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ANALYTICITY OF SOLUTIONS TO THE PRIMITIVE EQUATIONS

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ABSTRACT. This article presents the maximal regularity approach to the primitive equations. It is proved that the 3D primitive equations on cylindrical domains admit a unique, global strong solution for initial data lying in the critical solenoidal Besov space $B_{pq}^{2/p}$ for $p, q \in (1, \infty)$ with $1/p + 1/q \leq 1$. This solution regularize instantaneously and becomes even real analytic for $t > 0$.

1. INTRODUCTION

The primitive equations for the ocean and atmosphere are considered to be a fundamental model for geophysical flows which is derived from Navier-Stokes equations assuming a hydrostatic balance for the pressure term in the vertical direction. The mathematical analysis of the primitive equations commenced by Lions, Teman and Wang in the series of articles [25–27]; for a survey of known results and further references to the literature, we refer to the recent article by Li and Titi [28].

In contrast to Navier-Stokes equations, the 3D primitive equations admit a unique, global, strong solution for arbitrary large data in H^1 . This breakthrough result was proved by Cao and Titi [6] in 2007 using energy methods. A different approach to the primitive equations, based on methods of evolution equations, has been presented in [20]. There a Fujita-Kato type iteration scheme was developed in addition to H^2 -a priori bounds for the solution.

It is the aim of this article to present a third approach to the primitive equations, this time based on techniques from the theory of maximal L^q -regularity. This approach has several advantages compared to the two other approaches. Let us note first that combining this approach with the so-called parameter-trick due to Angenent [2,3] we are able to rigorously prove the immediate smoothing effect of solution. In particular, the solution regularizes instantaneously to become *real analytic in time and space*, a property, which is interesting for its own sake.

Real analyticity of solutions of certain classes of partial differential equations is usually difficult to prove directly, however, our approach allows to employ the implicit function theorem and thus yields an elegant strategy for solving this problem. For first results in this direction concerning the Navier-Stokes equations, we refer to the work of Masuda [30]; for the general theory concerning quasilinear systems and refinements, we refer to [32]. Let us remark that C^∞ -smoothness properties of the solutions to the primitive equations have also been obtained by Li and Titi in [29] by very different methods.

The above regularizing effect plays an important role when extending local solutions to global ones by means of certain a priori bounds. So far, in order to control the existence time in L^p -spaces, H^2 -a priori bounds have been used in [20,21]. In the following, we show that a priori bounds in the maximal regularity space $L^2(0, T; H^2) \cap H^1(0, T; L^2)$ are already sufficient to prove the global existence of a solution in L^q - L^p -spaces. Smoothing properties of the solution play also a very important role in the proof of the recent results on the existence of global, strong solutions to the primitive equations for rough initial data lying in the anisotropic and scaling invariant spaces $L^\infty(L^1)$ and $L^\infty(L^p)$; see [15].

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Second, our approach allows to prove the existence and uniqueness of a global, strong solution for initial values lying in critical spaces, which in the given situation are the Besov spaces

$$B_{pq}^\mu \quad \text{for } p, q \in (1, \infty) \quad \text{with } 1/p + 1/q \leq \mu \leq 1.$$

Here, we use in an essential way the concept of time weights for maximal L^p -regularity, see [32] and [35]. The above spaces seem to be the largest spaces of initial data for which one obtains the existence of a unique, global strong solution to the primitive equations when considering the problem within the $L^q - L^p$ -framework with $1 < p, q < \infty$.

Note that the above spaces are critical function spaces, where by critical is understood in the sense discussed e.g. in [34]. These spaces correspond in the situation of the Navier-Stokes equations to the critical function spaces $B_{pq}^{n/p-1}$ introduced by Cannone [5] for the full space case \mathbb{R}^n , and by Prüss and Wilke [35] for bounded domains. Also, for other solution classes for the 3-d Navier-Stokes equations initial conditions in Besov spaces occur such as in the works by Farwig, Giga and Hsu [10–12] which investigate solutions which are continuous in time and taking values in the class of Besov spaces $B_{p,q}^{3/p-1}$ for suitable coefficients p, q including the case $q = \infty$.

Choosing in particular $p = q = 2$ and $\mu = 1$ and noting that $B_{22}^1 = H^1$, we rediscover in particular the celebrated result by Cao and Titi [6]. Furthermore, choosing $p, q > 2$ allows us to enlarge the space of admissible initial values $H^{2/p,p}$ as constructed in [20, 21] to the above more general Besov space setting since $H^{2/p,p} \subset B_{pq}^{2/p}$. This class of initial values is also used in our works [16] on rough initial data lying in $L^\infty(L^p)$ for the case of mixed Dirichlet and Neumann boundary conditions. There, the solutions obtained in this article serve as reference solutions belonging to initial values in B_{pq}^μ and where parameters are chosen in such a way that $B_{pq}^\mu \hookrightarrow C^1$.

Third, our approach allows us to treat various types boundary conditions, such as Dirichlet, Neumann, and mixed Dirichlet and Neumann boundary conditions in a unified way. The corresponding $L^2(0, T; H^2) \cap H^1(0, T; L^2)$ a priori bounds for the case of mixed Dirichlet and Neumann boundary conditions rely on results obtained in [13, 20] and for Dirichlet boundary conditions these bounds can be obtained similarly. For the remaining case of pure Neumann boundary conditions, we present a proof of these bounds in Section 6. We hence obtain the existence of a unique, global, strong solution to the primitive equations for all of these boundary conditions.

This article is organized as follows: In Section 2 we describe the setting in detail and the main results are presented in Section 3. Some information to the linear theory is recapped and supplemented in Section 4. We collect the relevant results on maximal L^q -regularity in Section 5; they will be then applied in the proofs of our main results given in Section 6. Finally, the various approaches are compared in Section 7.

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2. PRELIMINARIES

Consider a cylindrical domain $\Omega = G \times (-h, 0) \subset \mathbb{R}^3$ with $G = (0, 1) \times (0, 1)$, $h > 0$. For simplicity, we investigate the primitive equations in the isothermal setting and denote by $v: \Omega \rightarrow \mathbb{R}^2$ the vertical velocity of the fluid and $\pi_s: G \rightarrow \mathbb{R}$ its surface pressure. There exist several equivalent formulations of the primitive equations, depending on whether the horizontal velocity $w = w(v)$ is completely substituted by the vertical velocity v and the full pressure by the surface pressure, respectively, compare e.g. [20]. For the purpose of this article the following representation of the primitive equations is the most convenient,

$$(2.1) \quad \begin{cases} \partial_t v + v \cdot \nabla_H v + w(v) \cdot \partial_z v - \Delta v + \nabla_H \pi_s = f, & \text{in } \Omega \times (0, T), \\ \operatorname{div}_H \bar{v} = 0, & \text{in } \Omega \times (0, T), \\ v(0) = v_0, & \text{in } \Omega, \end{cases}$$

where denoting by $x, y \in G$ horizontal coordinates and by $z \in (-h, 0)$ the vertical one, we use the notations

$$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2, \quad \nabla_H = (\partial_x, \partial_y)^T, \quad \operatorname{div}_H v = \partial_x v_1 + \partial_y v_2 \quad \text{and} \quad \bar{v} := \frac{1}{h} \int_{-h}^0 v(\cdot, \cdot, \xi) d\xi.$$

Here the horizontal velocity $w = w(v)$ is given by

$$w(v)(x, y, z) = - \int_{-h}^z \operatorname{div}_H v(x, y, \xi) d\xi, \quad \text{where} \quad w(x, y, -h) = w(x, y, 0) = 0.$$

The equations (2.1) are supplemented by the mixed boundary conditions on

$$\Gamma_u = G \times \{0\}, \quad \Gamma_b = G \times \{-h\} \quad \text{and} \quad \Gamma_l = \partial G \times (-h, 0),$$

i.e. the upper, bottom and lateral parts of the boundary $\partial\Omega$, respectively, given by

$$(2.2) \quad \begin{aligned} &v, \pi_s \text{ are periodic on } \Gamma_l \times (0, \infty), \\ &v = 0 \text{ on } \Gamma_D \times (0, \infty) \quad \text{and} \quad \partial_z v = 0 \text{ on } \Gamma_N \times (0, \infty). \end{aligned}$$

where Dirichlet, Neumann and mixed boundary conditions are comprised by the notation

$$\Gamma_D \in \{\emptyset, \Gamma_u, \Gamma_b, \Gamma_u \cup \Gamma_b\} \quad \text{and} \quad \Gamma_N = (\Gamma_u \cup \Gamma_b) \setminus \Gamma_D.$$

In the literature several sets of boundary conditions are considered. So, in [25, Equation (1.37) and (1.37)'] Dirichlet and mixed Dirichlet Neumann boundary conditions are considered, respectively, while in [6] Neumann boundary conditions are assumed.

Similarly to the Navier-Stokes equations, one may consider *hydrostatically solenoidal vector fields* as a subspace of $L^p(\Omega)^2$ for $p \in (1, \infty)$ which, following the approach developed in [20, Sections 3 and 4] is defined by

$$(2.3) \quad L_{\bar{\sigma}}^p(\Omega) = \overline{\{v \in C_{per}^\infty(\bar{\Omega})^2 : \operatorname{div}_H \bar{v} = 0\}}^{\|\cdot\|_{L^p(\Omega)^2}},$$

where horizontal periodicity is modeled by the function spaces $C_{per}^\infty(\bar{\Omega})$ and $C_{per}^\infty(\bar{G})$ is defined as in [20, Section 2], where smooth functions are periodic only with respect to x, y coordinates and not necessarily in the z coordinate.

Furthermore, there exists a continuous projection P_p , called the *hydrostatic Helmholtz projection*, from $L^p(\Omega)^2$ onto $L_{\bar{\sigma}}^p(\Omega)$, see [14] and [20] for details. In particular, P_p annihilates the pressure term $\nabla_H \pi_s$.

For $p \in (1, \infty)$ and $s \in [0, \infty)$ define the spaces

$$H_{per}^{s,p}(\Omega) := \overline{C_{per}^\infty(\bar{\Omega})}^{\|\cdot\|_{H^{s,p}(\Omega)}} \quad \text{and} \quad H_{per}^{s,p}(G) := \overline{C_{per}^\infty(\bar{G})}^{\|\cdot\|_{H^{s,p}(G)}},$$

where $H_{per}^{0,p} := L^p$. Here $H^{s,p}(\Omega)$ denotes the Bessel potential spaces, which are defined as restrictions of Bessel potential spaces on the whole space to Ω , compare e.g [36, Definition 3.2.2.]. It is well known that the space $H^{s,p}(\Omega)$ coincides with the classical Sobolev space $W^{m,p}(\Omega)$ provided $s = m \in \mathbb{N}$.

Also, we define for $p, q \in (1, \infty)$ and $s \in [0, \infty)$ the Besov spaces

$$B_{pq,per}^s(\Omega) := \overline{C_{per}^\infty(\bar{\Omega})}^{\|\cdot\|_{B_{pq}^s(\Omega)}} \quad \text{and} \quad B_{pq,per}^s(G) := \overline{C_{per}^\infty(\bar{G})}^{\|\cdot\|_{B_{pq}^s(G)}},$$

where B_{pq}^s denotes Besov spaces, which are defined as restrictions of Besov spaces on the whole space $B_{p,q}^s(\mathbb{R}^3)$, compare e.g. [36, Definitions 3.2.2].

