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NORMS OF INNER DERIVATIONS FOR MULTIPLIER ALGEBRAS OF C^*-ALGEBRAS AND GROUP C^*-ALGEBRAS, II.

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Abstract. The derivation constant $K(A) \geq \frac{1}{2}$ has been extensively studied for *unital* non-commutative C^*-algebras. In this paper, we investigate properties of $K(M(A))$ where $M(A)$ is the multiplier algebra of a non-unital C^*-algebra A. A number of general results are obtained which are then applied to the group C^*-algebras $A = C^*(G_N)$ where G_N is the motion group $\mathbb{R}^N \rtimes SO(N)$. Utilising the rich topological structure of the unitary dual $\widehat{G_N}$, it is shown that, for $N \geq 3$,

$$K(M(C^*(G_N))) = \frac{1}{2}\left\lceil \frac{N}{2} \right\rceil .$$

Keywords. C^*-algebra, multiplier algebra, derivation, motion group, unitary dual, graph structure.

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1. Introduction.

For a C*-algebra A, an elementary application of the triangle inequality shows that

$$\|D(a, A)\| \leq 2d(a, Z(A))$$

for all $a \in A$, where $D(a, A)$ is the inner derivation generated by a and $d(a, Z(A))$ is the distance from a to $Z(A)$, the centre of A. This leads naturally to the definition of $K(A)$ as the smallest number in $[0, \infty]$ such that

$$K(A)\|D(a, A)\| \geq d(a, Z(A))$$

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for all $a \in A$ [3, 28]. If the elements a are restricted to be self-adjoint then the corresponding constant is denoted by $K_s(A)$. If $A = B(H)$ (or, more generally, a non-commutative von Neumann algebra on a Hilbert space $H \neq \mathbb{C}$) then $K(A) = \frac{1}{2}$ [37, 38]. For unital non-commutative C^*-algebras, $K_s(A) = \frac{1}{2}\operatorname{Orc}(A)$ [35], where the *connecting order* $\operatorname{Orc}(A) \in \mathbb{N} \cup \{\infty\}$ is determined by a graph structure in the primitive ideal space $\operatorname{Prim}(A)$ (see Section 2), and for the constant $K(A)$ it has been shown that the only possible positive values less than or equal to $\frac{1}{2} + \frac{1}{\sqrt{3}}$ are:

$$\frac{1}{2}, \quad \frac{1}{\sqrt{3}}, \quad 1, \quad \frac{3+8\sqrt{2}}{14}, \quad \frac{4}{\sqrt{15}}, \quad \frac{1}{2} + \frac{1}{\sqrt{3}}$$

[36, 10, 11]. These results use the fine structure of the topology on $\operatorname{Prim}(A)$ together with spectral constructions and the constrained optimization of the bounding radii of planar sets.

If A is a non-unital C^*-algebra then, as discussed in [6], the multiplier algebra $M(A)$ is the natural unitization to consider in the context of inner derivations. For example, it is well-known that if A is a primitive C^*-algebra then so is $M(A)$ (cf. [6, Example 5.5]) and so $K(M(A)) = \frac{1}{2}$ [37, Theorem 5]. In particular, $K(M(A)) = \frac{1}{2}$ for every simple C^*-algebra A.

In general, in order to apply to $M(A)$ the results for unital algebras, there is a prima facie requirement for more detailed information on $\operatorname{Prim}(M(A))$. However, this space is usually much larger and more complicated than the dense open subset $\operatorname{Prim}(A)$. This is illustrated by the complexity of the Stone-Čech compactification $\beta\mathbb{N}$ of the natural numbers $\mathbb{N}$ and also by the results in [13], which apply to the motion group C^*-algebras considered in this paper (see the remarks after Theorem 3.3). However, when A is σ-unital, the normality of the complete regularization of $\operatorname{Prim}(A)$ enables ideal structure in $M(A)$ to be linked to ideal structure in A without having full knowledge of $\operatorname{Prim}(M(A))$ (Proposition 2.1). It follows from this that, in several cases of interest, the value of $K(M(A))$ is determined by the ideal structure in A itself and hence by the topological properties of the T_0-space $\operatorname{Prim}(A)$ [20, 3.1]. This allows the possibility of computing $K(M(A))$ for $A = C^*(G)$ in cases where G is a locally compact group whose unitary dual $\widehat{G}$ is well-understood as a topological space. In [6], we obtained two general C^*-theoretic results for $K(M(A))$ which enabled us to show that

$$K(M(C^*(G))) = K_s(M(C^*(G))) = 1$$

for a number of well-known locally compact groups G, including $SL(2,\mathbb{R})$, $SL(2,\mathbb{C})$ and the classical motion group of the plane $G_2 = \mathbb{R}^2 \rtimes SO(2)$.

In this paper, we focus on $C^*(G_N)$ where G_N is the motion group $\mathbb{R}^N \rtimes SO(N)$ $(N \geq 3)$. Since G_N is a Type I group, $\operatorname{Prim}(C^*(G_N))$ is homeomorphic to the unitary dual $\widehat{G_N}$, which is known to have a rich topological structure [15, 29, 22]. Indeed, the C^*-algebra $C^*(G_N)$ has recently been identified, via the Fourier transform, with an explicit algebra of operator fields over $\widehat{G_N}$ [1]. We use the topological structure of $\widehat{G_N}$ in showing that, for $N \geq 3$,

$$K(M(C^*(G_N))) = K_s(M(C^*(G_N))) = \frac{1}{2}\left\lceil \frac{N}{2} \right\rceil.$$

Somewhat surprisingly, this formula is not valid for the case $N = 2$, despite the fact that $C^*(G_2)$ is the most well-behaved of the motion group C^*-algebras (by [29] it is the only one which is quasi-standard in the sense of [8]). In contrast to the results above, note that it follows from [6, Proposition 2.1] that $K(C^*(G_N)) = K_s(C^*(G_N)) = 1$ for all $N \geq 2$. Thus $K(M(A))$ gives much more information than $K(A)$ for the algebras $A = C^*(G_N)$.

In the course of determining the values for $K(M(C^*(G_N)))$, we obtain in Sections 4, 5 and 7 several new results for general C^*-algebras A, sometimes under the assumption that A is σ-unital and that the complete regularization map on $\mathrm{Prim}(A)$ is closed. For example, in Theorem 4.6 we give sufficient conditions (which are satisfied in the case $A = C^*(G_N)$) for the inequality

$$\mathrm{Orc}(A) \leq \mathrm{Orc}(M(A)).$$

Section 5 gives some general upper bounds for $K(M(A))$, including a new result for unital C^*-algebras (Theorem 5.2). The inequality $\mathrm{Orc}(A) \leq \mathrm{Orc}(M(A))$ from Section 4 then combines with the results from Section 5 to determine $K(M(C^*(G_N)))$ in the case where N is even (Section 6).

When N is odd, it turns out that we need a sharper estimate for $\mathrm{Orc}(M(C^*(G_N)))$. Accordingly, in Section 7, we introduce a new constant $D(A)$ arising from a graph structure on $\mathrm{Sub}(A)$, a subset of the set of primal ideals of a C^*-algebra A. This is closely linked to the way in which $\mathrm{Orc}(A)$ is obtained from the graph structure on $\mathrm{Prim}(A)$. Indeed, either $|\mathrm{Orc}(A) - D(A)| \in \{0, 1\}$ or $\mathrm{Orc}(A) = D(A) = \infty$. In Theorem 7.6, we give sufficient conditions (which are satisfied by $A = C^*(G_N)$) for $N \geq 3$) for the equality

$$\mathrm{Orc}(M(A)) = D(A) + 1.$$

This exemplifies our earlier contention that, when A is σ-unital, ideal structure in $M(A)$ can be usefully related to ideal structure in A. This equality then combines with the results from Section 5 to determine $K(M(C^*(G_N)))$ in the case where N is odd (Section 8).

2. Preliminaries.

We begin by recalling some terminology from [35]. Let X be a topological space. For $x, y \in X$ we write $x \sim y$ if x and y cannot be separated by disjoint open sets. The relation $\sim$ is reflexive and symmetric but it is not always transitive. We will view X as a graph in which two points x and y are adjacent if and only if $x \sim y$. For $x, y \in X$ let $d(x, y)$ denote the distance from x to y in the graph $(X, \sim)$. If there is no walk from x to y we write $d(x, y) = \infty$. We define the *diameter* of a $\sim$-connected component of X to be the supremum of the distances between pairs of points in the component, except that we adopt the non-standard convention that the diameter of a singleton component is 1 (rather than 0). Define $\mathrm{Orc}(X)$, the connecting order of X, to be the supremum of the diameters of $\sim$-connected components of X. By virtue of our non-standard convention, $\mathrm{Orc}(X) = 1$ when X is a Hausdorff space. In the case when X is the primitive ideal space of a C*-algebra A we write $\mathrm{Orc}(A)$ instead of $\mathrm{Orc}(\mathrm{Prim}(A))$; and sometimes we write d_A, in place of d, for the distance function when we need to emphasize the algebra we are working in. If $\sim$ is an open equivalence relation on $\mathrm{Prim}(A)$ (that is, $\sim$ is an equivalence relation and the corresponding quotient map is open) then the C^*-algebra A is said to be *quasi-standard* (see [8], where several equivalent conditions and examples are given). Note that if A is quasi-standard then $\mathrm{Orc}(A) = 1$.

It was shown in [35, Theorem 4.4] that, if A is a unital C^*-algebra, $K_s(A) = \frac{1}{2}\operatorname{Orc}(A)$. It follows that if A is any C^*-algebra then $K_s(M(A)) = \frac{1}{2}\operatorname{Orc}(M(A))$ and so $\frac{1}{2}\operatorname{Orc}(M(A)) \leq K(M(A))$. It turns out that equality holds in the case where $A = C^*(G_N)$ ($N \geq 3$). We shall show this by establishing, for the four cases modulo 4, that

$$K(M(A)) \leq \frac{1}{2}\left\lceil \frac{N}{2} \right\rceil \qquad \text{and also} \qquad \left\lceil \frac{N}{2} \right\rceil \leq \operatorname{Orc}(M(A)).$$

We now recall some properties of the complete regularization of $\operatorname{Prim}(A)$ for a C^*-algebra A (see [16] for further details). For $P, Q \in \operatorname{Prim}(A)$ let $P \approx Q$ if and only if $f(P) = f(Q)$ for all $f \in C^b(\operatorname{Prim}(A))$. Then $\approx$ is an equivalence relation on $\operatorname{Prim}(A)$ and the equivalence classes are closed subsets of $\operatorname{Prim}(A)$. It follows that there is a one-to-one correspondence between $\operatorname{Prim}(A)/\approx$ and a set of closed two-sided ideals of A given by

$$[P] \to \bigcap [P] \qquad (P \in \operatorname{Prim}(A)),$$

where $\bigcap[P]$ is the intersection of the ideals in the equivalence class $[P]$ of P. The set of ideals obtained in this way is denoted by $\operatorname{Glimm}(A)$ and we identify this set with $\operatorname{Prim}(A)/\approx$ by the correspondence above. If A is unital then $\operatorname{Glimm}(A)$ consists of the ideals of A generated by the maximal ideals of the centre of A, as studied by Glimm [26]. The quotient map $\phi_A : \operatorname{Prim}(A) \to \operatorname{Glimm}(A)$ is called *the complete regularization map*. The standard topology on $\operatorname{Glimm}(A)$ is the topology τ_{cr}, which is the weakest topology for which the functions on $\operatorname{Glimm}(A)$ induced by $C^b(\operatorname{Prim}(A))$ are all continuous. This topology is completely regular, Hausdorff, weaker than the quotient topology (and equal to it when A is σ-unital [31, Theorem 2.6]) and hence makes ϕ_A continuous. The ideals in $\operatorname{Glimm}(A)$ are called *Glimm ideals* and the equivalence classes for $\approx$ in $\operatorname{Prim}(A)$ will sometimes be referred to as *Glimm classes*.

Note that if $P, Q \in \operatorname{Prim}(A)$, $G \in \operatorname{Glimm}(A)$ and $P \supseteq G = \bigcap[Q]$ then, since $[Q]$ is closed, $P \in [Q]$ and so $\phi_A(P) = \phi_A(Q) = G$. It follows that, for $P \in \operatorname{Prim}(A)$ and $G \in \operatorname{Glimm}(A)$, $P \supseteq G$ if and only if $\phi_A(P) = G$. For $P, Q \in \operatorname{Prim}(A)$, it is clear that $P \sim Q$ implies that $P \approx Q$. The converse implication holds whenever A is quasi-standard [8, Proposition 3.2]. In general, a Glimm class is said to be $\sim$-*connected* if it consists of a single $\sim$-component.

We recall that A is said to be σ-*unital* if it contains a strictly positive element or, equivalently, a countable approximate unit [34, 3.10.5]. If A is σ-unital with a strictly positive element u then $\operatorname{Prim}(A)$ is the union of the compact sets $\{P \in \operatorname{Prim}(A) : \|u + P\| \geq 1/n\}$ ($n \geq 1$). Since ϕ_A is continuous, Glimm(A) is σ-compact, hence Lindelöf, and therefore normal by (complete) regularity (see [25, 3D] or [33, Ch.2, Proposition 1.6]).

There is a homeomorphism ι from $\beta\operatorname{Glimm}(A)$ onto $\operatorname{Glimm}(M(A))$ such that $\iota(\phi_A(P)) = \phi_{M(A)}(\tilde{P})$ where, for $P \in \operatorname{Prim}(A)$, $\tilde{P}$ is the unique primitive ideal of $M(A)$ such that $\tilde{P} \cap A = P$ (see, for example, [2, p. 88] and [12, Proposition 4.7]). For $G \in \operatorname{Glimm}(A)$, we write $H_G = \iota(G)$. The next result [6, Proposition 3.2] is a technical step which is used to move from a general element of $\operatorname{Glimm}(M(A))$ to an element of the dense subset $\iota(\operatorname{Glimm}(A))$. We are grateful to the referee for pointing out that the proof in [6] applies to the case $n = 1$ as well as to the case $n \geq 2$.

Proposition 2.1. *Let A be a C^*-algebra with* $\operatorname{Glimm}(A)$ *normal. Let $n \geq 1$, $H \in \operatorname{Glimm}(M(A))$ and $Q_i \in \operatorname{Prim}(M(A)/H)$ ($1 \leq i \leq n$). For $1 \leq i \leq n$, let N_i be* *an open neighbourhood of* Q_i *in* $\mathrm{Prim}(M(A))$. *Then there exists* $K \in \mathrm{Glimm}(A)$ *and* $Q'_i \in \mathrm{Prim}(M(A)/H_K)$ *such that* $Q'_i \in N_i$ $(1 \leq i \leq n)$.

