λ, respectively, let $\widetilde{\theta}$ be such that $M_\lambda = \langle J^{E_\lambda}_{\widetilde{\theta}}, E_\lambda, F_\lambda\rangle$.

Now note that by Σ_n separation we may apply Łos' theorem to Σ_n formulæ, so these ultrapowers preserve Σ_n sentences, and hence C is a set of Σ_n indiscernibles for M_λ.

We use the notation $X \searrow p$, for sets of ordinals X, p to mean $X \setminus (\sup p + 1)$.

Lemma 3.4. *Let* $p = \{c_0, \ldots, c_k\}$ *and* $e \in [C]^{<\omega}$ *be a finite set of iteration points with* $\max e < c_k$*. Let* $X \in F_\lambda$ *be* Σ_n*-definable in* M_λ *from* $\pi(f)$ *(for* $f \in M^\kappa$*),* e, p*. Then* $X \supseteq C \searrow p$*.*

Proof. By the properties of iterated ultrapowers, the measure 1 set X must include an end-segment from C, which, as remarked above, is a set of Σ_n-indiscernibles for M_λ. Thus X contains at least one indiscernible greater than all the indiscernible parameters in its definition, and so by indiscernibility, it contains all indiscernible greater than $\max p$. □

We need to know more, namely that KP_n is preserved in the ultrapower.

Theorem 3.5. $M_\lambda \models \mathsf{KP}_n$.

Proof. First suppose that we know that the iterate $M_\alpha \models \mathsf{KP}_n$. We use the standard fact that the theory of KP_n is equivalent to there being no Σ_n-definable partial function, whose image of a set in the model is unbounded in the ordinals of the model.

Thus let $\varphi(\xi, \gamma, p)$ define a Σ_n-partial function f; $f(\xi) = \gamma \leftrightarrow \varphi(\xi, \gamma, p)$, with f and $A \in M_{\alpha+1}$ witnessing the failure of KP_n, i.e. $f``A$ is unbounded in $On \cap M_{\alpha+1}$.

Expanding $f``A$ being unbounded, we get the Π_{n+1} formula:

$$M_{\alpha+1} \models \forall\tau\exists\gamma > \tau\exists\xi \in A(\varphi(\xi, \gamma, p)).$$

Fix $p = \pi_{\alpha\alpha+1}(g)(\kappa_\alpha)$ for some $g \in M_\alpha$. By cofinality of the ultrapower embedding $\pi_{\alpha\alpha+1}$, let $\bar{A}$ be such that $A \subseteq \pi_{\alpha\alpha+1}(\bar{A})$. Then fix an arbitrary $\bar{\tau} \in M_\alpha$, and by applying Łoś' theorem to the above for $\tau = \pi_{\alpha\alpha+1}(\bar{\tau})$:

$$M_\alpha \models \{\beta < \kappa_\alpha \mid \exists\gamma > \bar{\tau}\exists\xi \in \bar{A}(\varphi(\xi, \gamma, g(\beta)))\} \in F^\alpha.$$

Hence:

$$M_\alpha \models \forall\bar{\tau}[\exists q\exists\xi \in \bar{A}\exists\gamma > \bar{\tau}(\varphi(\xi, \gamma, q))]$$

and hence the formula $\psi(\xi, \gamma) \equiv \gamma = \sup\{\gamma' \mid \exists q\varphi(\xi, \gamma', q)\}$ defines a Σ_n partial function $\bar{f}$, since the γ's are bounded by KP_n. But then:

$$\bar{f}``\bar{A} = \{\gamma \mid \exists\xi \in \bar{A}(\psi(\xi, \gamma))\}$$

is unbounded, thus contradicting KP_n in M_α.

Thus we have shown that if $M_\alpha \models \mathsf{KP}_n$, so too does $M_{\alpha+1}$, and we just have to show that the limit models behave similarly. But if ν is limit and $p, X \in M_\nu$ then take θ such that $p = \pi_{\theta,\nu}(\bar{p})$ and $X = \pi_{\theta,\nu}(\bar{X})$. By induction we may assume that if φ is Σ_n, $\bar{B} = \{\alpha \in \bar{X} \mid \varphi(\alpha, \bar{p})\}$ is an element of M_θ. Hence

$$\{\alpha \in X \mid \varphi(\alpha, p)\} = \{\alpha \in \pi_{\theta,\nu}(\bar{X}) \mid \varphi(\alpha, \pi_{\theta,\nu}(\bar{p}))\} = \pi_{\theta,\nu}(\bar{B}) \in M_\nu$$

by elementarity, and we are done. □

3.2. The Forcing

M_λ will be the ground model for Prikry forcing, which will add the set of indiscernibles $\vec{c}$ to the generic extension, whilst preserving KP_n.

Rowbottom's theorem will be needed for several of the proofs in this section:

Lemma 3.6. KP *proves that if κ is a measurable cardinal with normal measure F, and f is a partition of $[\kappa]^{<\omega}$ into less than κ pieces, then there is a $\Delta_1(f)$ set $H \in F$ homogeneous for f, i.e. such that for each n, f is constant on $[H]^n$.*

Proof. This is essentially [7], Theorem 10.22. Let F be a normal measure on κ, and let f partition $[\kappa]^{<\omega}$ into less than κ pieces. If we can find measure one sets H_n such that f is homogeneous on $[H_n]^n$, then f is also homogeneous for $H = \bigcap H_n$.

For $n = 1$ consider the intersection $\bigcap\{\kappa \setminus f^{-1}``\{\alpha\} \mid \alpha \in \operatorname{ran} f\}$. This set is empty, so by κ-completeness one of the terms must have been measure zero, i.e. some α must have pre-image of measure one, which we take to be H_1.

Now proceed by induction: let $f : [\kappa]^{n+1} \to I$ for some $|I| < \kappa$. Now for each $\alpha < \kappa$ we define f_α with domain $[\kappa \setminus \{\alpha\}]^n$ by $f_\alpha(x) = f(\{\alpha\} \cup x)$. By hypothesis there exist Δ_1 sets $X_\alpha \in F$ such that f_α is constant on $[X_\alpha]^n$. We fix this constant value to be i_α. By KP the sequence $\{X_\alpha \mid \alpha < \kappa\}$ exists.

Let X be the diagonal intersection of the X_αs, $\{\alpha < \kappa \mid \alpha \in \bigcap_{\gamma<\alpha} X_\gamma\}$, which is in F by normality, and is a Δ_1 subset of κ. Now if $\gamma < \alpha_1 < \ldots < \alpha_n$ are in X, then $\{\alpha_1, \ldots, \alpha_n\} \in [X_\gamma]^n$, so that $f(\{\gamma, \alpha_1, \ldots, \alpha_n\}) = f_\gamma(\{\alpha_1, \ldots, \alpha_n\}) = i_\gamma$. Now, $\gamma \mapsto i_\gamma$ constitutes a partition of $[X]^n$ into $\gamma < \kappa$ pieces, so again by hypothesis we know there is $i \in I$ and a Δ_1 set $H \subseteq X$ in F such that $i_\gamma = i$ for all $\gamma \in H$, hence $f(x) = i$ for all $x \in [H]^{n+1}$. □

The way we will use Rowbottom's theorem is, if we have a set $A \subseteq [\kappa]^{<\omega}$, we view the characteristic function of A as a partition of $[\kappa]^{<\omega}$ into 2. There is then a $\Delta_1(A)$ set $Z \in F$ such that either $[Z]^{<\omega} \subseteq A$ or $[Z]^{<\omega} \cap A = \varnothing$.

We now set up the basic forcing definitions in a way suitable for this purpose. This will require a return to ramified forcing, as originally conceived by Cohen (see [1]) and used in [15]. Since the forcing we use will be a class forcing from the perspective of the model, our setting is more general than that used by Cohen in [1], and the use of theories stronger than KP means we extend Welch's work, too. In addition, accounts since Cohen's of ramified forcing have mostly been sketched, without full definitions or proofs, and the notation of Cohen's account is now outdated. For these two reasons, we set out the full definitions and proofs here. We will need to check that the usual properties of forcing hold in spite of our weak setting and the fact that we are using a class forcing.

Definition 3.7. Let $\langle \mathbb{P}, \leq \rangle$ be Prikry forcing defined over M_λ, that is:

$$\mathbb{P} := \{\langle p, X\rangle \mid p \in [\kappa]^{<\omega}, X \in F_\lambda \cap M_\lambda\}$$
$$\langle p, X\rangle \leq \langle q, Y\rangle \leftrightarrow q \text{ is an initial segment of } p \wedge X \cup (p \setminus q) \subseteq Y.$$

This class is Δ_1-definable over M_λ, but since F_λ is not a set in the model, it is not a set forcing and so most of the work will be towards proving that KP_n is preserved.

Definition 3.8. We define a ranked forcing language, $\mathcal{L}_{F_\lambda}$. First of all, we define constants for the language, which are representations of class terms and are intended to name sets in the generic extension.

$\mathcal{C}_0$ consists of $\{\dot{\vec{c}}\} \cup \kappa_\lambda$, where $\dot{\vec{c}}$ is a constant symbol.

If $\mathcal{C}_\alpha$ is defined then $\mathcal{C}_{\alpha+1}$ consists of the class terms $\{x^\alpha \mid \varphi(x^\alpha)\}$ where φ is a formula built up from: symbols $\in, =, \dot{\vec{c}}$; any element of $\mathcal{C}_\beta$ for $\beta < \alpha$; logical connectives and where any quantified variable is of the form $\exists x^\beta$ for $\beta \lesssim \alpha$.

Take unions at limit stages, and let $\mathcal{C} = \bigcup_{\alpha<\widetilde{\theta}} C_\alpha$. (Recall that $\widetilde{\widetilde{\theta}}$ is the height of M_λ.)

The *rank* of one of these constants is the ordinal α such that it is an element of $\mathcal{C}_\alpha$.

$\mathcal{L}_{F_\lambda}$ then consists of all formulæ built from the symbols: $\in, =$; unranked variables $x, y, z, \ldots$; ranked variables $x^\alpha, y^\alpha, z^\alpha$ for $\alpha < \widetilde{\theta}$; connectives; quantifiers; and elements of $\mathcal{C}_\alpha$ for $\alpha < \widetilde{\theta}$.

If $\varphi \in \mathcal{L}_{F_\lambda}$ then we say it is *ranked* if every variable in it is ranked, in which case its rank is the maximum of the ranks of quantified variables in φ and of the ranks of any constant terms occurring in φ.

We now define the weak forcing relation $\mathbf{p} \Vdash^* \varphi$ for $\mathbf{p} = \langle p, X\rangle \in \mathbb{P}$ and φ a sentence in $\mathcal{L}_{F_\lambda}$:

1. $\mathbf{p} \Vdash^* x \in y$ iff $x \in \mathcal{C}_\alpha, y \in \mathcal{C}_\beta$ and:
    (a) $\alpha = \beta = 0$ and either $x \in y \in \kappa$ or $x \in p \wedge y = \dot{\vec{c}}$; or
    (b) $\alpha < \beta$, $y = \{z^\gamma \mid \varphi(z^\gamma)\}$ and $\mathbf{p} \Vdash^* \varphi(x)$; or else
    (c) $\alpha \geq \beta$ and $\exists z \in \mathcal{C}_\gamma$ for some γ, either $\beta > \gamma$ or $\beta = \gamma = 0$ and
    $$\mathbf{p} \Vdash^* z = x \wedge z \in y$$
2. $\mathbf{p} \Vdash^* x = y$ iff $\mathbf{p} \Vdash^* \forall z^\alpha(z \in x \leftrightarrow z \in y)$ for α the maximum of the ranks of x and y.
3. $\mathbf{p} \Vdash^* \varphi \wedge \psi$ iff $\mathbf{p} \Vdash^* \varphi$ and $\mathbf{p} \Vdash^* \psi$.
4. $\mathbf{p} \Vdash^* \neg\varphi$ iff $\forall \mathbf{q} \in \mathbb{P}(\mathbf{q} \leq \mathbf{p} \implies \mathbf{q} \nVdash^* \varphi)$.
5. $\mathbf{p} \Vdash^* \exists x^\alpha(\varphi(x^\alpha))$ iff there is some $t \in \mathcal{C}_\alpha$ such that $\mathbf{p} \Vdash^* \varphi(t)$.
6. $\mathbf{p} \Vdash^* \exists x(\varphi(x))$ iff there is some $t \in \bigcup_\alpha \mathcal{C}_\alpha$ such that $\mathbf{p} \Vdash^* \varphi(t)$.

We note that another notion of rank may be defined for formulæ of the forcing language that are ranked as defined above, so that whenever the definition of $\mathbf{p} \Vdash^* \varphi$ refers to a formula ψ of the forcing language, this new rank strictly decreases. This makes the above recursive definition legitimate, and allows arguments about the forcing relation "by induction on φ."

Definition 3.9. A filter $G \subset \mathbb{P}$ is *M_λ-generic* for $\mathbb{P}$ if it meets all $\Sigma_n^{M_\lambda}$-definable subsets of $\mathbb{P}$ *and* if, for every Σ_n sentence φ of the forcing language, there is a $\mathbf{p} \in G$ such that $\mathbf{p} \Vdash^* \varphi \vee \mathbf{p} \Vdash^* \neg\varphi$.

Remark. Note that the latter requirement also implies that such a generic decides every Π_n sentence.

Usually these two forms of genericity are equivalent, but here we will have to prove them both.

If G is M_λ-generic for $\mathbb{P}$ then let $\vec{c} = \bigcup\{p \in [\kappa_\lambda]^{<\omega} \mid \exists X \in F_\lambda(\langle p, X\rangle \in G)\}$. If we define an appropriate G, this will be the sequence $\vec{c}$ of iteration points we picked above. The generic extension is defined as $M_\lambda[G] = \langle J^{\vec{c}}_{\tilde{\theta}}, \vec{c}\rangle$, and if there is only one G under discussion we write $M_\lambda[\vec{c}]$.

This forcing relation obeys the usual properties:

Lemma 3.10.