Following [20], we define the *hydrostatic Stokes operator* A_p in $L_{\bar{\sigma}}^p(\Omega)$ as

$$A_p v := P_p \Delta v, \quad D(A_p) := \{v \in H_{per}^{2,p}(\Omega)^2 : \partial_z v|_{\Gamma_N} = 0, v|_{\Gamma_D} = 0\} \cap L_{\bar{\sigma}}^p(\Omega).$$

In [14] it has been shown that A_p has the property of maximal L^q -regularity. Following [33, Theorem 3.2] this is equivalent to maximal L^q -regularity of A_p in time-weighted spaces, which are defined for

$\mu \in (1/q, 1]$ and for $k \in \mathbb{N}$ recursively by

$$\begin{aligned} L_\mu^q(J; D(A_p)) &= \{v \in L_{loc}^1(J; D(A_p)) : t^{1-\mu}v \in L^q(J; D(A_p))\}, \\ H_\mu^{1,q}(J; L_\sigma^p(\Omega)) &= \{v \in L_\mu^q(J; L_\sigma^p(\Omega)) \cap H^{1,1}(J; L_\sigma^p(\Omega)) : t^{1-\mu}v_t \in L^q(J; L_\sigma^p(\Omega))\}, \\ &\vdots \\ H_\mu^{k+1,q}(J; L_\sigma^p(\Omega)) &= \{v \in H_\mu^{k,q}(J; L_\sigma^p(\Omega)) : v_t \in H^{k,q}(J; L_\sigma^p(\Omega))\}. \end{aligned}$$

Here v_t stands for the time derivative of v and $J = (0, T)$ denotes for $0 < T \leq \infty$ a time interval.

The natural trace spaces of these spaces are determined by real interpolation $(\cdot, \cdot)_{\theta, q}$ for $\theta \in (0, 1)$ and $p, q \in (1, \infty)$. They can be computed explicitly in terms of Besov spaces, as we will prove in Section 4.

Lemma 2.1. *Let $\theta \in (0, 1)$ and $p, q \in (1, \infty)$. Then for $X_{\theta, q} := (L_\sigma^p(\Omega), D(A_p))_{\theta, q}$ it holds that*

$$X_{\theta, q} = \begin{cases} \{v \in B_{pq, per}^{2\theta}(\Omega) \cap L_\sigma^p(\Omega) : \partial_z v|_{\Gamma_N} = 0, v|_{\Gamma_D} = 0\}, & \frac{1}{2} + \frac{1}{2p} < \theta < 1, \\ \{v \in B_{pq, per}^{2\theta}(\Omega) \cap L_\sigma^p(\Omega) : v|_{\Gamma_D} = 0\}, & \frac{1}{2p} < \theta < \frac{1}{2} + \frac{1}{2p}, \\ B_{pq, per}^{2\theta}(\Omega) \cap L_\sigma^p(\Omega), & 0 < \theta < \frac{1}{2p}. \end{cases}$$

3. MAIN RESULTS

Our first main result is the global, strong well-posedness of the primitive equations for arbitrarily large data in the critical Besov spaces defined above in Lemma 2.1.

Theorem 3.1 (Global well-posedness). *Let $p, q \in (1, \infty)$ such that $1/p + 1/q \leq 1$. For $0 < T < \infty$ let $\mu \in [1/p + 1/q, 1]$,*

$$v_0 \in X_{\mu-1/q, q} \quad \text{and} \quad P_p f \in H_\mu^{1,q}(0, T; L_\sigma^p(\Omega)) \cap H^{1,2}(\delta, T; L_\sigma^2(\Omega))$$

for some $\delta > 0$ sufficiently small.

Then there exists a unique, strong solution v to the primitive equations (2.1) satisfying

$$v \in H_\mu^{1,q}(0, T; L_\sigma^p(\Omega)) \cap L_\mu^q(0, T; D(A_p)).$$

Considering in particular the case $p = q = 2$, we are not only reproducing the known global existence result [6], but state furthermore that additional time regularity of the forcing term improves the regularity of these solutions.

Proposition 3.2. *Let $0 < T < \infty$ and $v_0 \in \{H^1 \cap L_\sigma^2(\Omega) : v|_{\Gamma_D} = 0\}$.*

(a) *If $P_2 f \in L^2(0, T; L_\sigma^2(\Omega))$, then there exists a unique, strong solution v to the primitive equations (2.1) in*

$$v \in H^1(0, T; L_\sigma^2(\Omega)) \cap L^2(0, T; D(A_2)).$$

(b) *If in addition $t \mapsto t \cdot P_2 f_t(t) \in L^2(0, T; L_\sigma^2(\Omega))$, then*

$$t \cdot v_t \in H^1(0, T; L_\sigma^2(\Omega)) \cap L^2(0, T; D(A_2)).$$

The following second main theorem deals with the parabolic smoothing effect and the real analyticity of the solution. Note that the additional regularity assumption on f is needed, together with Proposition 3.2, to prove the global existence of the solution in Theorem 3.1 above. We set $v^{(j)} := \partial_t^j v$ and denote by C^ω the space of real analytic functions.

Theorem 3.3 (Regularity). *Let $v \in H_\mu^{1,q}(0, T; L_\sigma^p(\Omega)) \cap L_\mu^q(0, T; D(A_p))$ be the solution to the primitive equations for $v_0 \in X_{\mu-1/q, q}$ and $P_p f \in L_\mu^q(0, T; L_\sigma^p(\Omega))$ for p, q, μ as in Theorem 3.1.*

(a) *If $P_p f \in H_\mu^{k,q}(0, T; L_\sigma^p(\Omega))$ for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then for any $0 < T' < T$*

$$t^j \cdot v^{(j)} \in H_\mu^{1,q}(0, T'; L_\sigma^p(\Omega)) \cap L_\mu^q(0, T'; D(A_p)), \quad j = 0, \dots, k,$$

$$v \in H_{loc}^{k+1,p}(0, T; L_\sigma^p(\Omega)) \cap H_{loc}^{k,q}(0, T; D(A_p)) \cap C^k((0, T); X_{1-1/q, q});$$

(b) If $P_p f \in C^\infty((0, T); L_\sigma^p(\Omega))$ or $P_p f \in C^\omega((0, T); L_\sigma^p(\Omega))$, then

$$v \in C^\infty((0, T); D(A_p)) \quad \text{or} \quad v \in C^\omega((0, T); D(A_p)),$$

respectively;

(c) If $P_p f \in C^\infty((0, T); C_{per}^\infty(\Omega)^2)$ or $P_p f \in C^\omega((0, T); C_{per}^\omega(\Omega)^2)$, then

$$v \in C^\infty((0, T); C_{per}^\infty(\Omega)^2) \quad \text{or} \quad v \in C^\omega((0, T); C_{per}^\omega(\Omega)^2),$$

respectively.

The proof of the above regularity results relies on the implicit function theorem. Masuda [30] introduced first an extra parameter to prove the spatial analyticity of solutions to the Navier-Stokes equations using the implicit function theorem. Angenent [2, 3] systematically developed the nowadays called parameter trick valid for general quasilinear evolution equations. For proofs within the L^p -setting we are working in and for further refinements of this method, we refer e.g. to [32].

Remarks 3.4. (a) The solution class given in Theorem 3.1 admits the following embeddings

$$\begin{aligned} H_\mu^{1,q}(0, T; L_\sigma^p(\Omega)) \cap L_\mu^q(0, T; D(A_p)) &\hookrightarrow C([0, T]; X_{\mu-1/q,q}), \\ H_\mu^{1,q}(0, T; L_\sigma^p(\Omega)) \cap L_\mu^q(0, T; D(A_p)) &\hookrightarrow C((\delta, T]; X_{1-1/q,q}), \quad \delta > 0. \end{aligned}$$

- (b) Replacing in Theorem 3.1 the condition on $P_p f$ by $P_p f \in L_\mu^q(0, T; L_\sigma^p(\Omega))$ one still obtains local solutions, see Proposition 6.3 below. However, to extend this solution to a global one, one needs a priori bounds in $X_{\bar{\mu},q}$ with $\mu < \bar{\mu} \leq 1$, that is, a slightly more regular space, than the space of admissible initial values $X_{\mu,q}$, compare Theorem 5.3 below. The additional assumption on the time regularity of $P_p f$ in Theorem 3.1 assures that this solution is also an L^2 solution, and by Proposition 3.2 $v \in H^1(\delta, T; D(A_2)) \hookrightarrow C([\delta, T]; D(A_2))$ for $\delta \in (0, T)$.
- (c) Concerning analyticity of solutions, for solutions u to the Navier-Stokes equations in \mathbb{R}^n with initial values in $u_0 \in L^n(\mathbb{R}^n)^n$ estimates on $\|D^\beta u\|_{L^q(\mathbb{R}^n)}$, $\beta \in \mathbb{N}_0^n$, have been established in [18] using heat kernel estimates. However, the method used there is not applicable in the presence of boundaries.
- (d) The surface pressure π_s can be reconstructed from v , compare [21, Equation (6.2)]. This gives $\pi_s \in L_\mu^q(0, T; H_{per,0}^{1,p}(G))$, where $H_{per,0}^{1,p}(G) = \{\pi_s \in H_{per}^{1,p}(G) : \int_G \pi_s = 0\}$.

4. LINEAR THEORY

Consider the linear problem

$$(4.1) \quad \begin{cases} \partial_t v - \Delta v + \nabla_H \pi_s = f, \\ \operatorname{div}_H \bar{v} = 0, \\ v(0) = v_0 \end{cases}$$

subject to boundary conditions (2.2). As described in [14], the key observation for solving this problem is to solve first the equation (4.1) for the surface pressure and to consider the hydrostatic Stokes operator defined by

$$(4.2) \quad A_p v = \Delta v - (\mathbb{1} - P_p)(\partial_z v)|_{\Gamma_D}, \quad \text{for } v \in D(A_p).$$

This allows us to analyze the above linear problem by perturbation methods for the Laplacian. In particular, it was shown in [14] that $-A_p$ admits a bounded H^∞ -calculus and thus also maximal L^q -regularity.

The analysis in [14] makes use of exponential stability of the hydrostatic semigroup generated by A_p . The latter fact has been obtained first in [20]. We give here an alternative proof of this property via the positivity of the spectrum of $-A_p$ and collect in addition further spectral properties of A_p .

Proposition 4.1. *Let $p \in (1, \infty)$ and $\lambda \geq 0$ if $\Gamma_D \neq \emptyset$ and $\lambda > 0$ if $\Gamma_D = \emptyset$.*

- (a) Let $v \in D(A_p)$ and $(A_p - \lambda)v = f$. If $P_p f \in H^{s,p}(\Omega)^2$ for $s \geq 0$, then $v \in H^{s+2,p}(\Omega)^2$ and there exists a constant $C > 0$ such that

$$\|v\|_{H^{s+2,p}(\Omega)^2} \leq C \|f\|_{H^{s,p}(\Omega)^2}.$$

- (b) The spectrum of A_p is purely discrete and all the eigenfunctions of A_p belong to $C^\infty(\overline{\Omega})^2$. In particular, the spectrum of $-A_p$ is independent of p , $\sigma(-A_p) \subset [0, \infty)$ and $\sigma(-A_p) \subset [c, \infty)$ for some $c > 0$ provided $\Gamma_D \neq \emptyset$.
- (c) The semigroup generated by $A_p - \lambda$ is exponentially stable.