3. Topological properties of $\widehat{G_N}$.

In this section, we collect some facts concerning the fine structure of $\widehat{G_N}$ $(N \geq 2)$. Recall that $G_N = \mathbb{R}^N \rtimes SO(N)$ where $SO(N)$ acts on $\mathbb{R}^N$ by rotation. We embed $SO(N-1)$ into $SO(N)$ by $SO(N-1) \to \mathrm{diag}(1, SO(N-1))$. Thus $SO(N-1)$ is the stability group of characters χ_t of $\mathbb{R}^N$ corresponding to vectors $(t, 0, \dots, 0) \in \mathbb{R}^N$, $t \neq 0$. For $t > 0$ and $\sigma \in SO(N-1)^\wedge\}$, let $\pi_{t,\sigma} = \mathrm{ind}_{\mathbb{R}^N \rtimes SO(N-1)}^{G_N} \chi_t \times \sigma$, the irreducible representation of G_N induced by the representation $\chi_t \times \sigma$ of $\mathbb{R}^N \rtimes SO(N-1)$. Then

$$\widehat{G_N} = SO(N)^\wedge \cup \{\pi_{t,\sigma} : t > 0,\ \sigma \in SO(N-1)^\wedge\},$$

where $SO(N)^\wedge$ is considered as a subset of $\widehat{G_N}$ since $G_N/\mathbb{R}^N = SO(N)$.

The topology on $\widehat{G_N}$ is described in [15] (see also [29], [22], [1]). Let

$$\mathcal{U}_N := \{\pi_{t,\sigma} : t > 0,\ \sigma \in SO(N-1)^\wedge\} = \widehat{G_N} \setminus SO(N)^\wedge.$$

The relative topology on $\mathcal{U}_N$ is the topology induced from the product topology on $(0, \infty) \times SO(N-1)^\wedge$ and of course the relative topology on the closed subset $SO(N)^\wedge$ is discrete. Furthermore, a sequence $(\pi_{t_n,\sigma_n})_{n\geq 1}$ in $\mathcal{U}_N$ is convergent to some $\pi \in SO(N)^\wedge$ if and only if $t_n \to 0$ as $n \to \infty$ and eventually σ_n is contained in $\pi|_{SO(N-1)}$ (see, for example, [22, Theorem 3.4]).

If $\pi_{t,\sigma}$ belongs to the Hausdorff space $\mathcal{U}_N$ then $B := \{\pi_{s,\sigma} : s \geq t/2\}$ is a neighbourhood of $\pi_{t,\sigma}$ and is closed in $\widehat{G_N}$. It follows that $\pi_{t,\sigma}$ is a separated point of $\widehat{G_N}$ and also that $\ker \pi_{t,\sigma}$ is a Glimm ideal of $C^*(G_N)$ (cf. [17, Proposition 7] and [29, Proposition 4.9]). In summary, $\widehat{G_N}$ consists of the closed subset $SO(N)^\wedge$ which is relatively discrete, together with a dense open subset of separated points which is the disjoint union of a countably infinite collection of open half-lines.

Bearing in mind that $SO(2)^\wedge = \widehat{\mathbb{T}} = \mathbb{Z}$, we next recall the representation theory of the groups $SO(N)$ for $N \geq 3$ [32] (see also [1, 22, 30]). Let $k = \lfloor \frac{N}{2} \rfloor$. The irreducible representations of $SO(N)$ are parametrized by signatures $(m_1, \dots, m_k) \in \mathbb{Z}^k$, where

$$m_1 \geq m_2 \geq \dots \geq m_{k-1} \geq |m_k| \text{ if } N = 2k$$

$$\text{and } m_1 \geq m_2 \geq \dots \geq m_k \geq 0 \text{ if } N = 2k+1.$$

Moreover, if $N = 2k$, then

$$(m_1, \dots, m_k)|_{SO(N-1)} = \sum_{m_i \geq q_i \geq |m_{i+1}|\ (1 \leq i \leq k-1)} (q_1, \dots, q_{k-1}),$$

and, if $N = 2k+1$, then

$$(m_1, \dots, m_k)|_{SO(N-1)} = \sum_{m_i \geq p_i \geq m_{i+1}\ (1 \leq i \leq k-1),\ m_k \geq p_k \geq -m_k} (p_1, \dots, p_k).$$

Note that in both cases, the combinatorial condition shows that the number of summands on the right-hand side is finite. This is, of course, consistent with the fact that the representation on the left-hand side is finite dimensional.

Since $\widehat{G_N} \setminus SO(N)^\wedge$ consists of separated points of $\widehat{G_N}$, the next result gives a full description of the relation $\sim$ of inseparability on $\widehat{G_N}$.

Lemma 3.1. *Let $N \geq 2$ and $\pi_1, \pi_2 \in SO(N)^\wedge \subseteq \widehat{G_N}$. Then $\pi_1 \sim \pi_2$ if and only if $\pi_1|_{SO(N-1)}$ and $\pi_2|_{SO(N-1)}$ have a common irreducible subrepresentation.*

Proof. Suppose that $\pi_1|_{SO(N-1)}$ and $\pi_2|_{SO(N-1)}$ both contain some $\sigma \in SO(N-1)^\wedge$. Let $t_n = 1/n$ $(n \geq 1)$. Then $\pi_{t_n,\sigma} \to \pi_1, \pi_2$ as $n \to \infty$ and so $\pi_1 \sim \pi_2$.

Conversely, suppose that $\pi_1 \sim \pi_2$. Since $C^*(G_N)$ is separable and $\mathcal{U}_N$ is dense in $\widehat{G_N}$, it follows from [7, Lemma 1.2] that there is a sequence $(\pi_{t_n,\sigma_n})_{n\geq 1}$ in $\mathcal{U}_N$ which is convergent to both π_1 and π_2. Then eventually σ_n is contained in both $\pi_1|_{SO(N-1)}$ and $\pi_2|_{SO(N-1)}$. □

We can now determine $\mathrm{Orc}(A)$ in the case where $A = C^*(G_N)$. Since $C^*(G_2)$ is quasi-standard, $\mathrm{Orc}(C^*(G_2)) = 1$.

Proposition 3.2. *Let G_N be a motion group ($N \geq 3$) and suppose that $k = \lfloor N/2 \rfloor$.*

(i) *$d((m_1, \dots, m_k), (n_1, \dots, n_k)) \leq k$ for $(m_1, \dots, m_k), (n_1, \dots, n_k) \in \widehat{SO(N)}$;*
(ii) *$d((1, \dots, 1), (0, \dots, 0)) \geq k$ for $(0, \dots, 0), (1, \dots, 1) \in SO(N)^\wedge$.*

Proof. (i) For $1 \leq j \leq k$, let $s_j = \max\{m_j, n_j\}$. Suppose first of all that N is even, so that $N = 2k$. Since $m_{k-1} \geq |m_k|$,

$$(m_1, \dots, m_k) \sim (s_1, m_2, \dots, m_{k-1}, 0)$$

because the restrictions to $SO(N-1)$ contain $(m_1, \dots, m_{k-1})$. Similarly,

$$(s_1, m_2, \dots, m_{k-1}, 0) \sim (s_1, s_2, m_3 \dots, m_{k-2}, 0, 0)$$

because the restrictions to $SO(N-1)$ contain $(s_1, m_2, \dots, m_{k-2}, 0)$.

Suppose that k is even, say $k = 2r$ where $r \geq 1$. Then, proceeding as above, we obtain an r-step $\sim$-walk in $\widehat{G_N}$ from $(m_1, \dots, m_k)$ to $(s_1, \dots, s_r, 0, \dots, 0)$ and so $d((m_1, \dots, m_k), (n_1, \dots, n_k)) \leq 2r = k$.

Now suppose that k is odd, say $k = 2r + 1$ where $r \geq 1$. In this case, we obtain an r-step $\sim$-walk from $(m_1, \dots, m_k)$ to $(s_1, \dots, s_r, m_{r+1}, 0, \dots, 0)$ and an r-step $\sim$-walk from $(n_1, \dots, n_k)$ to $(s_1, \dots, s_r, n_{r+1}, 0, \dots, 0)$. But

$$(s_1, \dots, s_r, m_{r+1}, 0, \dots, 0) \sim (s_1, \dots, s_r, n_{r+1}, 0, \dots, 0)$$

because the restrictions to $SO(N-1)$ contain $(s_1, \dots, s_r, 0, \dots, 0)$. Hence

$$d((m_1, \dots, m_k), (n_1, \dots, n_k)) \leq 2r + 1 = k.$$

We now turn to the case where N is odd, so that $N = 2k + 1$. If $k = 1$, $(m_1) \sim (n_1)$ in $\widehat{G_3}$ because the restrictions to $SO(2)$ contain the trivial representation (0) of $SO(2)$. So we now suppose $k \geq 2$. Since $0 \geq -m_k$,

$$(m_1, \dots, m_k) \sim (s_1, m_2, \dots, m_{k-1}, 0)$$

because the restrictions to $SO(N-1)$ contain $(m_1, \dots, m_{k-1}, 0)$. Similarly,

$$(s_1, m_2, \dots, m_{k-1}, 0) \sim (s_1, s_2, m_3 \dots, m_{k-2}, 0, 0)$$

because the restrictions to $SO(N-1)$ contain $(s_1, m_2, \ldots, m_{k-2}, 0, 0)$. Thus if $k = 2r$, we obtain an r-step $\sim$-walk in $\widehat{G_N}$ from $(m_1, \ldots, m_k)$ to $(s_1, \ldots, s_r, 0, \ldots, 0)$ and similarly from $(n_1, \ldots, n_k)$ to $(s_1, \ldots, s_r, 0, \ldots, 0)$. Hence $d((m_1, \ldots, m_k), (n_1, \ldots, n_k)) \leq 2r = k$.

If $k = 2r + 1$ (with $r \geq 1$), we obtain an r-step $\sim$-walk from $(m_1, \ldots, m_k)$ to $(s_1, \ldots, s_r, m_{r+1}, 0, \ldots, 0)$ and an r-step $\sim$-walk from $(n_1, \ldots, n_k)$ to $(s_1, \ldots, s_r, n_{k+1}, 0, \ldots, 0)$. But

$$(s_1, \ldots, s_r, m_{r+1}, 0, \ldots, 0) \sim (s_1, \ldots, s_r, n_{r+1}, 0, \ldots, 0)$$

because the restrictions to $SO(N-1)$ contain $(s_1, \ldots, s_r, 0, \ldots, 0)$. Hence

$$d((m_1, \ldots, m_k), (n_1, \ldots, n_k)) \leq 2r + 1 = k.$$

(ii) We consider first the case $N = 2k$ (so that $k \geq 2$). Suppose that $0 \leq i \leq k-2$, $\pi = (M_1, \ldots, M_k) \in SO(N)^\wedge$, $\pi' = (M_1', \ldots, M_k') \in SO(N)^\wedge$, $\pi \sim \pi'$ in $\widehat{G_N}$ and $M_j = 0$ for $i < j \leq k$. By Lemma 3.1, there exists $\sigma = (q_1, \ldots, q_{k-1}) \in SO(N-1)^\wedge$ such that

$$M_1 \geq q_1 \geq M_2 \geq \ldots \geq M_i \geq q_i \geq 0 \geq q_{i+1}$$

and

$$M_1' \geq q_1 \geq \ldots \geq M_{i+1}' \geq q_{i+1} \geq \ldots \geq |M_k'|.$$

Thus $q_{i+1} = \ldots = q_{k-1} = 0$ and so $M_{i+2}' = \ldots = M_k' = 0$. Applying this with $i = 0, 1, \ldots, k-2$ in turn, we obtain that

$$d((0, \ldots, 0), (1, \ldots, 1)) \geq k.$$

Now suppose that $N = 2k+1$. If $k = 1$, then $(0) \sim (1)$ in $\widehat{G_3}$ because the restrictions to $SO(2)$ contain the trivial representation (0) of $SO(2)$. So we now assume $k \geq 2$. Suppose that $0 \leq i \leq k-2$, $\pi = (M_1, \ldots, M_k) \in SO(N)^\wedge$, $\pi' = (M_1', \ldots, M_k') \in SO(N)^\wedge$, $\pi \sim \pi'$ in $\widehat{G_N}$ and $M_j = 0$ for $i < j \leq k$. By Lemma 3.1, there exists $\sigma = (p_1, \ldots, p_k) \in SO(N-1)^\wedge$ such that

$$M_1 \geq p_1 \geq M_2 \geq \ldots \geq M_i \geq p_i \geq 0 \geq p_{i+1} \geq \ldots \geq M_k \geq p_k \geq -M_k$$

and

$$M_1' \geq p_1 \geq \ldots \geq M_{i+1}' \geq p_{i+1} \geq \ldots \geq M_k' \geq p_k \geq -M_k.$$

Thus $p_{i+1} = \ldots = p_k = 0$ and so $M_{i+2}' = \ldots = M_k' = 0$. It follows that

$$d((0, \ldots, 0), (1, \ldots, 1)) \geq k,$$

as required. $\square$

Part (ii) of the next result is contained in [29, Proposition 4.9].

Theorem 3.3. *Let G_N be a motion group with $N \geq 3$ and set $A = C^*(G_N)$.*

(i) *$SO(N)^\wedge$ is a $\sim$-connected subset of $\widehat{G_N}$ and* $\mathrm{Orc}(A) = \lfloor N/2 \rfloor$.

(ii) $\mathrm{Glimm}(A)$ *consists of the ideals* $\ker \pi_{t,\sigma}$ $(t > 0,\ \sigma \in SO(N-1)^\wedge)$ *and the ideal*

$$I_0 := \bigcap \{\ker \pi : \pi \in SO(N)^\wedge\}.$$

(iii) *The complete regularization map* $\phi_A : \mathrm{Prim}(A) \to \mathrm{Glimm}(A)$ *is closed.*

Proof. (i) This follows from parts (i) and (ii) of Proposition 3.2.

(ii) Let $\pi \in SO(N)^\wedge$. As noted earlier, each ideal $\ker \pi_{t,\sigma}$ is a Glimm ideal and, in particular, $\ker \pi_{t,\sigma} \not\approx \ker \pi$. On the other hand, since $SO(N)^\wedge$ is $\sim$-connected, $\ker \pi' \approx \ker \pi$ for all $\pi' \in SO(N)^\wedge$. Thus I_0 is the only other Glimm ideal.

(iii) Let C be a closed subset of $\mathrm{Prim}(A)$. Then $\phi_A^{-1}(\phi_A(C))$ is either C or the union of C with the closed set $\{\ker \pi : \pi \in SO(N)^\wedge\}$, depending on whether or not C is disjoint from the latter set. Thus ϕ_A is closed with respect to the quotient topology τ_q on $\mathrm{Glimm}(A)$. Since G_N is second countable, A is separable and so τ_q coincides with τ_{cr} on $\mathrm{Glimm}(A)$ [31, Theorem 2.6] (see also [29, Proposition 4.9]). □

We remark here that it can be easily seen that the Glimm ideal I_0 has no compact neighbourhood in the space $\mathrm{Glimm}(A)$. Thus, even though $C^*(G_N)$ is separable, $\mathrm{Glimm}(A)$ is neither locally compact nor first countable [31, Theorem 3.3]. In particular, $C^*(G_N)$ is not a $\mathcal{CR}$-algebra in the sense of [21]. Moreover, it follows from Theorem 3.3(ii) that A/G is non-unital for all $G \in \mathrm{Glimm}(A)$ and so the results of [13] show that there is substantial complexity in the ideal structure of $M(A)$. In particular there is an injective map from the lattice of z-filters on $\mathrm{Glimm}(A)$ to the lattice of closed ideals of $M(A)$ [13, Theorem 3.2] and each Glimm class of $\mathrm{Prim}(M(A))$ which meets the canonical image of $\mathrm{Prim}(A)$ contains at least 2^c maximal ideals of $M(A)$ [13, Theorem 5.3].