1. *For no* $\mathbf{p}, \varphi$ *does* $\mathbf{p} \Vdash^* \varphi \wedge \neg\varphi$*;*
2. *If* $\mathbf{p} \Vdash^* \varphi$ *and* $\mathbf{q} \leq \mathbf{p}$ *then* $\mathbf{q} \Vdash^* \varphi$*;*
3. $\mathbf{p} \Vdash^* \varphi \iff \forall \mathbf{q} \leq \mathbf{p} \exists \mathbf{r} \leq \mathbf{q}(\mathbf{r} \Vdash^* \varphi)$*;*

Proof.

1–2 are standard and can be found in [1] Section IV.4.

3. The forward implication is obvious by 2 and the backwards implication is proved by induction on φ: Suppose first that φ is $x \in y$. If $x, y \in \mathcal{C}_0$ and $y \in \kappa$, then, since forcing this doesn't depend on $\mathbf{p}$, there's nothing to do. If φ is $x \in \dot{\vec{c}}$ then we have to show that $x \in p$ where $\mathbf{p} = \langle p, X\rangle$. But if not, then we can find $q \in X \setminus \{x\}$ and then $\langle p \cup \{q\}, X\rangle \Vdash^* \neg\varphi$ for a contradiction.

   For every other case we use the induction hypothesis in the obvious way.

□

The reason for using the ranked forcing language is that it allows the forcing relation to be simply definable, as we prove in the following lemma:

Lemma 3.11. *If* $\mathbf{p} \in \mathbb{P}$ *and* $\varphi \in \mathcal{L}_{F_\lambda}$ *is a ranked sentence, then:*

1. *If* $\mathbf{p} = \langle p, X\rangle$ *there is a* $\Delta_1^{M_\lambda}(\mathbf{p}, \varphi)$ *set* $Y \in F_\lambda$ *such that* $\langle p, Y\rangle \| \varphi$*, that is either* $\langle p, Y\rangle \Vdash^* \varphi$ *or* $\langle p, Y\rangle \Vdash^* \neg\varphi$*; and*
2. $\langle p, Y\rangle \Vdash^* \varphi$ *is* $\Delta_1^{M_\lambda}$*.*

Proof. The first assertion is a re-statement of the Prikry lemma, but we will prove both statements in a simultaneous induction on the complexity of φ.

First suppose φ is $x \in y$ for $x, y \in \mathcal{C}_0$. Then certainly $\mathbf{p} \Vdash^* \varphi$ is Δ_1-definable. If $\mathbf{p} \Vdash^* \varphi$, then set $Y = X$ and we are also done with part 2. If not, then:

$$(x \notin y \vee y \notin \kappa) \wedge (x \notin p \vee y \neq \dot{\vec{c}}).$$

This is preserved when strengthening $\mathbf{p}$, so in fact $\mathbf{p} \Vdash^* \neg\varphi$, and again we can take $Y = X$ for part 2.

Now suppose we have proved the statement of the lemma for all formulæ occurring in the definition of $\mathbf{p} \Vdash^* \varphi$. We shall not carry out the proof in every case, but the following illustrate the most important and involved parts:

1. If φ is $x \in y$ with x, y having rank $\alpha < \beta$ respectively, then let $y = \{z^\gamma \mid \psi(z^\gamma)\}$. Then the induction hypothesis says that for each $d \in \mathcal{C}_\gamma$ there is an extension of $\mathbf{p}$ to $\langle p, Y_d\rangle$ which decides $\psi(d)$. The condition $\mathbf{p}_x = \langle p, Y_x\rangle$ therefore decides $\psi(x)$. By definition, $\mathbf{p}_x \Vdash^* x \in y \iff \mathbf{p}_x \Vdash^* \psi(x)$, so we are done.
2. If φ is $\neg\psi$, then by hypothesis extend $\mathbf{p}$ to $\langle p, Y\rangle$ in a Δ_1 way such that $\langle p, Y\rangle \| \psi$. But then $\langle p, Y\rangle$ also decides φ.
   Likewise if we know $\langle p, Y\rangle \Vdash^* \psi$ then $\langle p, Y\rangle \nVdash^* \varphi$ and vice-versa, and if the former is Δ_1 definable over M_λ then so is the latter.
3. If φ is $\exists x^\alpha(\psi(x^\alpha))$ then define:

   $$S^+ = \{q \in [X]^{<\omega} \mid \exists d, \exists X^d \subseteq X(\langle p \cup q, X^d\rangle \Vdash \psi(d))\}.$$

   By hypothesis, for each q, d there is a $\Delta_1^{M_\lambda}$-definable set Y such that $\langle p\cup q, Y\rangle \| \psi(d)$, and such that $\langle p\cup q, Y\rangle \Vdash^* \psi(d)$ is Δ_1. This means that the existence of X^d in the definition of S^+ is equivalent to $\langle p \cup q, Y\rangle \Vdash^* \psi(d)$ since, if $\langle p \cup q, Y\rangle \Vdash^* \neg\psi(d)$ there can be no such X^d as any $\langle p \cup q, X^d\rangle$ is compatible with $\langle p \cup q, Y\rangle$.
   Thus S^+ is a $\Sigma_1^{M_\lambda}$ subset of $[\kappa_\lambda]^{<\omega}$ and by Rowbottom's theorem there is a set $A \in F_\lambda \cap M_\lambda$ such that either $[A]^{<\omega} \subseteq S^+$ or $[A]^{<\omega} \cap S^+ = \varnothing$.
   Suppose $\langle p, A\rangle$ does not decide φ. Hence there are conditions $\langle r, Y\rangle, \langle s, Z\rangle$, each stronger than $\langle p, A\rangle$ such that one forces φ and the other forces $\neg\varphi$. Extending the shorter sequence if necessary, assume $\ell h(r) = \ell h(s) = k$. Then $\langle r, Y\rangle \leq \langle p, A\rangle$ so $r \in [A]^k$, but similarly $s \in [A]^k$. Hence $[A]^k$ contains elements in S^+ and out of it. This is a contradiction and so $\langle p, A\rangle$ decides φ.

□

Corollary 3.12. *The relation $\mathbf{p} \Vdash^* \varphi$ is $\Delta_1^{M_\lambda}$ definable for ranked sentences φ, $\Sigma_n^{M_\lambda}$ for Σ_n sentences and $\Pi_n^{M_\lambda}$ for Π_n sentences of the forcing language.*

Proof. Let $\mathbf{p} = \langle p, X\rangle$. If φ is ranked then for each $q \in [X]^{<\omega}$ let Y^q be the set produced by the above lemma given $\langle p \cup q, X\rangle, \varphi$. Then:

$$\mathbf{p} \Vdash \varphi \iff \forall q \in [X]^{<\omega}(\langle p \cup q, Y^q\rangle \Vdash \varphi).$$

The forward direction is obvious since $\langle p \cup q, Y^q\rangle \leq \mathbf{p}$, and the backward direction holds because then, taking any $X' \subseteq X$ we have that $\langle p \cup q, X' \cap Y^q\rangle \Vdash \varphi$, i.e. the set of extensions of $\mathbf{p}$ which force φ is dense.

This proves the first part, since the given formula is Δ_1, and for unranked formulæ the result is a simple induction on Lévy rank. □

When interpreting a formula of $\mathcal{L}_{F_\lambda}$ in $M_\lambda[\vec{c}]$, we interpret constant symbols from $\mathcal{C}_\alpha$ as the corresponding class term, $\dot{\vec{c}}$ as the predicate $\vec{c}$ and quantification over ranked variables $\exists x^\alpha$ as bounded quantification $\exists x \in J_\alpha^{\vec{c}}$.

Lemma 3.13 (Truth Lemma)**.** *If G is M_λ generic for $\mathbb{P}$ and φ is a Σ_n or a Π_n sentence of $\mathcal{L}_{F_\lambda}$, then $M_\lambda[G] \vDash \varphi \iff \exists \mathbf{p} \in G(\mathbf{p} \Vdash^* \varphi)$.*

Proof. Again this is proved by induction on φ, first for ranked formulæ.

1. If φ is $x \in y$, we must check several cases, firstly assuming that $\mathbf{p} \in G, \mathbf{p} \Vdash^* \varphi$. If x, y in $\mathcal{C}_\alpha, \mathcal{C}_\beta$, respectively, then firstly consider the case when $\alpha = \beta = 0$. Here we only need to check that $\mathbf{p} \Vdash^* x \in \dot{\vec{c}} \implies M_\lambda[\vec{c}] \vDash x \in \vec{c}$, which is true since by definition $x \in p$, and $\mathbf{p} \in G$ so $p \subseteq \vec{c}$.
   If $\alpha < \beta$ then $\mathbf{p} \Vdash^* \psi(x)$ for ψ the defining formula of y. By induction, $M_\lambda[\vec{c}] \vDash \psi(x)$, so $M_\lambda[\vec{c}] \vDash x \in y$.
   If $\alpha \geq \beta$ then $\mathbf{p} \Vdash^* z = x \wedge z \in y$ for some $z \in \mathcal{C}_\gamma$, so inductively, $M_\lambda[\vec{c}] \vDash z = x \wedge z \in y$ which completes the forward direction.
   If $M_\lambda[\vec{c}] \vDash x \in y$ then first suppose $y = \vec{c}$. Then $\exists \langle p, X \rangle \in G(x \in p)$, i.e. $\langle p, X \rangle \Vdash^* x \in y$. If y is an ordinal then every condition will force $x \in y$. Otherwise, find α, β and $\bar{x} \in \mathcal{C}_\alpha, \bar{y} \in \mathcal{C}_\beta$ such that $\bar{x}, \bar{y}$ evaluate to x, y, respectively, in $M_\lambda[\vec{c}]$. When $\alpha < \beta$, the proof is straightforward by induction. If $\alpha \geq \beta$, we need to find $\gamma < \beta$, $\bar{z} \in \mathcal{C}_\gamma$ such that $\mathbf{p} \Vdash^* \bar{z} = \bar{x} \wedge \bar{z} \in \bar{y}$. But if $x \in y$ then x occurs in the $J^{\vec{c}}$ hierarchy strictly before y, and so is realised as a class term in $\mathcal{C}_\gamma$ for $\gamma < \beta$. From here induction gives us what we want.
2. If φ is $x = y$ then
   $$\begin{aligned} M_\lambda[\vec{c}] \vDash x = y &\iff \exists \alpha < \widetilde{\theta}(\forall z \in J^{\vec{c}}_\alpha(z \in x \leftrightarrow z \in y)) \\ &\iff \exists \mathbf{p} \in G(\mathbf{p} \Vdash^* (\forall z^\alpha(z \in x \leftrightarrow z \in y))) \\ &\iff \exists \mathbf{p} \in G(\mathbf{p} \Vdash^* (x = y)), \end{aligned}$$
   where we use the induction hypothesis in the second equivalence.
3. If φ is $\psi_0 \wedge \psi_1$ then the argument is standard.
4. If φ is $\neg\psi$ and $\mathbf{p} \in G$, $\mathbf{p} \Vdash^* \neg\psi$, then suppose ψ held in $M_\lambda[\vec{c}]$. Then, by the induction hypothesis there is $\mathbf{q} \in G(\mathbf{q} \Vdash^* \psi)$. But there would then be a $\mathbf{r} \leq \mathbf{p}, \mathbf{q}$ with $\mathbf{r} \Vdash^* \psi \wedge \neg\psi$, which is impossible, completing the proof of the right-to-left direction.
   On the other hand if $M_\lambda[\vec{c}] \vDash \neg\psi$ then there is by genericity at least a condition $\mathbf{p} \in G$ such that $\mathbf{p} \| \psi$. If $\mathbf{p} \Vdash^* \neg\psi$ then we are done, and otherwise the inductive hypothesis for ψ would imply that $M_\lambda[\vec{c}] \vDash \psi$, which is impossible.
5. If φ is $\exists x^\alpha \psi(x^\alpha)$ and $\mathbf{p} \in G$, $\mathbf{p} \Vdash^* \varphi$ then there is some $x \in \mathcal{C}_\alpha$ such that $\mathbf{p} \Vdash^* \psi(x)$, and hence $M_\lambda[\vec{c}] \vDash \psi(x)$, with x interpreted appropriately, hence $M_\lambda[\vec{c}] \vDash \exists x \in J^{\vec{c}}_\alpha(\psi(x))$. The opposite direction is identical.
6. If φ is $\exists x \psi(x)$ and $\mathbf{p} \in G$, $\mathbf{p} \Vdash^* \varphi$ then there is some α such that $\mathbf{p} \Vdash^* \exists x^\alpha \psi(x^\alpha)$, and we can use the previous case. If $M_\lambda[\vec{c}] \vDash \exists x \psi(x)$ then, by the definition of the generic extension, $M_\lambda[\vec{c}] \vDash x \in J^{\vec{c}}_\alpha(\psi(x))$, and again we proceed as above.

$\square$

We adopt the usual generalisation of diagonal intersection:

$$\Delta_p A_p = \Delta_\alpha \left(\bigcap \{A_p \mid \max p < \alpha\} \right).$$

Theorem 3.14. *The filter*

$$G_{\vec{c}} = \{\langle p, X\rangle \in \mathbb{P} \mid p \textit{ is an initial segment of } \vec{c} \textit{ and } \vec{c} \setminus p \subseteq X\}$$

is M_λ-generic for $(\mathbb{P})^{M_\lambda}$.

Proof. We first prove that $G_{\vec{c}}$ intersects every Σ_n dense class of M_λ. Let D be a $\Sigma_n^{M_\lambda}$ subset of $\mathbb{P}$. Let $S_p = \{q \in [\kappa_\lambda]^{<\omega} \mid \exists X(\langle p \cup q, X\rangle \in D)\}$. This is Σ_n, so by Rowbottom's theorem there is a measure 1 set $A_p \in F_\lambda \cap M_\lambda$ such that either $[A_p]^{<\omega} \subseteq S_p$ or $[A_p]^k \cap S_p = \varnothing$ for some k. If there is some $X \in F_\lambda$ such that $\langle p, X\rangle \in D$ then let X_p be the M_λ-least and set $B_p = A_p \cap X_p$, otherwise let $B_p = A_p$. Then let B be the diagonal intersection of the B_p's. Being measure 1, B includes a final segment of $\vec{c}$, so let q be such that $\vec{c} \setminus q \subseteq B$. By density there must be an extension of $\langle q, B\rangle$, say $\langle q \cup t, X\rangle$, lying in D.