Remarks 4.2. (a) In the quantitative analysis of the primitive model the Coriolis force, which may be incorporated by replacing f in 2.1 by $f_0 k \times v$, plays an important role. For the qualitative analysis we omit this term since it is a zero order term, which can be included easily into our analysis by setting

$$A_p v = P_p \Delta + P_p (f_0 k \times v).$$

- (b) In [7] the hydrostatic Stokes operator with Robin boundary conditions is considered. The strategy to solve first for the surface pressure can be applied in this case as well.

Proof of Proposition 4.1. Let $v \in D(A_p) \subset H^{2,p}$ and $P_p f \in H^{s,p}(\Omega)^2 \subset L^p(\Omega)^2$. We then obtain

$$(\Delta - \lambda)v = P_p f + (\mathbb{1} - P_p)(\partial_z v)|_{\Gamma_D} \in H^{\min\{s, 1-1/p-\epsilon\}}$$

for some $\epsilon > 0$ and elliptic regularity for the Laplacian implies $v \in H^{2+\min\{s, 1-1/p-\epsilon\}}(\Omega)$. Iterating this argument, the assertion follows.

The discreteness of the spectrum follows from the compactness of the embedding $D(A_p) \subset H^{2,p}(\Omega) \cap L^p_\sigma(\Omega) \hookrightarrow L^p_\sigma(\Omega)$. Again, by elliptic regularity of A_p , eigenfunctions of A_p are in $C^\infty(\overline{\Omega})$ and thus independent of p . Hence, it is sufficient to compute the spectrum in the case $p = 2$, where $-A_2 = -P_2 \Delta$ is associated to the closed symmetric form defined by

$$\mathfrak{a}[v, v'] := \langle \nabla v, \nabla v' \rangle_{L^2(\Omega)^{2 \times 2}}, \quad \text{where } v, v' \in \{H^1_{per}(\Omega) \cap L^2_\sigma(\Omega) : v|_{\Gamma_D} = 0\}.$$

Eigenvalues are characterized by the min-max principle, and therefore enlarging the form domain to $\{v \in H^{1,2}_{per}(\Omega)^2 : v|_{\Gamma_D} = 0\}$, we conclude that the eigenvalues of $-A_p$ can be inferred by those of $-\Delta$, and in the proof of (a) one can choose any $\lambda \geq 0$ for $\Gamma_D \neq \emptyset$ and any $\lambda > 0$ for $\Gamma_D = \emptyset$. The exponential stability of the hydrostatic semigroup follows from the fact that for analytic semigroups the growth bound coincides with the spectral bound. \square

Proof of Lemma 2.1. Note that the cylindrical domain Ω can be extended to the 3 dimensional full torus, and therefore we may equivalently define the function spaces appearing in Lemma 2.1 as restrictions of the function spaces on the full torus, compare e.g. [36, Chapter 9]. Then, the real interpolation space $X_{\theta,q}$ can be computed by means of these retractions and co-retractions, which are related to function spaces on the full torus, compare [37, Theorem 1.2.4] and [37, 1.7.1 Theorem 1]. Since $D(A_p)$ consists of functions with Dirichlet, Neumann or mixed boundary conditions one can extend these by odd and even extensions to the full torus, which defines a co-retraction to the full torus, where real interpolation spaces are known, compare e.g. [37, Section 4.11.1]. Since odd and even function have vanishing traces and traces of the derivatives, respectively, the traces can be found in the interpolation spaces as long as they are well-defined. Alternatively, the retractions and co-retractions given in [21, Section 4] can be used. \square

5. SEMILINEAR EVOLUTION EQUATIONS AND MAXIMAL L^q -REGULARITY

In this section we present results on semilinear evolution equations, which then will be applied to the primitive equations in Section 6. We also present only a simplified version of a more general result for quasilinear systems due to Prüss and Wilke [35], which, however, will be sufficient for our purposes.

Let X_0, X_1 be Banach spaces such that $X_1 \hookrightarrow X_0$ is densely embedded, and let $A: X_1 \rightarrow X_0$ be bounded. The aim is to solve the semi-linear problem for $0 < T \leq \infty$

$$(5.1) \quad u' + Au = F(u) + f, \quad 0 < t < T, \quad u(0) = u_0.$$

For a Banach space X , a time weight $\mu \in (1/q, 1]$, a time interval $J \subset [0, \infty)$ and $k \in \mathbb{N}$, we set

$$\begin{aligned} L_\mu^q(J; X) &= \{u \in L_{loc}^1(J; X) : t^{1-\mu}u \in L^q(J; X)\}, \\ H_\mu^{1,q}(J; X) &= \{u \in L_{loc}^1(J; X) \cap H^{1,1}(J; X) : t^{1-\mu}u' \in L^q(J; X)\}, \\ &\vdots \\ H_\mu^{k+1,q}(J; X) &= \{u \in L_{loc}^1(J; X) \cap H^{k+1,1}(J; X) : u' \in H_\mu^{k,q}(J; X)\}, \end{aligned}$$

Here u' denotes the time derivative of u in the distributional sense.

The problem (5.1) is considered for initial data within the real interpolation space

$$u_0 \in X_{\gamma,\mu} = (X_0, X_1)_{\mu-1/q,q} \quad \text{and for } f \in \mathbb{E}_{0,\mu}(J) := L_\mu^q(J; X_0),$$

where $q \in (1, \infty)$. We aim for solutions lying in the maximal regularity space

$$\mathbb{E}_{1,\mu}(J) := H_\mu^{1,q}(J; X_0) \cap L_\mu^q(J; X_1)$$

and define for $\beta \in [0, 1]$ the space X_β as the complex interpolation space $[X_0, X_1]_\beta$.

The following existence and uniqueness results are based on the following assumptions:

- (H1) A has maximal L^q -regularity for $q \in (1, \infty)$.
- (H2) $F: X_\beta \rightarrow X_0$ satisfied the estimate

$$\|F(u_1) - F(u_2)\|_{X_0} \leq C(\|u_1\|_{X_\beta} + \|u_2\|_{X_\beta})(\|u_1 - u_2\|_{X_\beta})$$

for some $C > 0$ independent of u_1, u_2 .

- (H3) $\beta - (\mu - 1/q) \leq \frac{1}{2}(1 - (\mu - 1/q))$, that is $2\beta - 1 + 1/q \leq \mu$.
- (S) X_0 is of class UMD, and the embedding

$$H^{1,q}(\mathbb{R}; X_0) \cap L^q(\mathbb{R}; X_1) \hookrightarrow H^{1-\beta,q}(\mathbb{R}; X_\beta)$$

is valid for each $\beta \in (0, 1)$ and $q \in (1, \infty)$.

Theorem 5.1. [35, Theorem 1.2]. *Assume that the assumptions (H1), (H2), (H3) and (S) hold and let*

$$u_0 \in X_{\gamma,\mu} \quad \text{and} \quad f \in L^q(0, T; X_0).$$

Then there exists a time $T' = T'(u_0)$ with $0 < T' \leq T$ such that problem (5.1) admits a unique solution

$$u \in H_\mu^{1,q}(0, T'; X_0) \cap L_\mu^q(0, T'; X_1).$$

Furthermore, the solution u depends continuously on the data.

- Remarks 5.2.**
- (a) Condition (S) holds true whenever X_0 is of class UMD and there is an operator $A_\# \in \mathcal{H}^\infty(X_0)$ with domain $D(A_\#) = X_1$ satisfying $\phi_{A_\#}^\infty < \pi/2$, see Remark 1.1 of [35].
 - (b) To verify condition (S) in the situation considered here, i.e., $X_0 = L_\sigma^p(\Omega)$ and $X_1 = D(A_p)$, note first that $L_\sigma^p(\Omega)$ is of class UMD as closed subspace of $L^p(\Omega)^2$, and second, considering $A_\# = A_p - \lambda$ with $\lambda > 0$, it was proved in [14] that $A_\# \in \mathcal{H}^\infty(X_0)$ with $\phi_{A_\#}^\infty = 0 < \pi/2$.
 - (c) Due to the embeddings

$$\mathbb{E}_{1,\mu}(0, T') \hookrightarrow C([0, T']; X_{\gamma,\mu}) \quad \text{and} \quad \mathbb{E}_{1,\mu}(\delta, T') \hookrightarrow C([\delta, T']; X_\gamma), \quad \delta > 0,$$

there is an instantaneous smoothing effect typical for parabolic equations, compare e.g. [32, Section 3.5.2].

When investigating the question of a global solution, we consider

$$t_+(u_0) := \sup\{T' > 0: \text{equation (5.1) admits a solution on } (0, T')\}.$$

By the above Theorem 5.1, this set is non-empty, and we say that (5.1) has a *global solution* if for $f \in L^q(0, T; X_0)$ one has $t_+(u_0) = T$, where $0 < T \leq \infty$. Global existence results can be derived from suitable a priori bounds following [32, Theorem 5.7.1]. The statement of [32, Theorem 5.7.1] and the relevant corollary [32, Corollary 5.1.2] are not formulated for the optimal time-weight used in [35, Theorem 1.2], however they carry over to this situation directly and without modifications. Therefore we state them here without proof. In the following, we denote by C_b bounded continuous functions.

Theorem 5.3. [32, Theorem 5.7.1] *Assume in addition to the assumptions of Theorem 5.1 that for $\mu < \bar{\mu} \leq 1$ the embedding*

$$X_{\gamma, \bar{\mu}} \hookrightarrow X_{\gamma, \mu}$$

is compact, and that for some $\tau \in (0, t_+(u_0))$ the solution of (5.1) satisfies

$$u \in C_b([\tau, t_+(u_0)]; X_{\gamma, \bar{\mu}}),$$

then there is a global solution to (5.1), i.e. $T' = T$.

Additional time and space-time regularity solutions of (5.1) for more regular right hand side f can be derived using the parameter trick based on the implicit function theorem. Here for the time regularity, the version of the parameter trick is adapted to time-weighted spaces, see e.g. [8, Theorem 9.1], [32, Section 9.4]

Let $F: X_1 \rightarrow X_0$ be a continuously differentiable function and f an integrable function. We consider the problem

$$(5.2) \quad u' + F(u) = f.$$

Theorem 5.4. *Compare [8, Theorem 9.1] Let $q \in (1, \infty)$. Assume that for $k \in \mathbb{N}$, the composition operator*

$$\mathcal{F}: H_\mu^{1,q}(0, T; X_0) \cap L_\mu^q(0, T; X_1) \rightarrow L_\mu^q(0, T; X_0), \quad u \mapsto F(u)$$

is k times continuously differentiable. Let $f \in L_\mu^q(0, T; X_0)$ and let $u \in H_\mu^{1,q}(0, T; X_0) \cap L_\mu^q(0, T; X_1)$ be a solution to (5.2) on $(0, T)$. Assume that for the differential DF of F the linear problem

$$u' + DF(u)v = g, \quad v(0) = 0,$$

admits for every $g \in L_\mu^q(0, T'; X_0)$, $0 < T' < T$ a unique solution $v \in H_\mu^{1,q}(0, T'; X_0) \cap L_\mu^q(0, T'; X_1)$.

