Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 916 (2017) 279–303

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

Sasakian quiver gauge theory on the Aloff–Wallach space $X_{1,1}$

Jakob C. Geipel

Institut für Theoretische Physik, Leibniz Universität Hannover, Appelstraße 2, 30167 Hannover, Germany

Received 10 August 2016; accepted 8 January 2017

Available online 11 January 2017

Editor: Professor Leonardo Rastelli

Abstract

We consider the SU(3)-equivariant dimensional reduction of gauge theories on spaces of the form $M^d \times X_{1,1}$ with d-dimensional Riemannian manifold M^d and the Aloff–Wallach space $X_{1,1} = \mathrm{SU}(3)/\mathrm{U}(1)$ endowed with its Sasaki–Einstein structure. The condition of SU(3)-equivariance of vector bundles, which has already occurred in the studies of Spin(7)-instantons on cones over Aloff–Wallach spaces, is interpreted in terms of quiver diagrams, and we construct the corresponding quiver bundles, using (parts of) the weight diagram of SU(3). We consider three examples thereof explicitly and then compare the results with the quiver gauge theory on $Q_3 = \mathrm{SU}(3)/(\mathrm{U}(1) \times \mathrm{U}(1))$, the leaf space underlying the Sasaki–Einstein manifold $X_{1,1}$. Moreover, we study instanton solutions on the metric cone $C\left(X_{1,1}\right)$ by evaluating the Hermitian Yang–Mills equation. We briefly discuss some features of the moduli space thereof, following the main ideas of a treatment of Hermitian Yang–Mills instantons on cones over generic Sasaki–Einstein manifolds in the literature.

1. Introduction

The emergence of extra dimensions in string theory and the typical ansatz for compactifications make a detailed understanding of higher-dimensional gauge theories desirable. Inspired by the seminal investigation of four-dimensional manifolds by self-dual connections [1], general-

E-mail address: jakob.geipel@itp.uni-hannover.de.

http://dx.doi.org/10.1016/j.nuclphysb.2017.01.006

ized self-duality equations and instantons in higher dimensions have been studied [2–5]. Their significance in physics is evident in heterotic string theory where an instanton equation is part of the BPS equations [5,6].

Often the manifolds modeling the internal degrees of freedom are chosen as coset spaces G/H, and dimensional reduction of the gauge theory on $M^d \times G/H$ to a theory on M^d is known as *coset space dimensional reduction* [7]. On those spaces one can demand G-equivariance of the vector bundles the gauge connection takes values in, and this *equivariant dimensional reduction* yields systematic restrictions which can be depicted as *quiver diagrams*, i.e. directed graphs. A detailed mathematical treatment for Kähler manifolds can be found in [8,9] and short physical reviews are given e.g. in [10,11].

These quiver gauge theories have been studied for the Kähler cosets $\mathbb{C}P^1$ [9,12–14], $\mathbb{C}P^1 \times \mathbb{C}P^1$ [15], and $\mathrm{SU}(3)/H$ [14,16]. The odd-dimensional counterparts of Kähler spaces are Sasaki manifolds [17], and among them Sasaki–Einstein manifolds [18] are of particular interest for compactifications in string theory because, by definition, their metric cones are Calabi–Yau [19,20]. In the literature, *Sasakian quiver gauge theory* has been studied on the orbifold S^3/Γ [22], on orbifolds $S^5/\mathbb{Z}_{q+1}$ of the five-sphere [23] and on the space $T^{1,1}$ [24], the base space of the conifold. The five-dimensional Sasaki–Einstein coset spaces as well as the new examples [25,26] are of interest for versions of the AdS/CFT correspondence. In dimension seven, one can encounter the following typical examples: the seven-sphere S^7, the Aloff–Wallach space $X_{1,1}$ [27], and also a new class of spaces constructed in [26]. They could play a role for compactifications of 11-dimensional supergravity. In this article we will consider the Sasakian quiver gauge theory on the Aloff–Wallach space $X_{1,1}$. The mathematical properties of the generic Aloff–Wallach spaces $X_{k,l}$ [27] – basically their G_2 and Spin(7) structure and, for the special case of $X_{1,1}$, being Sasaki–Einstein and even 3-Sasakian – are well known [28,29]. Moreover, instanton solutions on these spaces have been constructed in [30,31]. Due to the special geometry, more precisely the existence of Killing spinors, they have been intensively studied in M-theory or supergravity [32].