If $r \subseteq \vec{c}$ extends q then $r \setminus q \subseteq B$ by the way we picked q. By the definition of B, $B \smallsetminus q \subseteq A_q$ and so $B \smallsetminus q$ is homogeneous for S_q. Thus since $t \in S_q \cap B \smallsetminus q$ by definition, $[B \smallsetminus q]^{<\omega} \subseteq S_q$. Hence $r \in S_q$, and so for some Y the condition $\langle q \cup r, Y\rangle \in D$. Since $q \cup r \subseteq \vec{c}$, $G \cap D \neq \varnothing$, completing the proof of the first requirement for genericity.

Now we prove that $G_{\vec{c}}$ decides every Σ_n sentence of $\mathcal{L}_{F_\lambda}$. If φ is ranked then by Lemma 3.11 for any $p \in [\kappa_\lambda]^{<\omega}$ there is a $\Delta_1^{M_\lambda}(p, \varphi)$-definable set X such that $\langle p, X\rangle \| \varphi$. Since X is measure 1 it includes an end-segment of indiscernibles, and since it is definable from p and φ, it contains every indiscernible beyond p. Thus if $p \subseteq \vec{c}$ then $\langle p, X\rangle \in G_{\vec{c}}$, completing the proof in this case.

Suppose φ is $\exists x \psi(x)$, with ψ a Π_{k-1} formula of $\mathcal{L}_{F_\lambda}$, $k \leq n$, and fix $p \in [\kappa_\lambda]^{<\omega}$. Suppose inductively that, for each $t \in \mathcal{C}$, there is an $X \in F_\lambda$ such that $\langle p, X\rangle \in G_{\vec{c}}$ and $\langle p, X\rangle \| \psi(t)$. Let:

$$\theta(X) \equiv \langle p, X\rangle \Vdash^* \psi(t) \vee \langle p, X\rangle \Vdash^* \neg\psi(t).$$

This is Δ_k, so for each t, let X_t be the M_λ-least set satisfying θ, hence $\Delta_k^{M_\lambda}$:

$$\begin{aligned} X = X_t &\iff \theta(X) \wedge \forall Y(Y \not<_{M_\lambda} X \vee \neg\theta(Y)) \\ &\iff \theta(X) \wedge \exists u(u = \mathrm{pr}(X) \wedge \forall Y \in u(\neg\theta(Y))). \end{aligned}$$

Here pr is the Δ_1 function returning the set of all $<_{M_\lambda}$-predecessors of a set, following [2] where this is defined for L in II.3.5. Thus $t \mapsto X_t$ is $\Delta_k^{M_\lambda}$ and the following set is $\Pi_k^{M_\lambda}$:

$$Z = \{\alpha \in \kappa_\lambda \mid \forall t \in \mathcal{C}(\alpha \in X_t)\}.$$

Z is, by Σ_k separation, an element of M_λ and $Z \supseteq (\vec{c} \smallsetminus p)$, so $\mathbf{p} = \langle p, Z\rangle \in G_{\vec{c}}$. $\mathbf{p} \| \psi(t)$ for each t, so $\mathbf{p} \| \varphi$, completing the induction and the proof. $\square$

Remark. Note that we may add finitely many elements of κ_λ to $\vec{c}$, and the above proof still works. This means that, if $\mathbf{p} = \langle p, X\rangle \in \mathbb{P}$ then $G_{p \cup \vec{c}}$ is M_λ-generic and contains $\mathbf{p}$.

Theorem 3.15. *$\mathbf{p} \Vdash \varphi \iff \mathbf{p} \Vdash^* \varphi$, for Σ_n formulæ φ.*

Proof. By $\mathbf{p} \Vdash \varphi$ we mean the usual (semantic) forcing relation:

$$\forall G (G \text{ is } M_\lambda\text{-generic for } \mathbb{P} \wedge \mathbf{p} \in G \to M_\lambda[G] \vDash \varphi).$$

The right-to-left implication of the theorem is provided by the previous lemma.

For the forward direction suppose $\mathbf{p} \Vdash \varphi$. Suppose for a contradiction that the Σ_n class $D = \{\mathbf{q} \mid \mathbf{q} \Vdash^* \varphi\}$ is not dense below $\mathbf{p}$. In other words, suppose $\mathbf{q} \leq \mathbf{p}$ is such that $\forall \mathbf{r} \leq \mathbf{q} (\mathbf{r} \nVdash^* \varphi)$, i.e. $\mathbf{q} \Vdash^* \neg\varphi$. But then by the right-to-left implication, $\mathbf{q} \Vdash \neg\varphi$. Let G be M_λ-generic for $\mathbb{P}$ with $\mathbf{q} \in G$ so $M_\lambda[G] \vDash \neg\varphi$. But $\mathbf{p} \in G$, too, so $M_\lambda[G] \vDash \varphi$, which is a contradiction. $\square$

We have now checked that the usual properties of forcing hold, and can use the $\Vdash$ relation as we normally would, safe in the knowledge that $\mathbf{p} \Vdash \varphi$ is Σ_n as long as φ is.

3.2.1. The Generic Extension

We now want to prove that, in the Σ_n case of the Truth Lemma, it doesn't matter which initial segment of $\vec{c}$ we pick; we may always find a suitable measure-1 set Y to force the statement.

Lemma 3.16. *Let $p = \{c_0, \ldots, c_l\}$ and y an arbitrary constant Σ_n-definable from $\operatorname{ran}\pi$ and finitely many elements of C, with the maximum such element $c_j < c_l$. Suppose ψ is Π_{n-1}. Then:*

$$M_\lambda[\vec{c}] \vDash \exists z \psi(z, y) \Leftrightarrow M_\lambda \vDash \exists Y \langle p, Y \rangle \Vdash \exists z \psi(z, y).$$

Proof. Fixing p, suppose the left hand side holds, and work in M_λ. Then define:

$$A := \{q \in [\kappa_\lambda \smallsetminus p]^{<\omega} \mid \exists z \exists X \langle p \cup q, X \rangle \Vdash \psi(z, y)\}.$$

A is $\Sigma_n^{M_\lambda}(y, p)$, since $\mathbf{p} \Vdash \psi$ is $\Pi_{n-1}^{M_\lambda}$ if $n \geq 2$ (or $\Delta_1^{M_\lambda}$ if $n = 1$.) By hypothesis and the truth lemma, there is some extension of $\mathbf{p}$ in the generic that forces $\exists z \psi(z, y)$. Thus let $q \subseteq \vec{c} \smallsetminus p, X \in F_\lambda, z$ be such that $\mathbf{p'} = \langle p \cup q, X \rangle \in G_{\vec{c}}$, $\mathbf{p'} \Vdash \psi(z, y)$. Hence by indiscernibility, A contains any element of $[C]^{<\omega}$ the same length as q, and thus $M_\lambda \vDash A \in (F_\lambda)^k$. Hence by Rowbottom in M_λ, there is $Y \in F_\lambda$ with $[Y]^k \subseteq A$. For such a $q \in A$ let Y be the M_λ-least such set, so that Y is Σ_n-definable in M_λ from y and p. Then, if $q \in [Y]^k$ for some k, let z^q, X^q be witnesses that $q \in A$.

The map $q \mapsto X^q$ is therefore $\Sigma_n(y, p)$-definable, allowing us to define:

$$Y' = \Delta_q (X^q \smallsetminus q) \cap Y.$$

Y' is then the diagonal intersection of measure 1 sets and is Σ_n-definable, so $M_\lambda \vDash Y' \in F_\lambda$. Y' also is such that if $q \in [Y']^k$ then $Y' \smallsetminus q \subseteq X^q$, since suppose $\alpha \in Y' \smallsetminus q$ (hence $\alpha > \sup q$), then $\alpha \in X^{q'} \smallsetminus q'$ for all q' with supremum smaller than α. But $\sup q < \alpha$ so $\alpha \in X^q$. But we have now established that, for any

$\langle p \cup q, \bar{Y}\rangle \leq \langle p, Y'\rangle$, we have $\langle p \cup q, \bar{Y}\rangle \leq \langle p \cup q, X^q\rangle \Vdash \psi(z^q, y)$ and we are done with the forward direction.

Hence suppose the latter. Then let $\mathbf{p} = \langle p, Y\rangle$ be the M_λ-least such condition, hence Σ_n-definable in M_λ from y, p. But by indiscernibility in the form of Lemma 3.4, $Y \supseteq C \smallsetminus p$, so $\mathbf{p} \in G_{\vec{c}}$ by definition of the latter, and we are done. □

As mentioned above, we need to check that the generic extension is a model of KP_n.

Theorem 3.17. $M_\lambda[\vec{c}] \vDash \mathsf{KP}_n$.

Proof. Since κ_λ is the largest cardinal in M_λ, we only need to consider functions defined (partially) on κ_λ. Thus for any function f with $\operatorname{dom} f \subseteq \kappa_\lambda$ defined by $f(\xi) = y \Leftrightarrow \exists z \varphi(y, \xi, z, d)$ (where $\varphi \in \Pi_{n-1}^{M_\lambda[\vec{c}]}$ and d is some parameter, thus φ may refer to the generic via the predicate symbol $\dot{G}$) we wish to prove:

$$M_\lambda[\vec{c}] \vDash \exists \zeta \forall \xi \in \kappa_\lambda (\xi \in \operatorname{dom} f \to f(\xi) < \zeta).$$

Thus let $\mathbf{p} = \langle p, X\rangle = \langle \{c_0, \dots, c_k\}, X\rangle \in G_{\vec{c}}$ be a condition which forces the following:

$$f \text{ is a function} \wedge \operatorname{ran} f \subseteq On.$$

The parameter d must, in M_λ, be definable from $\vec{a} \in \operatorname{ran}\pi$ and finitely many indiscernibles, $\vec{e} \in [C]^{<\omega}$ since M_λ is an iterated ultrapower model. By strengthening $\mathbf{p}$ if necessary, assume $\max \vec{e} \leq \max p$. For $\xi \leq \max p$, define y_ξ to be such that:

$$\exists X_\xi (\langle p, X_\xi\rangle \Vdash f(\xi) = y_\xi)$$

wherever such a y_ξ exists. We claim this defines a $\Sigma_n^{M_\lambda}(p, d)$ function defined on $\operatorname{dom} f \cap (\max p + 1)$. First note that, if there is such a X_ξ, then any other $\langle p, X'_\xi\rangle$ is compatible with $\langle p, X_\xi\rangle$ and hence y_ξ is well defined.

For the domain, suppose $M_\lambda[\vec{c}] \vDash \xi \in \operatorname{dom} f$. Then by Lemma 3.16 there is an $X' \in F_\lambda$ such that $\langle p, X'\rangle \Vdash \xi \in \operatorname{dom} f$. Hence $\langle p, X' \cap X\rangle \Vdash \xi \in \operatorname{dom} f$, i.e. $\langle p, X' \cap X\rangle \Vdash \exists y f(\xi) = y$, and so y_ξ must be defined.

The map $\xi \mapsto y_\xi$ is $\Sigma_n^{M_\lambda}(p, d)$ and so by KP_n has a bound, let's say:

$$\forall \xi \in \operatorname{dom} f (\xi \leq \max p \to y_\xi < \tau_p).$$

Hence if $\xi \in \operatorname{dom} f, \xi \leq \max p$, we have $\langle p, X_\xi\rangle \Vdash f(\xi) = y_\xi$ and $\langle p, X_\xi\rangle \Vdash f(\xi) < \tau_p$. Let $D = \operatorname{dom} f \cap (\max p + 1)$, a Σ_n element of M_λ by KP_n. Define:

$$A = \{r \in [\kappa_\lambda \smallsetminus p]^{<\omega} \mid \exists \tau_{p \cup r}, Y_r \forall \xi \in D (\langle p \cup r, Y_r\rangle \Vdash f(\xi) < \tau_{p \cup r})\}.$$

Now, A is Σ_n definable and for each l we know we can find some $r = \langle c_{k+1}, \dots, c_{k+l}\rangle$ such that $r \in A$, since we can find the corresponding $\tau_{p \cup r}$ as we found τ_p. Thus by indiscernibility, $A \in (F_\lambda)^l$. But then by Rowbottom in M_λ, there is a set $A_l \in F_\lambda$ such that $[A_l]^l \subseteq A$, and since all parameters in its definition are at most $\max p$, $A_l \supseteq C \setminus p$. Then if we let $\tilde{A} = \bigcap_l A_l$, we have that $\tilde{A} \supseteq C \setminus p$. By KP_n, let τ bound all the ordinals $\tau_{p\cup r}$ for $r \in [\tilde{A}]^{<\omega}$.

