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The Dynamic Model of Plane Mechanism with Variable Ratio

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The paper contains the proposal how to reduce many–elements plane mechanism with one degree of freedom to chosen axis or line as one–element model of mechanism. Mostly the place of reduction is driving element in rotary motion (for example, shaft of electric motor) or element in linear motion (for example, piston rod of hydraulic cylinder). The way of determining reduced load and reduced mass of the model is described. Presented mathematical description let determine: firstly, required driving torque or force to provide the suitable acceleration when loads of element are known and secondly, the acceleration (angular or linear) of driving element as result of known driving torque or force and loads of element.

Keywords: plane mechanism, dynamic analysis, variable ratio.

1. Introduction

The knowledge about mechanisms is still developing since many years before the Middle Ages. The necessity of wider range dynamic analysis of mechanisms grew up after steam engine invent.

Among the different kinds of mechanisms the special group represent mechanisms with variable ratio. The structures and dynamics of mechanisms with variable ratio have been analyzed by many researchers.

In work [1] an analytical procedure for synthesizing crank–rocker mechanisms with optimum transmission angle over the working stroke is presented. In paper [2] optimum transmission angle for given values of the slider stroke and corresponding crank rotation is presented. In this study complex algebra is used to solve this classical problem

Work [3] describes literature on transmission angle in a planar 4-, 5-, 6- and

7-bar linkages and spatial linkages and shows a survey of synthesis of mechanism with transmission angle.

In paper [4] the dynamic equations of a slider–crank mechanism driven by a servomotor are derived by using Hamilton's principle, Lagrange multiplier, geometric constraints and partitioning method. The dynamic responses between the experimental results and numerical simulation are compared. In this paper, a new identification method based on the real–coded genetic algorithm (RGA) is presented to identify the parameters of mechanism.

A chain continuously variable transmission that offers a continuum of gear ratios between desired limits is described in paper [5]. Dynamic performance and torque capacity relying significantly on the friction characteristic of the contact patch between the chain and the pulley are taken into consideration. Two different mathematical models of friction, the computational scheme, and the results corresponding to different loading scenarios are discussed to define the influence of friction characteristics on the nonlinear dynamics and torque transmitting capacity of a chain CVT drive.

Paper [6] describes the analysis of stability of periodical elastic motion of a flexible four bar crank rocker mechanism using the first approximation of Liapunov's stability theorem and Floquet theory. A procedure for predicting the stability is presented. Carried out experimental investigation on the stability confirm the theoretical researches.

Paper [7] shows analysis of the infinitely variable transmissions (IVT). Experimental tests let measure input and output power (a circulating one as well). The IVT efficiency curves, in relation to the torque and the transmission ratio variation, are presented.

Work [8] presents the kinematic and dynamic analysis of a very interesting modified slider–crank mechanism which has an additional eccentric link between connecting rod and crank pin, as distinct from a conventional mechanisms. The modified mechanism has a bigger output torque than that of the conventional mechanism. In work [9] the transmission angle of a compliant slider–crank mechanism is introduced.

Paper [10] describes efficiency of infinitely variable transmission (IVT) where the transmission ratio may achieve zero and compares the efficiency of possible IVT configurations consisting of a conventional CVT (continuously variable transmission) coupled to a planetary gear train and a fixed ratio mechanism.

Paper [11] investigates kinematic and dynamic analyses of a novel intermittent slider–crank mechanism, which consists of four parts: a crank, a connecting rod associated with a pneumatic cylinder, a slider and two stops at both ends of a stroke

In paper [12] authors are interested in the study of the dynamic behavior of a planar flexible slider–crank mechanism with clearance. Simulation and experimental tests are carried out.

The historical review of the gears with variable transmission ratio is presented in paper [13].

In work [14] a triple pendulum with damping, external forcing and impact is investigated. Some numerical examples for three coupled identical rods with horizontal barrier are shown.

The literature review shows that the slider crank mechanisms are well recognized. All mentioned mechanisms are investigated by using relatively complicated numerical methods.

However, there are another plane mechanisms with v ariable ratio and design engineers have to solve calculate problems connected with them especially, when mass forces and torques are taken into consideration. As examples jib mechanism of harbor jib crane (Fig. 1) and jib mechanism of truck mounted crane (Fig. 2) are chosen.

Figure 1 Harbor jib crane

Figure 2 Track mounted crane

For analyzing mechanism, particularly for selection of driving engine there is comfortably to reduce many–element plane mechanism with one degree of freedom to one–element mechanism model (most often to engine driving shaft or to moving element of linear motor).

The following assumptions are made:

- elements of mechanisms are rigid (not deformable),
- elements connections are realized as slidable or rotational kinematic pairs,
- considered mechanisms are simple, it means there is only one place (point or element) of power delivery and only one place (point or element) of power receiving,
- all elements of mechanisms move only in one plane; in progressive motion, or rotation motion, or plane motion,
- the mechanism has only one degree of freedom, it means: the position, velocity and acceleration of one element determines the position, velocity and acceleration of each other element.

In the paper the method of creating the one-element dynamic model of mechanism reduced to chosen element (mostly driving one) is described. The relationships between velocities and accelerations (linear and angular) of different elements are shown. The way of reducing loads and masses of any element to chosen driven element is explained.

2. Mechanism in plane motion with rotary driving element

2.1. *Example of mechanism*

In Fig. 3 the four–jointed (S, A, B, D) mechanism is showed. The driving crank 1 rotates round stationary point S and drives other elements of mechanism; element 2 in plane motion and element 3 rotating round stationary point D. The element 2 has center of gravity in point C, temporary center of rotation – in point C^I. There is taken assumption that only element 2 has mass m_C and moment of inertia I_C in respect of center of gravity C.

The element 2 is loaded by force $\bar{\mathbf{F}}_C$ applied to its gravity center C and by torque T_C. The driving torque T_S is applied to crank 1.

Other designations in Fig. 1 are as follows:

ω_S – angular velocity of crank 1,

ε_S – angular acceleration of crank 1,

$\bar{\mathbf{v}}_C$ – vector of element 2 gravity center C velocity,

a_C^v – point C acceleration tangent component (in direction of velocity vector $\bar{\mathbf{v}}_C$),

$\bar{\mathbf{F}}_C^v$ – vector of force $\bar{\mathbf{F}}_C$ projection on direction of velocity vector $\bar{\mathbf{v}}_C$,

ω_C – angular velocity of element 2,

ε_C – angular acceleration of element 2.

In analysis of motion of mechanism showed in Fig. 3 two cases are considered.