Then for every $j = 0, \dots, k$

$$\begin{aligned} u &\in H_{loc}^{k+1,q}(0, T'; X_0) \cap H_{loc}^{k,q}(0, T'; X_1), \\ t &\mapsto t^j u^{(j)}(t) \in H_\mu^{1,q}(0, T'; X_0) \cap L_\mu^q(0, T'; X_1). \end{aligned}$$

If \mathcal{F} and f are of class C^∞ or C^ω , then $u \in C^\infty((0, T); X_1)$ or $u \in C^\omega((0, T); X_1)$, respectively.

Remark 5.5. In many versions of such regularity theorems the mapping property $\mathcal{F}: X_{\gamma, \mu} \rightarrow \mathcal{L}(X_1, X_0)$ is assumed, while the condition imposed in [8, Theorem 9.1] is weaker compared to other versions.

6. PROOFS OF THE MAIN RESULTS

In this section we prove our main results. For $1 < p < \infty$ we set

$$X_0 = L_{\bar{\sigma}}^p(\Omega) \quad \text{and} \quad X_1 = D(A_p).$$

6.1. Local well-posedness. Similarly to [20, Section 5], we define for $1 < p < \infty$ the bilinear map F_p by

$$F_p(v, v') := P_p(v \cdot \nabla_H v' + w(v) \partial_z v'),$$

and set $F_p(v) := F_p(v, v)$. Since $u' = (v', w(v'))$ is divergence-free, we also obtain the representation

$$F_p(v, v') = P_p \operatorname{div}(u' \otimes v).$$

We start by collecting various facts concerning the map F_p .

Lemma 6.1. *There exists a constant $C > 0$, depending only on Ω , $p \in (1, \infty)$ and $s \geq 0$, such that for $v, v' \in H^{s+1+1/p, p}(\Omega)^2$*

$$\|F_p(v, v')\|_{H^{s, p}(\Omega)^2} \leq C \|v\|_{H^{s+1+1/p, p}} \|v'\|_{H^{s+1+1/p, p}},$$

i.e., $F_p(\cdot, \cdot): H^{s+1+1/p, p}(\Omega)^2 \times H^{s+1+1/p, p}(\Omega)^2 \rightarrow H^{s, p}(\Omega)^2$ is a continuous bilinear map.

Proof. The assertion is proved first inductively for the case $s = m \in \mathbb{N}_0$. A complex interpolation result for non-linear operators due to Bergh [4], which, due to the bilinear structure of $F_p(\cdot, \cdot)$, is applicable completes the proof.

The induction basis $m = 0$ follows as in [20, Lemma 5.1] using anisotropic estimates and the bilinearity of $F_p(\cdot, \cdot)$. To prove the induction step, observe that

$$\|F_p(v, v')\|_{H^{m+1, p}} \leq \|\nabla F_p(v, v')\|_{H^{m, p}} + \|F_p(v, v')\|_{H^{m, p}}$$

by using

$$\partial_i F_p(v, v') = F_p(\partial_i v, v') + F_p(v, \partial_i v'), \quad i \in \{x, y, z\}$$

and $\|\partial_i v\|_{H^{m+1+1/p, p}} \leq C \|v\|_{H^{m+2+1/p, p}}$. □

Recalling that X_β is given by $X_\beta = [L_\sigma^p(\Omega), D(A_p)]_\beta$ we obtain

$$[L_\sigma^p(\Omega), D(A_p)]_{(1+1/p)/2} \subset H^{1+1/p, p}(\Omega)^2,$$

which yields the following corollary of Lemma 6.1.

Corollary 6.2. *Let $\beta = \frac{1}{2}(1 + 1/p)$. Then there exists a constant $C > 0$, independent of v, v' , such that*

$$\|F_p(v) - F_p(v')\|_{L_\sigma^p(\Omega)} \leq C (\|v\|_{X_\beta} + \|v'\|_{X_\beta}) \|v - v'\|_{X_\beta}.$$

Local well-posedness for the primitive equations follows now from Theorem 5.1 by using Remarks 5.2 (a) and (b) for the conditions (S) and (H1) and Corollary 6.2 for the conditions (H2) and (H3).

Proposition 6.3 (Local well-posedness). *Let $p, q \in (1, \infty)$ with $1/p + 1/q \leq 1$, $\mu \in [1/p + 1/q, 1]$ and $T > 0$. Assume that*

$$v_0 \in X_{\mu-1/q, q} \quad \text{and} \quad P_p f \in L_\mu^q(0, T; L_\sigma^p(\Omega)).$$

Then there exists $T' = T'(v_0)$ with $0 < T' \leq T$ and a unique, strong solution v to (2.1) on $(0, T')$ with

$$v \in H_\mu^{1, q}(0, T'; L_\sigma^p(\Omega)) \cap L_\mu^q(0, T'; D(A_p)).$$

6.2. Time and space regularity. Define $F(v) := A_p v + F_p(v)$.

Lemma 6.4. *Let $p, q \in (1, \infty)$ with $1/p + 1/q \leq 1$, $\mu \in [1/p + 1/q, 1]$, $T > 0$. Then the mapping*

$$\mathcal{F}: H_\mu^{1, q}(0, T; L_\sigma^p(\Omega)) \cap L_\mu^q(0, T; D(A_p)) \rightarrow L_\mu^q(0, T; L_\sigma^p(\Omega)), \quad v \mapsto F(v)$$

is continuously differentiable and even real analytic with

$$DF(v)h = A_p h + F_p(v, h) + F_p(h, v).$$

Moreover, for any $g \in L_\mu^q(0, T; L_\sigma^p(\Omega))$ the equation

$$\partial_t h - DF(v)h = g, \quad h(0) = 0,$$

admits a unique solution $h \in H_\mu^{1,q}(0, T; L_\sigma^p(\Omega)) \cap L_\mu^q(0, T; D(A_p))$ with

$$\|h\|_{H_\mu^{1,q}(0, T; L_\sigma^p(\Omega)) \cap L_\mu^q(0, T; D(A_p))} \leq C(v, g)$$

where $C(v, g) > 0$ remains bounded for $v \in H_\mu^{1,q}(0, T; L_\sigma^p(\Omega)) \cap L_\mu^q(0, T; D(A_p))$ and $g \in L_\mu^q(0, T; L_\sigma^p(\Omega))$.

Proof. In order to prove the existence of the Fréchet derivative of \mathcal{F} , note that the bilinearity of $F_p(\cdot)$ yields

$$\mathcal{F}(v + h) = \mathcal{F}(v) + A_p h + F_p(v, h) + F_p(h, v) + F_p(h, h).$$

Proceeding similarly to [35] we set for $h \in \mathbb{E}_{1,\mu}(0, T)$

$$h_0^* := e^{tA_p} h(0) \quad \text{and} \quad h' := h - h_0^*.$$

Hence, $h = h_0^* + h'$ with $h_0^*(0) = h(0) \in X_{\gamma,\mu}$ and $h'(0) = 0$. As observed in [35], the assumption (S), Sobolev's embeddings and Hardy's inequality, imply

$$\begin{aligned} \{u \in \mathbb{E}_{1,\mu}(0, T) \mid u(0) = 0\} &\hookrightarrow \{u \in H_\mu^{1-\beta,p}(0, T; X_\beta) \mid u(0) = 0\} \\ &\hookrightarrow \{u \in H_\mu^{1-\beta-1/2p, 2p}(0, T; X_\beta) \mid u(0) = 0\} \hookrightarrow L_\sigma^{2p}(0, T; X_\beta), \end{aligned}$$

where we use that $1 - \beta - 1/2p + \mu = \frac{1}{2}(1 + \mu) = \vartheta$. Note that the embedding constants are independent of T . Hence,

$$\begin{aligned} \|F_p(h, h)\|_{\mathbb{E}_{0,\mu}} &\leq \|F_p(h', h')\|_{\mathbb{E}_{0,\mu}} + \|F_p(h', h_0^*)\|_{\mathbb{E}_{0,\mu}} + \|F_p(h_0^*, h')\|_{\mathbb{E}_{0,\mu}} + \|F_p(h_0^*, h_0^*)\|_{\mathbb{E}_{0,\mu}} \\ &\leq C \left(\|h'\|_{L_\vartheta^{2q}(0, T; X_\beta)}^2 + \|h'\|_{L_\vartheta^{2q}(0, T; X_\beta)} \|h_0^*\|_{L_\vartheta^{2q}(0, T; X_\beta)} + \|h_0^*\|_{L_\vartheta^{2q}(0, T; X_\beta)}^2 \right) \\ &\leq C \left(\|h'\|_{\mathbb{E}_{1,\mu}}^2 + \|h'\|_{\mathbb{E}_{1,\mu}} \|h(0)\|_{X_{\gamma,\mu}} + \|h(0)\|_{X_{\gamma,\mu}}^2 \right) \leq C \|h\|_{\mathbb{E}_{1,\mu}}^2, \end{aligned}$$

using $\|h_0^*\|_{L_\vartheta^{2q}(0, T; X_\beta)} \leq C \|h(0)\|_{X_{\gamma,\mu}} \leq C \|h\|_{\mathbb{E}_{1,\mu}}$, and $\|h'\|_{\mathbb{E}_{1,\mu}} \leq \|h\|_{\mathbb{E}_{1,\mu}} + \|h_0^*\|_{\mathbb{E}_{1,\mu}} \leq C \|h\|_{\mathbb{E}_{1,\mu}}$, since $\|h_0^*\|_{\mathbb{E}_{1,\mu}} \leq C \|h(0)\|_{X_{\gamma,\mu}} \leq C \|h\|_{\mathbb{E}_{1,\mu}}$, compare [32, Proof of Proposition 3.4.2 and 3.4.3, Theorem 3.4.8]. For this reason, $\|F(h, h)\|_{\mathbb{E}_{0,\mu}} / \|h\|_{\mathbb{E}_{1,\mu}} \rightarrow 0$ for $\|h\|_{\mathbb{E}_{1,\mu}} \rightarrow 0$. Similarly, we show that the map $v \mapsto DF(v)$ is continuous. Clearly, since \mathcal{F} is quadratic in v , it is real analytic.

It remains to prove the (global) solvability in $\mathbb{E}_{1,\mu}(0, T)$ of

$$\partial_t h - DF(v)h = g, \quad h(0) = 0.$$

It seems that [35, Theorem 1.2] cannot be applied directly because $v \mapsto DF(v)$ is well defined for $v \in \mathbb{E}_{1,\mu}(0, T)$, but it is not for $v \in X_{\gamma,\mu}$. However, the assertion can be proven by adapting the methods used in [35] and by using the linearity of the equation.

First, we define the reference function $h_0^* \in \mathbb{E}_{1,\mu}(0, T)$ for $h_0 \in X_{\gamma,\mu}$ as the solution of the inhomogeneous linear problem

$$u' + A_p u = g, \quad u(0) = h_0, \quad \text{i.e.} \quad h_0^*(t) = e^{tA_p} h_0 + \int_0^t e^{(t-s)A_p} g(s) ds.$$

Then, defining the ball

$$\mathbb{B}_{r, T', h_0} := \{u \in \mathbb{E}_{1,\mu}(0, T) : u(0) = h_0 \text{ and } \|u - h_0^*\|_{\mathbb{E}_{1,\mu}(0, T')} \leq r\} \subset \mathbb{E}_{1,\mu}(0, T), \quad r \in (0, 1].$$

we consider the map

$$\mathcal{J}_{h_0} : \mathbb{B}_{r, T', h_0} \rightarrow \mathbb{E}_{1,\mu}(0, T), \quad \mathcal{J}_{h_0} h = u,$$

where u is the unique solution in $\mathbb{E}_{1,\mu}(0, T)$ to the linear problem

$$u' + A_p u = g - F_p(v, h) - F_p(h, v), \quad u(0) = h_0.$$

Choosing $r \in (0, 1]$ and $0 < T' \leq T$ appropriately, the mapping \mathcal{J}_{h_0} restricts to a contractive self-mapping on \mathbb{B}_{r, T', h_0} .