Hence $\tilde{\mathbf{p}} := \langle p, \tilde{A} \cap X\rangle \in G_{\vec{c}}$ and for any $\xi \in \operatorname{dom} f$, if $\langle p \cup q, B\rangle$ is an arbitrary extension of $\tilde{\mathbf{p}}$, we can find some $r \in [B]^{<\omega}$ (and so $q \cup r \in [\tilde{A}]^{<\omega}$) such that $\xi \leq \max r$ and so $\langle p \cup q \cup r, Y_{q\cup r}\rangle \Vdash f(\xi) < \tau_{p\cup q\cup r} < \tau$. Hence $\tilde{\mathbf{p}} \Vdash \sup \operatorname{ran} f < \tau$ and we are done. $\square$

Theorem 3.18. *If M is the $<_*$-least mouse such that $M \vDash \mathsf{KP}_n$, then $M_\lambda[\vec{c}]$ is the smallest transitive model of KP_n containing $\vec{c}$ as an element. Thus $M_\lambda[\vec{c}] = \mathcal{A}_{\mathsf{KP}_n}[\vec{c}] = L_{\widetilde{\theta}}[\vec{c}]$.*

Proof. By Separation in $M_\lambda[\vec{c}]$, we have that $\vec{c}$ is an element of $M_\lambda[\vec{c}]$. All that remains to prove is minimality. Let N be a transitive model of KP_n with $\vec{c} \in N$. We claim that it is sufficient to consider N of the form $J^{\vec{c}}_\alpha$: Otherwise, if $J^{\vec{c}}_\alpha \nvDash \mathsf{KP}_n$ then there is a Σ_n-definable partial function definable over $J^{\vec{c}}_\alpha$ unboundedly into the ordinals of $J^{\vec{c}}_\alpha$. But this function would then be unbounded and $\Sigma^{\bar{N}}_n$ for any admissible set $\bar{N}$ with $\vec{c} \in \bar{N}$ of height α (since the $J^{\vec{c}}_\beta$ hierarchy is Δ_1 over any structure containing $\vec{c}$) including N if $On \cap N = \alpha$.

Now, we defined $M_\lambda[\vec{c}]$ to be $J^{\vec{c}}_{\widetilde{\theta}}$. Hence we need to show that $J^{\vec{c}}_\alpha$ is not a model of KP_n for $\alpha < \widetilde{\theta}$.

But by leastness of M, M_λ is also the least $J^{\widetilde{F}}_\alpha$ such that $J^{\widetilde{F}}_\alpha \vDash \mathsf{KP}_n$. Thus if $\alpha < \widetilde{\theta}$, $J^{\widetilde{F}}_\alpha$ is not a model of KP_n and hence $J^{\vec{c}}_\alpha$ cannot be a model of KP_n.

Since $J^{\vec{c}}_{\widetilde{\theta}}$ is admissible, $\widetilde{\theta} = \omega\alpha$ and hence in fact $M_\lambda[\vec{c}] = L_{\widetilde{\theta}}[\vec{c}]$. $\square$

Proof of Theorem 3.2. Since the class C of iteration points is a club, we just have to prove the indiscernibility.

Here we abbreviate $\mathcal{A}_{\mathsf{KP}_n}[\vec{c}]$, defined in Definition 3.1, by $\mathcal{A}_n[\vec{c}]$. Since a sentence of the language of $\mathcal{A}_n[\vec{c}]$ or $\mathcal{A}_n[\vec{d}]$ is a formula of set theory possibly mentioning the predicate symbol $\dot{G}$, it is sufficient to take such a Σ_n sentence φ and show that $\mathcal{A}_n[\vec{c}]$ and $\mathcal{A}_n[\vec{d}]$ agree on φ. Let M_λ (respectively $M_{\lambda'}$) be the iterate whose measurable cardinal is the supremum of $\vec{c}$ (respectively $\vec{d}$.) Then, using that forcing φ is Σ_n:

$$\begin{aligned}
\mathcal{A}_n[\vec{d}] \vDash \varphi &\Leftrightarrow M_{\lambda'} \vDash \exists Y(\langle \varnothing, Y\rangle \Vdash \varphi) && \text{By Lemma 3.16}\\
&\Leftrightarrow M_\lambda \vDash \exists Y(\langle \varnothing, Y\rangle \Vdash \varphi) && \text{By } \Sigma_n\text{-elementarity}\\
&\Leftrightarrow \mathcal{A}_n[\vec{c}] \vDash \varphi && \text{By Lemma 3.16.}
\end{aligned}$$

$\square$

3.3. ZFC

The natural extension of these results to stronger mice will also be interesting. In this section, M is a non-trivial ZFC-mouse with normal measure F. We will prove:

Theorem 3.19. *If there is a non-trivial mouse* $M \models \mathsf{ZFC}$ *then there is a class* C *of* Σ_ω *generating indiscernibles for* ZFC*.*

Thus these are full indiscernibles and any two models generated by an ω-sequence of them will be fully elementarily equivalent.

The procedure here is much simpler than in the preceding case. Since $M \models \mathsf{ZFC}$ again in the language of set theory augmented with a predicate for the filter F on κ, F is a definable set in M by Separation and Power Set. Thus the forcing is a set forcing and we can dispense with most of the analogues of the previous section. Define again M_λ to be the λth iterate and have measurable cardinal $\sup \vec{c}$.

Lemma 3.20. $M_\lambda[\vec{c}] = \mathcal{A}_\omega[\vec{c}]$*, the smallest transitive model of* ZFC *with* $\vec{c} \in \mathcal{A}_\omega[\vec{c}]$*.*

Proof. Since the forcing is set-sized, this is standard: we know that $M_\lambda[\vec{c}] \models \mathsf{ZFC}$, $\vec{c} \in M_\lambda[\vec{c}]$, and that $M_\lambda[\vec{c}]$ is the smallest transitive model of ZFC containing $\vec{c}$. $\square$

Proof of Theorem 3.19. Let φ be any sentence in the language of $\mathcal{A}_\omega[\vec{c}]$. Then since we have full elementarity between any two iterates of M,

$$\mathcal{A}_\omega[\vec{c}] \models \varphi \iff M_\lambda \models (\Vdash_{\mathbb{P}} \varphi) \iff M_{\lambda'} \models (\Vdash_{\mathbb{P}} \varphi) \iff A_\omega[\vec{d}] \models \varphi$$

using the same notation as in the proof of Theorem 3.2. $\square$

3.4. Preserving substructures

Now let us abandon our previous notation and fix $M = \langle J^F_\alpha, F\rangle$ to be an admissible structure of the kind discussed in [15], that is, the Q-structure of a "clever" mouse, M_0. Throughout this section, "admissible" will mean a transitive model of KP in the language containing the predicate F. Then $M \models \mathsf{KP} \wedge$ "F is a normal measure on κ" and M satisfies the following Σ_1-Rowbottom property:

$$\{\xi < \kappa \mid M \models \varphi(\xi, p)\} \in F^{M_0} \implies \exists \tau < \alpha \{\xi < \kappa \mid J^F_\tau \models \varphi(\xi, p)\} \in F^M$$

for Σ_1 formulæ φ. Under these conditions it was proved in [15] that Prikry forcing over M preserves KP. Originally this was used under the hypothesis that M_0 is the least clever mouse, yielding that $M_\lambda[\vec{c}]$ (the forcing extension of the Q-structure of the λth iterate of M_0) is the least admissible set containing $\vec{c}$ as an element. If M_0 is, say, the second clever mouse, then the situation changes as we would hope.

Lemma 3.21. *Let* $\kappa_\lambda \in N \in M_\lambda$ *with* N *admissible. Let* $D \in N$ *be predense in* N*, in the partial order* $(\mathbb{P})^N$*. Then* D *is predense in* $(\mathbb{P})^{M_\lambda}$*. Hence* $\vec{c}$ *is* $\mathbb{P}$*-generic over* N

Proof. So let $\mathbf{p} = \langle p, X\rangle \in (\mathbb{P})^{M_\lambda}$, and we need to find $\mathbf{r} \leq \mathbf{q} \in D$ such that $\mathbf{r} \leq \mathbf{p}$. Now, define the set of extensions of p which extend an element of D:

$$H = \{r \in [\kappa \smallsetminus p]^{<\omega} \mid \exists \langle q, Y_r\rangle \in D(p \cup r \supseteq q \wedge p \cup r \setminus q \subseteq Y_r)\}.$$

This is $\Delta_1^N(D, p)$, so by admissibility, $H \in N$. By Rowbottom, $\exists Z \in N \cap F_\lambda$ such that either $[Z]^{<\omega} \subseteq H$ or $[Z]^{<\omega} \cap H = \varnothing$. The latter case is impossible since, considering the condition $\langle p, Z\rangle \in (\mathbb{P})^N$, by predensity there is $r \in [Z]^{<\omega}$ and some $Z' \subseteq Z$ such that $\langle p \cup r, Z'\rangle$ extends an element of D. That is to say, $\langle p \cup r, Z'\rangle \leq \langle q, Y_r\rangle$ for some condition $\langle q, Y_r\rangle \in D$, i.e. $r \in H$.

But now $\langle p, Z \cap X\rangle$ is a condition in M_λ such that, for any $r \in Z \cap X$, the condition $\mathbf{r} := \langle p \cup r, Z \cap X \cap Y_r\rangle \leq \langle p \cup r, Y_r\rangle$ extends a condition in D, so $\mathbf{p}$ is compatible with an element of D, which is what we wanted. □

Lemma 3.22. *If N in addition is the Q-structure of a clever mouse then $M_\lambda[\vec{c}] \models \exists A(A$ is admissible$)$.*

Proof. By the results of [15], Section 3, $N[\vec{c}] \in M_\lambda[\vec{c}]$ is admissible. □

Now, if N, M are the least and second-least admissible-in-F structures containing κ_λ as an element, we want to know that $N_\lambda[\vec{c}], M_\lambda[\vec{c}]$ are similarly minimal. For $N_\lambda[\vec{c}]$ this is already known, and for $M_\lambda[\vec{c}]$ it is a minor modification:

Lemma 3.23. *If N, M are as in the preceding paragraph then $M_\lambda[\vec{c}] \models$ "$N_\lambda[\vec{c}]$ is the only admissible set $\bar{N}$ with $\vec{c} \in \bar{N}$".*

Proof. Suppose not, i.e. there is a $\gamma \in On \cap M_\lambda[\vec{c}]$ such that $\gamma > \kappa_\lambda$, $J_\gamma^{\vec{c}} \models \mathsf{KP}$ and $N_\lambda[\vec{c}] \in J_\gamma^{\vec{c}}$. Then as in the proof of the corresponding Lemma 3.10 of [15], J_γ^F (F the measure on κ_λ) is $Q_{\kappa_\lambda}^{N'}$ for some mouse $N' \in H_{\kappa_\lambda}^{M_\lambda}$. $N <_* N' <_* M$, so N' is therefore not clever as there are no clever mice between N and M. Then obtain a contradiction with the admissibility of $J_\gamma^{\vec{c}}$ as in [15]. □

Theorem 3.24. *If M is as above, there is a closed-unbounded class of Σ_1 generating indiscernibles for the theory "$T = \mathsf{KP} +$ there is an admissible set N with $\vec{c} \in N$."*

Proof. Thus with this theory T, the model $\mathcal{A}_T[\vec{c}]$ is the second-least admissible set containing $\vec{c}$.

The proof is exactly the same as the above indiscernibility proofs, relying on the fact that forcing relation for Σ_1 sentences is Σ_1 over M_λ, as proved in [15], and that the iterates are Σ_1 elementary. □

4. Determinacy from Indiscernibility

Theorems 3.2, 3.19 and 3.24 provide the indiscernibles that are used, via the usual argument, to prove determinacy in this section. From now on, fix a pointclass Γ, $n \in \omega$ and a theory T, and suppose the hypotheses of Theorem 1.2 are satisfied, letting C be the set of generating indiscernibles.

First of all we need a kind of remarkability for generating indiscernibles. For $\vec{c}$ a set of ordinals, let c_m denote the mth ordinal in the increasing enumeration of $\vec{c}$.

Lemma 4.1. *Suppose:*

1. $\varphi(x)$ *is a* Σ_n*-formula with one free variable and there is a* $m \in \omega$ *such that, for any* $\vec{c} \in [C]^\omega$, φ *defines an ordinal* γ *over* $\mathcal{A}_T[\vec{c}]$ *with* $c_{m-1} \leq \gamma < c_m$*;*
2. $\vec{c}, \vec{d} \in [C]^\omega$ *with* $\{i \in \omega : c_i \neq d_i\}$ *finite and* $c_i = d_i$ *for* $i < m$*.*
3. $\mathcal{A}_T[\vec{c}] \vDash \varphi(\gamma_1)$ *and* $\mathcal{A}_T[\vec{d}] \vDash \varphi(\gamma_2)$*.*

Then $\gamma_1 = \gamma_2$*.*

Proof. Without loss of generality suppose that $c_m < d_m$. Assume further that $\vec{c}$ and $\vec{d}$ are cardinals and hence limit points of C. Doing so does not affect the result; if $\gamma_1 \neq \gamma_2$ satisfied the conditions of the Lemma, "spreading out" $\vec{c}$ and $\vec{d}$ does not change this. Let $\vec{e} = \vec{c} \cup \vec{d}$. By 2., $e_i = c_i = d_i$ for $i < m$, and $\vec{c}, \vec{d}$ are definable in $\mathcal{A}_T[\vec{e}]$ (from $\vec{e}$, which is in our language) by just writing down the points where $\vec{c}$ and $\vec{d}$ differ, of which there are finitely many.

They are hence Δ_0 definable from $\vec{e}$, and so $\mathcal{A}_T[\vec{c}], \mathcal{A}_T[\vec{d}]$ are Σ_1-definable inner models of $\mathcal{A}_T[\vec{e}]$. Thus the following is expressible as a Σ_n-sentence of $\mathcal{A}_T[\vec{e}]$:

$$\exists \gamma_1 < c_m \exists \gamma_2 < d_m \left(\mathcal{A}_T[\vec{c}] \vDash \varphi(\gamma_1) \wedge \mathcal{A}_T[\vec{d}] \vDash \varphi(\gamma_2) \wedge \gamma_1 < \gamma_2 \right)$$

as is the same sentence but with $\gamma_1 > \gamma_2$ instead of $\gamma_1 < \gamma_2$.

So, first suppose that $\gamma_1 > \gamma_2$, and let $\vec{d'} \in [C]^\omega$ have the same order relationship with $\vec{d}$ as $\vec{d}$ does with $\vec{c}$. Note this is possible because $\vec{c}$ and $\vec{d}$ consist of limit points of C and hence there is enough room to find such a $\vec{d'}$ from the remaining elements of C. By indiscernibility we also have $\gamma_2 > \gamma_2'$ where γ_2' is defined in $\mathcal{A}_T[\vec{e}]$ by $\varphi(\gamma_2')^{\mathcal{A}_T[\vec{d'}]}$. Hence by picking successive $\vec{d'}$s in the same way, we end up with a decreasing sequence of ordinals — contradiction.