1. The angular acceleration ε_S is known and required driving torque T_S is the result.

2. The driving torque T_S is known and angular acceleration ε_S is the result.

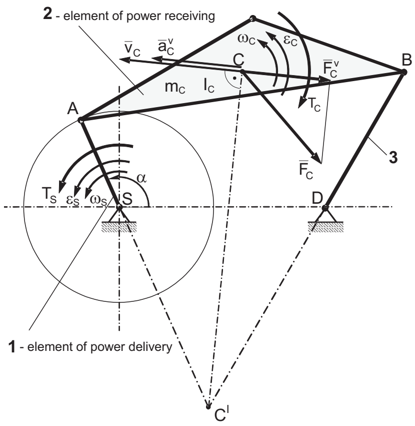

Figure 3 Four-jointed (S, A, B, D) mechanism with driving crank 1 and driven element 2 in plane motion

2.2. *Velocity and acceleration of element 2*

There is interdependence between velocities and accelerations of elements 1 and 2. The connections between velocities are given by formulas (1) and (2).

$$\frac{|\bar{\mathbf{v}}_C|}{\omega_S} = \frac{v_C}{\omega_S} = R_z \tag{1}$$

R_z – effective radius of mechanism for elements 1 and 2,
$|\bar{\mathbf{v}}_C| = v_C$ – value of point C velocity vector.

$$\frac{\omega_S}{\omega_C} = i_m \tag{2}$$

i_m – ratio of mechanism between elements 1 and 2.

The effective radius R_z and ratio i_m are v ariable but always can be determined dependently on position of mechanism given by angle α of crank 1 turn.

For further consideration the v alue of point C acceleration tangent component a_C^v in direction of velocity vector $\bar{\mathbf{v}}_C$ is important. Other components of point C acceleration don't have influence on power balance of mechanism.

The component a_C^v is determined as follows:

$$\begin{aligned}
&a_C^v = \frac{d}{dt}\left|\bar{\mathbf{v}}_C\right| = \frac{d}{dt}\left(R_z\omega_S\right) = R_z\frac{d\omega_S}{dt} + \omega_S\frac{dR_z}{dt} \\
&\frac{d\alpha}{dt} = \omega_S \qquad \frac{d\omega_S}{dt} = \varepsilon_S \qquad \frac{dR_z}{dt} = \frac{dR_z}{d\alpha}\frac{d\alpha}{dt} = \frac{dR_z}{d\alpha}\omega_S \\
&a_C^v = R_z\varepsilon_S + \omega_S^2\frac{dR_z}{d\alpha} \qquad a_{C0}^v = R_z\varepsilon_S \qquad \Delta a_C^v = \omega_S^2\frac{dR_z}{d\alpha} \\
&a_C^v = a_{C0}^v + \Delta a_C^v
\end{aligned} \tag{3}$$

The value of acceleration a_C^v is the sum of a_{C0}^v depending on angular acceleration ε_S of crank 1 and Δa_C^v depending on angular velocity of crank 1 ω_S squared. This component is equal zero when R_z = const.

The value of element 2 angular acceleration is determined as follows.

$$\begin{aligned}
&\varepsilon_C = \frac{d\omega_C}{dt} = \frac{d}{dt}\left(\frac{\omega_S}{i_m}\right) = \frac{\frac{d\omega_S}{dt}i_m - \omega_S\frac{di_m}{dt}}{i_m^2} = \frac{1}{i_m}\frac{d\omega_S}{dt} - \frac{\omega_S}{i_m^2}\frac{di_m}{dt} \\
&\frac{d\alpha}{dt} = \omega_S \qquad \frac{d\omega_S}{dt} = \varepsilon_S \qquad \frac{di_m}{dt} = \frac{di_m}{d\alpha}\frac{d\alpha}{dt} = \frac{di_m}{d\alpha}\omega_S \\
&\varepsilon_C = \frac{\varepsilon_S}{i_m} - \frac{\omega_S^2}{i_m^2}\frac{di_m}{d\alpha} \qquad \varepsilon_{C0} = \frac{\varepsilon_S}{i_m} \qquad \Delta\varepsilon_C = -\frac{\omega_S^2}{i_m^2}\frac{di_m}{d\alpha} \\
&\varepsilon_C = \varepsilon_{C0} + \Delta\varepsilon_C
\end{aligned} \tag{4}$$

The value of angular acceleration ε_C is the sum of ε_{C0} depending on angular acceleration ε_S of crank 1 and $\Delta\varepsilon_C$ depending on angular velocity of crank 1 ω_S squared. This component is equal zero when i_m = const.

2.3. *Motion equations of mechanism*

The formula (5) determines efficiency of mechanism between elements 1 and 2 and describes power balance of mechanism.

$$\eta_m = \frac{P_u}{P_{in}} \tag{5}$$

P_u – power necessary to move element 2 (including mass forces),
P_{in} – power delivered to element 1.

$$\begin{aligned}
P_u &= -\bar{\mathbf{F}}_C\bar{\mathbf{v}}_C + m_C a_C^v v_C + T_C\omega_C + I_C\varepsilon_C\omega_C \\
&= F_C^v v_C + m_C a_C^v v_C + T_C\omega_C + I_C\varepsilon_C\omega_C
\end{aligned} \tag{6}$$

$\left|\bar{\mathbf{F}}_C^v\right| = F_C^v$ – value of force vector $\bar{\mathbf{F}}_C$ projection on direction of velocity vector $\bar{\mathbf{v}}_C$,
$F_C^v v_C$ – power necessary to overcome force $\bar{\mathbf{F}}_C$,
$m_C a_C^v v_C$ – power necessary to accelerate element 2,
$T_C\omega_C$ – power necessary to overcome torque T_C,
$I_C\varepsilon_C\omega_C$ – power necessary to set angular acceleration of element 2.

$$P_{in} = T_S\omega_S \tag{7}$$

By using formulas (6) and (7) in (5) formula (8) can be determined.

$$\eta_m = \frac{F_C^v v_C + m_C a_C^v v_C + T_C \omega_C + I_C \varepsilon_C \omega_C}{T_S \omega_S} \tag{8}$$

The torque T_S needful for driving crank 1 is determined from relation (8).

$$T_S = F_C^v \frac{v_C}{\omega_S} \frac{1}{\eta_m} + m_C a_C^v \frac{v_C}{\omega_S} \frac{1}{\eta_m} + T_C \frac{\omega_C}{\omega_S} \frac{1}{\eta_m} + I_C \varepsilon_C \frac{\omega_C}{\omega_S} \frac{1}{\eta_m} \tag{9}$$

After using relations (1) $\div$ (4) the following formulas are correct.

$$\begin{aligned} T_S &= F_C^v \frac{R_z}{\eta_m} + m_C \frac{R_z}{\eta_m} a_C^v + \frac{T_C}{i_m \eta_m} + \frac{I_C}{i_m \eta_m} \varepsilon_C \\ &= F_C^v \frac{R_z}{\eta_m} + m_C \frac{R_z}{\eta_m} (a_{C0}^v + \Delta a_C^v) + \frac{T_C}{i_m \eta_m} + \frac{I_C}{i_m \eta_m} (\varepsilon_{C0} + \Delta \varepsilon_C) \\ &= F_C^v \frac{R_z}{\eta_m} + m_C \frac{R_z}{\eta_m} \left(R_z \varepsilon_S + \omega_S^2 \frac{dR_z}{d\alpha} \right) + \frac{T_C}{i_m \eta_m} \\ &+ \frac{I_C}{i_m \eta_m} \left(\frac{\varepsilon_S}{i_m} - \frac{\omega_S^2}{i_m^2} \frac{di_m}{d\alpha} \right) = F_C^v \frac{R_z}{\eta_m} + m_C \frac{R_z^2}{\eta_m} \varepsilon_S + m_C \frac{\omega_S^2}{\eta_m} R_z \frac{dR_z}{d\alpha} \\ &+ \frac{T_C}{i_m \eta_m} + \frac{I_C}{i_m^2 \eta_m} \varepsilon_S - \frac{I_C \omega_S^2}{\eta_m} \frac{1}{i_m^3} \frac{di_m}{d\alpha} \end{aligned} \tag{10}$$

When the system is massless (mass m_C and moment of inertia I_C can be neglected) the relation (10) describing torque T_S simplifies.

$$T_S = P_C^v \frac{R_z}{\eta_m} + \frac{T_C}{i_m \eta_m} \qquad T_S = T_{Su} \qquad T_{Su} = F_C^v \frac{R_z}{\eta_m} + \frac{T_C}{i_m \eta_m} \tag{11}$$

The torque T_{Su} (11) is named the static load torque reduced to crank 1 (exactly to shaft S of crank 1).