As above, writing $h = h' + h_0^*$ and $v = v' + v_0^*$, we obtain

$$\|F_p(v, h) + F_p(h, v)\|_{\mathbb{E}_{0,\mu}} \leq \|F_p(v, h)\|_{\mathbb{E}_{0,\mu}} + \|F_p(h, v)\|_{\mathbb{E}_{0,\mu}} \leq C \|v\|_{\mathbb{E}_{1,\mu}} (\|h - h_0^*\|_{\mathbb{E}_{1,\mu}} + \|h(0)\|_{X_{\gamma,\mu}}),$$

and similarly

$$\begin{aligned} \|\mathcal{T}_{h_0}h_1 - \mathcal{T}_{h_0}h_2\|_{\mathbb{E}_{1,\mu}} &\leq C\|F(v, h_1 - h_2) - F(h_1 - h_2, v)\|_{\mathbb{E}_{0,\mu}} \\ &\leq C\|v\|_{\mathbb{E}_{1,\mu}} (\|h_1 - h_2\|_{\mathbb{E}_{1,\mu}}). \end{aligned}$$

Now, $\|v\|_{\mathbb{E}_{1,\mu}(0,T)}$ is finite and $\|v\|_{\mathbb{E}_{1,\mu}(T_1,T_2)} \rightarrow 0$ for $|T_2 - T_1| \rightarrow 0$. Therefore, for $\varepsilon > 0$ there is a finite partition $0 = T_0 < T_1 < \dots < T_n = T$ of $(0, T)$ such that

$$\|v\|_{\mathbb{E}_{1,\mu}(T_i, T_{i+1})} < \varepsilon, \quad i \in \{0, \dots, n-1\}.$$

Choosing $\varepsilon < 1/2C$ and by replacing – using the linearity – $h(T_i)$ and g by $h_n(T_i) = h(T_i)/n$, $g_n = g/n$, respectively, $n \in \mathbb{N}$, and making relevant norms sufficiently small, we prove global existence iteratively in finitely many steps, where the norm of h is controlled by r in each step. Therefore, $\|h\|_{\mathbb{E}_{1,\mu}(0,T)}$ depends on the partition used above, in particular on the number of steps to reach the global solution and the norms of $h(T_i)$ and g in $X_{\gamma,\mu}$ and $L_\mu^q(0, T; L_\sigma^p(\Omega))$, respectively. \square

Proof of Theorem 3.3. The assertions (a) and (b) follow from Theorem 5.4 by using Lemma 6.4.

Note that Theorem 5.4 is based on the Banach space version of the implicit function theorem applied to maps of the type $(u(t, \cdot), \lambda) \mapsto \lambda F(u)(\lambda t, \cdot)$. Now, in order to prove (c), the implicit function theorem needs to be applied to both space and time variables. For the most direct approach we need that

$$\Phi_{\lambda,\eta}: (0, \infty) \times \Omega \rightarrow (0, \infty) \times \Omega, \quad (t, x) \mapsto (\lambda \cdot t, x + t\eta), \quad \lambda \in (0, \infty), \quad \eta \in \mathbb{R}^3,$$

defines an isomorphism for parameters satisfying $|\lambda - 1| < \epsilon$ and $\|\eta\| < \epsilon$, $\epsilon > 0$, see e.g. [31, Section 5]. This is not true for general domains, but it is true for the whole space and also for the torus taking into account periodicity.

The strategy applied here is to first prove analyticity with respect to the horizontally periodic x, y variables, thus proving analyticity of the pressure, and secondly, to apply a localization procedure in z direction close to $\Gamma_D \neq \emptyset$, while on Γ_N solutions are extended by even reflection onto a larger domain.

To this end, let $\Omega_{per} = \Omega \cup \Gamma_l$ be equipped with the topology of $S^1 \times S^1 \times (-h, 0)$, where $S^1 = \mathbb{R}/\mathbb{Z}$, that is, taking into account lateral periodicity which induces a group structure in the lateral direction. Then

$$\Phi_{\lambda,\eta_H} \quad \text{for} \quad \eta_H = (\mu_x, \mu_y, 0), \quad \eta_x, \eta_y \in S^1$$

defines an isomorphism on Ω_{per} and for $v_{\lambda,\eta_H} = v \circ \Phi_{\lambda,\eta_H}$ we obtain by the chain rule

$$\partial_t v_{\lambda,\eta_H} = \lambda(\partial_t v) \circ \Phi_{\lambda,\eta_H} + \eta_H \cdot (\nabla v) \circ \Phi_{\lambda,\eta_H}.$$

Moreover, for $\epsilon > 0$, we define the real analytic map

$$H: \mathbb{E}_{1,\mu} \times (1 - \epsilon, 1 + \epsilon) \times (-\epsilon, \epsilon)^2 \rightarrow \mathbb{E}_{0,\mu} \times X_{\gamma,\mu}$$

by

$$H(\lambda, \eta_H, v) := (\partial_t v_{\lambda,\eta_H} - \lambda(A_p v + F_p(v))_{\lambda,\eta_H} - \eta_H \cdot \nabla v_{\lambda,\eta_H}, v_0 - v),$$

where the solution v to (2.1) with initial data v_0 solves $H(1, 0, v) = (0, 0)$.

Note that $(A_p v + F_p(v))_{\lambda,\eta_H} = A_p v_{\lambda,\eta_H} + F_p(v_{\lambda,\eta_H})$. The Fréchet derivative $\partial_v H$ is then an isomorphism by arguments similar to the ones given in the proof of Lemma 6.4 and by using that H is polynomial in v . Therefore, the implicit function theorem yields that $v(\lambda t, x + t\eta_H)$ is real analytic around $(1, 0)$ in η_H and λ . From this we deduce real analyticity of v around (x, t) with respect to time and the horizontal directions, compare e.g. [31, Section 5]. One can also adapt the approach in [9] for locally symmetric spaces to the situation of a symmetry in only two space directions. In particular, this proves analyticity of the surface pressure π_s .

Concerning the z -direction, we note first that (2.1) is compatible with even reflections along the Neumann part of the boundary. Thus for $\Gamma_D = \emptyset$ solutions v may be extended to the full torus – a feature used in the literature dealing with Neumann boundary values, see e.g. [28] – and replacing η_H by general $\eta \in \mathbb{R}^3$ in the above arguments implies analyticity of solutions including the boundary.

If $\Gamma_D \neq \emptyset$ we need to apply a localization procedure with respect to z -variable. The details of this method are neglected here and we refer to [32, Section 9] for details.

Since the main non-locality in the primitive equation arise from the pressure term, we consider finally

$$\partial_t v - \Delta v + v \cdot \nabla_H v + w(v) \cdot \partial_z v = f_s, \quad f_s = f - \nabla_H \pi_s, \quad v(0) = v_0,$$

where f_s is real analytic by the considerations above and the assumption on f . The above proofs can now be adapted by using the fact that the non-linearity $v \mapsto w(v)\partial_z + v \cdot \nabla_H v$ is real analytic. \square

Remarks 6.5. (a) A different strategy to prove smoothness, but not real analyticity, of solutions is to consider higher order time derivatives, which are well defined according to Theorem 3.3 (a) and (b). Then, by Lemma 6.1

$$F_p(\cdot): H^{s,p}(\Omega)^2 \rightarrow H^{s-(1+1/p),p}(\Omega)^2, \quad s \geq (1/2 + 1/2p), \quad p \in (1, \infty),$$

is well-defined and bounded and we write

$$\partial_t^{(n)} v = A_p^{-1}(\partial_t^{(n)} F_p(v) - \partial_t^{(n+1)} v).$$

Since $\partial_t^{(n)} v \in D(A_p)$ for all $n \in \mathbb{N}_0$, we conclude first, that $\partial_t^{(n)} F_p(v) \in H^{2-(1+1/p),p}(\Omega)^2$. Secondly, applying elliptic regularity from Proposition 4.1, we conclude that $\partial_t^{(n)} v \in H^{(3-1/p),p}(\Omega)^2$. Iterating this argument, that is, 'trading time for space regularity' and using Sobolev embeddings we arrive at

$$v \in C^\infty((0, \infty); C_{per}^\infty(\bar{\Omega})^2),$$

thereby proving smoothness including the boundary.

(b) Another strategy for smoothness of solutions, namely proving first additional space regularity and deriving therefrom additional time regularity has been developed in [17] in the case of the Navier-Stokes equations.

The following elementary lemma is needed to extend regularity of solutions from $(0, T')$ for any $0 < T' < T$ to $(0, T)$.

Lemma 6.6. *Let $v \in \mathbb{E}_{1,\mu}(0, T')$ for any $0 < T' < T$, and $\sup_{0 < T' < T} \|v\|_{\mathbb{E}_{1,\mu}(0, T')} < C$ for some constant $C > 0$. Then $v \in \mathbb{E}_{1,\mu}(0, T)$.*

Proof. First, note that v_t and $A_p v$ are measurable functions on $(0, T)$ by considering these as point-wise limits of the extensions by zero of $v_t|_{(0, T')}$ and $A_p v|_{(0, T')}$, respectively. Secondly, by dominated convergence

$$\int_0^T t^{(1-\mu)q} \|v_t - A_p v\|_{L_\sigma^p(\Omega)}^q = \lim_{T' \rightarrow T} \int_0^{T'} t^{(1-\mu)q} \|v_t - A_p v\|_{L_\sigma^p(\Omega)}^q < \infty.$$

\square

Lemma 6.7. *Let p, q, μ and $v_0, P_p f$ be as in Proposition 6.3. Assume that*

$$v \in H_\mu^{1,q}(0, T; L_\sigma^p(\Omega)) \cap L_\mu^q(0, T; D(A_p))$$

is a solution to (2.1). If in addition $t \mapsto t \cdot P_p f_t \in L_\mu^q(0, T; L_\sigma^p(\Omega))$, then

$$t \cdot v_t \in H_\mu^{1,q}(0, T; L_\sigma^p(\Omega)) \cap L_\mu^q(0, T; D(A_p)), \quad \|t \cdot v_t\|_{H_\mu^{1,q}(0, T; L_\sigma^p(\Omega)) \cap L_\mu^q(0, T; D(A_p))} \leq C(v, f, f_t, T),$$

for some finite constant C .