Now suppose that $\gamma_1 < \gamma_2$. There are two further sub-cases to consider, the first being when $\gamma_2 < c_j$ for some j. In this case, we can consider successive $\vec{d'}, \vec{d''}, \ldots$ where $\vec{d'}, \vec{d}$ have the same order relations as $\vec{d}, \vec{c}$ and so on. Hence γ_2' (defined as before) is greater than γ_2. But we can do this greater than c_j-many times as C is a proper class, after which we necessarily have $\gamma_2^{(c_j)} > c_j$. But this is impossible by indiscernibility.

Finally suppose $\gamma_2 > c_i$ for all i. This is still expressible as a Σ_n formula inside $\mathcal{A}_T[\vec{e}]$ and so this time take successive sequences $\vec{c'}$ so that $\sup c_i'$ approaches d_i. Thus we can increase $\vec{c'}$ d_n-many times, so that γ_2, which is less than d_n, must be less than some c_i', which cannot happen by indiscernibility.

Thus the only possibility we are left with is the one we wanted: $\gamma_1 = \gamma_2$. $\square$

We now commence with the proof of Theorem 1.2, following the method established by Martin for proving determinacy in the Π^1_1 difference hierarchy. Suppose $A \subseteq \omega^\omega$ is ω^2-$\Pi^1_1 + \Gamma$ and let $\langle B_\beta \mid \beta < \omega^2 + 1 \rangle$ with $B_{\omega^2} \in \Gamma$ witness this. Let $\mathcal{A} = \mathcal{A}_T[\langle \aleph_n \mid n < \omega\rangle]$. We set up an auxiliary game G^* as in, for example, [11]. For $x \in \omega^\omega$, let $\prec^\beta_x$ be a linear ordering of ω with maximal element 0, such that $\prec^\beta_{x\restriction n}$ depends only on $x \restriction n$ and:

$$\prec^\beta_x \text{ is a well-ordering if and only if } x \in B_\beta$$

We know such orderings exist by the general theory of Π^1_1 and we can in fact make the function $x \mapsto \prec^\beta_x$ recursive.

Next let $\langle \beta, n\rangle \to \xi^\beta_n$ be a bijection between $\omega^2 \times \omega$ and ω such that:

1. ξ^β_n is even iff β is even;
2. ξ^β_n is increasing in n for fixed β;
3. For natural numbers i and $a < b$, $\xi^{\omega i + a}_0 < \xi^{\omega i+b}_0$

Note this mapping can also be taken to be recursive.

In the auxiliary game G^*, each move is a pair (a_i, η_i) such that $a_i \in \omega$ and $\eta_i \in \aleph_\omega$:

$$\begin{tikzcd}
I & \langle a_0, \eta_0\rangle \arrow[dr] & & \langle a_2, \eta_2\rangle \arrow[dr] & & \cdots \\
II & & \langle a_1, \eta_1\rangle \arrow[ur] & & \langle a_3, \eta_3\rangle \arrow[ur] &
\end{tikzcd}$$

defining an auxiliary game tree, T^*. The tree is definable in a $\Delta^{\mathcal{A}}_0$ way, and so by admissibility is an element of $\mathcal{A}$.

For $\beta < \omega^2$ and a play of this game, $x^* = \langle (a_i, \eta_i) \mid i \in \omega\rangle$, define the function $F^\beta : \omega \to \aleph_\omega$ by:

$$F^\beta(n) = \eta_{\xi^\beta_n}$$

Then we say that a play of G^* is *badly lost* if some $F^{\omega\cdot i+b}$ is not an order-preserving embedding of $\langle \omega, \prec^{\omega\cdot i+b}_x\rangle$ into $\langle \aleph_i, <\rangle$. A badly lost position is defined in the same way. If x^* is a badly lost play, then there is a shortest badly lost position $p^* \subseteq x^*$ witnessing that some $F^{\omega\cdot i+b}$ is not order preserving. We then say x^* is badly lost *for I* if the least such b is even, otherwise it is badly lost for II, and denote the sets of such plays as B^I and B^{II}, respectively.

Then, if x^* is badly lost for I, II wins, and vice-versa. If x^* is not badly lost then I wins if and only if $x^* \in B_{\omega^2}$.

This implicitly defines the winning set of the game G^*, which we denote A^*. We now use the work of Section 2.2 to calculate the complexity of A^*.

Lemma 4.2. *A^* is $\widetilde{\Sigma}^0_1 \wedge \widetilde{\Pi}^0_1$ if $n = 1$ and $\widetilde{\Sigma}^0_n$ if $n > 1$ in the space $[T^*]$.*

Proof. A^* is defined as:

$$x \in A^* \iff x \notin B^I \wedge (x \in B^{II} \vee \pi(x) \in A_{\omega^2})$$

where π is the projection function from $[T^*]$ to ω^ω:

$$\pi(\langle\langle a_0, \eta_0\rangle, \langle a_1, \eta_1\rangle, \langle a_2, \eta_2\rangle, \ldots\rangle) = \langle a_0, a_1, a_2, \ldots\rangle.$$

Observe that π is continuous, and in fact the relation $p^* \subseteq \pi(x^*)$ (for $p^* \in T^*$, $x^* \in [T^*]$) is generalised semi-recursive. Hence, since A_{ω^2} is $\widetilde{\Sigma}^0_n$ considered as a subset of $[T^*]$ by Proposition 2.8, the assertion "$\pi(x) \in A_{\omega^2}$" is $\widetilde{\Sigma}^0_n$ by Lemma 2.7.

Now, B^I and B^{II} are $\widetilde{\Sigma}^0_1$: Let $\varphi(p)$ be the formula:

$$\exists b, i \left(F_p^{\omega\cdot i+b} \text{ is not an order-preserving map from } \langle \omega, \prec_p^{\omega\cdot i+b}\rangle \text{ to } \aleph_i\right)$$
$$\wedge \text{ the least such } b \text{ is even.}$$

This is $\Delta^{\mathcal{H}}_1$ and if $\varphi(p)$ with $p \subseteq q$ then $\varphi(q)$. Then $x \in B^I$ if and only if $\exists m(\varphi(x \restriction m))$, so B^I is $\widetilde{\Sigma}^0_1$. Similarly for B^{II}. Hence A^* is $\widetilde{\Pi}^0_1 \wedge (\widetilde{\Sigma}^0_1 \vee \widetilde{\Sigma}^0_n)$. If $n = 1$ this is $\widetilde{\Pi}^0_1 \wedge \widetilde{\Sigma}^0_1$ and if $n > 1$ then this is $\widetilde{\Sigma}^0_n$.

□

Now suppose that, by the hypothesis of Theorem 1.2, σ^* is a Σ_n-definable winning strategy for G^*.

We argue that whichever player has the strategy σ^* has a winning strategy for G. The strategy for G is defined by having the player "pretend" their opponent is playing G^*, using indiscernibility to find that the integer moves returned by σ^* are independent of the ordinals they choose to pretend were played. The auxiliary game is constructed so that the integers played constitute a winning play in G, so a strategy so defined will also be winning. We prove this below in the case when II has a winning strategy; the case for I is identical.

When constructing the pretend moves, we will only be considering sequences where I *plays well*, i.e. he plays so as not to be badly lost and further:

1. $F^{\omega i+2j}(n) > F^{\omega i+j'}(0)$ for $j' < 2j$, $i, j \in \omega$;
2. $F^{\omega i+2j}(n) > \aleph_i$ for $i, j \in \omega$;
3. $F^{\omega i+2j}(n) \in C$ for $i, j \in \omega$.

We now seek to establish the independence of moves from choice of indiscernibles. Suppose $\langle \eta_0, \ldots, \eta_{2n}\rangle$ and $\langle \eta'_0, \ldots, \eta'_{2n}\rangle$ are two sequences of indiscernibles and

$$p = \langle\langle a_0, \eta_0\rangle, \ldots, \langle a_{2n}, \eta_{2n}\rangle\rangle, \qquad p' = \langle\langle a_0, \eta'_0\rangle, \ldots, \langle a_{2n}, \eta'_{2n}\rangle$$

are two positions consistent with σ^* in which I has played well. We want that the number components of $\sigma^*(p)$ and $\sigma^*(p')$ agree, and further that the ordinal components match, subject to certain conditions. To be precise:

Lemma 4.3. *With p, p' as above, if $\sigma^*(p) = \langle a_{2n+1}, \eta_{2n+1}\rangle$ and $\sigma^*(p') = \langle a'_{2n+1}, \eta'_{2n+1}\rangle$ then:*

1. $a_{2n+1} = a'_{2n+1}$
2. *If $2n+1 = \xi^\gamma_r$ and whenever $i \le n$ such that $2i = \xi^\beta_l$, (for $l, r \in \omega$ and $\beta < \gamma$) we have that $\eta_{2i} = \eta'_{2i}$, then $\eta_{2n+1} = \eta'_{2n+1}$. That is, the ordinal part of the move σ^* outputs is not dependent on I's ordinal moves $\eta_{\xi^\beta_r}$ for $\beta \ge \gamma$.*

Proof. This is directly analogous to [4], Lemmas 0.8.1, 0.8.2. Let $\vec{\kappa} = \langle \kappa_m \mid m < \omega \rangle$ be the increasing enumeration of $\{\eta_{2i} \mid i \leq n\} \cup \{\aleph_i \mid i < \omega\}$, with $\vec{\kappa}'$ defined analogously but with the η'_{2i}s. Also let J be the set $\{j < \omega \mid \exists i(\kappa_j = \eta_{2i})\}$, the indices in the enumeration which are ordinals played by I. Since there have only been finitely many moves so far, this set is finite. Also, $J = \{j < \omega \mid \exists i(\kappa'_j = \eta'_{2i})\}$ since I plays well, so all his ordinals must be played into the appropriate block at the appropriate turn.

Since the ordinal sequences differ only finitely, $\mathcal{A}_T[\vec{\kappa}]$ and $\mathcal{A}_T[\vec{\kappa}']$ have the same domains as $\mathcal{A}$. In all of these structures, the game G^* is definable from the $\aleph_i$s, and the sequence $\langle \aleph_i \mid i < \omega \rangle$ is definable (uniformly in each structure) from J, which is finite. Likewise, the set of non-losing positions for II is Π_1 over all these structures (we already showed it is Π_1 over $\mathcal{A}$), and is the same set in each.

II's integer move a_{2n+1} is just defined by the Σ_n formula $a_{2n+1} = (\sigma^*(p))_0$ (that is, the first component of the output of σ^*). Further, a'_{2n+1} is an integer with the same Σ_n-definition in $\mathcal{A}_T[\vec{\kappa}']$, which is Σ_n-elementarily equivalent to $\mathcal{A}_T[\vec{\kappa}]$, so $a_{2n+1} = a'_{2n+1}$.

For the second part we are dealing with ordinals, not integers, and therefore use the remarkability established in Lemma 4.1. Fix m to be the least number such that there exists $i \leq n$ and $\beta > \gamma$ with $\kappa_m = \eta_{2i} = \eta_{\xi^\beta_l}$. Note that if such an η_{2i} does not exist then the result is trivial since all I's ordinal moves were the same between p and p'. Now, if $j < m$ then $\kappa_j = \kappa'_j$ (by hypothesis) and:

$$\kappa_{m-1} \leq \eta_{2n+1} < \kappa_m; \qquad \kappa'_{m-1} \leq \eta'_{2n+1} < \kappa'_m \tag{1}$$

recalling that $2n+1 = \xi^\gamma_r$. But, together with the Σ_n-definition of η_{2n+1} in $\mathcal{A}_T[\vec{\kappa}]$ from I's moves, which are in turn definable from $\{\kappa_j \mid j \in J\}$. η'_{2n+1} has the same Σ_n-definition in $\mathcal{A}_T[\vec{\kappa}']$ from $\{\kappa'_j \mid j \in J\}$ and so we can apply Lemma 4.1 and $\eta_{2n+1} = \eta'_{2n+1}$. □

So we define II's strategy in G as usual: Suppose we have defined $\sigma(p)$ for p of length at most $2n$. Then let $p = \langle a_0, \ldots, a_{2n} \rangle$ be a position in G compatible with σ as defined so far, and suppose we can find $p^* = \langle \langle a_0, \eta_0 \rangle, \ldots, \langle a_{2n}, \eta_{2n} \rangle \rangle$, a position in G^* consistent with σ^* where I has played well. Then, set $\sigma(p) = (\sigma^*(p))_0$. That this is well-defined will be shown in the following lemma:

Lemma 4.4. *If σ^* is a winning strategy for II in G^* from the perspective of $\mathcal{A}$, then σ is, in V, winning for II in G.*

Proof. Let $x = \langle a_0, a_1, \ldots \rangle$ be a play in G consistent with σ. If we can show that for any even $\gamma < \omega^2$, $x \in \bigcap_{\beta \leq \gamma} B_\beta \to x \in B_{\gamma+1}$, and that if $x \in \bigcap_{\beta < \omega^2} B_\beta$ then $x \in B_{\omega^2}$, then we will have shown that $x \notin A$, which is what we want.

We do this by constructing the ordinals $F^\beta(i)$ in such a way that each F^β is order-preserving. Inductively assume we have defined $F^{\beta'}(i)$ for all i and all $\beta' < \beta$, for some $\beta \leq \gamma + 1$, and we will define the ordinals $F^\beta(i)$.

Start by picking I's ordinals, i.e. when β is even: For each $i < \omega$ and $\xi = \sup_{\beta' < \beta} F^{\beta'}(0)$, let $F^\beta : \omega \to \aleph_{k+1}$ (where $\omega \cdot k \leq \beta < \omega \cdot k + 1$) embed

$\prec_x^\beta$ order-preservingly into $C \cap \aleph_{k+1} \setminus \xi$, i.e. let I play well. We know this is possible because $x \in B_\beta$ and C is closed and unbounded below $\aleph_{k+1}$.