When mass m_C and moment of inertia I_C have to be taken into consideration but the effective radius R_z and the ratio i_m are constant:

$$\left(R_z = \text{const} \ \wedge \ i_m = \text{const} \ \Rightarrow \ \frac{dR_z}{d\alpha} = 0 \ \wedge \ \frac{di_m}{d\alpha} = 0 \right)$$

the relation (10) describing torque T_S changes.

$$\begin{aligned} T_S &= T_{Su} + \left(m_C \frac{R_z^2}{\eta_m} + \frac{I_C}{i_m^2 \eta_m} \right) \varepsilon_S \\ T_S &= T_{Su} + I_{zS} \varepsilon_S \\ I_{zS} &= m_C \frac{R_z^2}{\eta_m} + \frac{I_C}{i_m^2 \eta_m} \end{aligned} \tag{12}$$

The moment of inertia I_{zS} is named the effective moment of inertia reduced to crank 1 (exactly to shaft S of crank 1).

In general case the torque T_S can be determined by relation (13).

$$T_S = T_{Su} + I_{zS} \varepsilon_S + m_C \frac{\omega_S^2}{\eta_m} R_z \frac{dR_z}{d\alpha} - \frac{I_C \omega_S^2}{\eta_m} \frac{1}{i_m^3} \frac{di_m}{d\alpha} \tag{13}$$

The two last components of relation (13) are named the torque (-ΔM_S) and developed as follows.

$$
\begin{aligned}
-\Delta M_S &= m_C \frac{\omega_S^2}{\eta_m} R_z \frac{dR_z}{d\alpha} - \frac{I_C \omega_S^2}{\eta_m} \frac{1}{i_m^3} \frac{di_m}{d\alpha} \\
&= m_C \frac{\omega_S^2}{\eta_m} \frac{1}{2} \frac{d}{d\alpha}\left(R_z^2\right) - \frac{I_C \omega_S^2}{\eta_m}\left(-\frac{1}{2}\right) \frac{d}{d\alpha}\left(\frac{1}{i_m^2}\right) \\
&= \frac{1}{2}\omega_S^2 \left[\frac{d}{d\alpha}\left(m_C \frac{R_z^2}{\eta_m}\right) + \frac{d}{d\alpha}\left(\frac{I_C}{i_m^2 \eta_m}\right)\right] \\
&= \frac{1}{2}\omega_S^2 \frac{d}{d\alpha}\left(m_C \frac{R_z^2}{\eta_m} + \frac{I_C}{i_m^2 \eta_m}\right) = \frac{1}{2}\omega_S^2 \frac{dI_{zS}}{d\alpha}
\end{aligned}
\tag{14}
$$

2.4. *One–element dynamic model of mechanism*

The component (14) has measure of torque. The additional torque ΔM_S is described by relation (15).

$$
\Delta T_S = -\frac{1}{2}\omega_S^2 \frac{dI_{zS}}{d\alpha} \tag{15}
$$

Using relations (14) and (15) in (13) the equation describing motion of mechanism can be presented.

$$
T_S = T_{Su} + I_{zS}\varepsilon_S - \Delta T_S \tag{16}
$$

The equation (16) describes the motion of one–element dynamic model of mechanism reduced to shaft S of the crank 1. It is shown in Fig. 4.

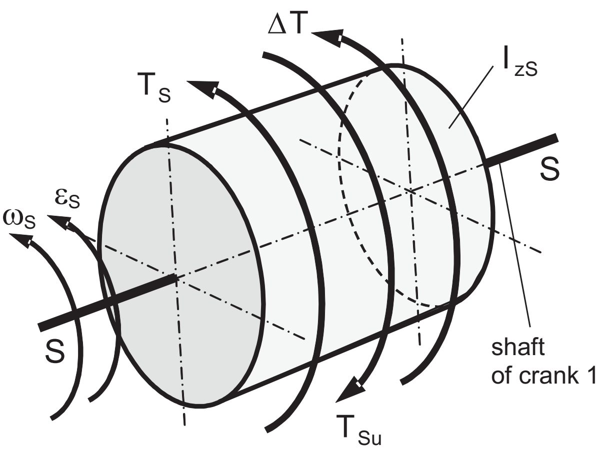

Figure 4 One–element dynamic model of mechanism reduced to shaft S of crank 1

The relation (16) is useful when angular crank acceleration ε_S is known and calculation of driving torque T_S is demanded.

When driving torque T_S is known and calculation of angular crank acceleration ε_S is necessary there is comfortably to transform equation (16) to the form (17):

$$I_{zS}\varepsilon_S = T_S - T_{Su} + \Delta T_S \tag{17}$$

All formulas above are correct by assumption that power is delivered to crank 1 and received from element 2 (the power transfer from crank 1 to element 2). This case is characteristic by positive value of power $P_{in} = T_S \cdot \omega_S$ ($P_{in} > 0$).

When the power is transferred in opposite direction from element 2 to crank 1 ($P_{in} = T_S\omega_S < 0$) the formulas (6) $\div$ (16) are some different. It is easy to show that in this case the factor of efficiency η_m appears in numerator and not in denominator of adequate fractions.

2.5. *Analytical example*

In Fig. 5 the slider–crank mechanism containing driving crank 1 (length r), connecting–rod 2 (length l) and piston 3 is shown.

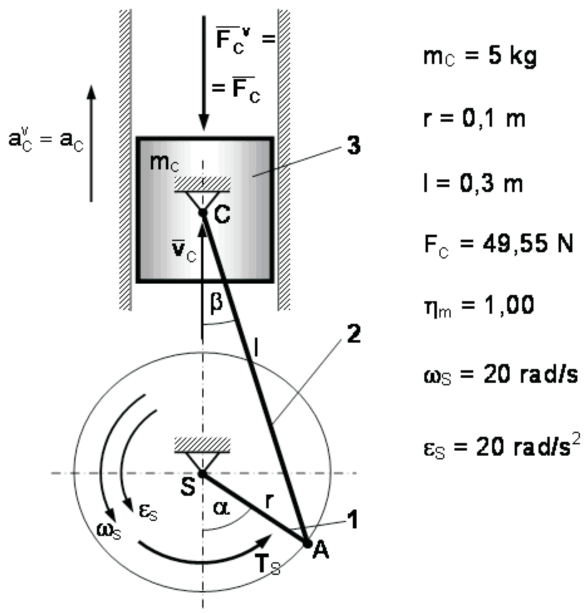

Figure 5 Example of slider–crank mechanism

The driving torque T_S is applied to crank 1, only the piston 3 has mass m_C and can move along v ertical line without rotation. The piston 3 is loaded by vertical force $|\bar{\mathbf{F}}_C|$. The crank 1 turns round point (shaft) S with determined angular velocity ω_S and acceleration ε_S.