Proof. Let v be a solution to 2.1 in $\mathbb{E}_{1,\mu}(0, T)$. Consider for $0 < \varepsilon < \frac{T-T'}{T'}$, $0 < T' < T$, the map

$$G: (-\varepsilon, \varepsilon) \times \mathbb{E}_{1,\mu}(0, T') \rightarrow \mathbb{E}_{0,\mu}(0, T') \times X_{\gamma,\mu}, \quad (\lambda, \nu) \mapsto (\nu' + (1 + \lambda)F(\nu) - (1 + \lambda)f_\lambda, \nu(0) - v(0)),$$

where $f_\lambda(t, \cdot) := f((1 + \lambda)t, \cdot)$. As in [8, Section 9.2] one can prove that the implicit function theorem applies and there is an implicit function $g_\lambda(-\varepsilon', \varepsilon') \rightarrow \mathbb{E}_{1,\mu}(0, T')$, $\varepsilon' \leq \varepsilon$ which solves $G(\lambda, g_\lambda(\lambda)) = (0, 0)$. By uniqueness we conclude that $g_\lambda = v_\lambda$. The implicit derivative at $\lambda = 0$ is

$$\partial_\lambda g_\lambda|_{\lambda=0} = t \cdot v_t = -(\partial_v G)(0, v)(A_p v + F_p(v) - f - t \cdot f_t, 0),$$

i.e., $t \mapsto -t \cdot v_t$ is the solution to the equation

$$\partial_t h - A_p h + F_p(h, v) + F_p(v, h) = A_p v + F_p(v) - f - t \cdot f_t, \quad h(0) = 0,$$

and by Lemma 6.4

$$\|t \cdot v_t\|_{\mathbb{E}_{1,\mu}(0, T')} \leq C(v, \|A_p v + F_p(v) - f - t \cdot f_t\|_{\mathbb{E}_{0,\mu}(0, T')}).$$

Note that $\|A_p v + F_p(v) - f - t \cdot f_t\|_{\mathbb{E}_{0,\mu}} \leq C \left(\|v\|_{\mathbb{E}_{1,\mu}} + \|v\|_{\mathbb{E}_{1,\mu}}^2 + \|f\|_{\mathbb{E}_{0,\mu}} + \|t \cdot f_t\|_{\mathbb{E}_{0,\mu}} \right)$. Now since $\|v\|_{\mathbb{E}_{1,\mu}(0, T)} + \|f\|_{\mathbb{E}_{0,\mu}(0, T)} + \|t \cdot f_t\|_{\mathbb{E}_{0,\mu}(0, T)}$ are bounded by assumption, $\sup_{T' < T} \|t \cdot v_t\|_{\mathbb{E}_{1,\mu}(0, T')}$ is bounded as well, and therefore $t \cdot v_t \in \mathbb{E}_{1,\mu}(0, T)$ by Lemma 6.6. \square

Remark 6.8. Note that in Lemma 6.7 regularity of $t \cdot v_t$ is derived on $(0, T)$ while in Theorem 3.3 it is proven on $(0, T')$ for any $T' < T$. Extending the regularity onto $(0, T)$ is possible due to the control on the implicit derivative by Lemma 6.4.

6.3. *A priori* bounds in $H^1(0, T; L^2) \cap L^2(0, T; H^2)$.

Theorem 6.9 (*A priori bounds*). *There exists a continuous function B satisfying the following property: for any solution of (2.1) such that for $0 < T < \infty$*

$$v \in H^1(0, T; L^2_{\bar{\sigma}}(\Omega)) \cap L^2(0, T; D(A_2)), \quad v_0 \in \{H^1 \cap L^2_{\bar{\sigma}}(\Omega) : v|_{\Gamma_D} = 0\}, \quad P_2 f \in L^2(0, T; L^2_{\bar{\sigma}}(\Omega))$$

one has

$$\|v\|_{H^1(0, T; L^2_{\bar{\sigma}}(\Omega)) \cap L^2(0, T; D(A_2))} \leq B(\|v_0\|_{H^1(\Omega)}, \|P_2 f\|_{L^2(0, T; L^2(\Omega))}, T).$$

Proof. In [13] global a priori bounds in $L^\infty(0, T; H^1(\Omega))$ and $L^2(0, T; H^2(\Omega))$ have been derived for the case of mixed Dirichlet and Neumann boundary condition. The case of pure Dirichlet boundary conditions can be treated similarly. Here, we supplement the corresponding proof for Neumann boundary conditions, where we even prove $L^\infty(0, T; H^2(\Omega))$ -bounds.

A standard procedure yields the energy equality:

$$(6.1) \quad \|v(t)\|_{L^2(\Omega)}^2 + 2 \int_0^t \|\nabla v(s)\|^2 ds = \|a\|_{L^2(\Omega)}^2 + 2 \int_0^t \int_\Omega f(s) \cdot v(s) ds.$$

We subdivide our proof into seven steps. The solution of (2.1) splits, compare [20, (6.3) and (6.4)], into

$$(6.2) \quad \bar{v}_t - \Delta_H \bar{v} + \nabla_H p = -\bar{v} \cdot \nabla_H \bar{v} - \frac{1}{h} \int_{-h}^0 \tilde{v} \cdot \nabla_H \tilde{v} + \operatorname{div}_H \tilde{v} \tilde{v} + \bar{f}, \quad \operatorname{div}_H \bar{v} = 0,$$

$$(6.3) \quad \tilde{v}_t - \Delta \tilde{v} + \tilde{v} \cdot \nabla_H \tilde{v} + w \partial_z \tilde{v} = -\bar{v} \cdot \nabla_H \tilde{v} - \tilde{v} \cdot \nabla_H \bar{v} + \frac{1}{h} \int_{-h}^0 \tilde{v} \cdot \nabla_H \tilde{v} + \operatorname{div}_H \tilde{v} \tilde{v} + \tilde{f}.$$

The proof presented below basically follows the steps of [20, Section 6]. However, the Neumann boundary condition make the proof different in two ways. One is that the extra term $\partial_z v|_{\Gamma_D}$ appearing in the equations for \bar{v} and $\tilde{v} = v - \bar{v}$ is now absent (see (6.3) and (6.4) of [20]), and the other is that Poincaré's inequality for v is no longer available. However, we still have

$$(6.4) \quad \|\tilde{v}\|_{L^2(\Omega)} \leq h \|\partial_z \tilde{v}\|_{L^2(\Omega)}.$$

Step 1. We derive an estimate for $\tilde{v} := v - \bar{v} \in L_t^\infty(L_x^4)$. As in [20, (6.8)] and [13, Step 3], by multiplying (6.3) with $|\tilde{v}|^2\tilde{v}$ and integrating over Ω , we obtain by integrating by parts

$$\begin{aligned} & \frac{1}{4}\partial_t\|\tilde{v}\|_{L^4(\Omega)}^4 + \frac{1}{2}\|\nabla|\tilde{v}|^2\|_{L^2(\Omega)}^2 + \|\tilde{v}\|\nabla\tilde{v}\|_{L^2(\Omega)}^2 \\ &= -\int_{\Omega}(\tilde{v}\cdot\nabla_H\tilde{v})\cdot|\tilde{v}|^2\tilde{v} + \frac{1}{h}\int_{\Omega}\int_{-h}^0(\tilde{v}\cdot\nabla_H\tilde{v} + \operatorname{div}_H v\tilde{v})dz\cdot|\tilde{v}|^2\tilde{v} + \int_{\Omega}\tilde{f}\cdot|\tilde{v}|^2\tilde{v} =: I_1 + I_2. \end{aligned}$$

We estimate

$$I_1 \leq C\|\nabla_H\tilde{v}\|_{L^2(G)}\|\tilde{v}\|_{L^4(\Omega)}^4 + C\|\nabla_H\tilde{v}\|_{L^2(G)}^2\|\tilde{v}\|_{L^4(\Omega)}^4 + \frac{1}{4}\|\nabla_H|\tilde{v}|^2\|_{L^2(\Omega)}^2,$$

and similarly to [13, Section 4, Step 3]

$$I_2 \leq C(1 + \|\tilde{f}\|_{L^2(\Omega)}^2)\|\tilde{v}\|_{L^4(\Omega)}^4 + \|\tilde{f}\|_{L^2(\Omega)}^{8/5}\|\tilde{v}\|_{L^4(\Omega)}^{12/5} + \frac{1}{4}\|\nabla_H|\tilde{v}|^2\|_{L^2(\Omega)}^2$$

as well as

$$\|\tilde{f}\|_{L^2(\Omega)}^{8/5}\|\tilde{v}\|_{L^4(\Omega)}^{12/5} \leq C(1 + \|\tilde{v}\|_{L^4(\Omega)}^4)\|\tilde{f}\|_{L^2(\Omega)}^2 + 1 + \|\tilde{v}\|_{L^4(\Omega)}^4.$$

Combining these estimates we obtain

$$\begin{aligned} \|\tilde{v}(t)\|_{L^4(\Omega)} + \int_0^t\|\tilde{v}(s)\|\nabla\tilde{v}(s)\|_{L^2(\Omega)}^2 ds &\leq \|\tilde{a}\|_{L^4(\Omega)} + C\int_0^t(1 + \|\tilde{f}\|_{L^2(\Omega)}^2) \\ &\quad + C\left(1 + \|\nabla_H\tilde{v}\|_{L^2(G)}^2 + \|\tilde{f}\|_{L^2(\Omega)}^2\right)\|\tilde{v}(t)\|_{L^4(\Omega)} ds, \end{aligned}$$

and Gronwall's inequality yields

$$\|\tilde{v}(t)\|_{L^4(\Omega)} \leq \left(\|\tilde{a}\|_{L^4(\Omega)} + C(t + \int_0^t\|\tilde{f}\|_{L^2(\Omega)}^2)\right) \exp\left(C\int_0^t\left(1 + \|\nabla_H\tilde{v}\|_{L^2(G)}^2 + \|\tilde{f}\|_{L^2(\Omega)}^2\right) ds\right),$$

where $\int_0^t\|\nabla_H\tilde{v}\|_{L^2(G)}^2 ds$ is bounded by (6.1)

This implies that there exists a continuous function $B_1 = B_1(\|a\|_{H^1(\Omega)})$ such that

$$(6.5) \quad \|\tilde{v}(t)\|_{L^4(\Omega)} + \int_0^t\|\tilde{v}(s)\|\nabla\tilde{v}(s)\|_{L^2(\Omega)}^2 ds \leq B_1(\|a\|_{H^1(\Omega)}, \|f\|_{L^2(0,T;L^2(\Omega))}), \quad t \in [0, T].$$

Step 2. We derive an estimate for $\nabla_H\bar{v} \in L_t^\infty(L_x^2)$. As in [20, p. 1103] we obtain

$$8\partial_t\|\nabla_H\bar{v}\|_{L^2(G)}^2 + \|\Delta_H\bar{v}\|_{L^2(G)}^2 + \|\nabla_H\pi\|_{L^2(G)}^2 \leq C\|\tilde{v}\|\nabla_H\bar{v}\|_{L^2(G)}^2 + C\|\tilde{v}\|\nabla_H\tilde{v}\|_{L^2(\Omega)}^2 + \|\tilde{f}\|_{L^2(G)}^2.$$

By an interpolation inequality, $\|\tilde{v}\|_{H^1(G)} = \|\tilde{v}\|_{L^2(G)} + \|\nabla_H\tilde{v}\|_{L^2(G)}$ etc., and $\|\nabla_H^2\bar{v}\|_{L^2(G)} \leq C\|\Delta_H\bar{v}\|_{L^2(G)}$, we obtain

$$\begin{aligned} \|\tilde{v}\|\nabla_H\bar{v}\|_{L^2(G)}^2 &\leq C\|\tilde{v}\|_{L^4(G)}^2\|\nabla_H\bar{v}\|_{L^4(G)}^2 \leq C\|\tilde{v}\|_{L^2(G)}\|\tilde{v}\|_{H^1(G)}\|\nabla_H\bar{v}\|_{L^2(G)}\|\nabla_H\bar{v}\|_{H^1(G)} \\ &\leq C(1 + \|\tilde{v}\|_{L^2(G)}^2 + \|\tilde{v}\|_{L^2(G)}^4)\|\nabla_H\bar{v}\|_{L^2(G)}^2 + C\|\tilde{v}\|_{L^2(G)}^2\|\nabla_H\bar{v}\|_{L^2(G)}^4 + \frac{1}{2}\|\Delta_H\bar{v}\|_{L^2(G)}^2. \end{aligned}$$

It then follows from Gronwall's inequality and (6.5) that

$$\|\nabla_H\bar{v}(t)\|_{L^2(G)}^2 + \int_0^t\|\nabla_H\pi(s)\|_{L^2(G)}^2 ds \leq B_2(\|a\|_{H^1(\Omega)}, \|f\|_{L^2(0,T;L^2(\Omega))}), \quad t \in [0, T]$$

for some continuous function B_2 .