4. **A lower bound for** $\mathrm{Orc}(M(A))$**.**

In this section we establish a lower bound for $\mathrm{Orc}(M(A))$, showing that $\mathrm{Orc}(M(A)) \geq \mathrm{Orc}(A)$ under fairly general conditions (Theorem 4.6). It follows from this that when $A = C^*(G_N)$, where G_N is a motion group, then $\mathrm{Orc}(M(A))) \geq \lfloor N/2 \rfloor$ (Corollary 4.7).

For subsets Y and Z of a topological space X, let

$$d(Y, Z) = \inf\{d(y, z) : y \in Y,\ z \in Z\}$$

where $d(y, z)$ is as defined at the start of Section 2. For $n \geq 0$, let

$$Y^n = \{x \in X : d(\{x\}, Y) \leq n\}$$

and $Y^\infty = \{x \in X : d(\{x\}, Y) < \infty\}$. Note that $Y^0 = Y$, and that $Y^\infty = Y^{\mathrm{Orc}(A)}$ if $\mathrm{Orc}(A) < \infty$.

We shall be interested in C*-algebras A which have the property that X^1 is closed in $\mathrm{Prim}(A)$ whenever X is a closed subset of $\mathrm{Prim}(A)$. An elementary compactness argument shows that this property holds when $\mathrm{Prim}(A)$ is compact, see [35, Corollary 2.3], and hence holds whenever A is unital or is the stabilization of a unital C*-algebra. It also often holds when $\mathrm{Prim}(A)$ is non-compact.

Lemma 4.1. *Let A be a C*-algebra and suppose that there is a closed, relatively discrete subset Y of $\mathrm{Prim}(A)$ such that Y contains all non-singleton $\sim$-components of $\mathrm{Prim}(A)$. Then X^1 is closed whenever X is a closed subset of $\mathrm{Prim}(A)$.*

Proof. Let X be a closed subset of $\mathrm{Prim}(A)$. Then $X^1 = X \cup (X \cap Y)^1$ since Y contains all the non-singleton $\sim$-components of $\mathrm{Prim}(A)$. But $(X \cap Y)^1$ is contained in Y and hence is closed in $\mathrm{Prim}(A)$, because every subset of Y is closed in Y and therefore in $\mathrm{Prim}(A)$. Thus X^1 is the union of two closed sets. □

When $A = C^*(G_N)$, with G_N a motion group, we may take Y to be the closed, relatively discrete set $\widehat{SO(N)}$. Then Lemma 4.1 implies that X^1 is closed whenever X is a closed subset of $\mathrm{Prim}(A)$.

If Y and Z are compact subsets of a topological space X such that $d(Y,Z) \geq 2$ then a routine compactness argument shows that there exist disjoint open subsets U and V of X with $Y \subseteq U$ and $Z \subseteq V$ [35, Lemma 2.2]. The next result extends this argument. We say that a topological space is *locally compact* if every point has a neighbourhood base of compact sets.

Lemma 4.2. [14, Lemma 4.1] *Let X be a locally compact space and let Y and Z be subsets of X which are Lindelöf in the relative topology. Then the following are equivalent:*

(i) *the closure of Y^1 does not meet Z and the closure of Z^1 does not meet Y;*
(ii) *there exist disjoint open subsets U and V of X with $Y \subseteq U$ and $Z \subseteq V$.*

We will apply Lemma 4.2 to $X = \mathrm{Prim}(A)$, where A is a σ-unital C*-algebra. Then $\mathrm{Prim}(A)$ is σ-compact, and hence Lindelöf, so every closed subset of X is Lindelöf. Every compact subset of X is Lindelöf too, of course.

When $\mathrm{Prim}(A)$ is compact and $\mathrm{Orc}(A) < \infty$, it is automatic that every Glimm class is $\sim$-connected [35, Corollary 2.7]. When $\mathrm{Prim}(A)$ is non-compact, however, there may be Glimm classes made up of more than one $\sim$-component, even when $\mathrm{Orc}(A) < \infty$. Furthermore, the complete regularization map ϕ_A, which is automatically closed when $\mathrm{Prim}(A)$ is compact, need not be closed in the general case.

The next result shows that these difficulties do not arise if $\mathrm{Orc}(A) < \infty$ and X^1 is closed for every closed subset X of $\mathrm{Prim}(A)$.

Proposition 4.3. *Let A be a σ-unital C*-algebra. Then the following are equivalent:*

(i) *X^∞ is closed whenever X is a closed subset of* $\mathrm{Prim}(A)$*;*
(ii) *ϕ_A is a closed map and every Glimm class is $\sim$-connected.*

In particular, if $\mathrm{Orc}(A) < \infty$ *and X^1 is closed for every closed subset X of* $\mathrm{Prim}(A)$ *then conditions (i) and (ii) both hold.*

Proof. (i) $\Rightarrow$ (ii). For $P, Q \in \mathrm{Prim}(A)$ define $P \diamond Q$ if $d_A(P,Q) < \infty$. Then $\diamond$ is an equivalence relation on $\mathrm{Prim}(A)$ and, for $P, Q \in \mathrm{Prim}(A)$, $P \diamond Q$ implies $P \approx Q$. Set $W = \mathrm{Prim}(A)/\diamond$ equipped with the quotient topology, and let $q : \mathrm{Prim}(A) \to W$ be the quotient map. If X is a closed subset of $\mathrm{Prim}(A)$ then $q^{-1}(q(X)) = X^\infty$, which is closed by (i). Hence q is a closed map. If $P \in \mathrm{Prim}(A)$ and $Q \in \overline{\{P\}}$ then $Q \sim P$. Thus $\{q(P)\} = q(\overline{\{P\}})$, which is closed in W, and so W is a T_1-space.

Let Y and Z be non-empty, disjoint closed subsets of W. Then $Y' := q^{-1}(Y)$ and $Z' := q^{-1}(Z)$ are disjoint closed $\sim$-saturated subsets of $\mathrm{Prim}(A)$. Since A is σ-unital, $\mathrm{Prim}(A)$ is σ-compact and hence Lindelöf. Thus Y' and Z' are Lindelöf, so Lemma 4.2 implies the existence of disjoint open sets U and V containing Y' and Z' respectively.

We now use a standard characterization (see [27, 7.2.14]): a quotient map $p : X \to D$ is closed if and only if whenever $d \in D$ and G is an open set containing $p^{-1}(d)$ then there exists a saturated open set H such that $p^{-1}(d) \subseteq H \subseteq G$ (where H is saturated if $H = p^{-1}(p(H))$). Applying this characterization in the present case to each of the points of Y and Z relative to U and V we obtain $\diamond$-saturated open sets U' and V' such that $Y' \subseteq U' \subseteq U$ and $Z' \subseteq V' \subseteq V$.

Hence $q(U')$ and $q(V')$ are disjoint open sets of W containing Y and Z respectively. Hence W is normal.

Since W is normal and T_1, any two distinct points in W can be separated by a continuous real-valued function. It follows that $\diamond$ coincides with $\approx$ on $\mathrm{Prim}(A)$ so that $W = \mathrm{Glimm}(A)$ as sets and $q = \phi_A$. Thus each Glimm class is $\sim$-connected and ϕ_A is closed for the quotient topology on $\mathrm{Glimm}(A)$. Since we have seen that the quotient topology is normal and T_1, hence completely regular, it coincides with τ_{cr} (see [31, Theorem 2.6] for a more general result).

(ii) $\Rightarrow$ (i). (This does not need the σ-unital hypothesis). Let X be a closed subset of $\mathrm{Prim}(A)$. Since ϕ_A is a closed and continuous map, $\phi_A^{-1}(\phi_A(X))$ is closed. But since each Glimm class is $\sim$-connected, $\phi_A^{-1}(\phi_A(X)) = X^\infty$.

Finally, suppose that $\mathrm{Orc}(A) < \infty$ and that X^1 is closed for every closed subset X of $\mathrm{Prim}(A)$. Then, for such X, the sets $X^1, (X^1)^1, \ldots, X^{\mathrm{Orc}(A)} (= X^\infty)$ are all closed subsets of $\mathrm{Prim}(A)$. $\square$

For a topological space X, we now define a *chain of length* n on X to be a collection of n closed subsets $X_1, \ldots, X_n$ with the following properties:

(i) $\bigcup_{i=1}^n X_i = X$;
(ii) X_i and X_j are disjoint if $|i - j| > 1$;
(iii) if $n > 1$ then the (open) sets $X_1 \setminus X_2$ and $X_n \setminus X_{n-1}$ are non-empty.

A chain of length n is said to be *admissible* if there exist $x \in X_1 \setminus X_2$ and $y \in X_n \setminus X_{n-1}$ such that $d(x, y) < \infty$. Note that this further condition (in addition to (i) and (ii)) implies that $X_i \cap X_{i+1}$ is non-empty for $i = 1, \ldots, n-1$, for otherwise x and y would belong to different clopen subsets of X. It was shown in [35, Lemma 2.1] that if $X_1, \ldots, X_n$ is a chain on X of length $n > 1$ and $x, y \in X$ with $x \in X_1 \setminus X_2$ and $y \in X_n \setminus X_{n-1}$ then $d(x, y) \geq n$. Hence $\mathrm{Orc}(X)$ is greater than or equal to the length of any admissible chain on X.

Lemma 4.4. *Let A be a σ-unital C*-algebra and suppose that X^1 is closed whenever X is a closed subset of* $\mathrm{Prim}(A)$*. Let $\mathcal{S}$ be the class of subsets of* $\mathrm{Prim}(A)$ *which are either compact or closed. Suppose that $X, Y \in \mathcal{S}$ with $d(X, Y) \geq k \geq 2$ and with $X \cup Y$ contained in a single $\sim$-component. Then there is an admissible chain $X_1, \ldots, X_k$ of closed subsets of* $\mathrm{Prim}(A)$ *with $X \subseteq X_1 \setminus X_2$ and $Y \subseteq X_k \setminus X_{k-1}$.*

Proof. We follow the method of [35, Lemma 2.4]. First note that if $X \in \mathcal{S}$ then X^1 is closed, either by assumption if X is closed, or by [35, Corollary 2.3] if X is compact. Thus the hypotheses imply that the sets X^n and Y^n are closed for all $n \geq 1$. Since $d(X, Y^{k-2}) \geq 2$, and X and Y^{k-2} are Lindelöf, it follows from Lemma 4.2 that there are disjoint open sets $U_1 \supseteq X$ and $V_1 \supseteq Y^{k-2}$. Set $X_1 = \mathrm{Prim}(A) \setminus V_1$ and $Y_2 = \mathrm{Prim}(A) \setminus U_1$ and note that X_1 and Y_2 are closed and that $X \subseteq X_1 \setminus Y_2$ and $Y^{k-2} \subseteq Y_2 \setminus X_1$. If $k = 2$ then X_1 and $X_2 = Y_2$ have the required properties.

Otherwise, if $k > 2$ then $d(X_1, Y^{k-3}) \geq 2$ since X_1 is disjoint from Y^{k-2}, and X_1 and Y^{k-3} are closed and hence Lindelöf. For $k > 2$ we define inductively, for $i = 2, \ldots, k-1$, $X_i = (\mathrm{Prim}(A) \setminus V_i) \cap Y_i$ and $Y_{i+1} = \mathrm{Prim}(A) \setminus U_i$, where U_i and V_i are disjoint open sets containing $X_1 \cup \ldots \cup X_{i-1}$ and $Y^{k-(i+1)}$ respectively. Note that for $2 \leq i \leq k-2$, $d((X_1 \cup \ldots \cup X_i), Y^{k-(i+2)}) \geq 2$, and $X_1 \cup \ldots \cup X_i$ and $Y^{k-(i+2)}$ are Lindelöf, so the induction can proceed. Finally set $X_k = Y_k$. Then $Y \subseteq X_k \setminus X_{k-1}$, and it is easy to check that $X_1, \ldots, X_k$ is an admissible chain of length k. $\square$

The next lemma asserts a sort of 'normality' for $\mathrm{Prim}(A)$ when A is a σ-unital C^*-algebra. Recall from Section 2 that, for $P \in \mathrm{Prim}(A)$, $\tilde{P} \in \mathrm{Prim}(M(A))$ satisfies $\tilde{P} \cap A = P$. For $X \subseteq \mathrm{Prim}(A)$, let $\tilde{X} = \{\tilde{P} : P \in X\}$.

Lemma 4.5. [14, Lemma 2.1] *Let A be a σ-unital C^*-algebra and let X and Y be disjoint closed subsets of* $\mathrm{Prim}(A)$*. Then the closures of $\tilde{X}$ and $\tilde{Y}$ are disjoint in* $\mathrm{Prim}(M(A))$.

Theorem 4.6. *Let A be a σ-unital C^*-algebra and suppose that X^1 is closed whenever X is a closed subset of* $\mathrm{Prim}(A)$*. Then* $\mathrm{Orc}(M(A)) \geq \mathrm{Orc}(A)$.

Proof. We may suppose that $\mathrm{Orc}(A) \geq 2$ for otherwise the result is trivial. Let $k \in \mathbb{N}$ with $2 \leq k \leq \mathrm{Orc}(A)$. Then there exist $P, Q \in \mathrm{Prim}(A)$ such that $k \leq d_A(P, Q) < \infty$. By Lemma 4.4, applied to the compact sets $\{P\}$ and $\{Q\}$, there is an admissible chain $X_1, \ldots X_k$ on $\mathrm{Prim}(A)$, of length k. For each $1 \leq i \leq k$, let Y_i be the closure of $\tilde{X}_i = \{\tilde{R} : R \in X_i\}$ in $\mathrm{Prim}(M(A))$. It then follows, using Lemma 4.5 to check the preservation of disjointness, that $Y_1, \ldots, Y_k$ is a chain on $\mathrm{Prim}(M(A))$ of length k. Since $d_{M(A)}(\tilde{P}, \tilde{Q}) \leq d_A(P, Q) < \infty$, this chain is admissible. Hence $\mathrm{Orc}(M(A)) \geq k$ by [35, Lemma 2.1]. $\square$

Corollary 4.7. *Let G_N be a motion group with $N \geq 3$ and set $A = C^*(G_N)$. Then* $\mathrm{Orc}(M(A)) \geq \lfloor N/2 \rfloor$.

Proof. This follows from Theorem 3.3 and Theorem 4.6, and from Lemma 4.1 and the remark following. $\square$

5. Upper bounds for $K(M(A))$.

In this section, motivated by the motion groups G_N, we obtain some general C^*-theoretic results for the constants $K(A)$ and $K(M(A))$.

Let $\mathrm{Id}(A)$ be the set of all closed two-sided ideals of a C^*-algebra A. This is a compact Hausdorff space for the topology defined by Fell in [24, Section II]. We denote this topology by τ_s and we recall that a net (J_α) is τ_s-convergent to J in $\mathrm{Id}(A)$ if and only if $\|a + J_\alpha\| \to \|a + J\|$ for all $a \in A$ (see [24, Theorem 2.2]). A (closed two-sided) ideal J of A is said to be *primal* if whenever $n \geq 2$ and $J_1, J_2, \ldots, J_n$ are ideals of A with product $J_1 J_2 \ldots J_n = \{0\}$ then at least one of the J_i is contained in J. This concept arose in [5] where it was shown that a state of A is a weak*-limit of factorial states if and only if the kernel of the associated GNS-representation is primal. It follows from [5, Proposition 3.2] that an ideal J of A is primal if and only if there exists a net in $\mathrm{Prim}(A)$ which is convergent to every point in some dense subset of $\mathrm{Prim}(A/J)$. The set of all primal ideals of A is τ_s-closed in Id(A) (see [4, p.531]). $\mathrm{Primal}'(A)$ is the set of proper primal ideals of A, $\mathrm{Min\text{-}Primal}(A)$ is the set of minimal primal ideals of A and $\mathrm{Sub}(A)$ is the τ_s-closure of $\mathrm{Min\text{-}Primal}(A)$ in $\mathrm{Id(A)} \setminus \{\mathrm{A}\}$ and is therefore contained in $\mathrm{Primal}'(A)$ [9, p.84].