For data showed in Fig. 5 it is necessary to calculate:

a_{C0} – acceleration of piston 3 caused by angular acceleration ε_S of crank 1,

Δa_C – additional acceleration of piston 3 caused by variation of effective radius R_z,

a_C – total acceleration of piston 3 ($a_C = a_{C0} + \Delta a_C$),

T_{Su} –static load torque reduced to crank 1 (exactly to shaft S),

$I_{zS} \cdot \varepsilon_S$ – load torque caused by angular acceleration ε_S of crank 1,

ΔT_S – additional driving torque caused by variation of effective radius R_z,

T_S – total driving torque applied to crank 1 (shaft S) ($T_S = T_{Su} + I_{zS}\varepsilon_S - \Delta T_S$).

To simplify the problem the value of efficiency of mechanism is set as $\eta_m = 1$. Therefore there is no necessary to test the sign of crank power $P_{in} = M_S\omega_S$ in every moment of the motion.

Using obvious laws of geometry and kinematics basing on designations in Fig. 5 the following formulas can be determined.

Effective radius R_z of mechanism for elements 3 (piston) and 1 (crank).

$$R_z = \frac{v_C}{\omega_S} = r\left[\sin\alpha - \frac{1}{2}\frac{\sin 2\alpha}{\sqrt{\frac{r^2}{l^2} - \sin^2\alpha}}\right] \tag{18}$$

The derivative of radius R_z in respect to angle α of crank 1 rotation.

$$\frac{dR_z}{d\alpha} = r\left[\cos\alpha - \frac{1}{2}\frac{2\cos 2\alpha\left(\frac{r^2}{l^2} - \sin^2\alpha\right) + \frac{1}{2}\sin^2 2\alpha}{2\left(\frac{r^2}{l^2} - \sin^2\alpha\right)^{\frac{3}{2}}}\right] \tag{19}$$

The radius R_z and its derivative $\frac{dR_z}{d\alpha}$ are functions of variable angle α of crank 1 rotation. Their dependences on angle for range $\langle 0, 360^o\rangle$ are shown in Fig. 6.

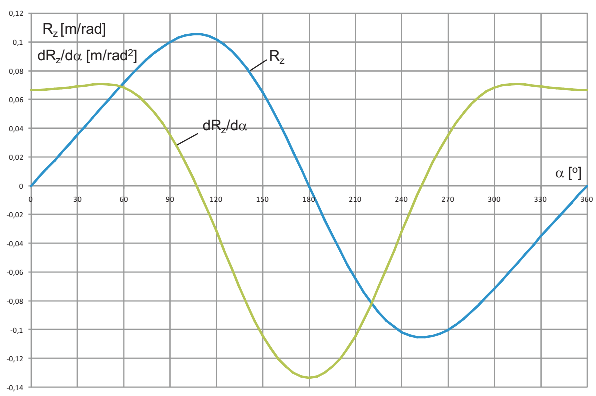

Figure 6 Radius R_z and its derivative $\frac{dR_z}{d\alpha}$ as functions of angle α of crank 1 rotation

The effective moment of inertia I_{zS} reduced to crank 1 (to shaft S) caused by piston 3 mass m_C.

$$I_{zS} = m_C \frac{R_z^2}{\eta_m} = \frac{m_C r^2}{\eta_m} \left[\sin\,\alpha - \frac{1}{2} \frac{\sin\,2\alpha}{\sqrt{\frac{r^2}{l^2} - \sin^2\alpha}} \right]^2 \tag{20}$$

The derivative of inertia moment I_{zS} in respect to angle α of crank 1 rotation.

$$\frac{dI_{zS}}{d\alpha} = 2\frac{m_C r^2}{\eta_m} \left[\sin\,\alpha - \frac{1}{2} \frac{\sin\,2\alpha}{\sqrt{\frac{r^2}{l^2} - \sin^2\alpha}} \right] \left[\cos\,\alpha - \frac{2\cos\,2\alpha \left(\frac{r^2}{l^2} - \sin^2\alpha \right) + \frac{1}{2}\sin^2 2\alpha}{2\left(\frac{r^2}{l^2} - \sin^2\alpha \right)^{\frac{3}{2}}} \right] \tag{21}$$

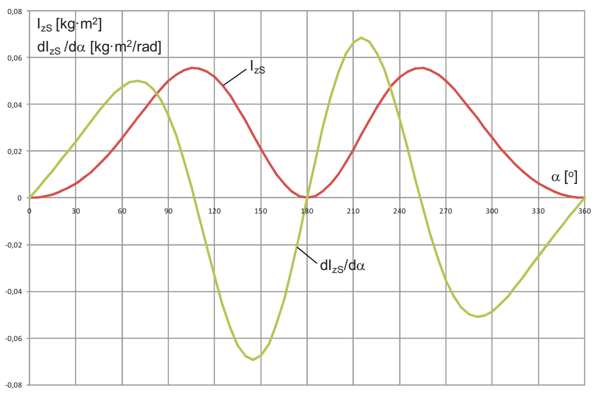

Figure 7 The effective moment of inertia I_{zS} and its derivative $\frac{dI_{zS}}{d\alpha}$ as functions of angle α of crank 1 rotation

The effective moment of inertia I_{zS} and its derivative $\frac{dI_{zS}}{d\alpha}$ are functions of variable angle α of crank 1 rotation. Their dependences on angle for range $\langle 0,\, 360^o \rangle$ are shown in Fig. 7.

Using formulas (3) the acceleration a_{C0}, Δa_C and total acceleration a_C of piston 3 can be determined as functions of angle α of crank 1 rotation. These dependences are shown in Fig. 8.

The torque T_S necessary to drive crank 1 with constant angular acceleration ε_S and still constant angular velocity ω_S and its components T_{Su}, $(I_{zS} \cdot \varepsilon_S)$ and ΔT_S can be determined from relations 11 ÷ 16. Their dependences on angle α of crank 1 rotation for range $\langle 0,\, 360^o \rangle$ are shown in Fig. 9.