Step 3. We derive an estimate for $v_z := \partial_z v \in L_t^\infty(L_x^2)$. As in [20, (6.6)] testing with $-\partial_z^2 v$ we obtain

$$\begin{aligned} \frac{1}{2} \partial_t \|v_z\|_{L^2(\Omega)}^2 + \|\nabla v_z\|_{L^2(\Omega)}^2 &= - \int_{\Omega} v_z \cdot \nabla_H \bar{v} \cdot v_z + \int_{\Omega} \operatorname{div}_H v_z \tilde{v} \cdot \tilde{v}_z \\ &\quad + \int_{\Omega} v_z \cdot \nabla_H v_z \cdot \tilde{v} - 2 \int_{\Omega} \tilde{v} \cdot \nabla_H v_z \cdot v_z - \int_{\Omega} f \cdot v_{zz} \\ &\leq C \|\nabla_H \bar{v}\|_{L^2(G)} \|v_z\|_{L^4(\Omega)}^2 + C \|\nabla_H v_z\|_{L^2(\Omega)} \|\tilde{v}\|_{L^2(\Omega)} \|v_z\|_{L^2(\Omega)} + \frac{1}{4} \|v_{zz}\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2 \\ &\leq C \|\nabla_H \bar{v}\|_{L^2(G)}^4 \|v_z\|_{L^2(\Omega)}^2 + C \|\tilde{v}\|_{L^2(\Omega)} \|\nabla \tilde{v}\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla v_z\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2, \end{aligned}$$

where we have used $\|v_z\|_{L^4(\Omega)} \leq C \|v_z\|_{L^2(\Omega)}^{1/4} \|\nabla v_z\|_{L^2(\Omega)}^{3/4}$ (note that $v_z = 0$ on $\Gamma_u \cup \Gamma_b$) and $v_z = \tilde{v}_z$. It follows from (6.5) that

$$\|v_z(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla v_z(s)\|_{L^2(\Omega)}^2 ds \leq B_3(\|a\|_{H^1(\Omega)}, \|f\|_{L^2(0,T;L^2(\Omega))}), \quad t \in [0, T].$$

Step 4. We derive an estimate for $\nabla v \in L_t^\infty(L_x^2)$. As in [20, (6.13)] we obtain

$$(6.6) \quad \begin{aligned} \partial_t \|\nabla v\|_{L^2(\Omega)}^2 + \|\Delta v\|_{L^2(\Omega)}^2 &\leq C(\|\bar{v} \cdot \nabla_H \bar{v}\|_{L^2(G)}^2 + \|\bar{v} \cdot \nabla_H \tilde{v}\|_{L^2(\Omega)}^2 + \|\tilde{v} \cdot \nabla_H \bar{v}\|_{L^2(\Omega)}^2 + \|w \partial_z v_z\|_{L^2(\Omega)}^2 \\ &\quad + \|\tilde{v} \cdot \nabla_H \tilde{v}\|_{L^2(\Omega)}^2 + \|\nabla_H \pi\|_{L^2(G)}^2 + \|f\|_{L^2(\Omega)}^2). \end{aligned}$$

In view of interpolation inequalities, elliptic regularity for Δ , and anisotropic estimates, we may bound the first four terms on the right-hand side as

- $\|\bar{v} \cdot \nabla_H \bar{v}\|_{L^2(G)}^2 \leq C \|\bar{v}\|_{L^2(G)} \|\bar{v}\|_{H^1(G)} \|\nabla_H \bar{v}\|_{L^2(G)} \|\nabla_H \bar{v}\|_{H^1(G)}$
 $\leq C \|\bar{v}\|_{L^2(G)} \|\bar{v}\|_{H^1(G)} (1 + \|\bar{v}\|_{L^2(G)} \|\bar{v}\|_{H^1(G)}) \|\nabla_H \bar{v}\|_{L^2(G)}^2 + \frac{1}{8} \|\Delta v\|_{L^2(\Omega)}^2,$
- $\|\tilde{v} \cdot \nabla_H \tilde{v}\|_{L^2(\Omega)}^2 = \|\bar{v} \cdot \nabla_H \tilde{v}\|_{L^2(\Omega)}^2 \leq C \|\bar{v}\|_{L^6(G)}^2 \|\nabla_H \tilde{v}\|_{L^2(\Omega)} \|\nabla_H \tilde{v}\|_{H^1(\Omega)}$
 $\leq C \|\bar{v}\|_{H^1(G)}^2 \|\nabla v\|_{L^2(\Omega)} \|\partial_z \nabla_H v\|_{L^2(\Omega)}$
 $\leq C \|\bar{v}\|_{H^1(G)}^2 \|\nabla v\|_{L^2(\Omega)}^2 + \frac{1}{8} \|\Delta v\|_{L^2(\Omega)}^2,$
- $\|\tilde{v} \cdot \nabla_H \bar{v}\|_{L^2(\Omega)}^2 \leq C \|\tilde{v}\|_{L^4(\Omega)}^2 \|\nabla_H \bar{v}\|_{L^4(G)}^2 \leq C \|\tilde{v}\|_{L^4(\Omega)}^2 (1 + \|\tilde{v}\|_{L^4(\Omega)}^2) \|\nabla_H \bar{v}\|_{L^2(G)}^2 + \frac{1}{8} \|\Delta v\|_{L^2(\Omega)}^2,$
- $\|w v_z\|_{L^2(\Omega)}^2 \leq C \|w\|_{L_{xy}^4 L_z^\infty} \|v_z\|_{L_{xy}^4 L_z^2} \leq C \|\operatorname{div}_H v\|_{L^2(\Omega)} \|\operatorname{div}_H v\|_{H^1(\Omega)} \|v_z\|_{L^2(\Omega)} \|\nabla v_z\|_{L^2(\Omega)}$
 $\leq C \|\operatorname{div}_H v\|_{L^2(\Omega)}^4 \|v_z\|_{L^2(\Omega)}^2 + C \|\operatorname{div}_H v\|_{L^2(\Omega)}^2 \|v_z\|_{L^2(\Omega)}^2 \|\nabla v_z\|_{L^2(\Omega)}^2 + \frac{1}{8} \|\Delta v\|_{L^2(\Omega)}^2.$

Combining these with (6.6) and applying Gronwall's inequality, we conclude

$$(6.7) \quad \|\nabla v(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\Delta v(s)\|_{L^2(\Omega)}^2 ds \leq B_4(\|a\|_{H^1(\Omega)}, \|f\|_{L^2(0,T;L^2(\Omega))}), \quad t \in [0, T].$$

Having now established $L^\infty(0, T; H^1(\Omega))$ and $L^2(0, T; H^2(\Omega))$ -a priori bounds B_4 for the all boundary conditions 2.2, we conclude, by using maximal regularity of A_2 that

$$\|v\|_{\mathbb{E}_{1,1}(0,T)} \leq c \|(\partial_t - (A_2 - \lambda)v)\|_{\mathbb{E}_{0,1}(0,T)} \leq C \|F_2(v) - \lambda v + f\|_{\mathbb{E}_{0,1}(0,T)}, \quad \lambda > 0,$$

and by using Lemma 6.1, interpolation inequality and Hölder's inequality

$$\int_0^T \|F_p(v(s))\|_{L^2(\Omega)}^2 ds \leq C \int_0^T \|v(s)\|_{H^1}^2 \|v(s)\|_{H^2}^2 ds \leq C \|v\|_{L^\infty(0,T;H^1(\Omega))}^2 \|v\|_{L^2(0,T;H^2(\Omega))}^2 \leq B_4^4 =: B,$$

an a priori bound in the maximal regularity space.

To derive in addition an $L^\infty(0, T; H^2(\Omega))$ -bounds we proceed as follows.

Step 5. We derive an estimate for $v_t := \partial_t v \in L_t^\infty(L_x^2)$. Taking the time derivative of (2.1), multiplying by v_t , and integrating over Ω , we obtain using the divergence free condition

$$(6.8) \quad \frac{1}{2} \partial_t \|v_t\|_{L^2(\Omega)}^2 + \|\nabla v_t\|_{L^2(\Omega)}^2 = - \int_{\Omega} (v_t \cdot \nabla_H v + w_t \partial_z v) \cdot v_t + \int_{\Omega} f_t \cdot v_t.$$

In view of interpolation inequalities and anisotropic estimates, the first two terms on the right-hand side may be bounded by

$$\|\nabla_H v\|_{L^2(\Omega)} \|v_t\|_{L^4(\Omega)}^2 \leq C(\|\nabla_H v\|_{L^2(\Omega)} + C\|\nabla_H v\|_{L^2(\Omega)}^4) \|v_t\|_{L^2(\Omega)}^2 + \frac{1}{4} \|\nabla v_t\|_{L^2(\Omega)}^2,$$

and by

$$\begin{aligned} & \|w_t\|_{L_{xy}^2 L_z^\infty} \|v_z\|_{L_{xy}^3 L_z^2} \|v_t\|_{L_{xy}^6 L_z^2} \\ & \leq C \|\operatorname{div}_H v_t\|_{L^2(\Omega)} \|v_z\|_{H^{1/3}(\Omega)} \|v_t\|_{H^{2/3}(\Omega)} \\ & \leq C \|\nabla v_t\|_{L^2(\Omega)} \|v_z\|_{L^2(\Omega)}^{2/3} \|\nabla v_z\|_{L^2(\Omega)}^{1/3} \|v_t\|_{L^2(\Omega)}^{1/3} \|v_t\|_{H^1(\Omega)}^{2/3} \\ & \leq C \|\nabla v_z\|_{L^2(\Omega)}^2 \|v_t\|_{L^2(\Omega)}^2 + C \|v_z\|_{L^2(\Omega)}^4 \|\nabla v_z\|_{L^2(\Omega)}^2 \|v_t\|_{L^2(\Omega)}^2 + \frac{1}{4} \|\nabla v_t\|_{L^2(\Omega)}^2, \end{aligned}$$

respectively. Therefore, integrating (6.8) with respect to t and noting that $\|v_t(0)\|_{L^2(\Omega)} \leq C(\|a\|_{H^2(\Omega)}^2 + \|a\|_{H^2(\Omega)})$ (see [20, p. 1111]), we conclude $\|v_t(t)\|_{L^2(\Omega)} \leq B_6(\|a\|_{H^2(\Omega)}, \|f\|_{W^1(0,T;L^2(\Omega))}^2)$ for all $t \in [0, T]$.