This article is organized as follows: Section 2 reviews the geometry of the space $X_{1,1}$, providing local coordinates, the structure equations, the Sasaki–Einstein properties as well as a comment on the closely related Kähler space $Q_3 := \mathrm{SU}(3)/\mathrm{U}(1) \times \mathrm{U}(1)$. The subsequent section begins with a short review of equivariant vector bundles over homogeneous spaces and the arising quiver diagrams. Then we study the equivariant gauge theory on $X_{1,1}$, placing the focus on the evaluation of the equivariance condition, already known from [30,31], in terms of quiver diagrams. We discuss the general construction for the quiver diagrams associated to $X_{1,1}$ and clarify it by considering three examples with a small number of vertices. The resulting Yang–Mills functional of the equivariant gauge theory is provided, and the reduction to the quiver gauge theory on Q_3 is discussed in the last part of Section 3. Subsequently, we study instanton solutions of the quiver gauge theory by evaluating the Hermitian Yang–Mills equations on the metric cone $C\left(X_{1,1}\right)$. We briefly sketch the techniques used by Donaldson [33] and Kronheimer [34] for the discussion of the Nahm equations and the application of those methods to Hermitian Yang–Mills instantons on generic Calabi–Yau cones [35]. We discuss the modifications that appear in our setup, due to using a different instanton connection in the ansatz for the gauge connection, in comparison with the general results of [35]. The appendix provides some technical details.

2. Geometry of the Aloff–Wallach space $X_{1,1}$

In this section we review the geometric properties of the Aloff–Wallach space $X_{1,1}$ and its metric cone $C\left(X_{1,1}\right)$ which are necessary for the discussion in this article. Among the huge number of articles on the geometry of Aloff–Wallach spaces $X_{k,l}$[27], we follow the exposition given in the article [30], in which G_2 and Spin(7)-instantons on the spaces have been considered. In particular, we employ their choice of SU(3) generators, structure constants and the ansatz for the gauge connections. Since we are aiming only at the Sasaki–Einstein structure of $X_{1,1}$, we will not consider general spaces $X_{k,l}$. For details on theses structures we refer to [30] and the references therein.

2.1. Local coordinates and structure equations

The Aloff–Wallach spaces [27], denoted as $X_{k,l}$, for coprime integers k and l, are defined as quotients

$$X_{k,l} = G/H := \mathrm{SU}\left(3\right)/\mathrm{U}\left(1\right)_{k,l} \tag{2.1}$$

where the embedding of elements $h \in \mathrm{U}\left(1\right)_{k,l}$ into SU(3) is given by

$$h = \mathrm{diag}\left(\mathrm{e}^{\,\mathrm{i}\,(k+l)\varphi},\, \mathrm{e}^{\,-\,\mathrm{i}\,k\varphi},\, \mathrm{e}^{\,-\,\mathrm{i}\,l\varphi}\right). \tag{2.2}$$

It is known that the homogeneous space $X_{1,1}$ is not only Sasaki–Einstein but moreover admits a 3-Sasakian structure.[1] Due to [36] a homogeneous 3-Sasakian manifold different from a sphere is an $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2$ bundle over a quaternionic Kähler manifold; in the case of $X_{1,1}$ the underlying space is $\mathbb{C}P^2$. Using this result, we can construct local coordinates[2] by starting from a local section of the fibration $\mathrm{SU}(3) \to \mathbb{C}P^2$, as it can be found e.g. in [16,21]. Given a local patch $\mathcal{U}_0 := \left\{[w_0 : w_1 : w_2] \in \mathbb{C}P^2 \mid w_0 \neq 0\right\}$ of $\mathbb{C}P^2$, one can introduce coordinates

$$Y := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \sim \left(1, \frac{w_1}{w_0}, \frac{w_2}{w_0}\right)^{\mathrm{T}}, \tag{2.3}$$

and a local section of the bundle $\mathrm{SU}(3) \to \mathbb{C}P^2$ is given by

$$\mathbb{C}P^2 \ni Y \longmapsto V := \frac{1}{\gamma}\begin{pmatrix} 1 & \bar{Y}^\dagger \\ -\bar{Y} & \Lambda \end{pmatrix} \in \mathrm{SU}(3) \tag{2.4}$$