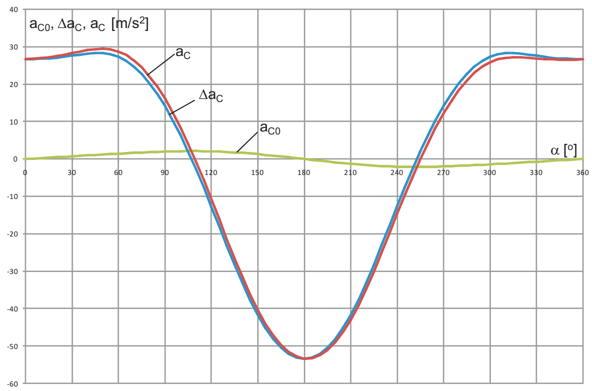

Figure 8 The acceleration a_{C0}, Δa_C and total acceleration a_C of piston 3 as functions of angle α of crank 1 rotation

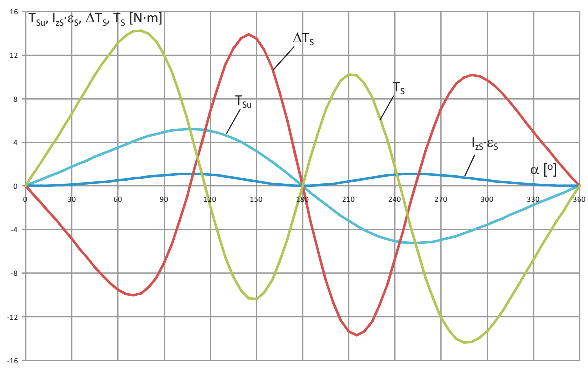

Figure 9 The torque T_S and its components T_{Su}, $(I_{zS} \cdot \varepsilon_S)$ and ΔT_S as functions of angle α of crank 1 rotation

3. Mechanism in plane motion with linear motor

3.1. *Example of mechanism*

In Fig. 10 the scheme of four–jointed (A, B, D, E) crane jib mechanism is showed. The jib of the crane 2 rotates round stationary point A and is driven by connected linear motor 1 (for example hydraulic cylinder) and drives other elements of mechanism; jib beak 3 in plane motion and connector 4 rotating round stationary point E. The beak 3 has center of gravity in point C, temporary center of rotation – in point C^I. There is taken assumption that only element 3 has mass m_C and moment of inertia I_C in respect of center of gravity C.

The beak 3 is loaded by force $\bar{\mathbf{F}}_C$ applied to its gravity center C and by torque T_C. The driving force $\bar{\mathbf{F}}_S$ is applied to moving element (piston rod of hydraulic cylinder) of linear motor 1 and has its direction.

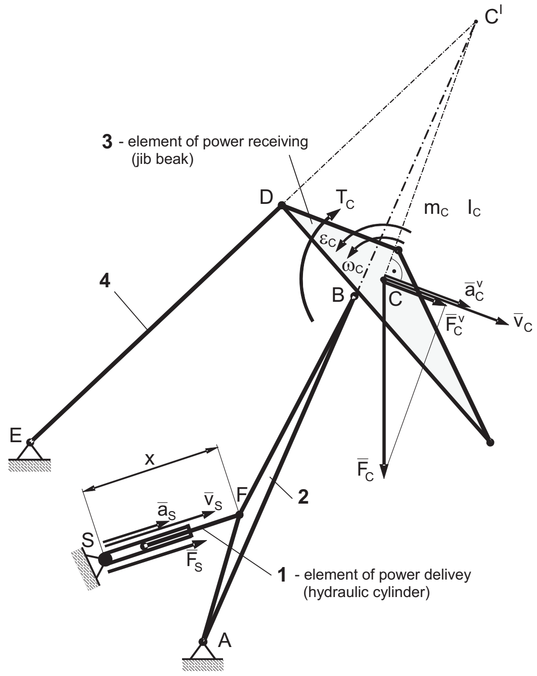

Figure 10 Four–jointed (S, A, B, D) mechanism with driving crank 1 and driven element 3 in plane motion

Other designations in Fig. 10 are as follows:

$\bar{\mathbf{v}}_S$ – vector of v elocity of linear motor 1 moving element (piston rod of hydraulic cylinder) in point S (vector $\bar{\mathbf{v}}_S$ is directed along axis of hydraulic cylinder),

$\bar{\mathbf{a}}_S^v$ – tangent component of acceleration of piston rod in point S (it has direction of vector $\bar{\mathbf{v}}_C$ velocity and its value is $|\bar{\mathbf{a}}_S^v| = a_S^v = \frac{dv_S}{dt}$),

$\bar{\mathbf{v}}_C$ – vector of beak 3 gravity center C velocity,

a_C^v – point C tangent acceleration component in direction of velocity vector $\bar{v}_C$,

$\bar{\mathbf{F}}_C^v$ – vector of force $\bar{F}_C$ projection on direction of velocity vector $\bar{v}_C$,

ω_C – angular velocity of beak 3,

ε_C – angular acceleration of beak 3.

In analysis of motion of mechanism showed in Fig. 10 two cases are considered.

1. The acceleration a_S^v is known and required driving force F_S is the result.

2. The driving force F_S is known and acceleration a_S^v is the result.

3.2. *Velocity and acceleration of element 3 (jib beak)*

There is interdependence between velocities and accelerations of elements 1 and 3. The connections between velocities are given by formulas (22) and (23).

$$\frac{|\bar{\mathbf{v}}_S|}{\omega_C} = \frac{v_S}{\omega_C} = R_e \tag{22}$$

R_e – effective radius of mechanism for elements 1 and 3,
$|\bar{\mathbf{v}}_S| = v_S$ – value of piston rod of hydraulic cylinder velocity vector in point S.

$$\frac{|\bar{\mathbf{v}}_S|}{|\bar{\mathbf{v}}_C|} = \frac{v_S}{v_C} = i_v \tag{23}$$

i_v – ratio of mechanism between elements 1 and 3,
$|\bar{\mathbf{v}}_C| = v_C$ – value of point C velocity vector.

The effective radius R_e and ratio i_v are variable but always can be determined dependently on position of mechanism given by length x of linear motor 1 (x = SF).

For further consideration the value of point C acceleration tangent component a_C^v in direction of velocity vector $\bar{\mathbf{v}}_C$ is important. Other components of point C acceleration don't have influence on power balance of mechanism. The component a_C^v is determined as follows.

$$\begin{aligned}
&a_C^v = \frac{d}{dt}|\bar{\mathbf{v}}_C| = \frac{d}{dt}\left(\frac{v_S}{i_v}\right) = \frac{\frac{dv_S}{dt} i_v - v_S \frac{di_v}{dt}}{i_v^2} = \frac{1}{i_v}\frac{dv_S}{dt} - \frac{v_S}{i_v^2}\frac{di_v}{dt} \\
&\frac{dx}{dt} = v_S \qquad \frac{dv_S}{dt} = a_S^v \qquad \frac{di_v}{dt} = \frac{di_v}{dx}\frac{dx}{dt} = \frac{di_v}{dx} v_S \\
&a_C^v = \frac{a_S^v}{i_v} - \frac{v_S^2}{i_v^2}\frac{di_v}{dt} \qquad a_{C0}^v = \frac{a_S^v}{i_v} \qquad \Delta a_C^v = -\frac{v_S^2}{i_v^2}\frac{di_v}{dt} \\
&a_C^v = a_{C0}^v + \Delta a_C^v
\end{aligned} \tag{24}$$

The value of acceleration a_C^v is the sum of a_{C0}^v depending on point S tangent acceleration a_S^v of piston rod 1 and Δa_C^v depending on point S velocity v_S of piston rod 1 squared. This component is equal zero when i_v = const.