Step 6. We derive an estimate for $v_z \in L_t^\infty(L_x^3)$. As in [20, p. 1109] we obtain testing with $-\partial_z(|v_z|v_z)$, now assuming $f_z \in L^2(0, T; L^2(\Omega))$

$$\begin{aligned} & \frac{1}{3} \partial_t \|v_z\|_{L^3(\Omega)}^3 + \frac{4}{9} \|\nabla |v_z|^{3/2}\|_{L^2(\Omega)}^2 + \| |v_z|^{1/2} |\nabla v_z| \|_{L^2(\Omega)}^2 \\ & = - \int_{\Omega} v_z \cdot \nabla_H v \cdot |v_z|v_z + \int_{\Omega} \operatorname{div}_H v |v_z|^3 - \int_{\Omega} f \cdot \partial_z(|v_z|v_z) \\ & \leq C \|\nabla_H v\|_{L^2(\Omega)}^4 \|v_z\|_{L^3(\Omega)}^3 + \frac{1}{9} \|\nabla |v_z|^{3/2}\|_{L^2(\Omega)}^2 + \|f_z\|_{L^2(\Omega)}^2 + \|v_z\|_{L^4(\Omega)}^4, \end{aligned}$$

where $\|v_z\|_{L^4(\Omega)}^4 \leq C(\|v_z\|_{L^3(\Omega)}^3 + 1)(\|\nabla v_z\|_{L^2(\Omega)}^2 + 1)$. Gronwall's inequality then implies $\|v_z(t)\|_{L^3(\Omega)} \leq B_7(\|a\|_{H^2(\Omega)}, \|f_z\|_{L^2(0,T;L^2(\Omega))}^2)$ for all $t \in [0, T]$.

Step 7. We now derive an estimate for $\nabla^2 v \in L_t^\infty(L_x^2)$. As in [20, p. 1111] we have

$$\begin{aligned} \|\nabla^2 v\|_{L^2(\Omega)} & \leq C \|\Delta v\|_{L^2(\Omega)} \leq C(\|v_t\|_{L^2(\Omega)} + \|v \cdot \nabla_H v\|_{L^2(\Omega)} + \|w \partial_z v\|_{L^2(\Omega)}) + \|f\|_{L^2(\Omega)}^2 \\ & \leq C \|v_t\|_{L^2(\Omega)} + C \|v\|_{L^6(\Omega)} \|v\|_{W^{1,3}(\Omega)} + C \|w\|_{L^6(\Omega)} \|v_z\|_{L^3(\Omega)} + \|f\|_{L^2(\Omega)}^2 \\ & \leq C \|v_t\|_{L^2(\Omega)} + C \|v\|_{H^1(\Omega)}^3 + C \|\nabla v\|_{L^2(\Omega)} (\|v_z\|_{L^3(\Omega)} + \|v_z\|_{L^3(\Omega)}^3) + \|f\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla^2 v\|_{L^2(\Omega)}, \end{aligned}$$

which implies the desired estimate

$$(6.9) \quad \|\nabla^2 v(t)\|_{L^2(\Omega)} \leq B_8(\|a\|_{H^2(\Omega)}, \|f\|_{W^1(0,T;L^2(\Omega))}^2, \|f_z\|_{L^2(0,T;L^2(\Omega))}^2) \quad \forall t \in [0, T].$$

Combining (6.1), (6.7), and (6.9), we completed the proof. \square

6.4. Global well-posedness.

Proof of Proposition 3.2. To prove assertion (a) consider

$$t_+(v_0) := \sup\{T' > 0: \text{Equation (2.1) has a solution in } \mathbb{E}_{1,1}(0, T')\}.$$

By Proposition 6.3 $t_+(v_0) > 0$ and the solutions in $\mathbb{E}_{1,1}(0, T')$ are unique. Indeed, if we assume that there are two solutions $v, v' \in \mathbb{E}_{1,1}(0, T')$, then setting

$$t_1(v_0) := \sup\{s > 0: \|(v - v')(s)\|_{X_{\gamma,1}} = 0\},$$

we see that $t_1(v_0) > 0$ by Proposition 6.3. Further, by continuity, $\mathbb{E}_{1,1}(0, T') \hookrightarrow C([0, T']; X_{\gamma,1})$ and the above supremum is attained. Assuming that $t_1(v_0) < T'$, again by Proposition 6.3, the solution with new initial value at $t_1(v_0)$ is unique on some time interval, thus contradicting the assumption.

Assume now, that $t_+(v_0) < T$. By Theorem 6.9 $\|v\|_{\mathbb{E}_{1,1}(0, T')} \leq B(\|v_0\|_{H^1(\Omega)}, \|P_2 f\|_{L^2(0, T'; L^2(\Omega))}, t_+(v_0))$ for any $0 < T' < t_+(v_0)$. Hence by Lemma 6.6 we have $v \in \mathbb{E}_{1,1}(0, t_+(v_0))$. Since the trace in $\mathbb{E}_{1,1}(0, t_+(v_0))$ is well-defined $v(t_+(v_0))$ can be taken as new initial value, thus extending the solution beyond $t_+(v_0)$ contradicting the assumption. Hence $t_+(v_0) = T$, and again combing Theorem 6.9 and Lemma 6.6 we have $v \in \mathbb{E}_{1,1}(0, T)$. This proves part (a).

Assertion (b) follows directly from Lemma 6.7. \square

Proof of Theorem 3.1. By Proposition 6.3 there is a local solutions, which by Theorem 3.3 (a) has additional time regularity, in particular $v \in H^{1,q}(\delta, T; D(A_p)) \hookrightarrow C^0(\delta, T; D(A_p))$ for some $0 < \delta \leq T'$ and $0 < T' < T$. Now, using $v(T')$ as new initial value, and taking advantage of the embedding $D(A_q) \subset (L^2_{\bar{\sigma}}(\Omega), D(A_2))_{1/2, q}$ for $q \in [6/5, \infty)$ and the additional assumption $P_2 f \in W^{1,2}(\delta, T; L^2_{\bar{\sigma}}(\Omega))$ we obtain that v is also an L^2 solution at least for $\delta > 0$. This holds for $q \in [6/5, \infty)$, and for $q \in (1, 6/5)$ this follows from a bootstrapping argument as in [21, Section 6.2]. By Proposition 3.2 there exists a global L^2 solution with $v \in C_b(\delta, D(A_2))$. Then using Lemma 2.1 and classical embedding results, see e.g. [36], we obtain

$$D(A_2) \hookrightarrow X_{\bar{\mu}, q} \quad \text{for } 0 \leq \bar{\mu} - \mu < 2 - \frac{2}{p},$$

and compactness of the embedding $X_{\bar{\mu}, q} \hookrightarrow X_{\mu, q}$ for $1/p < \mu < \bar{\mu} < 1$. Hence Theorem 5.3 applies since

$$\|v\|_{C_b(\delta, T; X_{\bar{\mu}, q})} \leq C \|v\|_{C_b(\delta, T; D(A_2))},$$

and therefore the solution exists globally, that is for any $T > 0$. \square

7. CONCLUDING REMARKS

The maximal regularity approach uses the contraction mapping principle to construct local solutions with initial values being traces of functions in the maximal L^q - L^p -regularity class which here reads as $v_0 \in B_{pq}^{2/p}$. Other methods to construct solutions for the primitive equations are the Fujita-Kato scheme as proposed in [20] for initial values $v \in H^{2/p, p}$, and the Galerkin method as used originally in [19] giving initial values $v_0 \in H^1$. Note that for $q = p = 2$ all results agree, and for $p, q \geq 2$ one has $H^{2/p, p} \subset B_{pq}^{2/p}$.

Comparing the maximal regularity and the Fujita-Kato approach, we see that, by using the maximal regularity approach, various boundary conditions can be treated simultaneously in the same way. The efficiency of this approach becomes furthermore obvious, when studying further couplings adding to the complexity of the equations. For instance, adding non-constant temperature τ one considers

$$\begin{cases} \partial_t v + v \cdot \nabla_H v + w \cdot \partial_z v - \Delta v + \nabla_H \pi_s + \nabla_H \int_{-h}^z \tau(\cdot, \xi) d\xi = f, & \text{in } \Omega \times (0, T), \\ \operatorname{div}_H \bar{v} = 0, & \text{in } \Omega \times (0, T), \\ \partial_t \tau + v \cdot \nabla_H \tau + w \cdot \partial_z \tau - \Delta \tau = g, & \text{in } \Omega \times (0, T), \end{cases}$$

compare [25], where the non-linearity

$$F_p(v, \tau) := \left(P_p(v \cdot \nabla_H v + w \cdot \partial_z v + \nabla_H \int_{-h}^z \tau(\cdot, \xi) d\xi), v \cdot \nabla_H \tau + w \cdot \partial_z \tau \right)$$

can be estimated as in [21, Lemma 5.1], and local well-posedness and regularity results follow directly.

Recently, the coupling to moisture and its analysis has come into focus, see [7, 22] and the references given therein. The equation for the moisture q is of the type

$$\partial_t q + v \cdot \nabla_H q + w \cdot \partial_z q - \Delta q = h + F(v, \tau, q)$$

with additional coupling term $F(v, \tau, q)$. In the model studied in [7] $F(v, \tau, q)$ involves some Heaviside functions and it is treated using variational methods. In [22] water vapor q_v , cloud water q_c and rain

water q_r mixing ratios are coupled to the temperature and velocity equations where the coupling terms involve expressions of the form

$$\tau(q_r^+)^{\beta}(q_{vs} - q_v), \quad \beta \in (0, 1], \quad q_r^+ = \max\{0, q_r\}$$

for fixed saturation mixing ratio q_{vs} . For $\beta = 1$ this is Lipschitz continuous, and hence maximal L^q -regularity can be used, while for $\beta < 1$ other methods must be applied.

On the other hand, the Fujita-Kato method is more flexibel in various situations compared to the approach presented here. This approach allows to include for example anisotropic spaces. Considering for simplicity the case of pure Neumann boundary conditions, where $A_p v = \Delta v$, we may split $e^{tA_p} = e^{t\Delta_H} \circ e^{t\Delta_z}$ into commuting semigroups generated by $\Delta_H = \partial_x^2 + \partial_y^2$ and $\Delta_z = \partial_z^2$. So, using the anisotropic estimate

$$\|F_p(v)\|_{L^p(\Omega)} \leq \|v\|_{H_z^{1,p} H_{xy}^{1/p,p}} \|v\|_{L_z^p H_{xy}^{1+1/p,p}},$$

and considering quantities of the form

$$K(v)(t) = \sup_{0 < s < t} s^{1/2+1/2p} \|v(s)\|_{H_z^{1,p} H_{xy}^{1/p,p}} \quad \text{and} \quad H(v)(t) = \sup_{0 < s < t} s^{1/2+1/2p} \|v(s)\|_{L_z^p H_{xy}^{1+1/p,p}}$$

we may distribute time weights anisotropically which leads to initial values

$$v_0 \in H_z^{1/p,p} H_{xy}^{1/p} \cap L_z^p H_{xy}^{2/p} \cap L_{\sigma}^p(\Omega), \quad p \in (1, \infty),$$

which for $p = 2$ is slightly better than the result presented here since $H^1(\Omega) \subset H_z^{1/2} H_{xy}^{1/2} \cap L_z^2 H_{xy}^1$.

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