When A is separable, $(\mathrm{Primal}'(A), \tau_s)$ is metrizable [19, Lemme 2] and the set of separated points of $\mathrm{Prim}(A)$ is τ_s-dense in $\mathrm{Min\text{-}Primal}(A)$ [4, Corollary 4.6], so for $I \in \mathrm{Sub}(A)$ there is a sequence (P_n) of separated points of $\mathrm{Prim}(A)$ such $P_n \to I$ (τ_s). Hence $\mathrm{Prim}(A/I)$ is precisely the set of limits of (P_n) in $\mathrm{Prim}(A)$ and every cluster point of (P_n) is a limit (see [23, Theorem 2.1] and [9, Lemma 1.4]). Conversely suppose that (P_n) is a convergent sequence of separated points in $\mathrm{Prim}(A)$ and that every cluster point of (P_n) is a limit. Let X be the set of limits of (P_n). Then $P_n \to I = \ker X$ (τ_s) and so $I \in \mathrm{Sub}(A)$. Thus, given a separable C^*-algebra A and a description of $\mathrm{Prim}(A)$ as a topological space, it is usually possible to identify $\mathrm{Sub}(A)$. If A is quasi-standard then $\mathrm{Glimm}(A) = \text{Min-Primal}(A) = \mathrm{Sub}(A)$ as sets and topological spaces [8, Theorem 3.3]. On the other hand, if $\mathrm{Glimm}(A) = \text{Min-Primal}(A)$ (as sets) and if A is not quasi-standard then $\mathrm{Sub}(A)$ strictly contains $\text{Min-Primal}(A)$ [8, Theorem 3.3 ((v)$\to$(i))]. This phenomenon occurs for $C^*(SL(2,\mathbb{C}))$ [6, Example 4.1]. We shall determine $\mathrm{Sub}(C^*(G_N))$ in Proposition 6.1.

The next result was obtained in [6, Theorem 5.2]. It will later be applied to the cases $N \equiv 0$ and $N \equiv 3 \pmod 4$.

Theorem 5.1. *Let A be a C^*-algebra with* $\mathrm{Glimm}(A)$ *normal and ϕ_A closed. Suppose that there exists $n \geq 0$ such that whenever $G \in \mathrm{Glimm}(A)$ and $I^{(i)} \in \mathrm{Sub}(A)$ $(1 \leq i \leq 3)$ with $I^{(1)} \cap I^{(2)} \cap I^{(3)} \supseteq G$ then there exist $S^{(i)} \in \mathrm{Prim}(A)$ $(1 \leq i \leq 3)$ with $I^{(i)} \subseteq S^{(i)}$ and $T \in \mathrm{Prim}(A)$ with $d_A(S^{(i)}, T) \leq n$ $(1 \leq i \leq 3)$. Then $K(M(A)) \leq n+1$.*

Our second general result on $K(M(A))$, Theorem 5.3 below, will subsequently be applied to the remaining cases $N \equiv 1$ and $N \equiv 2 \pmod 4$. But first of all, we need a new result for unital C^*-algebras.

Theorem 5.2. *Let A be a unital C^*-algebra and let n be a positive integer. Suppose that whenever P, Q, R are primitive ideals of A lying in the same Glimm class there exist $S, T, U \in \mathrm{Prim}(A)$ such that $d_A(P,S) \leq n$, $d_A(Q,T) \leq n$, $d_A(R,U) \leq n$, and $S \cap T \cap U$ is primal. Then $K(A) \leq n + 1/2$.*

Proof. Let $a \in A$ with $\|D(a,A)\| \leq 1$. We want to show that $d(a, Z(A)) \leq n + 1/2$. Let $K \in \mathrm{Glimm}(A)$. Then by [36, Theorem 2.3] it suffices to show that $\|a_K - \lambda(a_K)\| \leq n + 1/2$, where a_K is the canonical image of a in A/K and $\lambda(a_K)$ is the unique scalar multiple of the identity in A/K which is closest to a_K. Set $b = a - \lambda(a_K)$. Then $\|D(b,A)\| \leq 1$ and $\lambda(b_K) = 0$. We show that $\|b_K\| \leq n + 1/2$.

Write $r = \|b_K\|$. Let C be the circle of radius r centred at the origin. Then there exist $x, y, z \in C \cap U(b_K, A/K)$ and extreme points f', g', h' of $N(A/K)$ such that C is the bounding circle of $\{x, y, z\}$ and $f'(b_K) = x$, $g'(b_K) = y$, $h'(b_K) = z$. Let $\epsilon > 0$ be given. By Milman's Theorem there exist $f, g, h \in G(A/K) \subseteq G(A)$ with $f(1), g(1), h(1) > 0$ such that $|f'(b) - f(b)| < \epsilon$, $|g'(b) - g(b)| < \epsilon$, and $|h'(b) - h(b)| < \epsilon$. Set $P = \Gamma(f)$, $Q = \Gamma(g)$, and $R = \Gamma(h)$.

Since $f, g, h \in G(A/K)$, it follows that $P, Q, R \supseteq K$. Hence there exist $S, T, U \in \mathrm{Prim}(A)$ such that $d_A(P,S) \leq n$, $d_A(Q,T) \leq n$, $d_A(R,U) \leq n$, and $S \cap T \cap U$ primal. Let $V = S \cap T \cap U$.

Since $d_A(P,S) \leq n$, there exist $P_1, \ldots P_n \in \mathrm{Prim}(A)$ such that $P \sim P_1 \sim \ldots \sim P_n = S$. By [10, Proposition 1.3],

$$|\lambda(b_{P_n}) - \lambda(b_V)| \leq 1/2, \tag{1}$$

$$|\lambda(b_{P \cap P_1}) - \lambda(b_{P_1})| \leq 1/2, \tag{2}$$

and, for $1 \leq i \leq n-1$,

$$|\lambda(b_{P_i}) - \lambda(b_{P_{i+1}})| \leq |\lambda(b_{P_i}) - \lambda(b_{P_i \cap P_{i+1}})| + |\lambda(b_{P_i \cap P_{i+1}}) - \lambda(b_{P_{i+1}})| \leq 1. \tag{3}$$

Writing $\mu = \lambda(b_V)$, we obtain from (1), (2), and (3) that

$$|\lambda(b_{P\cap P_1}) - \mu| \leq 1/2 + (n-1) + 1/2 = n. \tag{4}$$

On the other hand, since $P \cap P_1$ is primal, it follows from [36, Proposition 2.6] that

$$\|b_{P\cap P_1} - \lambda(b_{P\cap P_1})\| \leq 1/2. \tag{5}$$

Since f factors through A/P and hence through $A/(P\cap P_1)$, it follows from (4) and (5) that

$$\begin{aligned}|x - f(1)\mu| &\leq |x - f(b)| + |f(b_{P\cap P_1}) - \lambda(b_{P\cap P_1})f(1)| + f(1)|\lambda(b_{P\cap P_1}) - \mu| \\ &< \epsilon + \|f\|/2 + f(1)n \leq n + 1/2 + \epsilon.\end{aligned}$$

Similarly $|y - g(1)\mu| \leq n + 1/2 + \epsilon$ and $|z - h(1)\mu| \leq n + 1/2 + \epsilon$.

If $\mu = 0$ then we have that $\|b_K\| = |x| \leq n + 1/2 + \epsilon$. If $\mu \neq 0$ then we produce the line from μ to 0 to meet the circle C at a point E, say. Let FH be the diameter of C perpendicular to OE. Then the semicircle FEH meets $\{x, y, z\}$ in x, say. Hence $\|b_K\| \leq |x - f(1)\mu| \leq n + 1/2 + \epsilon$. So in either case $\|b_K\| \leq n + 1/2 + \epsilon$. Since ϵ was arbitrary, $\|b_K\| \leq n + 1/2$ as required. □

Theorem 5.3. *Let A be a C^*-algebra such that* $\mathrm{Glimm}(A)$ *is normal and ϕ_A is closed. Suppose that there exists $n \geq 0$ such that whenever $G \in \mathrm{Glimm}(A)$ and $I^{(i)} \in \mathrm{Sub}(A)$ $(1 \leq i \leq 3)$ with $I^{(1)} \cap I^{(2)} \cap I^{(3)} \supseteq G$ then there exist $S^{(i)}, T^{(i)} \in \mathrm{Prim}(A)$ $(1 \leq i \leq 3)$ with $I^{(i)} \subseteq S^{(i)}$, $d_A(S^{(i)}, T^{(i)}) \leq n$ $(1 \leq i \leq 3)$, and $T^{(1)} \cap T^{(2)} \cap T^{(3)}$ primal. Then $K(M(A)) \leq n + 3/2$.*

Proof. We show that $M(A)$ satisfies the hypotheses of Theorem 5.2 with n replaced by $n+1$. Suppose that $H \in \mathrm{Glimm}(M(A))$ and $Q^{(i)} \in \mathrm{Prim}(M(A)/H)$ $(1 \leq i \leq 3)$. Let $\mathcal{L}$ (respectively $\mathcal{M}$, $\mathcal{N}$) be a base of open neighbourhoods of $Q^{(1)}$ (respectively $Q^{(2)}$, $Q^{(3)}$) in $\mathrm{Prim}(M(A))$. Let $\Delta = \mathcal{L} \times \mathcal{M} \times \mathcal{N}$ with the usual order.

Temporarily fix $\alpha = (L, M, N) \in \Delta$. Exactly as in the proof of Theorem 5.1 (see [6, Theorem 5.2]), we may apply Proposition 2.1 to obtain $K_\alpha \in \mathrm{Glimm}(A)$ and $I_\alpha^{(i)} \in \mathrm{Sub}(A)$ $(1 \leq i \leq 3)$ such that $K_\alpha \subseteq I_\alpha^{(1)} \cap I_\alpha^{(2)} \cap I_\alpha^{(3)}$. By hypothesis there exist $T_{\alpha,j}^{(i)} \in \mathrm{Prim}(A)$ $(1 \leq i \leq 3,\ 1 \leq j \leq n+1)$ such that

$$I_\alpha^{(i)} \subseteq T_{\alpha,1}^{(i)} \sim \ldots \sim T_{\alpha,n+1}^{(i)} \qquad (1 \leq i \leq 3) \tag{1}$$

and $\bigcap_{i=1}^3 T_{\alpha,n+1}^{(i)}$ is a primal ideal of A.

We now let α vary. By the compactness of $\mathrm{Prim}(M(A))$ and by passing to successive subnets, we obtain $T_j^{(i)} \in \mathrm{Prim}(M(A))$ and commonly indexed subnets $(T_{\alpha(\beta),j}^{(i)})$ in $\mathrm{Prim}(A)$ such that

$$\tilde{T}_{\alpha(\beta),j}^{(i)} \to T_j^{(i)} \qquad (1 \leq i \leq 3,\ 1 \leq j \leq n+1).$$

It follows from (1) that

$$T_1^{(i)} \sim \ldots \sim T_{n+1}^{(i)} \qquad (1 \leq i \leq 3).$$

We show next that $\bigcap_{i=1}^3 T_{n+1}^{(i)}$ is a primal ideal of $M(A)$. Let V_i be an open neighbourhood of $T_{n+1}^{(i)}$ in $\mathrm{Prim}(M(A))$ $(1 \leq i \leq 3)$. There exists β such that

$$\tilde{T}_{\alpha(\beta),n+1}^{(i)} \in V_i \qquad (1 \leq i \leq 3).$$

Since $\bigcap_{i=1}^{3} T^{(i)}_{\alpha(\beta),n+1}$ is primal, there is a net in $\mathrm{Prim}(A)$ convergent to all of $T^{(1)}_{\alpha(\beta),n+1}$, $T^{(2)}_{\alpha(\beta),n+1}$ and $T^{(3)}_{\alpha(\beta),n+1}$, and hence a net in $\mathrm{Prim}(M(A))$ convergent to all of $\tilde{T}^{(i)}_{\alpha(\beta),n+1}$ $(1 \leq i \leq 3)$. Hence $V_1 \cap V_2 \cap V_3$ is non-empty as required.

Exactly as in the proof of Theorem 5.1 (see [6, Theorem 5.2] again), we have $Q^{(i)} \sim T^{(i)}_1$ $(1 \leq i \leq 3)$. Thus we have shown that $d_{M(A)}(Q^{(i)}, T^{(i)}_{n+1}) \leq n+1$ $(1 \leq i \leq 3)$ and that $\bigcap_{i=1}^{3} T^{(i)}_{n+1}$ is a primal ideal of $M(A)$. It follows from Theorem 5.2 that $K(M(A)) \leq (n+1) + \frac{1}{2} = n + \frac{3}{2}$. □

6. The cases $N \equiv 0$ and $N \equiv 2 \mod 4$.

In order to apply the results of the previous section, we need to begin by determining $\mathrm{Sub}(C^*(G_N))$.

For $\sigma \in SO(N-1)^\wedge$, let

$$I_{0,\sigma} = \bigcap \{\ker \pi : \pi \in SO(N)^\wedge,\ \pi|_{SO(N-1)} \geq \sigma\}.$$

Since the closed subset $SO(N)^\wedge$ of $\widehat{G_N}$ is relatively discrete, the set

$$\{\ker \pi : \pi \in SO(N)^\wedge,\ \pi|_{SO(N-1)} \geq \sigma\}$$

is a closed subset of $\mathrm{Prim}(C^*(G_N))$ and therefore is the hull of the ideal $I_{0,\sigma}$. Let (t_n) be any null sequence in $(0,\infty)$. Then, for all P in the hull of $I_{0,\sigma}$, $\ker \pi_{t_n,\sigma} \to P$ as $n \to \infty$. On the other hand, suppose that $Q \in \mathrm{Prim}(C^*(G_N))$ is a cluster point of $(\ker \pi_{t_n,\sigma})_{n\geq 1}$. Since $C^*(G_N)$ is separable, Q has a countable base of neighbourhoods in $\mathrm{Prim}(C^*(G_N))$ and so there is a subsequence $(\ker \pi_{t_{n_k},\sigma})_{k\geq 1}$ convergent to Q. Since $t_{n_k} \to 0$ as $k \to \infty$, $Q = \ker \pi$ for some $\pi \in SO(N)^\wedge$ such that σ is contained in $\pi|_{SO(N-1)}$, and hence $Q \supseteq I_{0,\sigma}$. It now follows that $\ker \pi_{t_n,\sigma} \to_{\tau_s} I_{0,\sigma}$ as $n \to \infty$. Since each $\ker \pi_{t_n,\sigma}$ is a separated point of $\mathrm{Prim}(C^*(G_N))$, and hence a minimal primal ideal of $C^*(G_N)$ [4, Proposition 4.5], we obtain that $I_{0,\sigma} \in \mathrm{Sub}(C^*(G_N))$.