The value of element 3 angular acceleration is determined as follows.

$$\begin{aligned}
&\varepsilon_C = \frac{d\omega_C}{dt} = \frac{d}{dt}\left(\frac{v_S}{R_e}\right) = \frac{\frac{dv_S}{dt} R_e - v_S \frac{dR_e}{dt}}{R_e^2} = \frac{1}{R_e}\frac{dv_S}{dt} - \frac{v_S}{R_e^2}\frac{dR_e}{dt} \\
&\frac{dx}{dt} = v_S \qquad \frac{dv_S}{dt} = a_S^v \qquad \frac{dR_e}{dt} = \frac{dR_e}{dx}\frac{dx}{dt} = \frac{dR_e}{dx} v_S \\
&\varepsilon_C = \frac{a_S^v}{R_e} - \frac{v_S^2}{R_e^2}\frac{dR_e}{dx} \qquad \varepsilon_{C0} = \frac{a_S^v}{R_e} \qquad \Delta\varepsilon_C = -\frac{v_S^2}{R_e^2}\frac{dR_e}{dx} \\
&\varepsilon_C = \varepsilon_{C0} + \Delta\varepsilon_C
\end{aligned} \tag{25}$$

The value of angular acceleration ε_C is the sum of ε_{C0} depending on point S tangent acceleration a_S^v of piston rod 1 and $\Delta\varepsilon_C$ depending on point S velocity v_S of piston rod 1 squared. This component is equal zero when R_e = const.

3.3. *Motion equation of mechanism*

The formula (5) repeated as (26) determines efficiency of mechanism between elements 1 and 2 and describes power balance of mechanism.

$$\eta_m = \frac{P_u}{P_{in}} \tag{26}$$

P_u – power necessary to move jib beak 3 (including mass forces),
P_{in} – power delivered to cylinder 1.

Power P_u can be described by formula (6) repeated as (27).

$$\begin{aligned} P_u &= -\bar{\mathbf{F}}_C \bar{\mathbf{v}}_C + m_C a_C^v v_C + T_C \omega_C + I_C \varepsilon_C \omega_C \\ &= F_C^v v_C + m_C a_C^v v_C + T_C \omega_C + I_C \varepsilon_C \omega_C \end{aligned} \tag{27}$$

Variables present in right side of formula (27) are described in chapter *2.3* and are connected with jib beak 3:

$$P_{in} = F_S \cdot v_S \tag{28}$$

By using formulas (27) and (28) in (26) formula (29) can be determined:

$$\eta_m = \frac{P_C^v v_C + m_C a_C^v v_C + M_C \omega_C + I_C \varepsilon_C \omega_C}{F_S v_S} \tag{29}$$

The driving force F_S is determined from relation (29):

$$F_S = P_C^v \frac{v_C}{v_S} \frac{1}{\eta_m} + m_C a_C^v \frac{v_C}{v_S} \frac{1}{\eta_m} + M_C \frac{\omega_C}{v_S} \frac{1}{\eta_m} + I_C \varepsilon_C \frac{\omega_C}{v_S} \frac{1}{\eta_m} \tag{30}$$

After using relations (22) ÷ (25) the following formulas are correct.

$$\begin{aligned} F_S &= \frac{P_C^v}{i_v \eta_m} + \frac{m_C}{i_v \eta_m} a_C^v + \frac{M_C}{R_e \eta_m} + \frac{I_C}{R_e \eta_m} \varepsilon_C \\ &= \frac{P_C^v}{i_v \eta_m} + \frac{m_C}{i_v \eta_m} (a_{C0}^v + \Delta a_C^v) + \frac{M_C}{R_e \eta_m} + \frac{I_C}{R_e \eta_m} (\varepsilon_{C0} + \Delta \varepsilon_C) \\ &= \frac{P_C^v}{i_v \eta_m} + \frac{m_C}{i_v \eta_m} \left(\frac{a_S^v}{i_v} - \frac{v_S^2}{i_v^2} \frac{di_v}{dt} \right) + \frac{M_C}{R_e \eta_m} \\ &+ \frac{I_C}{R_e \eta_m} \left(\frac{a_S^v}{R_e} - \frac{v_S^2}{R_e^2} \frac{dR_e}{dx} \right) \\ &= \frac{P_C^v}{i_v \eta_m} + \frac{m_C}{i_v^2 \eta_m} a_S^v - \frac{m_C v_S^2}{\eta_m} \frac{1}{i_v^3} \frac{di_v}{dt} \\ &+ \frac{M_C}{R_e \eta_m} + \frac{I_C}{R_e^2 \eta_m} a_S^v - \frac{I_C v_S^2}{\eta_m} \frac{1}{R_e^3} \frac{dR_e}{dx} \end{aligned} \tag{31}$$

When the system is massless (mass m_C and moment of inertia I_C can be neglected) the relation (31) describing force F_S simplifies.

$$F_S = \frac{P_C^v}{i_v \eta_m} + \frac{M_C}{R_e \eta_m} \qquad F_S = F_{Su} \qquad F_{Su} = \frac{P_C^v}{i_v \eta_m} + \frac{M_C}{R_e \eta_m} \tag{32}$$

The force F_{Su} (32) is named the static load force reduced to linear motor 1 (exactly to piston rod of cylinder 1).

When mass m_C and moment of inertia I_C have to be taken into consideration but the effective radius R_e and the ratio i_v are constant:

$$\left(R_e = const \ \wedge \ i_v = const \ \Rightarrow \ \frac{dR_e}{dx} = 0 \ \wedge \ \frac{di_v}{dx} = 0\right)$$

the relation (32) describing force F_S changes.

$$\begin{aligned} F_S &= F_{Su} + \left(\frac{m_C}{i_v^2\eta_m} + \frac{I_C}{R_e^2\eta_m}\right) a_S^v \\ F_S &= F_{Su} + m_{zS} a_S^v \\ m_{zS} &= \frac{m_C}{i_v^2\eta_m} + \frac{I_C}{R_e^2\eta_m} \end{aligned} \tag{33}$$

The mass m_{zS} is named the effective mass reduced to linear motor 1 (exactly to point S of cylinder piston rod 1).

In general case the force F_S can be determined by relation (34).

$$F_S = F_{Su} + m_{zS} a_S^v - \frac{m_C v_S^2}{\eta_m} \frac{1}{i_v^3} \frac{di_v}{dt} - \frac{I_C v_S^2}{\eta_m} \frac{1}{R_e^3} \frac{dR_e}{dx} \tag{34}$$

The two last components of relation (34) are named the force (-ΔF$_S$) and developed as follows.

$$\begin{aligned} -\Delta F_S &= -\frac{m_C v_S^2}{\eta_m} \frac{1}{i_v^3} \frac{di_v}{dt} - \frac{I_C v_S^2}{\eta_m} \frac{1}{R_e^3} \frac{dR_e}{dx} \\ &= -\frac{m_C v_S^2}{\eta_m} \left(-\frac{1}{2}\right) \frac{d}{dx}\left(\frac{1}{i_v^2}\right) - \frac{I_C v_S^2}{\eta_m} \left(-\frac{1}{2}\right) \frac{d}{dx}\left(\frac{1}{R_e^2}\right) \\ &= \frac{1}{2} v_S^2 \left[\frac{d}{dx}\left(\frac{m_C}{i_v^2\eta_m}\right) + \frac{d}{dx}\left(\frac{I_C}{R_e^2\eta_m}\right)\right] \\ &= \frac{1}{2} v_S^2 \frac{d}{dx}\left(\frac{m_C}{i_v^2\eta_m} + \frac{I_C}{R_e^2\eta_m}\right) = \frac{1}{2} v_S^2 \frac{dm_{zS}}{dx} \end{aligned} \tag{35}$$