To pick II's ordinals, i.e. when β is odd, we use σ^* inductively. Let p^* be the position of length ξ_m^β whose number components come from x and whose ordinal components are picked as follows: η_i is already defined by induction for $i = \xi_{m'}^{\beta'}$ if $\beta' < \beta$. Otherwise if i is odd, σ^* specifies η_i, whilst if it is even we let η_i be an arbitrary element of C, maintaining that I plays well. This latter condition can always be met since C is closed and unbounded below each uncountable cardinal. Now, by Lemma 4.3, $\sigma^*(p^*)$ is the same regardless of the arbitrary indiscernibles we choose for I, so is well-defined, and we define $\eta_{\xi_m^\beta}$ as the ordinal component of $\sigma^*(p^*)$.

We have now defined all ordinals η_i^β for $\beta \leq \gamma + 1$. For $\beta = \gamma + 1$, since σ^* is winning in $\mathcal{A}$, this tells us that the function $F^{\gamma+1} : \prec_{x \restriction k}^{\gamma+1} \to \aleph_\omega$ given by $i \mapsto \eta_i^{\gamma+1}$ is order-preserving for each k; otherwise the shortest $x \restriction k$ where this was not possible would witness that II lost badly while following σ^*. Since σ^* is winning, this cannot happen. Hence $\prec_x^{\gamma+1} = \bigcup_{i\in\omega} \prec_{x\restriction i}^{\gamma+1}$ can be mapped order-preservingly into the ordinals, and $x \in B_{\gamma+1}$ as desired.

All that is left is then to show that if x is in all the B_βs up to ω^2, then $x \in B_{\omega^2}$. This is true in $\mathcal{A}$ since σ is defined from II's winning strategy σ^*, so we just need to know that it is true in V. But the sentence "there exists a real x consistent with σ in all B_βs except B_{ω^2}" is $\Sigma_2^1(\sigma)$, and thus by 2.15 is absolute for $\mathcal{A}$. Hence if σ were not winning in V, it would not be winning in $\mathcal{A}$, which is a contradiction.

Hence $x \notin A$, and σ is winning. $\square$

The corresponding proof for I is all but identical, and thus we have proved Theorem 1.2.

5. Individual Determinacy Proofs

To complete the proof of the main theorem, we need to show that determinacy of the class $\widetilde{\Gamma}$ holds in the models $\mathcal{A}_T[\vec{c}]$, for the relevant theories T. The class $\widetilde{\Gamma}$ is precisely the class given by Lemma 4.2, and we now prove the determinacy of each of these classes in the corresponding model, which are the forcing extensions found in Section 3.

5.1. Det($\widetilde{\Sigma}_1^0 \wedge \widetilde{\Pi}_1^0$)

Lemma 5.1. *Let N be admissible with $T^* \in N$, and τ be the canonical winning strategy for II in an open game $G(A; T^*)$. Then*

1. *τ is $\Sigma_1 \wedge \Pi_1$-definable over N; and*
2. *is still winning in any admissible $M \ni N$.*

Proof. Recall that, if II has a winning strategy in an open game, the canonical one is given by never playing so as to end up in a *ranked* position, where the rank of a position p is 0 if it is already lost for II (i.e. already in one of the basic open sets making up the winning set) and otherwise:

$$\mathrm{rk}(p) = \mu.\xi(\exists a \forall b(p^\frown\langle a,b\rangle \in T^* \wedge \mathrm{rk}(p^\frown\langle a,b\rangle) < \xi)).$$

By the recursion theorem, rk is a Σ_1^{KP} function. τ is then defined by

$$\tau(p) = a \leftrightarrow a \text{ is not ranked} \wedge \forall a' <_N a(a \text{ is ranked}).$$

"a is ranked" is Σ_1 (it is equivalent to $\exists\xi\, \mathrm{rk}(a) = \xi$) so this is $\Pi_1 \wedge \Sigma_1$, as desired for the first claim.

For the second claim, let $N \in M$ with the latter admissible. If τ is not winning in M then some position p consistent with τ must have a game rank in M but not in N, so suppose p is such a position, with rank least. Then by definition,

$$M \vDash \exists a \forall b(\mathrm{rk}(p^\frown\langle a,b\rangle) < \mathrm{rk}(p)).$$

But, fixing a, any such $p^\frown\langle a,b\rangle$ must thus also have a rank in N, since p is the rank-minimal position such that p does not have a rank in N. Hence by admissibility p can be ranked in N as $\sup_b \mathrm{rk}(p^\frown\langle a,b\rangle)$, which is a contradiction. □

The following is essentially from [14] but translated into a set-theoretical form. The original proof uses Π_1^1 comprehension in the Z_2 context, and our hypothesis essentially gives us $\widetilde{\Pi}_1^1$ comprehension for the second admissible above the game-tree.

Theorem 5.2. *Let M be an admissible set with $T^* \in M$ such that*

$$M \vDash \exists M_0(\mathrm{Trans}(M_0) \wedge T^* \in M_0 \wedge (\mathsf{KP})^{M_0}).$$

Then any $\widetilde{\Sigma}_1^0 \wedge \widetilde{\Pi}_1^0$ game has a winning strategy definable over M by a boolean combination of Σ_1 formulæ. Hence a further admissible set containing M will contain the winning strategy as an element.

Proof. Let A be $\widetilde{\Sigma}_1^0$ and B be $\widetilde{\Pi}_1^0$, and fix the game G where player I is trying to get into $A \cap B$. Now a winning strategy for the game with winning set A (and likewise for B) is definable over the admissible M_0 via a game-rank argument, but depending on who wins, is not necessarily a member.

For $p \in T^*$, $\lceil T_p^* \rceil$ is the open neighbourhood of p. Suppose player I has no winning strategy to play into $A \cap B$, and define the set $z \subseteq T^*$:

$$z = \{p \in T^* \mid |p| \text{ is even} \wedge \lceil T_p^* \rceil \subseteq A \wedge \exists\sigma(\sigma \text{ is a w-s for } I \text{ into } B \cap \lceil T_p^* \rceil)\}.$$

z is thus a $\widetilde{\Sigma}_1^1$ generalised real and an element of M by the generalised Spector-Gandy theorem. The class of extensions of elements of z is $\widetilde{\Sigma}_1^0(z)$ and thus has a winning strategy $\Pi_1 \wedge \Sigma_1$-definable over M since $z \in M$. In fact the strategy must be for II, since a winning play for I would also be in $A \cap B$, and I has no winning strategy there, by assumption. Call II's strategy τ_0.

Define

$$y = \{p \in T^* \mid |p| \text{ is even} \wedge \lceil T^*_p \rceil \subseteq A \wedge p \notin z\}.$$

So if $p \in y$ then by open determinacy, II has a winning strategy to play out of $B \cap \lceil T^*_p \rceil$. Let τ_p be this strategy for each such p. Such strategies are definable over M_0, thus Δ_1-definable elements of M and so $p \mapsto \tau_p$ is a Σ_1^M function by admissibility.

Then we define a winning strategy τ^* for II by setting $\tau^*(p) = \tau_0(p)$ whenever $\lceil T^*_p \rceil \not\subseteq A$, and $\tau^*(p^\frown r) = \tau_p(r)$ when p is the shortest position such that $\lceil T^*_p \rceil \subseteq A$. Thus if τ_0 leads II to a position p which will end up in A, she switches to τ_p which will keep her out of B forever, thus winning. Since τ_0 and τ_p are definable over M, so is τ.

$$\tau(p) = \begin{cases} \tau_0(p) & \text{if } \lceil T^*_p \rceil \not\subseteq A; \\ \tau_q(p) & \text{otherwise, where } q \subseteq p \text{ is shortest such that } \lceil T^*_q \rceil \subseteq A. \end{cases}$$

This is definable over M by a boolean combination of Σ_1 formulæ. If N is an admissible set with $M \in N$ then τ_0, τ_p will be in N, and still be winning for their respective subgames. Hence $\tau \in N$ and is also winning. ☐

Thus, since starting with the second-least clever mouse yields generating indiscernibles for such structures M, Theorem 1.2 applies, and we have proved Theorem 1.1 part 1.

5.2. Det($\widetilde{\Sigma}^0_2$)

We show that if A is $\widetilde{\Sigma}^0_2$ then $G(A; T^*)$ is determined in transitive models of KP_1. Let T be KP_1, then let $\mathcal{A}_T[\vec{c}]$ be the least transitive model of T containing $\vec{c}$ as an element, where we assume $\{\aleph_i \mid i \in \omega\} \subseteq \vec{c}$. Being a model of KP_1 implies that there are admissibles N_i such that:

$$N_1 \prec_{\Sigma_1} N_2 \prec_{\Sigma_1} \ldots \prec_{\Sigma_1} \mathcal{A}_T[\vec{c}].$$

Being a minimal transitive model implies that $\mathcal{A}_T[\vec{c}]$ has a parameterless Σ_1 Skolem function.

The determinacy proof is just the same as usual, first proved by Wolfe in [17], and also found in [9] as Theorem 1.3.3., but we have to pay attention to how much separation and replacement we're using. For convenience, fix $\mathcal{A}$ to be $\mathcal{A}_T[\vec{c}]$ for some arbitrary $\vec{c}$ (for example, the sequence of $\aleph_i$s).

We first need to know the complexities of some standard concepts. Most importantly, "being a winning strategy" is Δ_1 in a limit of admissibles:

Lemma 5.3. *The sentence "σ is a winning strategy for $G(A; T^*)$" (for I, II or either player) is Δ_1 over any admissible M satisfying:*

1. $T^* \in M$;
2. $\exists N \in M(N$ *is admissible and* $\sigma \in N)$

Hence the sentence "there exists a winning strategy for $G(A; T^*)$" (for I, II or either player) is Σ_1 over $\mathcal{A}$.

Proof. Let $\sigma \in N \in M$, with N, M admissible. Then certainly being winning is expressible by the Π_1 formula:

$$\forall x (x \in \lceil T^* \rceil \to \sigma * x \in A).$$

Note that checking $x \in A$ for a generalised-arithmetic A is Δ_1 over M since it is definable over $\mathcal{H}$.

Now we just need a Σ_1 formula. But note that the set $\{x \mid x * \sigma \in A\}$ is $\widetilde{\Sigma}^1_1$ and so by Theorem 2.14, if non-empty, has an element definable over N, and hence in M, witnessing that σ is not winning. Thus if M is as required, the following sentence will do:

$$\exists N (\mathrm{Trans}(N) \wedge N \vDash \mathsf{KP} \wedge T^* \in N \wedge \forall x \in \lceil T^* \rceil \cap \mathrm{Def}(N)(\sigma * x \in A)).$$

Again, everything inside the scope of the quantifier is Δ_1, so we are done with the first assertion of the lemma. The second assertion then follows trivially. □

Now let's examine quasi-strategies:

Lemma 5.4. *If $G(A; T^*)$ is not a win for I, then II's non-losing quasi-strategy is a Π_1 definable element of $\mathcal{A}$.*

Proof. "not being a win for I" is a $\Pi_1^{\mathcal{A}}$ property by the previous lemma, and in this case, II's non-losing quasi-strategy is $\Pi_1^{\mathcal{A}}$:

$$T' := \{p \in T^* \mid \forall n \leq |p| (G(A; T^*_{p \restriction n}) \text{ is not a win for } I)\}.$$

Being Π_1, T' is Δ_0 definable from its Σ_1 complement in T^*. Thus by Σ_1-Separation, T' is an element of $\mathcal{A}$. □

We now proceed with Wolfe's proof, starting with the following Lemma:

Lemma 5.5. *Let $B \subseteq A$ with $B \in \widetilde{\Pi}^0_1$ and A a generalised-arithmetic class of $\mathcal{A}$. If I has no winning strategy for $G(A; T^*)$, then in $\mathcal{A}$ there is a strategy τ for II such that for any play $x \in \lceil \tau \rceil$, x extends a position p such that:*

1. *$\lceil T^*_p \rceil \cap B = \varnothing$;*
2. *$G(A; T^*_p)$ has no winning strategy for I.*

Proof. Define C to be the set of plays such that no initial move satisfies both the assertions of the lemma, that is:

$$C := \{x \in \lceil T \rceil^* \mid \forall p \subseteq x (\lceil T^*_p \rceil \cap B \neq \varnothing \vee G(A; T^*_p) \text{ is a win for } I)\}.$$

Being a win for I is Σ_1 and the rest of the definition here is Δ_0, so C is Σ_1-definable over $\mathcal{A}$. To prove the lemma we just need a winning strategy for II in the game $G(C; T^*)$, so suppose there isn't one. But C is closed; it is the complement of the union of basic open sets and so I must have a winning strategy, σ. Now, since $G(A;T^*)$ is not a win for I, by the previous lemma II's non-losing quasi-strategy T' is a $\Pi_1^{\mathcal{A}}$ element of $\mathcal{A}$. Hence I has no winning strategy in $G(A;T')$. But since T' doesn't restrict I's moves, σ restricted to T' is still a winning strategy for $G(C;T')$.

Denote σ's restriction to T' by σ', and let x be any play consistent with σ'. Then for every $p \in T'$, and hence for any $p \subseteq x$, the game $G(A;T^*_p)$ is not a win for I since T' is II's non-losing quasi-strategy. Hence we have shown that the second condition above holds for every $p \subseteq x$. Since we assumed that not both are true, then, every $p \subseteq x$ satisfies $\lceil T^*_p \rceil \cap B \neq \varnothing$. But B is closed so this means it contains x. Hence $x \in B \subseteq A$, but x was an arbitrary play consistent with σ', so σ' is winning for $G(A;T')$, which is a contradiction. $\square$

The proof of the theorem now follows quite simply.

Theorem 5.6. $\mathcal{A} \vDash \mathrm{Det}(\widetilde{\Sigma}^0_2)$.

Proof. Let $A \in \widetilde{\Sigma}^0_2$, so that $A = \bigcup_{i\in\omega} A_i$ for closed A_i. Suppose $G(A;T^*)$ is not a win for I, and we will construct a winning strategy τ for II. The familiar idea is to build τ from countably many τ_i, each given by applying the lemma with B set to A_i.