Proposition 6.1. *Let $A = C^*(G_N)$ $(N \geq 2)$. Then*

$$\mathrm{Sub}(A) = \{I_{0,\sigma} : \sigma \in SO(N-1)^\wedge\} \cup \{\ker \pi_{t,\sigma} : t > 0, \sigma \in SO(N-1)^\wedge\}.$$

In particular, $\mathrm{Sub}(A) = \text{Min-Primal}(A)$ *if and only if N is even.*

Proof. Suppose that $I \in \mathrm{Sub}(A)$ and consider $\pi \in \widehat{A}$ such that $\ker \pi \supseteq I$. If $\pi \in \mathcal{U}_N$ then the minimal primal ideal $\ker \pi$ contains the primal ideal I and so $I = \ker \pi$. So we may assume from now on that $\mathrm{hull}(I) \subseteq \{\ker \pi : \pi \in SO(N)^\wedge\}$. By the discussion at the start of Section 5, there is a sequence (P_n) of separated points of $\mathrm{Prim}(A)$ such that $P_n \to_{\tau_s} I$ as $n \to \infty$. In particular, for $P \in \mathrm{Prim}(A)$, $P_n \to P$ as $n \to \infty$ if and only if $P \supseteq I$.

For each $n \geq 1$ there exists $t_n > 0$ and $\sigma_n \in SO(N-1)^\wedge$ such that $P_n = \ker \pi_{t_n,\sigma_n}$. Let $\pi \in SO(N)^\wedge$ such that $\ker \pi \supseteq I$. Then $\pi_{t_n,\sigma_n} \to \pi$ as $n \to \infty$ and so (t_n) is a null sequence and eventually σ_n is contained in $\pi|_{SO(N-1)}$. Replacing (P_n) by a subsequence (which will also be τ_s-convergent to I), we may assume that $\sigma_n = \sigma$ (say) for all n. Then $P_n = \ker \pi_{t_n,\sigma} \to_{\tau_s} I_{0,\sigma}$ as $n \to \infty$, as observed at the start of this section. Since τ_s is Hausdorff, $I = I_{0,\sigma}$ as required.

The final statement of the proposition follows from the characterization of $\text{Min-Primal}(A)$ in [29, Proposition 4.6]. □

The next two results deal with the cases $N \equiv 0$ and $N \equiv 2 \mod 4$. The strategies are similar in that both proofs use Corollary 4.7 for one estimate. However, for the other estimate, the first case uses Theorem 5.1 whereas the second case requires the more complicated Theorem 5.3. For the application of Theorems 5.1 and 5.3, we recall that, for the separable C^*-algebra $A = C^*(G_N)$ $(N \geq 3)$, Glimm(A) is normal (see Section 2) and ϕ_A is closed (Theorem 3.3).

Theorem 6.2. *Let* $A = C^*(G_N)$ *where* $N \equiv 0 \pmod 4$. *Then*

$$K(M(A)) = K_s(M(A)) = \frac{1}{2}\mathrm{Orc}(M(A)) = N/4.$$

Proof. We begin with the case $N > 4$ so that $N/4 > 1$. We aim to show that A satisfies the hypotheses of Theorem 5.1 with $n = \frac{N}{4} - 1$. Let $I^{(1)}, I^{(2)}, I^{(3)} \in \mathrm{Sub}(A)$ with $I^{(1)}, I^{(2)}, I^{(3)}$ all containing the same Glimm ideal G. If G is one of the separated points of $\mathrm{Prim}(A)$ then $I^{(1)} = I^{(2)} = I^{(3)} = G$ and we may take $S^{(1)} = S^{(2)} = S^{(3)} = T = G$.

We may suppose therefore that $G = I_0$ (see Theorem 3.3) and that $I^{(1)} = I_{0,\sigma}$, $I^{(2)} = I_{0,\sigma'}$, and $I^{(3)} = I_{0,\sigma''}$ for some $\sigma, \sigma', \sigma'' \in SO(N-1)^\wedge$ by Proposition 6.1. Let $\sigma = (q_1, \ldots, q_{k-1})$, $\sigma' = (q'_1, \ldots, q'_{k-1})$, and $\sigma'' = (q''_1, \ldots, q''_{k-1})$, where $k = N/2$. Set $Q_i = \max\{q_i, q'_i, q''_i\}$ for $1 \leq i \leq N/4$. Define $\pi = (Q_1, q_1, q_2, \ldots, q_{k-2}, 0) \in SO(N)^\wedge$ and similarly π' and π'' (replacing the q_j by q'_j and q''_j respectively). Since $\pi|_{SO(N-1)} \geq \sigma$, $\ker\pi \supseteq I^{(1)}$. Similarly, $\ker\pi' \supseteq I^{(2)}$ and $\ker\pi'' \supseteq I^{(3)}$.

There is an $(\frac{N}{4} - 1)$-step $\sim$-walk from each of π, π', π'' to $\rho := (Q_1, \ldots, Q_{\frac{N}{4}}, 0, \ldots, 0)$ in $\widehat{G_N}$. To see this, note that for $1 \leq i \leq \frac{N}{4} - 2$,

$$(Q_1, \ldots, Q_i, q_i, q_{i+1}, \ldots, q_{k-1-i}, 0, \ldots, 0) \sim (Q_1, \ldots, Q_i, Q_{i+1}, q_{i+1}, \ldots, q_{k-2-i}, 0, \ldots, 0)$$

because the restrictions to $SO(N-1)$ contain $(Q_1, \ldots, Q_i, q_{i+1}, \ldots, q_{k-1-i}, 0, \ldots, 0)$, and finally

$$(Q_1, \ldots, Q_{\frac{N}{4}-1}, q_{\frac{N}{4}-1}, q_{\frac{N}{4}}, 0, \ldots, 0) \sim (Q_1, \ldots, Q_{\frac{N}{4}}, 0, \ldots, 0) = \rho$$

because the restrictions to $SO(N-1)$ contain $(Q_1, \ldots, Q_{\frac{N}{4}-1}, q_{\frac{N}{4}}, 0, \ldots, 0)$. Similar arguments apply to π' and π'', replacing the q_j by q'_j and q''_j (respectively).

Thus taking $S^{(1)} = \ker\pi$, $S^{(2)} = \ker\pi'$, $S^{(3)} = \ker\pi''$, and $T = \ker\rho$ we have satisfied the hypotheses of Theorem 5.1. Thus $K(M(A)) \leq \frac{N}{4}$. Combining this with Theorem 3.3(i), Corollary 4.7 and [35, Theorem 4.4], we have

$$K(M(A)) \leq \frac{N}{4} \leq \frac{\mathrm{Orc}(A)}{2} \leq \frac{\mathrm{Orc}(M(A))}{2} = K_s(M(A)) \leq K(M(A))$$

and hence equality throughout.

For the simpler case $N = 4$, it again suffices to check the hypothesis of Theorem 5.1 for $G = I_0$ (the only non-maximal Glimm ideal of A). So let $I^{(1)} = I_{0,(q)}$, $I^{(2)} = I_{0,(q')}$ and $I^{(3)} = I_{0,(q'')}$ where $(q), (q'), (q'') \in SO(3)^\wedge$. Let $Q = \max\{q, q', q''\}$ and let $\pi = (Q, 0) \in SO(4)^\wedge$. Then $\pi|_{SO(3)}$ contains (q),(q') and (q'') and so $\ker\pi \supseteq I^{(1)} + I^{(2)} + I^{(3)}$. Using Theorem 5.1 in the case $n = 0$ and arguing as above, we have

$$K(M(A)) \leq 1 \leq \frac{\mathrm{Orc}(A)}{2} \leq \frac{\mathrm{Orc}(M(A))}{2} = K_s(M(A)) \leq K(M(A))$$

and hence equality throughout. □

Theorem 6.3. *Let $A = C^*(G_N)$ where $N \equiv 2 \pmod 4$ and $N \geq 6$. Then*

$$K(M(A)) = K_s(M(A)) = \frac{1}{2}\mathrm{Orc}(M(A)) = N/4.$$

Proof. Let $k = N/2$ and $m = (N-2)/4$, so that $k = 2m+1$. We begin with the case $N > 6$ so that $m > 1$. We aim to show that A satisfies the hypotheses of Theorem 5.3 with $n = m-1$. Let $I^{(1)}, I^{(2)}, I^{(3)} \in \mathrm{Sub}(A)$ with $I^{(1)}, I^{(2)}, I^{(3)}$ all containing the same Glimm ideal G. If G is one of the separated points of $\mathrm{Prim}(A)$ then $I^{(1)} = I^{(2)} = I^{(3)} = G$ and we may take $S^{(i)} = T^{(i)} = G$ $(1 \leq i \leq 3)$ so that $T^{(1)} \cap T^{(2)} \cap T^{(3)} = G$ which is primal.

We may suppose therefore that $G = I_0$ (see Theorem 3.3) and that $I^{(1)} = I_{0,\sigma}$, $I^{(2)} = I_{0,\sigma'}$, and $I^{(3)} = I_{0,\sigma''}$ for some $\sigma, \sigma', \sigma'' \in SO(N-1)^\wedge$ by Proposition 6.1. Let $\sigma = (q_1, \ldots, q_{k-1})$, $\sigma' = (q'_1, \ldots, q'_{k-1})$, and $\sigma'' = (q''_1, \ldots, q''_{k-1})$. Set $Q_i = \max\{q_i, q'_i, q''_i\}$ for $1 \leq i \leq m$. Define $\pi = (Q_1, q_1, q_2, \ldots, q_{k-2}, 0) \in SO(N)^\wedge$ and similarly π' and π'' (replacing the q_j by q'_j and q''_j respectively). Since $\pi|_{SO(N-1)} \geq \sigma$, $\ker\pi \supseteq I^{(1)}$. Similarly, $\ker\pi' \supseteq I^{(2)}$ and $\ker\pi'' \supseteq I^{(3)}$.

There is an $(m-1)$-step $\sim$-walk in $\widehat{G_N}$ from π to $\rho := (Q_1, \ldots, Q_m, q_{m+1}0, \ldots, 0) \in SO(N)^\wedge$. To see this (as in the proof of Theorem 6.2), note that for $1 \leq i \leq m-1$,

$$(Q_1, \ldots, Q_i, q_i, q_{i+1}, \ldots, q_{k-1-i}, 0, \ldots, 0) \sim (Q_1, \ldots, Q_i, Q_{i+1}, q_{i+1}, \ldots, q_{k-2-i}, 0, \ldots, 0)$$

because the restrictions to $SO(N-1)$ contain $(Q_1, \ldots, Q_i, q_{i+1}, \ldots, q_{k-1-i}, 0, \ldots, 0)$, Similarly, there is an $(m-1)$-step $\sim$-walk from π' to $\rho' := (Q_1, \ldots, Q_m, q'_{m+1}, 0, \ldots, 0)$ and from π'' to $\rho'' := (Q_1, \ldots, Q_m, q''_{m+1}, 0, \ldots, 0)$.

The restrictions of ρ, ρ' and ρ'' to $SO(N-1)$ contain $\mu := (Q_1, \ldots, Q_m, 0, \ldots, 0)$ as a common subrepresentation and so if (t_n) is any null sequence in $(0, \infty)$ then $\pi_{t_n,\mu} \to \rho, \rho', \rho''$ in $\widehat{G_N}$ as $n \to \infty$. Hence $\ker\rho \cap \ker\rho' \cap \ker\rho''$ is a primal ideal of A.

Taking $S^{(1)} = \ker\pi$, $S^{(2)} = \ker\pi'$, $S^{(3)} \ker\pi''$ and $T^{(1)} = \ker\rho$, $T^{(2)} = \ker\rho'$, $T^{(3)} \ker\rho''$, we have satisfied the hypotheses of Theorem 5.3. Thus $K(M(A)) \leq (m-1) + 3/2 = N/4$. Combining this with Theorem 3.3(i), Corollary 4.7 and [35, Theorem 4.4], we have

$$K(M(A)) \leq \frac{N}{4} \leq \frac{\mathrm{Orc}(A)}{2} \leq \frac{\mathrm{Orc}(M(A))}{2} = K_s(M(A)) \leq K(M(A))$$

and hence equality throughout.

For the simpler case $N = 6$, it again suffices to check the hypothesis of Theorem 5.3 for the Glimm ideal $G = I_0$. With notation as above, we have that the restrictions to $SO(N-1)$ of $\pi = (Q_1, q_1, 0)$, $\pi' = (Q_1, q'_1, 0)$ and $\pi'' = (Q_1, q''_1, 0)$ contain $(Q_1, 0)$ as a common subrepresentation and so the ideal $\ker\pi \cap \ker\pi' \cap \ker\pi''$ is primal. Applying Theorem 5.3 in the case $n = 0$, we have

$$K(M(A)) \leq \frac{3}{2} \leq \frac{\mathrm{Orc}(A)}{2} \leq \frac{\mathrm{Orc}(M(A))}{2} = K_s(M(A)) \leq K(M(A))$$

and hence equality throughout. □

7. The constant $D(A)$ and the value of $\mathrm{Orc}(M(A))$.

If we apply the strategy of the previous section to the case where N is odd, we find that $K(M(C^*(G_N))) \in [\frac{N-1}{4}, \frac{N+1}{4}]$ rather than obtaining an exact value. This forces us to improve on the estimate for $\mathrm{Orc}(M(C^*(G_N)))$ that was given in Corollary 4.7.

In the first part of this section we obtain an upper bound for $\mathrm{Orc}(M(A))$, in terms of the ideal structure of A, which is applicable to a fairly general class of C^*-algebras A. In the second part, we obtain the precise value of $\mathrm{Orc}(M(A))$ for a smaller class of C^*-algebras which does, however, contain the group C^*-algebras of the motion groups.

Recall from Section 5 that, for a C*-algebra A, $\mathrm{Sub}(A)$ is the τ_s-closure of Min-Primal(A) in $\mathrm{Primal}'(A)$. We now define a graph structure on $\mathrm{Sub}(A)$. For $I, J \in \mathrm{Sub}(A)$ write $I * J$ if $I + J \neq A$. The relation $*$ defines a graph structure on $\mathrm{Sub}(A)$ analogous to the graph structure on $\mathrm{Prim}(A)$ defined by the relation $\sim$. Let $d^*(I, J)$ denote the distance between I and J in the graph $(\mathrm{Sub}(A), *)$. As before, we define the diameter of a $*$-component to be the supremum of the distances between points in the component, with the exception that this time we define the diameter of a singleton to be 0 (rather than 1 as in $(\mathrm{Prim}(A), \sim)$). Let $D(A)$ be the supremum of the diameters of $*$-connected components of $\mathrm{Sub}(A)$.

For example, let $A = C^*(SL(2, \mathbf{C}))$ (see [6, Example 5.1] for notation). Then, apart from loops, the only edge in $(\mathrm{Sub}(A), *)$ is $J * P_{2,0}$. Hence $D(A) = 1$.

It is natural to begin by investigating the case $D(A) = 0$. This turns out to give a description of quasi-standard C*-algebras.

Lemma 7.1. *Let A be a C^*-algebra. Then $D(A) = 0$ if and only if A is quasi-standard.*

Proof. Suppose that A is quasi-standard. By [8, Theorem 3.3 ((i) $\Rightarrow$ (v))], $\mathrm{Sub}(A) =$ Min-Primal(A) and every primitive ideal of A contains a unique minimal primal ideal. Hence $D(A) = 0$.