3.4. *One–element dynamic model of mechanism*

The component (35) has measure of force. The additional force ΔF$_S$ is described by relation (36):

$$\Delta F_S = -\frac{1}{2} v_S^2 \frac{dm_{zS}}{dx} \tag{36}$$

Using relations (35) and (36) in (34) the equation describing motion of mechanism can be presented:

$$F_S = F_{Su} + m_{zS} a_S^v - \Delta F_S \tag{37}$$

The equation (37) describes the motion of one-element dynamic model of mechanism reduced to linear motor 1 (exactly to point S of cylinder piston rod 1). It is shown in Fig. 11.

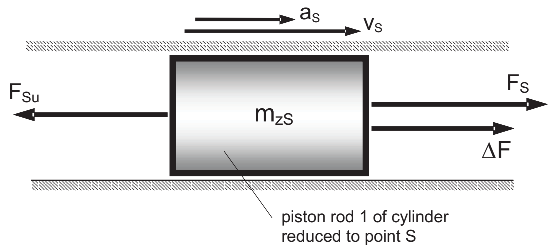

Figure 11 One–element dynamic model of mechanism reduced to linear motor 1

The relation (37) is useful when piston rod acceleration a_S is known and calculation of driving force F_S is demanded.

When driving force F_S is known and calculation of piston rod acceleration a_S is necessary there is comfortably to transform equation (37) to the form (38).

$$m_{zS}a_S^v = F_S - F_{Su} + \Delta F_S \tag{38}$$

All formulas above are correct by assumption that power is delivered to piston rod 1 and received from jib beak 3 (the power transfer from linear motor 1 to element 3). This case is characteristic by positive value of power $P_{in} = F_S v_S$ ($P_{in} > 0$).

When the power is transferred in opposite direction from element 3 to piston rod 1 ($P_{in} = F_S v_S < 0$) the formulas (27)–(35) are different. It is easy to show that in this case the factor of efficiency η_m appears in numerator and not in denominator of adequate fractions.

3.5. *Analytical example*

Fig. 12 shows the mechanism containing linear motor 1 driving jib 2 (length l) which rotates around stationary point A.

The driving force $\bar{\mathbf{F}}_S$ is applied to moving element of linear motor 1 and has its direction. With the end of the jib 2 is connected element 3 treated as material point C (moment of inertia $I_C = 0$) which as the only one has mass m_C and is loaded by its weight $\bar{\mathbf{F}}_C = m_C \cdot \bar{\mathbf{g}}$. The linear motor 1 can change length x, its moving part in point S has determined velocity $\bar{\mathbf{v}}_S$ and tangent acceleration component $\bar{\mathbf{a}}_S^v$. The velocity $\bar{\mathbf{v}}_S$ has direction SB and its value is $v_S = |\bar{\mathbf{v}}_S|$. Component $\bar{\mathbf{a}}_S^v$ has direction of vector $\bar{\mathbf{v}}_C$ velocity and its known value is $|\bar{\mathbf{a}}_S^v| = a_S^v = \frac{dv_S}{dt}$. For data showed in Fig. 12 it is necessary to calculate:

a_{C0}^v – tangent acceleration of point C caused by acceleration $\bar{\mathbf{a}}_S^v$ of linear motor 1 moving element,

Δa_C^v – additional tangent acceleration of point C caused by variation of ratio i_v,

a_C^v – total tangent acceleration of point C ($a_C^v = a_{C0}^v + \Delta a_C^v$),

F_{Su} – static load force reduced to linear motor 1,

$m_{zS}a_S^v$ – load force caused by acceleration a_S^v of linear motor 1 moving element,

ΔF_S – additional driving force caused by variation of ratio i_v,

F_S – total driving force applied to moving element of linear motor 1 ($F_S = F_{Su} + m_{zS} a_S^v - \Delta F_S$).

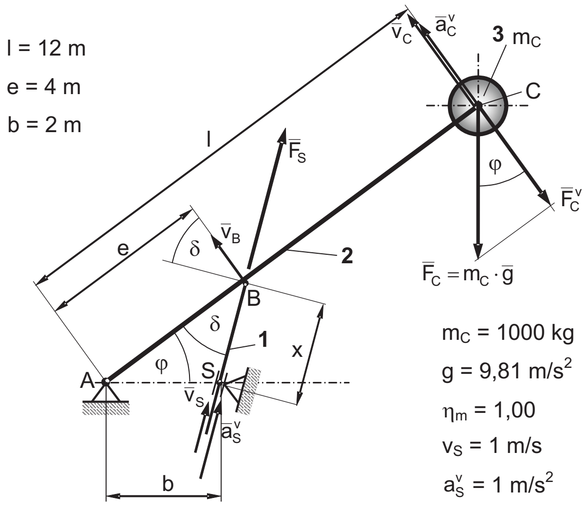

Figure 12 Example of jib mechanism

To simplify the problem the value of efficiency of mechanism is set as $\eta_m = 1$. Therefore there is no necessary to test the sign of linear motor power $P_{in} = F_S v_S$ in every moment of the motion.

Using obvious laws of geometry and kinematics basing on designations in Fig. 12 the following formulas can be determined.

Ratio i_v of mechanism for elements 3 and point S of linear motor 1 moving element.

$$i_v = \frac{v_S}{v_C} = \frac{e}{l}\sqrt{1 - \left(\frac{x^2 + e^2 - b^2}{2e \cdot x}\right)^2} \tag{39}$$

The derivative of ratio i_v in respect to length x of linear motor 1.

$$\frac{di_v}{dx} = -\frac{e^2}{l^2}\frac{x^4 - \left(e^2 - b^2\right)^2}{4e^2x^3 \cdot \frac{e}{l}\sqrt{1 - \left(\frac{x^2+e^2-b^2}{2e \cdot x}\right)^2}} \tag{40}$$

The effective mass m_{zS} reduced to point S of linear motor 1 moving element caused by element 3 mass m_C.

$$m_{zS} = m_C l^2 \cdot \frac{4x^2}{4e^2x^2 - \left(x^2 + e^2 - b^2\right)^2} \tag{41}$$

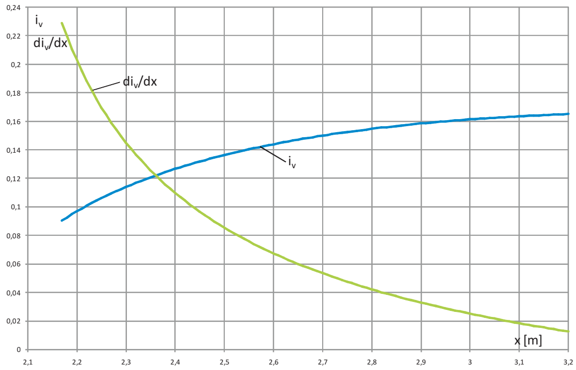

Figure 13 Ratio i_v and its derivative $\frac{di_v}{dx}$ as functions of linear motor 1 length x

The derivative of mass m_{zS} in respect to linear motor length x.

$$\frac{dm_{zS}}{dx} = \frac{1}{2} m_C \frac{l^2}{e^4} \frac{x^4 - \left(e^2 - b^2\right)^2}{x^3 \left[1 - \left(\frac{x^2+e^2-b^2}{2e \cdot x}\right)^2\right]^2} \tag{42}$$

The effective mass m_{zS} and its derivative $\frac{dm_{zS}}{dx}$ are functions of variable length x (segment SB) of linear motor 1. Their dependences on x for range $\langle 2{,}2\text{ m}, 3{,}2\text{ m}\rangle$ are shown in Fig. 14.