Let τ_0 be the strategy given by the previous lemma for $B = A_0$, and let p_0 be the shortest position consistent with τ_0 and q^I satisfying both conditions stated in the lemma. We then simultaneously define p_i and τ_i by induction, for as long as p_i is shorter than $2n$.

Firstly:

$$S(T^*_p) = \begin{array}{l}\text{the least strategy } \tau \text{ that wins } G(C;T^*_p)\text{, with}\\ C \text{ as defined in the previous lemma.}\end{array}$$

$$P(\tau_i, q^I) = \begin{array}{l}\text{the shortest position } p \subseteq \tau_i * q^I \text{ satisfying both}\\ \text{conditions of the previous lemma.}\end{array}$$

(Here we interpret $\tau_i * q^I$ for the finite sequence q^I as being all plays compatible with τ_i where I played from q^I his first n moves.) Then, if we have defined all strategies and positions up to τ_i, p_i:

$$\tau_{i+1}(p) = \begin{cases} \tau_i(p) & \text{if } p \subseteq p_i \\ S(T^*_{p_i}) & \text{otherwise} \end{cases}$$
$$p_{i+1} = P(\tau_{i+1}, q^I).$$

Note that p_{i+1} (and hence τ_{i+2}) will be undefined if the lemma would produce a position longer than $2n$.

Let $\tau(\varnothing) = \tau_0(\varnothing)$. We define a winning strategy for II by recursion on length of position.

Suppose we have defined τ on all positions of length up to $2n$, and let q be a position of length $2n+1$ compatible with τ. Let $q^I = \langle q_0, q_2, \ldots, q_{2n}\rangle$, and use it to define τ_i and p_i as above, with k least such that p_k is undefined. Then we define $\tau(q) = \tau_k(q)$. Note that S and P are Σ_1 and Δ_0 over $\mathcal{A}$, respectively, so this amounts to a Σ_1 recursion, giving τ a Σ_1-definable element of $\mathcal{A}$, so it just remains to show that τ is winning.

So let x be consistent with τ, so that we have positions $p_i \subseteq x$ for all $i \in \omega$, with $\lceil T^*_{p_i} \rceil \cap A_i = \varnothing$. Hence $x \in \lceil T^*_{p_i} \rceil$ for each i, and so $x \notin \bigcup_{i \in \omega} A_i = A$, so II wins, and τ is winning. $\square$

This gives us that, if I does not have a winning strategy in $\mathcal{A}$, II does, and in fact it is Σ_1 definable. If I does have the winning strategy, we can in fact find a Σ_1 definable one because, by Lemma 5.3, "being a winning strategy" is Δ_1 definable over $\mathcal{A}$ and thus the $\mathcal{A}$-least such strategy is $\Sigma_1^{\mathcal{A}}$.

Thus we have proved Theorem 1.1 part 2.

5.3. Det($\widetilde{\Sigma}^0_3$)

Let $\mathcal{T}$ be the theory $\mathsf{KP} + \Sigma_2\text{-}\mathsf{Sep}$, and $\mathcal{A}_{\mathcal{T}}[\vec{c}]$ the least transitive model of $\mathcal{T}$ containing $\vec{c}$ as an element. For simplicity of notation let $\mathcal{A}$ be $\mathcal{A}_{\mathcal{T}}[\vec{c}]$ for some $\vec{c}$ including $\{\aleph_i \mid i \in \omega\}$. The game tree T^* is a Δ_0 element of $\mathcal{A}$. It is a standard fact that models of $\mathsf{KP} + \Sigma_2\text{-}\mathsf{Sep}$ are limits of admissibles and more, so being a winning strategy is a $\Delta_1^{\mathcal{A}}$ predicate, and the non-losing quasi-strategy is $\Pi_1^{\mathcal{A}}$.

Let A be a $\widetilde{\Sigma}^0_3$ subset of $\lceil T^* \rceil$, and we will want to show that $\mathcal{A} \vDash$ "$G(A; T^*)$ is determined." We follow the specialised version of Davis' original proof as laid out in [16]. This in turn uses Martin's version, as in [9], of the proof closely, but explicitly minimising the required strength. We note that $\Sigma_2\text{-}\mathsf{Sep}$ is more than is necessary to effect this proof, so our proof is simpler than that in [16].

As with Martin's proof, we start with a lemma that is applied repeatedly to build up a strategy to prove the theorem. We will have to ensure that the objects defined are elements of the structure $\mathcal{A}$, and must allow for the added complexity coming from the use of the generalised pointclass $\widetilde{\Sigma}^0_3$.

Lemma 5.7. *Let $B \subseteq A$ be $\widetilde{\Pi}^0_2$ and T a game subtree of T^*. Suppose I has no winning strategy in $G(A; T)$, then II has a quasi-strategy $\bar{T}$ such that:*

1. $\lceil \bar{T} \rceil \cap B = \varnothing$*;*
2. $G(A; \bar{T})$ *is not a win for* I*.*

Proof. Let T' be II's non-losing quasi-strategy, which is a Π_1-definable set in $\mathcal{A}$. Define a position p to be *good* if there is a quasi-strategy $\bar{T} \subseteq T'_p$ satisfying the above two properties. Goodness is a Σ_2 property, asserting the existence of a tree with properties 1. and 2., with 1. being Δ_1 and 2. being Π_1. The set of good positions H is therefore an element of $\mathcal{A}$.

Denote by $\hat{T}$ the function with domain H, defined over $\mathcal{A}$ as: $\hat{T}(p)$ is the least quasi-strategy witnessing that p is good. This is Σ_2 definable by the existence of Σ_2 Skolem functions.

Let $B = \bigcap_n D_n$, with each $D_n \in \widetilde{\Sigma}^0_1$, and define the sets:

$$E_n = A \cup \{x \in \lceil T' \rceil \mid \exists p \subseteq x (\lceil T'_p \rceil \subseteq D_n \wedge p \text{ is not good})\}.$$

Thus each E_n is a Π_2 set in $\mathcal{A}$ once we make the $\exists p$ bounded in the usual way.

The goal, then, is to prove that the initial position $\varnothing$ is good. This will be accomplished by showing that there is at least one E_n for which I has no winning strategy in T'. Following Martin, we first show that this does what we want, so assume that there is no winning strategy for I in the game $G(E_n; T')$ in $\mathcal{A}$. Let T'' be II's non-losing quasi-strategy in this game.

We now define a quasi-strategy $\bar{T}$. $\bar{T}$ firstly contains all $p \in T''$ until $\lceil T'_p \rceil \not\subseteq D_n$. Then letting p be a minimal position such that $\lceil T'_p \rceil \subseteq D_n$, note that p must be good. This is because, if p were bad the definition of E_n implies I has a trivial winning strategy in $G(E_n; T'_p)$. But p is supposed to be in II's non-losing quasi-strategy, and hence upon reaching such a p we can include $\hat{T}(p)$ into $\bar{T}$.

Now we wish to show that $\bar{T}$ is a witness in $\mathcal{A}$ to the initial position being good. Note firstly that $\lceil \bar{T} \rceil \cap B = \varnothing$, because either a play through $\bar{T}$ remains in T'' (and so it is not in D_n, hence not in B) or it goes through $\hat{T}(p)$ which witnesses p's goodness, hence avoiding B.

We now need to show that $G(A; \bar{T})$ is not a win for I, so suppose otherwise and let $\sigma \in \mathcal{A}$ witness that. Every position p consistent with σ satisfies $\lceil T'_p \rceil \not\subseteq D_n$: if not, then there is such a p with $\bar{T}_p = \hat{T}(p)$. But $\hat{T}(p)$ is a witness to p's goodness in $\mathcal{A}$, so $G(A; \bar{T}_p)$ is not a win for I in $\mathcal{A}$.

By the definition of $\bar{T}$, then, all plays consistent with σ lie in $\lceil T'' \rceil$. Hence σ is also a winning strategy for $G(A; T'')$ and, since $A \subseteq E_n$, for $G(E_n; T'')$. But this allows us the contradiction we want: Define a strategy τ for I in $G(E_n; T')$ by following σ until possibly reaching a point $p \notin T''$. Then there is a strategy σ_p winning for I in $\mathcal{A}$ (since T' is non-losing in $\mathcal{A}$), so let τ follow this strategy. This procedure is definable from σ and the strategies σ_p, all of which are elements of $\mathcal{A}$, and thus the strategy is in $\mathcal{A}$ for the desired contradiction.

We have now shown that, under the assumption that at least one game $G(E_n; T')$ is not a win for I, we have that $\varnothing$ is good as desired, and we need to show that this assumption is valid. Note that the above argument can be adapted to show that, if $G(E_n^p; T'_p)$ is not a win for I, where:

$$E_n^p := A \cup \{x \in \lceil T'_p \rceil \mid \exists q \subseteq x (p \subseteq q \wedge \lceil T'_q \rceil \subseteq D_n \wedge q \text{ is not good})\}$$

then p is good. Hence suppose the lemma is false. We can thus assume that each $G(E_n^p; T'_p)$ is a win for I (in $\mathcal{A}$) since this is the only case in which we haven't proved the lemma. We then show that I has a winning strategy in $G(A; T')$ in $\mathcal{A}$, contradicting the hypothesis of the lemma.

I starts by playing with the strategy from the previous lemma for $G(E_0; T')$. If we never reach a point p_1 with $\lceil T'_{p_1} \rceil \subseteq D_0$ then, if the play ends up in E_0 it must be in A. Otherwise all plays above p_1 are in D_0, and I now must start playing according to the strategy for $G(E_1^{p_1}; T'_{p_1})$. We now either end up in A or use the strategy for $G(E_2^{p_2}; T'_{p_2})$. If this process terminates, then I has a winning strategy for some $G(E_i^{p_i}; T'_{p_i})$ and the only way for I to win in the game would be to get into A, giving a winning strategy in $G(A; T')$. If the process continues indefinitely, then I has landed in each D_n and hence is in $B \subseteq A$.

It remains to show that the strategy thus defined is an element of $\mathcal{A}$, so let's make our definitions above precise. Suppose q is an even-length position with

$q \supseteq p_i$, and that we have defined p_i and σ_i, but not p_{i+1} and σ_{i+1}.

$$\sigma(q) = \begin{cases} \sigma_i(q) & \text{if } \lceil T'_q \rceil \nsubseteq D_i \\ \sigma_{i+1}(q) & \text{otherwise} \end{cases}$$

where, if "otherwise," we set:

$$\sigma_{i+1} = I\text{'s least winning strategy in } G(E_i^q; T'_q).$$

Thus σ is defined by a recursion on ω and is an element of $\mathcal{A}$. But T' is I's non-losing quasi-strategy in $\mathcal{A}$ and so this is a contradiction.

□

Theorem 5.8. *For any $\widetilde{\Sigma}^0_3$ subset A of $\lceil T^* \rceil$, $\mathcal{A} \vDash$ "$G(A;T^*)$ is determined".*

Proof. Now we are ready to prove the theorem. Let $A = \bigcup_{n\in\omega} A_n$, with each A_n $\widetilde{\Pi}^0_2$, and suppose $G(A;T^*)$ is not a win for I, and we show that II has a winning strategy definable by a Σ_2-recursion over $\mathcal{A}$ which, by Σ_2-admissibility is therefore an element of $\mathcal{A}$. We apply the previous lemma repeatedly, substituting each A_n as an instance of B, and the resulting quasi-strategies as instances of T.

First apply the Lemma with $B = A_0$ and T^*, yielding a quasi-strategy $\bar{T}^* \in \mathcal{A}$, which we shall now call $T^{\varnothing}$. Taking $T^{\varnothing}$ to be the least such gives $T^{\varnothing}$ a Σ_2 element of $\mathcal{A}$.

The lemma tells us that $G(A;T^{\varnothing})$ is not a win for I, so for any length-1 position $p_1 \in T^*$, let $\tau(p_1)$ be some arbitrary, fixed move in II's non-losing quasi-strategy. Now let p_2 be a length-2 move consistent with τ as defined so far and apply the Lemma with $B = A_1$ and $T = (T^{\varnothing})_{p_2}$, yielding a quasi-strategy which we call T^{p_2}, Σ_2-definable from $(T^{\varnothing})_{p_2}$ and hence an element of $\mathcal{A}$. Since the Lemma guarantees I still has no winning strategy in $G(A;T^*_{p_2})$, for an arbitrary length-3 position p_3, let $\tau(p_3)$ be an arbitrary move consistent with II's non-losing quasi-strategy there.

Continuing in this way defines a strategy τ for II by Σ_2-recursion, and as noted above, this gives $\tau \in \mathcal{A}$. All that remains is to show that τ is winning. If x is consistent with τ then by construction it is in each $\lceil T^{p_{2n}} \rceil$, and $\lceil T^{p_{2n}} \rceil \cap B_n = \varnothing$ by the lemma, so $\forall n \in \omega (x \notin B_n)$, so $x \notin A$ and is a win for II. □

Thus we have proved Theorem 1.1 part 3.

5.4. Det(n-$\widetilde{\Pi}^0_3$)

To prove this, we refer to the work of Montalbán and Shore, [12], Section 4, which in turn is a version of Martin's proof in Section 1.4 of [9]. Since that proof is long, needs very little modification and we have already seen how the arguments transfer from the ordinary case to determinacy of these auxiliary games, we will be brief.

Working in a minimal model $\mathcal{A}$ with $T^* \in \mathcal{A}$ and satisfying KP_{n+1}, we have to ensure that all the objects defined in the proof of [12] exist in $\mathcal{A}$.