Conversely, suppose that $D(A) = 0$ and that $I \in \mathrm{Sub}(A)$. There exists a minimal primal ideal J of A such that $J \subseteq I$. Since $I * J$ and $D(A) = 0$, $I = J$. Hence Min-Primal(A) is τ_s-closed in $\mathrm{Primal}'(A)$. Furthermore each primitive ideal of A must contain a unique minimal primal ideal, for otherwise $D(A) \geq 1$. Hence A is quasi-standard by [8, Theorem 3.3 ((v) $\Rightarrow$ (i))]. □

The only motion group G_N for which the C*-algebra is quasi-standard is G_2, and this has been studied in [6, Example 4.1].

If $I * J$ then there exists $P \in \mathrm{Prim}(A)$ such that $P \supseteq I + J$, and if $P, Q \in \mathrm{Prim}(A)$ with $P \sim Q$ then there exists $I \in$ Min-Primal$(A) \subseteq \mathrm{Sub}(A)$ with $I \subseteq P \cap Q$. Hence if $I_0 * \ldots * I_k$ is a walk in $\mathrm{Sub}(A)$ of length $k \geq 2$ then there is a walk $P_1 \sim \cdots \sim P_k$ of length $k - 1$ in $\mathrm{Prim}(A)$, where $P_i \supseteq I_{i-1} + I_i$. Any strictly shorter walk between P_1 and P_k yields a strictly shorter walk between I_0 and I_k. In this way one sees that $D(A) \leq \mathrm{Orc}(A) + 1$, and a similar argument shows that $\mathrm{Orc}(A) \leq D(A) + 1$ (and also that $D(A)$ is infinite if and only if $\mathrm{Orc}(A)$ is infinite). Similarly, for $G \in \mathrm{Glimm}(A)$, $\{P \in \mathrm{Prim}(A) : P \supseteq G\}$ is a $\sim$-connected subset of $\mathrm{Prim}(A)$ if and only if $\{J \in \mathrm{Sub}(A) : J \supseteq G\}$ is a $*$-connected subset $\mathrm{Sub}(A)$.

Lemma 7.2. *Let A be a σ-unital C^*-algebra and let n be a positive integer. Suppose that for all $G \in \mathrm{Glimm}(A)$ and all $R, S \in \mathrm{Prim}(M(A)/H_G)$, $d_{M(A)}(R, S) \leq n$. Then $\mathrm{Orc}(M(A)) \leq n$.*

Proof. We may assume that $\mathrm{Orc}(M(A)) \geq 2$. Let H be a Glimm ideal of $M(A)$ and suppose that $R, S \in \mathrm{Prim}(M(A)/H)$ with $2 \leq d_{M(A)}(R, S) < \infty$. Set $d_{M(A)}(R, S) = k$. Then by [35, Lemma 2.4] or Lemma 4.4 (applied to $M(A)$), there is an admissible chain $X_1, \ldots, X_k$ of length k of closed subsets of $\mathrm{Prim}(M(A))$ such that $R \in X_1 \setminus X_2$ and $S \in X_k \setminus X_{k-1}$. Since

$X_1 \setminus X_2$ is an open subset of $\mathrm{Prim}(M(A))$, there exists $b \in M(A)$ such that $\|b\| = \|b_R\| = 1$ and $b + T = 0$ for all $T \in \mathrm{Prim}(M(A))$ with $T \notin X_1 \setminus X_2$. Similarly, there exists $c \in M(A)$ such that $\|c\| = \|c_S\| = 1$ and $c + T = 0$ for all $T \in \mathrm{Prim}(M(A))$ with $T \notin X_k \setminus X_{k-1}$.

Set $V = \{G \in \mathrm{Glimm}(A) : \|b+H_G\| \geq 1/2\}$ and $W = \{G \in \mathrm{Glimm}(A) : \|c+H_G\| \geq 1/2\}$. Using the canonical homeomorphism $\iota : \beta\,\mathrm{Glimm}(A) \to \mathrm{Glimm}(M(A))$ (see Section 2) and the upper semi-continuity of norm functions on $\mathrm{Glimm}(M(A))$, we obtain that V and W are closed subsets of $\mathrm{Glimm}(A)$. Furthermore, H lies in the closure of both $\iota(V)$ and $\iota(W)$. To see this, let (P_α) be a net in $\mathrm{Prim}(A)$ with $\tilde{P}_\alpha \to R$. Set $G_\alpha = \phi_A(P_\alpha)$. Then, by the continuity of $\phi_{M(A)}$,

$$H_{G_\alpha} = i(G_\alpha) = \phi_{M(A)}(\tilde{P}_\alpha) \to \phi_{M(A)}(R) = H.$$

Eventually $\|b+\tilde{P}_\alpha\| > 1/2$, by lower semi-continuity of norm functions on $\mathrm{Prim}(M(A))$, and hence eventually $\|b + H_{G_\alpha}\| \geq \|b + \tilde{P}_\alpha\| > 1/2$. Thus eventually $G_\alpha \in V$, so H lies in the closure of $\iota(V)$. Similarly H lies in the closure of $\iota(W)$.

Since A is σ-unital, $\mathrm{Glimm}(A)$ is normal (see Section 2) and so $V \cap W$ is non-empty by [6, Lemma 3.1]. Let $G \in V \cap W$. Then there exists $T \in \mathrm{Prim}(M(A)/H_G)$ such that $\|b+T\| > 0$, and hence such that $T \in X_1 \setminus X_2$. Similarly there exists $T' \in \mathrm{Prim}(M(A)/H_G)$ such that $T' \in X_k \setminus X_{k-1}$. But then $d_{M(A)}(T, T') \geq k$ by [35, Lemma 2.1]. Hence $k \leq n$, by hypothesis, so $\mathrm{Orc}(M(A)) \leq n$ as required. □

We are now ready for the first result linking $\mathrm{Orc}(M(A))$ and $D(A)$.

Theorem 7.3. *Let A be a σ-unital C^*-algebra such that X^1 is closed whenever X is a closed subset of* $\mathrm{Prim}(A)$*. If $D(A) \geq 1$ then*

$$\mathrm{Orc}(M(A)) \leq D(A) + 1 \leq \mathrm{Orc}(A) + 2.$$

Proof. Suppose that $D(A) \geq 1$. Since $D(A) \leq \mathrm{Orc}(A) + 1$, it suffices to show that $\mathrm{Orc}(M(A)) \leq D(A) + 1$. Without loss of generality we may assume that $D(A) < \infty$ and hence $\mathrm{Orc}(A) < \infty$. It follows from Proposition 4.3 that ϕ_A is a closed map and that every Glimm class in $\mathrm{Prim}(A)$ is $\sim$-connected.

Towards a contradiction, suppose that there exist $G \in \mathrm{Glimm}(A)$ and $Q, R \in \mathrm{Prim}(M(A)/H_G)$ such that $D(A) + 1 < d_{M(A)}(Q, R) \leq \infty$. Then there exists $k \in \mathbb{N}$ such that $D(A) + 1 < k \leq d_{M(A)}(Q, R)$. Since $D(A) \geq 1$, we have $k \geq 3$. By [35, Lemma 2.4] or Lemma 4.4 (applied to $M(A)$), there is a chain $X_1, \ldots, X_k$ on $\mathrm{Prim}(M(A))$ with $Q \in X_1 \setminus X_2$ and $R \in X_k \setminus X_{k-1}$. Set $W = X_1 \setminus X_2$ and let $V = \{P \in \mathrm{Prim}(A) : \tilde{P} \in W\}$. Then W is an open subset of $\mathrm{Prim}(M(A))$ containing Q and $\tilde{V} = \{\tilde{P} : P \in V\}$ is a dense open subset of W by the density of the open subset $\mathrm{Prim}(A)^\sim$ in $\mathrm{Prim}(M(A))$. We claim that $\overline{V}$ (the closure of V in $\mathrm{Prim}(A)$) meets $\mathrm{Prim}(A/G)$. To see this, let (Q_α) be a net in V such that $\tilde{Q}_\alpha \to Q$. Using the canonical homeomorphism $\iota : \beta\,\mathrm{Glimm}(A) \to \mathrm{Glimm}(M(A))$ and the continuity of $\phi_{M(A)}$, we have

$$\iota(\phi_A(Q_\alpha)) = \phi_{M(A)}(\tilde{Q}_\alpha) \to \phi_{M(A)}(Q) = H_G = \iota(G).$$

Hence $\phi_A(Q_\alpha) \to G$. Thus G belongs to the closure of $\phi_A(V)$ in $\mathrm{Glimm}(A)$. Since ϕ_A is a closed map this implies that there exists $T \in \overline{V}$ such that $\phi_A(T) = G$.

Now let (P_α) be a net in V such that $P_\alpha \to T$. For each α, let $I_\alpha \in \text{Min-Primal}(A)$ with $I_\alpha \subseteq P_\alpha$. By passing to a subnet, if necessary, we may assume that (I_α) is τ_s-convergent in $\mathrm{Id}(A)$, with limit I say. For $a \in I$,

$$0 = \|a + I\| = \lim \|a + I_\alpha\| \geq \liminf \|a + P_\alpha\| \geq \|a + T\|.$$

Thus $I \subseteq T$ and so $I \neq A$ and $I \in \mathrm{Sub}(A)$. Since each proper closed primal ideal of A contains a unique Glimm ideal [8, Lemma 2.2], and $T \supseteq I$ and $T \supseteq G$, it follows that $I \supseteq G$.

Set $X_i' = \{P \in \mathrm{Prim}(A) : \tilde{P} \in X_i\}$ $(1 \leq i \leq k)$. Then $X_1', \ldots, X_k'$ is a chain on $\mathrm{Prim}(A)$. We claim that $\mathrm{Prim}(A/I) \subseteq X_1'$. To see this, let $T' \in \mathrm{Prim}(A/I)$. Suppose that T' does not lie in the closed set X_1' and set $M = \ker X_1'$. Then $T' \not\supseteq M$ and so $I \not\supseteq M$. Since $I_\alpha \to I$ (τ_s), eventually $I_\alpha \not\supseteq M$ and so there exists $Q_\alpha \in \mathrm{Prim}(A/I_\alpha)$ such that $Q_\alpha \not\supseteq M$. Hence $Q_\alpha \notin X_1'$. On the other hand $P_\alpha \in \mathrm{Prim}(A/I_\alpha)$ and $P_\alpha \in V \subseteq X_1'$. Thus P_α and Q_α are in disjoint open subsets of $\mathrm{Prim}(A)$. This contradicts the primality of I_α, so T' must belong to X_1'.

In the same way, using $R \in X_k$, we may obtain $J \in \mathrm{Sub}(A)$ with $J \supseteq G$ such that $\mathrm{Prim}(A/J) \subseteq X_k'$. Since $\mathrm{Prim}(A/G)$ is a $\sim$-connected subset of $\mathrm{Prim}(A)$, I and J are in the same $*$-component of $\mathrm{Sub}(A)$. Let $I = I_0 * I_1 * \ldots * I_{n-1} * I_n = J$ be any walk in $\mathrm{Sub}(A)$ from I to J, of length n. Then $\mathrm{Prim}(A/I_1)$ meets X_1' and hence $\mathrm{Prim}(A/I_1) \subseteq X_1' \cup X_2'$, since $\mathrm{Prim}(A/I_1)$ is a limit set and $(X_1' \cup X_2') \setminus X_3'$ and $(X_3' \cup \ldots X_k') \setminus X_2'$ are disjoint open subsets of $\mathrm{Prim}(A)$. By induction, $\mathrm{Prim}(A/I_i) \subseteq X_1' \cup X_2' \cup \ldots \cup X_{i+1}'$ for $1 \leq i \leq k-1$. Thus the least i such that $\mathrm{Prim}(A/I_i)$ can meet X_k' is $i = k-2$ (recall that $X_{k-j}' \cap X_k'$ is empty for $2 \leq j \leq k-1$). But $\mathrm{Prim}(A/I_{n-1})$ does meet $\mathrm{Prim}(A/J)$ (and hence X_k') and thus $n-1 \geq k-2$. Hence $d^*(I,J) \geq k-1$ and so $D(A) \geq k-1 > D(A)$, a contradiction.

We have shown that for all $G \in \mathrm{Glimm}(A)$ and $R, S \in \mathrm{Prim}((M(A)/H_G)$, $d_{M(A)}(R,S) \leq D(A)+1$. It follows from Lemma 7.2 that $\mathrm{Orc}(M(A)) \leq D(A)+1$. ☐

If A is a σ-unital C^*-algebra with $D(A) = 0$ (i.e. A quasi-standard) then $\mathrm{Orc}(M(A)) \leq 2$ [6, Theorem 3.4]. Combining this with Theorem 7.3, we see that if A is a σ-unital C*-algebra such that X^1 is closed whenever X is a closed subset of $\mathrm{Prim}(A)$ then $\mathrm{Orc}(M(A)) \leq \mathrm{Orc}(A) + 2$.

Corollary 7.4. *Let A be a σ-unital C^*-algebra such that X^1 is closed whenever X is a closed subset of* $\mathrm{Prim}(A)$. *Then*

$$\mathrm{Orc}(A) \leq \mathrm{Orc}(M(A)) \leq \mathrm{Orc}(A) + 2.$$

Proof. By Theorem 4.6, $\mathrm{Orc}(A) \leq \mathrm{Orc}(M(A))$. The second inequality has been noted above. ☐

In the second part of this section, we shall consider C^*-algebras with the following properties:

(1) X^1 is closed whenever X is a closed subset of $\mathrm{Prim}(A)$;

(2) there is a dense subset $S \subseteq \mathrm{Prim}(A)$ such that: (a) each $P \in S$ is a Glimm ideal; (b) each $P \in S$ is a maximal ideal; (c) A/P is non-unital for all $P \in S$.

Property (1) has been discussed earlier, at the beginning of Section 4. Property (2) is a considerable restriction, but it is satisfied by the C^*-algebras of many locally compact groups including $SL(2,\mathbb{R})$, $SL(2,\mathbb{C})$, the motion groups G_N (see below) and all non-abelian, connected, simply connected nilpotent Lie groups.

If $\mathrm{Prim}(A)$ is a T_1-space, $\mathrm{Orc}(A)<\infty$, and A has property (1) then every Glimm class is $\sim$-connected by Proposition 4.3, so every separated point of $\mathrm{Prim}(A)$ automatically satisfies (a) and (b) of property (2). Hence if A is a separable unital C*-algebra with $\mathrm{Prim}(A)$ a T_1-space and $\mathrm{Orc}(A)<\infty$ then its stabilization $A\otimes\mathcal{K}$ satisfies properties (1) and (2) (where $\mathcal{K}$ is the C^*-algebra of compact linear operators on a separable Hilbert space of infinite dimension). Also, if A is a separable, liminal C^*-algebra such that $\mathrm{Orc}(A)<\infty$ and the set $S_1=\{P\in\mathrm{Prim}(A):A/P \quad \text{is non-unital}\}$ is dense in $\mathrm{Prim}(A)$, then the assumption of property (1) automatically leads to property (2). To see this note that S_1 is a G_δ subset of $\mathrm{Prim}(A)$ (see the proof of [17, Proposition 12]). On the other hand, the set S_2 of separated points of $\mathrm{Prim}(A)$ is also a dense G_δ subset [20, 3.9.4(b)]. As above, each P in S_2 satisfies (a) and (b). Since $\mathrm{Prim}(A)$ is a Baire space, the set $S=S_1\cap S_2$ has the required properties. Incidentally, the density of S_1 always forces $Z(A)=\{0\}$ and the converse is true when A is liminal [17, Proposition 12].