with

$$\gamma := \sqrt{1+\bar{Y}^\dagger\bar{Y}}, \quad \Lambda\bar{Y} = \bar{Y}, \quad \bar{Y}^\dagger\Lambda = \bar{Y}^\dagger, \quad \Lambda := \gamma\mathbf{1}_2 - \frac{1}{\gamma+1}\bar{Y}\bar{Y}^\dagger, \quad \Lambda^2 = \gamma^2\mathbf{1}_2 - \bar{Y}\bar{Y}^\dagger. \tag{2.5}$$

Furthermore, an arbitrary element g of SU(2) can be written as

$$g = \frac{1}{(1+z\bar{z})^{1/2}}\begin{pmatrix} 1 & -\bar{z} \\ z & 1 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{\,\mathrm{i}\,\varphi} & 0 \\ 0 & \mathrm{e}^{\,-\,\mathrm{i}\,\varphi} \end{pmatrix}, \tag{2.6}$$

[1] This means that the (Riemannian) holonomy of the metric cone $C\left(X_{1,1}\right)$ can be reduced from SU(4) to Sp(2).

[2] Since we will work entirely on Lie algebra level, a local description is sufficient for our purposes.

where z and $\bar{z}$ are stereographic coordinates on $\mathbb{C}P^1$. Putting both expressions (2.4) and (2.6) together, one gets a local section of the bundle $\mathrm{SU}(3) \longrightarrow X_{1,1}$ as

$$\begin{aligned}(y_1, y_2, z, \varphi) &\longmapsto \tilde{V} \\ &:= V \cdot g = \frac{1}{\gamma}\begin{pmatrix} 1 & \bar{Y}^\dagger \\ -\bar{Y} & \Lambda \end{pmatrix} \frac{1}{(1+z\bar{z})^{1/2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\bar{z} \\ 0 & z & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{\,\mathrm{i}\varphi} & 0 \\ 0 & 0 & \mathrm{e}^{-\mathrm{i}\varphi} \end{pmatrix}.\end{aligned} \tag{2.7}$$

Hence, the manifold can be locally described by the coordinates $\{y_1, \bar{y}_1, y_2, \bar{y}_2, z, \bar{z}, \varphi\}$, and the Maurer–Cartan form provides SU(3) left-invariant 1-forms Θ^α and e^i, defined by

$$\mathcal{A}_0 := \tilde{V}^{-1}\mathrm{d}\tilde{V} =: \begin{pmatrix} \frac{2\,\mathrm{i}}{\sqrt{3}}e^8 & \sqrt{2}\Theta^2 & -\sqrt{2}\Theta^{\bar{1}} \\ -\sqrt{2}\Theta^{\bar{2}} & -\frac{\mathrm{i}}{\sqrt{3}}e^8 - \mathrm{i}\,e^7 & -\Theta^{\bar{3}} \\ \sqrt{2}\Theta^1 & \Theta^3 & -\frac{\mathrm{i}}{\sqrt{3}}e^8 + \mathrm{i}\,e^7 \end{pmatrix}. \tag{2.8}$$

Here we have defined the forms such that the generators of SU(3) (see Appendix A.1) coincide with those from [30]. Due to the flatness of the connection, $\mathrm{d}\mathcal{A}_0 + \mathcal{A}_0 \wedge \mathcal{A}_0 = 0$, one obtains the structure equations

$$\begin{aligned} \mathrm{d}\Theta^1 &= -\mathrm{i}\,e^7 \wedge \Theta^1 + \sqrt{3}\,\mathrm{i}\,e^8 \wedge \Theta^1 - \Theta^{\bar{2}3}, \\ \mathrm{d}\Theta^2 &= -\mathrm{i}\,e^7 \wedge \Theta^2 - \sqrt{3}\,\mathrm{i}\,e^8 \wedge \Theta^2 + \Theta^{\bar{1}3}, \\ \mathrm{d}\Theta^3 &= -2\,\mathrm{i}\,e^7 \wedge \Theta^3 - 2\Theta^{12}, \\ \mathrm{d}e^7 &= -\mathrm{i}\left(\Theta^{1\bar{1}} + \Theta^{2\bar{2}} + \Theta^{3\bar{3}}\right), \\ \mathrm{d}e^8 &= \sqrt{3}\,\mathrm{i}\left(\Theta^{1\bar{1}} - \Theta^{2\bar{2}}\right), \end{aligned} \tag{2.9}$$