First, let A be as follows:

$$\begin{aligned} A \quad & \text{is } m\text{-}\widetilde{\Pi}^0_3 \text{ as witnessed by } \langle A_i \mid i \leq m\rangle, \text{ a descending sequence of } \widetilde{\Pi}^0_3 \text{ sets;} \\ A_i = & \bigcap_{j\in\omega} A_{i,j} \text{ for } \widetilde{\Sigma}^0_2 \text{ sets } A_{i,j}; \\ A_{i,j} = & \bigcup_{k\in\omega} A_{i,j,k} \text{ for } \widetilde{\Pi}^0_1 \text{ sets } A_{i,j,k}. \end{aligned}$$

Let s be a position in T^* of length at most m, and S a subtree of T^*. Let $l = m - |s|$ and fix player x to be I if l is even and II otherwise, with $\bar{x}$ the opposite player. Then let B^x be B if $x = I$ and B^c otherwise, for any set B (taking complementation inside the ambient space, which is $\lceil T^* \rceil$ in the case of A^x, but when considering the winning set of a game in the tree S, it is in $\lceil S \rceil$). We define the predicates $P^s(S)$ by recursion up to m, exactly as in [12]:

Definition 5.9.

$P^{\varnothing}(S)$ holds iff there is a winning strategy for player x in $G(A; S)$.

$P^s(S)$ holds for $|s| = n + 1$ iff there is a quasistrategy $U \subseteq S$ for player x such that

1. $\lceil U \rceil \subseteq A^x \cup A_{(m-n-1),s(n)}$; and
2. $P^{s\restriction n}(U)$ fails.

Definition 5.10.

A quasistrategy U *witnesses* that $P^s(S)$ if it is as required in Definition 5.9.

A quasistrategy U *locally witnesses* that $P^s(S)$ if either $s = \varnothing$ and U is a witness to $P^{\varnothing}(S)$, or $|s| > 0$, U is a quasistrategy for player x and there is a $D \subseteq S$ such that for every position $d \in D$, there is a quasistrategy $R^d \subseteq S_d$ for $\bar{x}$ such that:

1. $\forall d \in D \cap U (U_d \cap R^d$ witnesses that $P^s(R^d))$
2. $\lceil U \rceil \setminus \bigcup_{d\in D} \lceil R^d \rceil \subseteq A^x$
3. For a position $p \in S$ there is at most one position $d \in D$ such that $d \subseteq p \wedge p \in R^d$.

These are technical conditions that are used in the proof to ultimately ensure determinacy. Many sets are defined using these conditions, so we will need to know the complexity, and thus that the sets are elements of $\mathcal{A}$.

Proposition 5.11.

1. *$P^s(S)$ is* $\Sigma_{|s|+1}$
2. *"U witnesses that $P^s(S)$ holds" is* $\Pi_{|s|}$
3. *"U locally witnesses that $P^s(S)$ holds" is* $\Sigma_{|s|+1}$.

Proof. Since "there is a winning strategy for x in $G(A; S)$" is Σ_1 over $\mathcal{A}$ (recall Lemma 5.3), $P^{\varnothing}(S)$ is Σ_1. Then if $|s| = n + 1$, the definition requires checking that there is a U such that $\neg P^{s\restriction n}(U)$, so inductively the predicate overall is $\Sigma_{|s|+1}$.

The complexities of the other predicates follow straightforwardly as well. $\square$

Contrast this with Remark 4.3 of [12], where the same concepts are noted to be $\Sigma^1_{|s|+2}$, $\Pi^1_{|s|+1}$ and $\Sigma^1_{|s|+2}$, respectively; the Kleene-Basis theorem allows us to eliminate a quantifier in the base case.

With this observation, the remainder of Montalbán and Shore's proof can be followed in $\mathcal{A}$. Note that, whenever we define a set (of plays, or of positions) it is, in their proof, at most Σ^1_{n+2} using their Remark 4.3. Consequently by our Proposition 5.11 the same sets are at most Σ^1_{n+1}, indeed whenever their set is Σ_{m+1}, ours is Σ_m. When it comes to define a function returning such sets, we use the fact that the $\mathcal{A}$-least set satisfying a Σ_m property is again Σ_m.

Following their proof with these minor modifications, then, gives us:

Theorem 5.12. $\mathcal{A} \vDash \mathrm{Det}(n\text{-}\widetilde{\Pi}^0_3)$.

So we have proved Theorem 1.1, part 4.

5.5. Det($\widetilde{\Sigma}^0_\alpha$)

For the final part of Theorem 1.1 we need to make a minor modification to the picture; Section 2.2 and Lemma 4.2 only deal with sets of finite Borel rank.

We will first deal with the simpler case of part 6 of the main theorem, so let $\mathcal{A}$ be some $\mathcal{A}_{\mathsf{ZFC}}[\vec{c}]$, with $\vec{c} \supseteq \{\aleph_i \mid i \in \omega\}$.

Definition 5.13. Fix some $\Delta^{\mathcal{H}}_1$ encoding $p : \aleph_\omega \to T^*$ and let $A \subseteq \lceil T^* \rceil$.

1. A is $\widetilde{\Sigma}^0_1$ iff there is a partial $\Sigma_1(\mathcal{H})$ function $e : \aleph_\omega \to \aleph_\omega$ where $A = \bigcup_\alpha \lceil T^*_{p(e(\alpha))} \rceil$. The code for A is then $\langle 0, e \rangle$, having coded e as an element of $\aleph_\omega$. This is possible because there are only $\aleph_\omega$-many such functions that are $\Sigma_1(\mathcal{H})$ definable with parameters from $\mathcal{H}$.
2. A is $\widetilde{\Sigma}^0_\alpha$ iff there is a partial $\Sigma_1(\mathcal{H})$ function $e : \omega \to \aleph_\omega$ such that $e(a)$ is a code for a $\widetilde{\Pi}^0_\beta$ set for some $\beta < \alpha$ and $A = \bigcup_a e(a)$. The code for A is then $\langle 1, e \rangle$.
3. A is $\widetilde{\Pi}^0_\alpha$ iff there is a $\widetilde{\Sigma}^0_\alpha$ set such that $A = B^c$. If B has code c then the code for A is $\langle 2, c \rangle$.

Theorem 5.14. *For $\alpha < \omega$, this definition of $\widetilde{\Sigma}^0_\alpha$ agrees with the old definition.*

Proof. It is clear that the two definitions of $\widetilde{\Sigma}^0_1$ are equivalent: if we know the Σ_1 set X of the original definition, then setting $e(\alpha) = \alpha \iff p(\alpha) \in X$ and leaving e undefined otherwise, we have that e is $\Sigma_1(\mathcal{H})$ and the sets defined are the same. If we know e, and $e(\alpha) = \beta$, put $p(\beta) \in X$.

Now assume that we have shown the two definitions of $\widetilde{\Sigma}^0_n$ agree. Then if $A = \{x^* \mid \exists a R(x^*, a)\}$ for some $\widetilde{\Pi}^0_n$ predicate R, we take $e(a)$ to be (the code for) $\{x^* \mid R(x^*, a)\}$ and $A = \bigcup_a e(a)$. Noting that obtaining the codes for $R(\cdot, a)$ is $\Delta^{\mathcal{H}}_1$, we have that e is a $\Delta^{\mathcal{H}}_1$ map, satisfying the requirements for the new definition.

On the other hand if $A = \bigcup_a e(a)$ is $\widetilde{\Sigma}^0_{n+1}$ with the above definition we may assume that the relation $R(x^*, a)$ given by $x^* \in e(a)$ is $\widetilde{\Pi}^0_n$, and set $x^* \in A \iff \exists a(R(x^*, a))$. □

Thus if we fix $A^* \subseteq [T^*]$ to be $\widetilde{\Sigma}^0_\alpha$, we have $A^* \in \mathcal{A}$. Furthermore, by the way it is defined, such sets are all within $\mathbf{\Sigma}^0_\alpha$. Hence by the Borel Determinacy Theorem, $\mathcal{A} \vDash \mathrm{Det}(\widetilde{\Sigma}^0_\alpha)$. By slightly altering Lemma 4.2 we can see that, if $A \in \Sigma^0_\alpha$ then the corresponding winning set for the auxiliary game is $\widetilde{\Sigma}^0_\alpha$, and then the rest of the machinery of Theorem 1.2 works as before.

For part 5 of the main theorem we just need to modify things slightly in light of Martin's level-by-level analysis of the determinacy of the Borel sets:

Theorem 5.15 ([9], 2.3.5). *Let M be a model of $\mathsf{ZC}^- + \Sigma_1$ - $\mathsf{Replacement}$, and suppose $\mathcal{P}^\alpha(T) \in M$. Then, if α is finite, $M \vDash$ "all $\mathbf{\Delta}^0_{\alpha+4}$ games in T are determined". If α is an infinite countable ordinal of M then $M \vDash$ "all $\mathbf{\Delta}^0_{\alpha+3}$ games are determined".*

(Note that here we mean $\mathbf{\Delta}^0_\alpha$ in the sense of M, so while this may not be all of the true $\mathbf{\Delta}^0_\alpha$ sets we know that in our situation, with M transitive, it will be all of the $\widetilde{\Delta}^0_\alpha$ sets.)

Thus we will want $\mathcal{A}$ to be a model of $\mathsf{ZC}^- + \Sigma_1$ - $\mathsf{Replacement} + \mathcal{P}^\alpha(T^*)$ exists (where T^* is the auxiliary game tree, $(\omega \times \aleph_\omega)^{<\omega}$).

Now consider the argument again, starting with a model M with measurable cardinal κ, of $\mathsf{ZFC}^- + \mathcal{P}^\alpha(\kappa)$ exists. We iterate this model as in Section 3 to obtain M_λ, which is elementarily equivalent to M, and then force to produce models $\mathcal{A}[\vec{c}]$. By elementarity, $M_\lambda \vDash \mathsf{ZFC}^- + \mathcal{P}^\alpha(\kappa_\lambda)$ exists, and so by the usual argument for showing that the Power Set axiom holds in generic extensions, so does $M_\lambda[\vec{c}]$. Thus all the models $\mathcal{A}[\vec{c}] = M_\lambda[\vec{c}]$ which we use in the arguments of Section 4 see that $\mathcal{P}^\alpha(T^*)$ exists, since in that case $\kappa_\lambda = \aleph_\omega$, hence the hypotheses of Theorem 5.15 are satisfied when it comes to proving determinacy of the auxiliary game.

We have thus completed the proof Theorem 1.1.

6. Conclusions and Open Questions

The main result shows the adaptability of the technique originally invented by Martin for proving determinacy results on α - $\mathbf{\Pi}^1_1$. The basic results of Section 2.1 show that we automatically get the determinacy of ω^2 - $\Pi^1_1 + \Pi^0_2$ from KP_1 (and so on) as we would expect. In this way we have established new upper-bounds on the consistency strength of determinacy of classes strictly between ω^2 - Π^1_1 and (ω^2+1) - Π^1_1.

Open questions are of two kinds; the first is whether the hypotheses used here are minimal and the second is what hypotheses can we find that prove the determinacy of similar pointclasses. To the second question, we have essentially exhausted the possibilities for extending ω^2 - Π^1_1 by one more set, but we could investigate the modifications to the difference hierarchy in [4] to see if more results are provable.

To the first question, we should not expect optimality in parts 1–3 of 1.1, since for those we either know that the determinacy proof holds in models weaker than the forcing extensions we consider, or have no reason to suspect otherwise.

The problem is that we cannot preserve arbitrary theories when iterating and doing the forcing in Section 3, so even where we have optimal determinacy results (for instance, it is known that $\mathrm{Det}(\Sigma^0_2)$ is equivalent to closure under Σ^1_1-monotone inductive definitions) we cannot necessarily transfer them.

In the case of part 4, Montalbán and Shore show in [12] that we cannot even find an exact characterisation of Determinacy down in the Borel hierarchy, so we should not hope to find any in the refined difference hierarchy. However, we may rather hope to prove, as they did, that no exact correspondence can be found.

On the other hand we know that Borel determinacy requires almost the full strength of ZFC, with Martin's analysis showing that we should expect parts 5 and 6 to be nearly optimal.

References

[1] P. J. Cohen. *Set Theory and the Continuum Hypothesis.* W.A. Benjamin, Inc., New York, 1966.

[2] K. Devlin. *Constructibility.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1984.

[3] A. Dodd and R. Jensen. The core model. *Ann. Math. Logic*, 20(1):43–75, 1981.

[4] D. A. DuBose. Determinacy and iterated sharps. *The Journal of Symbolic Logic*, 55(2):502–525, June 1990.

[5] H. M. Friedman. Determinateness in the low projective hierarchy. *Fund. Math.*, 72:79–84, 1971.

[6] F. Hausdorff. *Mengenlehre.* Dover Publications, New York, 3rd edition, 1944.

[7] T. J. Jech. *Set Theory.* Springer-Verlag, 3rd millenium edition, 2002.

[8] A. Kanamori. *The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings.* Springer Monographs in Mathematics. Springer-Verlag, second edition, 2008.

[9] D. A. Martin. Borel and projective games. Book to be published.

[10] D. A. Martin. Measurable cardinals and analytic games. *Fund. Math.*, 66:287–291, 1970.

[11] D. A. Martin. An extension of borel determinacy. *Annals of Pure and Applied Logic*, 49(3):279 – 293, 1990.

[12] A. Montalbán and R. A. Shore. The limits of determinacy in second-order arithmetic. *Proc. Lond. Math. Soc.*, 104(2):223–252, 2012.

[13] Y. N. Moschovakis. *Descriptive set theory*, volume 155 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2009.

[14] K. Tanaka. Weak axioms of determinacy and subsystems of analysis I: Δ^0_2 games. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 36(6):481–491, 1990.

[15] P. D. Welch. Determinacy in the difference hierarchy of co-analytic sets. *Annals of Pure and Applied Logic*, 80(1):69–108, 1996.

[16] P. D. Welch. Weak systems of determinacy and arithmetical quasi-inductive definitions. *J. Symbolic Logic*, 76(2):418–436, 2011.

[17] P. Wolfe. The strict determinateness of certain infinite games. *Pacific J. Math.*, 5:841–847, 1955.

[18] M. Zeman. *Inner models and large cardinals*, volume 5 of *de Gruyter Series in Logic and its Applications*. Walter de Gruyter & Co., Berlin, 2002.