It follows from property (2) that every $P\in S$ is a separated point of $\mathrm{Prim}(A)$. If a net converges in $\mathrm{Prim}(A)$ to a separated point Q then every cluster point must contain Q and hence the net converges to Q in τ_s [23, Theorem 2.1]. It follows that S is τ_s-dense in the set of separated points of $\mathrm{Prim}(A)$. Thus, if A is also separable, every ideal in $\mathrm{Sub}(A)$ is the τ_s-limit of a sequence in S (cf. the discussion preceding Theorem 5.1). Now suppose that $I\in\mathrm{Sub}(A)$, that (P_n) is a sequence in S such that $P_n\to I$ (τ_s) and that a primitive ideal P lies in the closure of $\{P_n:n\geq 1\}$ but $P\neq P_n$ for all n. Let $a\in I$ and $\epsilon>0$. Since $P_n\to I$ (τ_s), there exists $N\in\mathbb{N}$ such that $\|a+P_n\|<\epsilon$ for all $n>N$. As P does not belong to the closed set $\{P_1,\dots,P_N\}$ it belongs to the closure of the set $\{P_n:n>N\}$. By lower semi-continuity of norm functions on $\mathrm{Prim}(A)$, $\|a+P\|\leq\epsilon$. Since ϵ was arbitrary, we obtain that $a\in P$ and hence that $P\supseteq I$. We shall use this fact several times in the proof of Theorem 7.6.

We now show that if $A=C^*(G_N)$ $(N\geq 2)$ then A satisfies both properties (1) and (2). Property (1) has been previously observed as a consequence of Lemma 4.1. The set $S=\{\ker\pi:\pi\in\mathcal{U}_N\}$ is a dense open subset of separated points of $\mathrm{Prim}(A)$ which are also Glimm ideals (see Section 3). For $P\in S$, A/P is *-isomorphic to the non-unital, simple C^*-algebra $\mathcal{K}$ and so S satisfies 2(b) and 2(c).

Next we need a technical lemma.

Lemma 7.5. *Let A be a separable C^*-algebra with* $\mathrm{Orc}(A)<\infty$ *and with properties (1) and (2) above. Let $G\in\mathrm{Glimm}(A)$ and let $I\in\mathrm{Sub}(A)$ with $G\subseteq I$. Set $X_G=\{P\in\mathrm{Prim}(A):P\supseteq G\}$. Let V be an open subset of $\mathrm{Prim}(A)\setminus X_G$ and suppose that there is a sequence $(P_n)_{n\geq1}$ in $S\cap V$ such that $P_n\to I$ (τ_s). Then there exists $Q\in\mathrm{Prim}(M(A))$ with $Q\supseteq H_G$ such that*

(i) *Q lies in the closure of $\{\tilde{P}_n:n\geq 1\}$;*
(ii) *if (Q_α) is any net in $\mathrm{Prim}(A)$ with $\tilde{Q}_\alpha\to Q$ then eventually $Q_\alpha\in V$;*
(iii) *$\{Q\}^1$ lies in the closure of $\tilde{V}=\{\tilde{P}:P\in V\}$ in $\mathrm{Prim}(M(A))$.*

Proof. Set $W=\mathrm{Prim}(A)\setminus V$ and $U=\mathrm{Glimm}(A)\setminus\phi_A(W)$. By Proposition 4.3, ϕ_A is closed and so U is an open subset of $\mathrm{Glimm}(A)$ and $G\notin U$. On the other hand, $P_n=\phi_A(P_n)\in U$ for each n since $\{P_n\}$ is a Glimm class. Also, for any $P\in\mathrm{Prim}(A/I)$, $P_n\to P$ in $\mathrm{Prim}(A)$

and so $P_n = \phi_A(P_n) \to \phi(P) = G$ in Glimm(A). Hence G lies in the boundary of the open set U. Since A is separable, it follows from [14, Lemma 3.9] that U is the cozero set of a continuous real-valued function f on Glimm(A). Replacing f by $|f|/(1+|f|)$, we may assume that $0 \le f \le 1$.

By [6, Theorem 2.2 (iii),(iv)], there exists an element $b \in M(A)$ with $0 \le b \le 1$ such that $(1-b) \in \tilde{K}$ for all $K \in \mathrm{Glimm}(A) \setminus U$ and $\|(1-b)+\tilde{K}\| = 1$ for all $K \in U$ such that A/K is non-unital. In particular, $\|(1-b)+\tilde{P}_n\| = 1$ for all $n \ge 1$. The set

$$N = \{R \in \mathrm{Prim}(M(A)) : \|(1-b)+R\| \ge 1\}$$

is compact and hence the sequence $(\tilde{P}_n)$ in N has a convergent subnet $(\tilde{P}_{n_\alpha})$ with a limit $Q \in N$. Since Glimm$(M(A))$ is Hausdorff,

$$\phi_{M(A)}(Q) = \lim \phi_{M(A)}(\tilde{P}_{n_\alpha}) = \lim \iota(\phi_A(P_{n_\alpha})) = \iota(G) = H_G$$

and so $Q \supseteq H_G$.

Set $Y = \{R \in \mathrm{Prim}(M(A)) : \|(1-b)+R\| > 0\}$. Then Y is a neighbourhood of Q in Prim$(M(A))$. If $P \in \mathrm{Prim}(A)$ and $K := \phi_A(P) \notin U$ then $1-b \in \tilde{K} \subseteq \tilde{P}$ and so $\tilde{P} \notin Y$. For (ii), suppose that $\tilde{Q}_\alpha \to Q$. Then eventually $\tilde{Q}_\alpha \in Y$ and so $Q_\alpha \in \phi_A^{-1}(U) \subseteq V$.

Finally, suppose that $T \in \mathrm{Prim}(M(A))$ with $T \sim Q$. By the density of $\{\tilde{P} : P \in \mathrm{Prim}(A)\}$ in Prim$(M(A))$ there exists a net (Q_α) in Prim(A) such that $\tilde{Q}_\alpha \to Q, T$. By (ii), eventually $Q_\alpha \in V$. Hence T lies in the closure of $\tilde{V}$ in Prim$(M(A))$. □

We come now to the main theorem of this section. In the course of this, we shall need the fact that if A is a C^*-algebra and G is a σ-unital Glimm ideal of A then Glimm$(A) \setminus \{G\}$ is a normal subspace of Glimm(A). To see this, first note that the complete regularity of the Hausdorff space Glimm(A) passes to any subspace. Secondly, Glimm$(A) \setminus \{G\}$ is the image under ϕ_A of the set $\{P \in \mathrm{Prim}(A) : P \not\supseteq G\}$ which is homeomorphic to Prim(G) and therefore σ-compact. Thus Glimm$(A) \setminus \{G\}$ is a σ-compact, completely regular Hausdorff space and is therefore normal (see [25, 3D] or [33, Ch.2, Proposition 1.6]). If A is separable, this applies to any $G \in \mathrm{Glimm}(A)$. Incidentally, it is always the case that Glimm$(A) \setminus \{G\}$ is canonically homeomorphic to Glimm(G) but we have avoided the need to prove that here.

Theorem 7.6. *Let A be separable C^*-algebra having properties (1) and (2) above, and with $D(A) \ge 1$. Then* Orc$(M(A)) = D(A) + 1$.

Proof. By Theorem 7.3, Orc$(M(A)) \le D(A)+1$. Thus it suffices to show that Orc$(M(A)) \ge D(A)+1$ and for this we may assume that Orc$(M(A)) < \infty$. Then it follows from Theorem 4.6 that Orc$(A) < \infty$ (and so $D(A) < \infty$) and from Proposition 4.3 that ϕ_A is a closed map.

We begin by considering the case $D(A) = 1$. Suppose that Orc$(M(A)) = 1$. By [6, Corollary 2.4] and property (2), Prim(A) is discrete and so $D(A) = 0$, a contradiction. Thus Orc$(M(A)) \ge 2$, as required.

Now suppose that $D(A) = 2$. Let $I, J \in \mathrm{Sub}(A)$ with $d^*(I,J) = 2$. Let G be the unique Glimm ideal such that $I \cap J \supseteq G$ and set $X_G = \mathrm{Prim}(A/G)$. Set $X = \mathrm{Prim}(A/I)$ and $Y = \mathrm{Prim}(A/J)$. Then X and Y are disjoint closed subsets of X_G. Since X_G is not a singleton, $G \notin S$ and $X_G \cap S = \emptyset$. Let (P_n) and (Q_n) be sequences in S with $P_n \to I$ (τ_s) and $Q_n \to J$ (τ_s). Since $I \neq J$ and τ_s is Hausdorff, we may assume that the sets

$X' = \{P_n : n \geq 1\}$ and $Y' = \{Q_n : n \geq 1\}$) are disjoint. By an observation preceding Lemma 7.5, $X' \cup X$ and $Y' \cup Y$ are closed subsets of $\mathrm{Prim}(A)$. Since ϕ_A is closed, $\phi_A(X')$ and $\phi_A(Y')$ are disjoint closed subsets of the normal space $\mathrm{Glimm}(A) \setminus \{G\}$. By two applications of normality, there exist disjoint open subsets U' and V' of $\mathrm{Glimm}(A) \setminus \{G\}$ such that $X' = \phi_A(X') \subseteq U'$ and $Y' = \phi_A(Y') \subseteq V'$ and such that the closures of U' and V' in $\mathrm{Glimm}(A) \setminus \{G\}$ are disjoint. Set $U'' = \phi_A^{-1}(U')$ and $V'' = \phi_A^{-1}(V')$.

If $P, Q \in \mathrm{Prim}(A) \setminus X$ and $P \sim Q$ relative to the space $\mathrm{Prim}(A) \setminus X$ then $P \sim Q$ in $\mathrm{Prim}(A)$ and hence $P \approx Q$ in $\mathrm{Prim}(A)$. Hence X' and $X_G \setminus X$ are relatively closed and $\sim$-saturated subsets of the space $\mathrm{Prim}(A) \setminus X$ which is homeomorphic to $\mathrm{Prim}(I)$. Since I is a separable C*-algebra, $\mathrm{Prim}(I)$ is σ-compact and so it follows from Lemma 4.2 that there exist disjoint open sets U''' and W in $\mathrm{Prim}(A) \setminus X$ such that $X' \subseteq U'''$ and $X_G \setminus X \subseteq W$. Similarly there exist disjoint open sets V''' and W' in $\mathrm{Prim}(A) \setminus Y$ such that $Y' \subseteq V'''$ and $X_G \setminus Y \subseteq W'$. Set $U = U'' \cap U'''$ and $V = V'' \cap V'''$. Then U and V are open subsets of $\mathrm{Prim}(A)$ with $P_n \in U$, and $Q_n \in V$ for all n. Since $U \subseteq \phi_A^{-1}(U') \subseteq \phi_A^{-1}(\mathrm{Glimm}(A) \setminus \{G\})$, we have $U \cap X_G = \emptyset$ and similarly $V \cap X_G = \emptyset$. It follows from Lemma 7.5 that there exist $R, T \in \mathrm{Prim}(M(A))$ with $R, T \supseteq H_G$ such that $\{R\}^1$ lies in the closure of $\tilde{U}$ in $\mathrm{Prim}(M(A))$ and $\{T\}^1$ lies in the closure of $\tilde{V}$.

Suppose that $P \in \overline{U} \cap \overline{V}$ (where the closures are taken in $\mathrm{Prim}(A)$). Then $\phi_A(P) \in \overline{U'} \cap \overline{V'} = \{G\}$ (where the closures are taken in $\mathrm{Glimm}(A)$) and so $P \supseteq G$. On the other hand, by the properties of U''' and W, $P \notin X_G \setminus X$ and similarly $P \notin X_G \setminus Y$. Since X and Y are disjoint subsets of X_G, we have $P \notin X_G$, a contradiction. Thus $\overline{U} \cap \overline{V} = \emptyset$ and hence the closures of $\tilde{U}$ and $\tilde{V}$ in $\mathrm{Prim}(M(A))$ are disjoint by Lemma 4.5. Hence $\{R\}^1$ and $\{T\}^1$ are disjoint and thus $d_{M(A)}(R, T) \geq 3$. Since $\mathrm{Orc}(M(A)) < \infty$, Glimm classes are $\sim$-connected in $\mathrm{Prim}(M(A))$ [35, Corollary 2.7], and hence $\mathrm{Orc}(M(A)) \geq 3$ as required in this case.

Finally suppose that $D(A) = k \geq 3$. Let $I, J \in \mathrm{Sub}(A)$ with and $d^*(I, J) = k$ and let G be the unique Glimm ideal such that $I \cap J \supseteq G$. Set $X = \mathrm{Prim}(A/I)$ and $Y = \mathrm{Prim}(A/J)$. Then X and Y are disjoint closed subsets of $X_G := \mathrm{Prim}(A/G)$, $d_A(X, Y) = k - 1$, $G \notin S$ and $X_G \cap S = \emptyset$. The sets X and Y are $\sim$-connected since I and J are primal, and hence $X \cup Y$ is $\sim$-connected since $d_A(X, Y) < \infty$. By Lemma 4.4, there is an admissible chain $X_1, \ldots, X_{k-1}$ of closed subsets of $\mathrm{Prim}(A)$ with $X \subseteq X_1 \setminus X_2$ and $Y \subseteq X_{k-1} \setminus X_{k-2}$. For $1 \leq i \leq k-1$, let Z_i be the closure of $\tilde{X}_i$ in $\mathrm{Prim}(M(A))$. Then as in the proof of Theorem 4.6 we have that $Z_1, \ldots, Z_{k-1}$ is a chain on $\mathrm{Prim}(M(A))$ of length $k - 1$.

Let (P_n) be a sequence in the dense subset S with $P_n \to I$ (τ_s). Since $X_1 \setminus X_2$ is an open set containing $\mathrm{Prim}(A/I)$ we may assume that $P_n \in X_1 \setminus X_2$ for all n. Set $X' = \{P_n : n \geq 1\}$. Then X' and X_2 are disjoint. As before, X' is a relatively closed and $\sim$-saturated subset of the space $\mathrm{Prim}(A) \setminus X$ which is homeomorphic to the σ-compact space $\mathrm{Prim}(I)$. Let $E = ((X_1 \cap X_G) \setminus X) \cup X_2 \cup \ldots \cup X_{k-1}$, a relatively closed subset of $\mathrm{Prim}(A) \setminus X$ disjoint from X'. So $E = F \cap (\mathrm{Prim}(A) \setminus X))$ for some closed subset F of $\mathrm{Prim}(A)$ disjoint from X'. Since X' is a $\sim$-saturated subset of $\mathrm{Prim}(A)$, F^1 is disjoint from X' and is closed in $\mathrm{Prim}(A)$ by property (1). Now suppose that $P \in \overline{E^1}$ (where both operations are taken relative to the space $\mathrm{Prim}(A) \setminus X$). Then, in $\mathrm{Prim}(A)$, $P \in \overline{F^1} = F^1$ and so $P \notin X'$. Hence by Lemma 4.2 there exist disjoint open subsets U and V of $\mathrm{Prim}(A) \setminus X$ such that $X' \subseteq U$ and $E = ((X_1 \cap X_G) \setminus X) \cup X_2 \cup \ldots \cup X_{k-1} \subseteq V$.

