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Basic Study of Synchronization in a Two Degree of Freedom, Duffing Based System

Artur Dąbrowski
Konrad Mnich
Andrzej Stefański
Department of Dynamics
Lodz University of Technology
Stefanowskiego 1/15
90–924 Łódź, Poland
konradmnich@gmail.com

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The issue of synchronization has been analyzed from many points of view since it was first described in 17th century. Nowadays the theory that stands behind it is so developed, it might be difficult to understand where it all comes from. This paper reminds the key concepts of synchronization by examination of two coupled Duffing oscillators. It describes also the origins of Duffing equation and proves its ability to behave chaotically.

Keywords: Synchronization, Duffing, oscillator, chaos, history.

1. Introduction

History shows, that the concepts developed from the purely theoretical divagations often find their place in applied sciences. It was also the case of synchronization. When a Dutch scientist Christiaan Huygens first observed two pendulum clocks adjusting their ticking, he probably didn't suspect how this could be useful in science. And yet, after three hundred years, the analysis of synchronization became an important aspect of engineering [1, 2, 3]. Although its applications in mechanics are still not evident, it plays an important role e.g. in signal processing [4]. Recently the idea has spread even further, into economics [5], biology [6] and other sciences that deal with the large networks of individuals (economies, ecosystems, neural networks) [7, 8], which under certain assumptions may be treated as dynamical systems that can synchronize. In general all the definitions of synchronization are connected with the adjustment of multiple processes in time. This article focuses on the complete synchronization, which occurs when the phase space trajectories of different oscillators ($x_1(t)$ and $x_2(t)$) converge, so that after a sufficiently long time they are coincident.

$$\lim_{t\to\infty} |x_1(t) - x_2(t)| = 0 \tag{1}$$

It happens only when the oscillators are identical and though can never take place in the real life systems. Therefore, the notion of imperfect complete synchronization was introduced, which applies whenever the difference between the trajectories is sufficiently small.

$$\lim_{t\to\infty} |x_1(t) - x_2(t)| < \varepsilon, \qquad \varepsilon \ll |x_1(0) - x_2(0)| \tag{2}$$

Where ε is an integer, small comparing to the initial distance between the trajectories. There are multiple types synchronization, like phase or generalized synchronization, but they will not be treated here [1]. In order for synchronization to occur, the oscillators must be mutually coupled. In the case of the complex, natural systems the number of possible couplings is huge and their nature difficult to define. For the sake of simplicity, the model presented in this paper is limited to a single conservative coupling, that in a mechanical system could be represented by an ideal spring element. The goal of a simulation presented in this article is to observe for which parameters of coupling the coupled oscillators achieve a complete synchronization.

2. Duffing oscillator

In this study, the authors could have used almost any type of oscillator to observe the synchronization phenomenon. Duffing oscillator was chosen for it describes a wide range of real life oscillations and contains a nonlinear term, which makes the dynamics of the problem more interesting. Its name comes from a German engineer Georg Duffing, who first presented this model of oscillation in 1918 [9]. Originally he used it to describe the movement of a forced pendulum (Fig. 1) as presented in Eq. 3:

$$\ddot{x} + \chi\dot{x} + \alpha x - \gamma x^3 = k\sin(\omega t) \tag{3}$$

where x is a function of t and χ, α, γ, k, ω are the constant coefficients, according to the original Duffing's notation.

One can easily see, that the term $\alpha x - \gamma x^3$ is just an approximation of the sine function with the first two terms of Tylor's expansion. This observation leads us to a conclusion that γ should be equal to $\frac{\alpha}{6}$ since:

$$\sin x \approx x - \frac{x^3}{3!} \tag{4}$$

Nowadays, the term "Duffing oscillator" does not necessarily refer to a pendulum and the implications of Eq. 4 are not always respected. For instance, it is frequently used to model a forced mass–spring–damper system, with a nonlinear spring, where

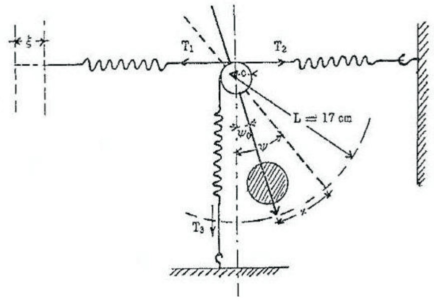

Figure 1 Original Duffing's drawing presenting a forced pendulum. Three linear springs provide a harmonic force of excitation

α and γ describe the spring stiffness (Fig. 2). In fact any dynamical system described by (3) is called Duffing oscillator, no matter what the coefficients are and whether it is physically realizable or not. For certain values of parameters it can even behave chaotically, which means that its trajectory is practically unpredictable in the long term. It is important to notice since later we will show that even such systems can synchronize [10].