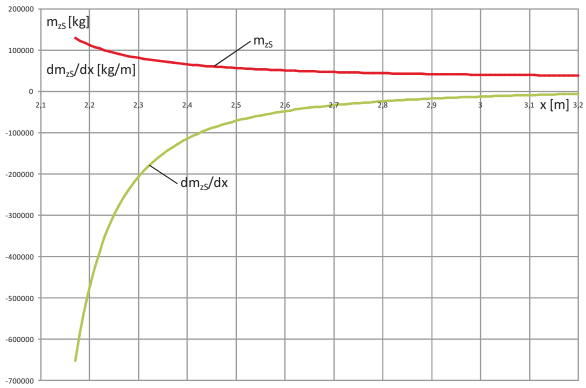

Figure 14 The effective mass m_{zS} and its derivative $\frac{dm_{zS}}{dx}$ as functions of linear motor 1 length x

The ratio i_v and its derivative $\frac{di_v}{dx}$ are functions of variable length x (segment SB) of linear motor 1. Their dependences on x for range $\langle 2{,}2\text{ m}, 3{,}2\text{ m}\rangle$ are shown in Fig. 13.

Using formulas (24) the acceleration a^v_{C0}, Δa^v_C and total tangent acceleration a^v_C of element 3 can be determined as functions of linear motor 1 length x. This dependence is shown in Fig. 15.

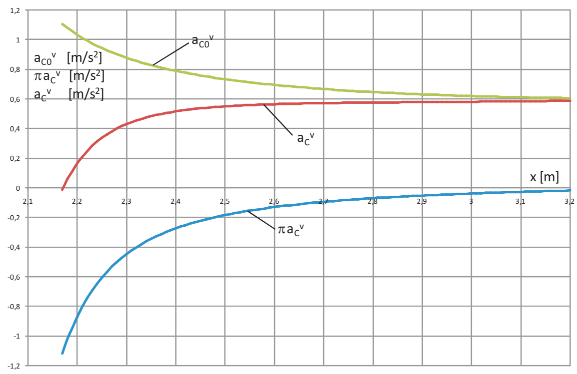

Figure 15 The tangent accelerations a^v_{C0}, Δa^v_C and total tangent acceleration a^v_C of element 3 as functions of linear motor 1 length x

The force F_S necessary to move linear motor 1 with constant tangent acceleration a^v_S and still constant velocity v_S of its moving part in point S and its components F_{Su}, $(m_{zS} \cdot a^v_S)$ and ΔF_S can be determined from relations (32) ÷ (37). Their dependences on length x of linear motor 1 for range ⟨2,2 m, 3,2 m⟩ are shown in Fig. 16.

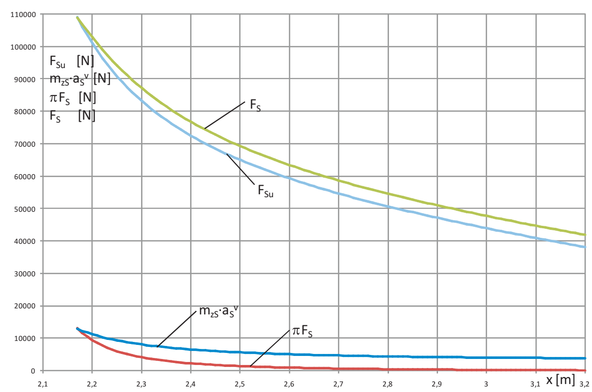

Figure 16 The force F$_S$ and its components F_{Su}, $\left(m_{zS} \cdot a^v_S\right)$ and ΔF_S as functions of length x of linear motor 1

4. Conclusions

1. Every one degree of freedom plane mechanism with rigid connections and variable ratio and effective radius can be reduced to driving element as one–element dynamic model:

   (a) with variable effective moment of inertia in case of rotating motor (Fig. 8) or

   (b) with variable effective mass in case of linear motor (Fig. 11).

2. It is necessary to know:

– For rotating driving element:

1. the ratio i_m and effective radius R_z between driving element and considered element in every position of mechanism given by angle α of driving shaft rotation,

2. their derivatives in respect to angle α of driving shaft rotation $\frac{di_m}{d\alpha}$, $\frac{dR_z}{d\alpha}$.

– For linear driving motor:

1. the ratio i_v and effective radius R_e between driving element and considered element in every position of mechanism given by length x of linear motor.

2. their derivatives in respect to length x of linear motor $\frac{di_v}{dx}$, $\frac{dR_e}{dx}$.

When the geometry of mechanism is determined it is possible.

1. The knowledge of variables specified above let calculate accelerations (linear and angular) of considered element; primary ones dependent on acceleration (angular ε_S or linear a_S^v) of driving element and additional ones dependent on velocity (angular ω_S or linear v_S) of driving element caused by variable ratio and effective radius of mechanism.

2. The derived formulas let determine the main parameters of dynamic model of mechanism reduced to:

- driving shaft;

1. the reduced moment of inertia I_{zS} as a function of angle α of driving shaft rotation,

2. its derivative $\frac{dI_{zS}}{d\alpha}$ in respect to angle α.

- linear motor;

1. the reduced mass m_{zS} as a function of linear motor length x,

2. its derivative $\frac{dm_{zS}}{dx}$ in respect to length x.

3. All loads of mechanism must be reduced to driving shaft (torques M_{Su} and ΔM in Fig. 8) or to linear motor (forces F_{Su} and ΔF in Fig. 11). The proper formulas are defined in the paper.

4. The relations derived in the paper cover two cases and let calculate:

- For rotating driving element:

1. driving torque M_S when angular driving shaft acceleration ε_S is known,

2. angular shaft acceleration ε_S when driving torque M_S is known.

– For linear driving motor:

1. driving force F_S when linear motor acceleration a_S^v is known,

2. linear motor acceleration a_S^v when driving force F_S is known.

3. In the paper there is described situation where only one element of mechanism has mass and moment of inertia. When more or all elements of mechanism with one degree of freedom have masses and moments of inertia the loads and masses can be reduced to driving element by using superposition method. Load and mass (and moment of inertia) of each element have to be reduced to driving element separately and then all reduced loads and masses should be summed up to reach resultant reduced load and mass of the model.

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