By Lemma 7.5 there exists $R \in \mathrm{Prim}(M(A))$ such that $R \supseteq H_G$ and $\{R\}^1$ is contained in the closure of $\tilde{U}$ in $\mathrm{Prim}(M(A))$. But $\overline{U} \subseteq \mathrm{Prim}(A) \setminus V$ (where $\overline{U}$ denotes the closure of U

in $\mathrm{Prim}(A)$), so $\overline{U} \cap (X_2 \cup \ldots \cup X_{k-1})$ is empty. Hence, by Lemma 4.5, $\{R\}^1$ does not meet $Z_2 \cup \ldots \cup Z_{k-1}$. Similarly there exists $T \in \mathrm{Prim}(M(A))$ such that $T \supseteq H_G$ and $\{T\}^1$ does not meet $Z_1 \cup \ldots \cup Z_{k-2}$. Since $Z_1, \ldots, Z_{k-1}$ is a chain on $\mathrm{Prim}(M(A))$, $\{R\}^1 \subseteq Z_1 \setminus Z_2$. But $Z_1 \setminus Z_2$ and $\mathrm{Prim}(M(A)) \setminus Z_1$ are disjoint open subsets of $\mathrm{Prim}(M(A))$ and so $\{R\}^2 \subseteq Z_1$. Proceeding inductively, we obtain that $\{R\}^i \subseteq Z_1 \cup \ldots \cup Z_{i-1}$ for $2 \le i \le k-1$. Thus $\{R\}^{k-1} \cap \{T\}^1 = \emptyset$ and so $d_{M(A)}(R,T) \ge k+1$. Since $\mathrm{Orc}(M(A) < \infty$, $\mathrm{Orc}(M(A)) \ge k+1$ as required. □

8. The cases $N \equiv 1$ and $N \equiv 3 \mod 4$.

In this section, we finally complete the computation of $K(M(C^*(G_N)))$. The next result will be used in Theorems 8.3 and 8.2 to show that if $A = C^*(G_N)$ and N is odd then

$$K(M(A)) = K_s(M(A)) = \frac{1}{2}\mathrm{Orc}(M(A)) = \frac{N+1}{4}.$$

Proposition 8.1. *Let $A = C^*(G_N)$ with N odd. Then $D(A) \ge \frac{N-1}{2}$.*

Proof. Let $N = 2k+1$ and suppose first of all that $k > 1$. Since $SO(N)^\wedge$ is a $\sim$-connected subset of $\widehat{G_N}$ (Theorem 3.3), the ideals $I_{0,(0,\ldots,0)}$ and $I_{0,(1,\ldots,1)}$ belong to the same $*$-component of $\mathrm{Sub}(A)$ (in fact, it is easy to exhibit a $*$-walk between them via the ideals $I_{0,(1,\ldots,1,0\ldots,0)}$).

Suppose that $0 \le i \le k-2$ and that $\sigma = (p_1, \ldots, p_k)$ and $\sigma' = (p'_1, \ldots, p'_k)$ are elements of $SO(N-1)^\wedge$ such that $I_{0,\sigma} * I_{0,\sigma'}$ and $p_j = 0$ for $i < j \le k$. Then there exists $\pi \in SO(N)^\wedge$ such that $\pi|_{SO(N-1)} \ge \sigma$ and $\pi|_{SO(N-1)} \ge \sigma'$. So there exist integers $m_1 \ge m_2 \ge \ldots \ge m_k \ge 0$ such that

$$m_1 \ge p_1 \ge m_2 \ge \ldots \ge p_i \ge m_{i+1} \ge 0 \ge m_{i+2}$$

and

$$m_1 \ge p'_1 \ge \ldots \ge m_{i+2} \ge p'_{i+2} \ge \ldots \ge m_k \ge p'_k \ge -m_k.$$

Thus $m_{i+2} = \ldots = m_k = 0$ and so $p'_{i+2} = \ldots = p'_k = 0$. It follows that

$$d^*(I_{0,(0,\ldots,0)}, I_{0,(1,\ldots,1)}) \ge k$$

and so $D(A) \ge k$ as required.

In the case $k = 1$, consideration of the inequalities $m_1 \ge 1 \ge -m_1$ and $m_1 \ge 0 \ge -m_1$ shows that the hull of $I_{0,(1)}$ is strictly contained in the hull of $I_{0,(0)}$. Thus $I_{0,(1)}$ strictly contains $I_{0,(0)}$ and so $D(A) \ne 0$. □

Our final two results are similar in nature to Theorems 6.2 and 6.3 but the proofs differ from these in the following respects. Firstly, since N is now odd, the checking of the hypotheses of Theorems 5.1 and 5.3 is somewhat different and so we give the details. Secondly, we use Proposition 8.1 and Theorem 7.6 in place of Corollary 4.7.

Theorem 8.2. *Let $A = C^*(G_N)$ where $N \equiv 3 \pmod 4$. Then*

$$K(M(A)) = K_s(M(A)) = \frac{1}{2}\mathrm{Orc}(M(A)) = \frac{N+1}{4}.$$

Proof. Let $k = \frac{N-1}{2}$ and $m = \frac{N-3}{4}$ so that $N = 2k+1$ and $k = 2m+1$. We begin with the case $m \ne 0$ ($N \ge 7$). We aim to show that A satisfies the hypotheses of Theorem 5.1 with $n = m$. Let $I^{(1)}, I^{(2)}, I^{(3)} \in \mathrm{Sub}(A)$ with $I^{(1)}, I^{(2)}, I^{(3)}$ all containing the same Glimm ideal

G. If G is one of the separated points of $\mathrm{Prim}(A)$ then $I^{(1)} = I^{(2)} = I^{(3)} = G$ and we may take $S^{(1)} = S^{(2)} = S^{(3)} = T = G$.

We may suppose therefore that $G = I_0$ (see Theorem 3.3) and that $I^{(1)} = I_{0,\sigma}$, $I^{(2)} = I_{0,\sigma'}$, and $I^{(3)} = I_{0,\sigma''}$ for some $\sigma, \sigma', \sigma'' \in SO(N-1)^\wedge$ by Proposition 6.1. Let $\sigma = (p_1, \ldots, p_k)$, $\sigma' = (p'_1, \ldots, p'_k)$, and $\sigma'' = (p''_1, \ldots, p''_k)$. Set $P_i = \max\{p_i, p'_i, p''_i\}$ for $1 \le i \le m+1$. Define $\pi = (P_1, p_1, p_2, \ldots, p_{k-1}) \in SO(N)^\wedge$ and similarly π' and π'' (replacing the p_j by p'_j and p''_j respectively). Since $\pi|_{SO(N-1)} \ge \sigma$, $\ker \pi \supseteq I^1$. Similarly, $\ker \pi' \supseteq I^2$ and $\ker \pi'' \supseteq I^3$.

There is an m-step $\sim$-walk in $\widehat{G_N}$ from each of π, π', π'' to $\rho := (P_1, P_2, \ldots, P_{m+1}, 0, \ldots, 0) \in SO(N)^\wedge$. To see this, note that for $1 \le i \le m-1$,

$$(P_1, \ldots, P_i, p_i, p_{i+1}, \ldots, p_{k-i}, 0, \ldots, 0) \sim (P_1, \ldots, P_i, P_{i+1}, p_{i+1}, \ldots, p_{k-1-i}, 0, \ldots, 0)$$

because the restrictions to $SO(N-1)$ contain $(P_1, \ldots, P_i, p_{i+1}, \ldots, p_{k-i}, 0, \ldots, 0)$, and finally

$$(P_1, \ldots, P_m, p_m, p_{m+1}, 0, \ldots, 0) \sim (P_1, \ldots, P_{m+1}, 0, \ldots, 0) = \rho$$

because the restrictions to $SO(N-1)$ contain $(P_1, \ldots, P_m, p_{m+1}, 0, \ldots, 0)$. Similar arguments apply to π' and π'', replacing the p_j by p'_j and p''_j (respectively).

Taking $S^{(1)} = \ker \pi$, $S^{(2)} = \ker \pi'$, $S^{(3)} = \ker \pi''$ and $T = \ker \rho$, we have satisfied the hypotheses of Theorem 5.1. Thus $K(M(A)) \le m + 1 = \frac{N+1}{4}$. Combining this with Proposition 8.1, Theorem 7.6 and [35, Theorem 4.4], we have

$$K(M(A)) \le \frac{N+1}{4} \le \frac{D(A)+1}{2} = \frac{\mathrm{Orc}(M(A))}{2} = K_s(M(A)) \le K(M(A))$$

and hence equality throughout.

For the simpler case $N = 3$, it suffices to check the hypothesis of Theorem 5.1 for $G = I_0$ (the only non-maximal Glimm ideal of A). So let $I^{(1)} = I_{0,(q)}$, $I^{(2)} = I_{0,(q')}$ and $I^{(3)} = I_{0,(q'')}$ where $(q), (q'), (q'') \in SO(2)^\wedge$. Let $Q = \max\{|q|, |q'|, |q''|\}$ and let $\pi = (Q) \in SO(3)^\wedge$. Then $\pi|_{SO(2)}$ contains (q),(q') and (q'') and so $\ker \pi \supseteq I^{(1)} + I^{(2)} + I^{(3)}$. Using Theorem 5.1 and arguing as above, we have

$$K(M(A)) \le 1 \le \frac{D(A)+1}{2} = \frac{\mathrm{Orc}(M(A))}{2} = K_s(M(A)) \le K(M(A))$$

and hence equality throughout. □

Theorem 8.3. *Let $A = C^*(G_N)$ where $N \equiv 1 \pmod 4$ and $N \ge 5$. Then*

$$K(M(A)) = K_s(M(A)) = \frac{1}{2}\mathrm{Orc}(M(A)) = \frac{N+1}{4}.$$

Proof. Let $k = \frac{N-1}{2}$ and $m = \frac{N-1}{4}$, so that $N = 2k+1$ and $k = 2m$. We begin with the case $N \ge 9$ so that $m > 1$. We aim to show that A satisfies the hypotheses of Theorem 5.3 with $n = m - 1$. Let $I^{(1)}, I^{(2)}, I^{(3)} \in \mathrm{Sub}(A)$ with $I^{(1)}, I^{(2)}, I^{(3)}$ all containing the same Glimm ideal G. If G is one of the separated points of $\mathrm{Prim}(A)$ then $I^{(1)} = I^{(2)} = I^{(3)} = G$ and we may take $S^{(i)} = T^{(i)} = G$ $(1 \le i \le 3)$ so that $T^{(1)} \cap T^{(2)} \cap T^{(3)} = G$ which is primal.

We may suppose therefore that $G = I_0$ (see Theorem 3.3) and that $I^{(1)} = I_{0,\sigma}$, $I^{(2)} = I_{0,\sigma'}$, and $I^{(3)} = I_{0,\sigma''}$ for some $\sigma, \sigma', \sigma'' \in SO(N-1)^\wedge$ by Proposition 6.1. Let $\sigma = (p_1, \ldots, p_k)$, $\sigma' = (p'_1, \ldots, p'_k)$, and $\sigma'' = (p''_1, \ldots, p''_k)$. Set $P_i = \max\{p_i, p'_i, p''_i\}$ for $1 \le i \le m$. Define $\pi = (P_1, p_1, p_2, \ldots, p_{k-1}) \in SO(N)^\wedge$ and similarly π' and π'' (replacing the p_j by p'_j and p''_j respectively). Since $\pi|_{SO(N-1)} \ge \sigma$, $\ker \pi \supseteq I^{(1)}$. Similarly, $\ker \pi' \supseteq I^{(2)}$ and $\ker \pi'' \supseteq I^{(3)}$.

There is an $(m-1)$-step $\sim$-walk in $\widehat{G_N}$ from π to $\rho := (P_1, \ldots, P_m, p_{m+1}, 0, \ldots, 0) \in SO(N)^\wedge$. To see this, note that for $1 \le i \le m-1$,

$$(P_1, \ldots, P_i, p_i, \ldots, p_{k-i}, 0, \ldots, 0) \sim (P_1, \ldots, P_i, P_{i+1}, p_{i+1}, \ldots, p_{k-1-i}, 0, \ldots, 0)$$

because the restrictions to $SO(N-1)$ contain $(P_1, \ldots, P_i, p_{i+1}, \ldots, p_{k-i}, 0, \ldots, 0)$. Similarly, there is an $(m-1)$-step $\sim$-walk from π' to $\rho' := (P_1, \ldots, P_m, p'_{m+1}, 0, \ldots, 0)$ and from π'' to $\rho'' := (P_1, \ldots, P_m, p''_{m+1}, 0, \ldots, 0)$.

The restrictions of ρ, ρ' and ρ'' to $SO(N-1)$ contain $(P_1, \ldots, P_m, 0, \ldots, 0)$ as a common subrepresentation and so $\ker\rho \cap \ker\rho' \cap \ker\rho''$ is a primal ideal of A, as in the proof of Theorem 6.3.

Taking $S^{(1)} = \ker\pi$, $S^{(2)} = \ker\pi'$, $S^{(3)} = \ker\pi''$ and $T^{(1)} = \ker\rho$, $T^{(2)} = \ker\rho'$, $T^{(3)} = \ker\rho''$, we have satisfied the hypotheses of Theorem 5.3. Thus $K(M(A)) \le (m-1) + 3/2 = \frac{N+1}{4}$. Combining this with Proposition 8.1, Theorem 7.6 (see the remark preceding Lemma 7.5) and [35, Theorem 4.4], we have

$$K(M(A)) \le \frac{N+1}{4} \le \frac{D(A)+1}{2} = \frac{\mathrm{Orc}(M(A))}{2} = K_s(M(A)) \le K(M(A))$$

and hence equality throughout.

For the simpler case $N = 5$, it again suffices to check the hypothesis of Theorem 5.3 for the Glimm ideal $G = I_0$. With notation as above, we have that the restrictions to $SO(4)$ of $\pi = (P_1, p_1,)$, $\pi' = (P_1, p'_1,)$ and $\pi'' = (P_1, p''_1,)$ contain $(P_1, 0)$ as a common subrepresentation and so the ideal $\ker\pi \cap \ker\pi' \cap \ker\pi''$ is primal. Applying Theorem 5.3 in the case $n = 0$, we have

$$K(M(A)) \le \frac{3}{2} \le \frac{D(A)+1}{2} = \frac{\mathrm{Orc}(M(A))}{2} = K_s(M(A)) \le K(M(A))$$

and hence equality throughout. □

In the case when N is odd, it can be shown by direct arguments that $D(A) \le \frac{N-1}{2}$ (so that $D(A) = \frac{N-1}{2}$). However, this inequality can also be obtained indirectly from Theorem 7.6 and the fact that $\mathrm{Orc}(M(A)) = \frac{N+1}{2}$ (Theorems 8.3 and 8.2). In the case when N is even, direct arguments show that $D(A) = \frac{N}{2} - 1$. It follows that Theorems 6.2 and 6.3 could be proved by using Theorem 7.6 rather than the more elementary Corollary 4.7. However, since Theorem 7.6 uses Theorem 4.6, no saving would be gained by adopting this more complicated approach.

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