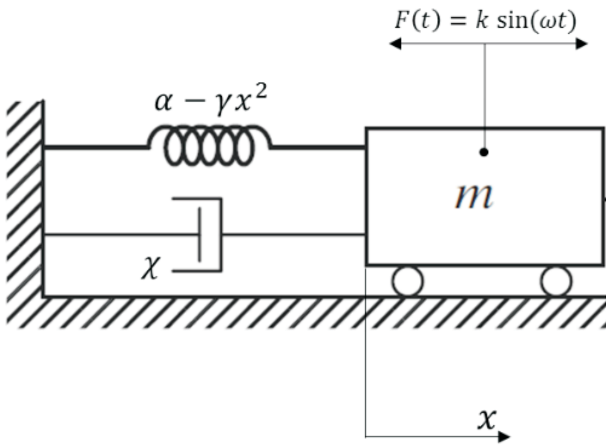

Figure 2 Forced mass–spring–damper system with a nonlinear spring

3. Simulation parameters

The system of our interest consists of two identical duffing oscillators coupled via conservative coupling. Their equations of motion can be presented as follows in a matrix form:

$$[I]\ddot{\mathbf{x}} + \chi[I]\dot{\mathbf{x}} + \alpha[I]\mathbf{x} - \gamma[I]\mathbf{x}^3 + s\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{x} = k\sin(\omega t)[I] \qquad (5)$$

Where $\mathbf{x} = [x_1, x_2]^T$ is a vector function representing the trajectories of two oscillators, $[I]$ is an identity matrix and s is a parameter of the coupling. The simulation verifies for which values of s the oscillators eventually synchronize, given that they start from different initial conditions and that the other parameters are fixed. Let us choose $\chi = 0.1$, $\alpha = 0$, $\gamma = -1$, $k = 10$ and $\omega = 1$, the values that are known to induce a chaotic response [1]. They are non–dimensional and do not represent any particular real life system. A Poincar map plotted for these parameters in Fig. 3 reveals the chaos involved, as a typical strange attractor appears.

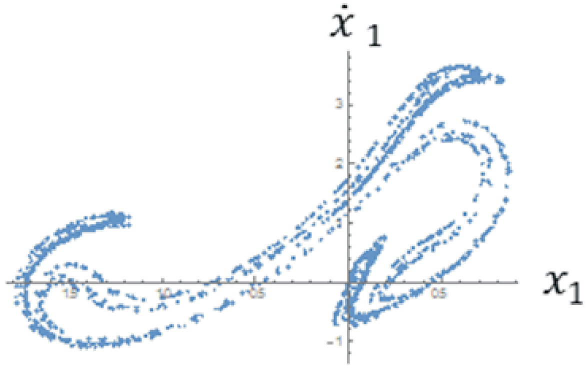

Figure 3 Poincar map of a single Duffing oscillator, plotted for 10‘000 periods of excitation with $\gamma = 0.1$

The simulations were prepared in Matlab 2014a and Mathematica 10.3.

4. Results

The results are gathered in form of a bifurcation diagram of parameter s, presented in Fig. 4. Its study reveals existence of three different types of behavior. In a zone where s ranges from 0 to 0.9 the oscillators do not synchronize at all, since a relative distance between them changes at each period of excitation. For s bigger than 0.14 the system achieves a complete synchronization and the relative distance remains constant. There is also a transitional interval $s \in (0.9, 0.14)$, in which the two possibilities coexist with an additional option of synchronization in counter phase. Figure 5 shows these observations in form of the $x_1 \, vs \, x_2$ graphs.

5. Conclusions

This study shows how one can estimate the thresholds of synchronization in a two degree of freedom system, with use of the basic mathematical tools and numerical simulations. It proves also that the chaotic oscillators can be synchronized by a suitable conservative coupling. At the same time, the simulation presented here reminds the fact, that a relatively easy model of oscillation can lead to the complex, chaotic responses.

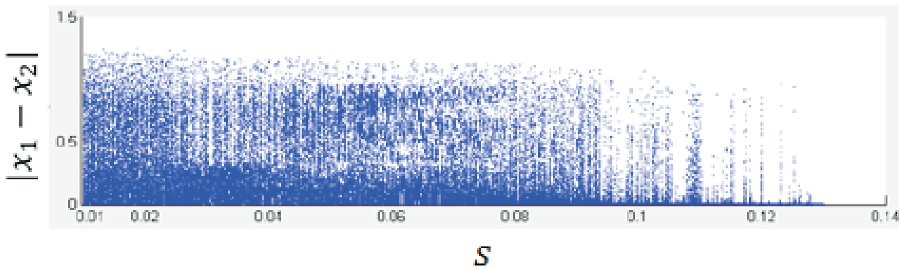

Figure 4 Bifurcation diagram of the conservative coupling parameter s

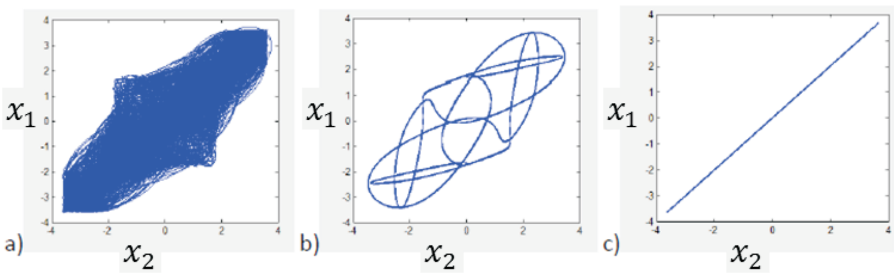

Figure 5 Three states of the system deduced from the bifurcation diagram: a) no synchronization, b) synchronization in counter phase, c) full synchronization

In such a case, all the computations need to be done cautiously as the chaotic systems are extremely sensitive to any changes in conditions. Even a round off error or an approximation made by a numerical solver can lead to the qualitative change in results.

As the title suggests, the paper does not tackle the more advanced issues connected with the analysis of synchronization. The next step could be to estimate a stability of the observed synchronous states, e.g. with use of Liapunov exponents [2, 3, 11] or to generalize the results on different networks of oscillators, but it would exceed the scope of this work.

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