together with the complex conjugated equations for $\Theta^{\bar{\alpha}}$, $\alpha = 1, 2, 3$. By construction, the group $\mathrm{U}(1)_{k,l}$ in the definition (2.1) is generated by I_8 in (A.1), and the remaining group U(1) inside $X_{1,1}$ is associated to I_7 and the local coordinate φ.

2.2. Sasaki–Einstein structure

Following [30], the Einstein metric is chosen to be

$$\mathrm{d}s^2_{X_{1,1}} = g_{\mu\nu}e^\mu \otimes e^\nu = \Theta^1 \otimes \Theta^{\bar{1}} + \Theta^2 \otimes \Theta^{\bar{2}} + \Theta^3 \otimes \Theta^{\bar{3}} + e^7 \otimes e^7, \tag{2.10}$$

and the Sasaki structure is defined by declaring the forms Θ^α to be holomorphic, $\tilde{J}\Theta^\alpha = \mathrm{i}\,\Theta^\alpha$. Here $\tilde{J}$ denotes the complex structure of the leaf space orthogonal to the contact direction e^7. Then the fundamental form ω associated to it satisfies the Sasaki condition

$$2\omega = \mathrm{d}\eta := \mathrm{d}e^7 = -\mathrm{i}\left(\Theta^{1\bar{1}} + \Theta^{2\bar{2}} + \Theta^{3\bar{3}}\right), \tag{2.11}$$

which implies that ω is the Kähler form of the leaf space. The metric cone $C\left(X_{1,1}\right)$ has by definition the metric

$$\mathrm{d}s^2_{C(X_{1,1})} = r^2\mathrm{d}s^2_{X_{1,1}} + \mathrm{d}r \otimes \mathrm{d}r = r^2\left(\mathrm{d}s^2_{X_{1,1}} + \frac{\mathrm{d}r}{r} \otimes \frac{\mathrm{d}r}{r}\right) = r^2 \sum_{\alpha=1}^{4} \Theta^\alpha \otimes \Theta^{\bar{\alpha}}, \tag{2.12}$$

where one has defined a fourth holomorphic form

$$\Theta^4 := \frac{\mathrm{d}r}{r} - \mathrm{i}\, e^7 . \tag{2.13}$$

Equation (2.12) establishes the correspondence between the metric cone and the conformally equivalent cylinder.[3] The definition of Θ^4 yields an integrable complex structure J on the metric cone whose fundamental form $\Omega(X, Y) := g(JX, Y)$ is then given by

$$\Omega = -\frac{\mathrm{i}}{2} r^2 \sum_{\alpha=1}^{4} \Theta^\alpha \wedge \Theta^{\bar{\alpha}} = r^2 \omega + r\mathrm{d}r \wedge e^7 . \tag{2.14}$$

Due to the Sasaki condition $\mathrm{d}e^7 = 2\omega$ this form is closed and the cone $C\left(X_{1,1}\right)$, thus, carries a Kähler structure. For the cone to be Calabi–Yau, the holonomy U(4) of the Kähler manifold must be reduced further to SU(4), which is ensured by the closure of the 4-form [30]

$$\Omega^{4,0} := r^4 \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge \Theta^4 . \tag{2.15}$$

Consequently, the geometric structure is that of a Calabi–Yau 4-fold, which implies the Sasaki–Einstein structure of $X_{1,1}$. As a Sasakian manifold, $X_{1,1}$ is a U(1)-bundle over an underlying Kähler manifold, namely the leaf space of the foliation along the Reeb vector field, with fundamental form ω. The Kähler manifold underlying $X_{1,1}$ is denoted as Q_3 or $\mathbb{F}_3$ [16,21],

$$X_{1,1} \xrightarrow{\quad \mathrm{U}(1) \quad} Q_3 := \frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times \mathrm{U}(1)} \tag{2.16}$$

From the (local) section in (2.7) one has locally $Q_3 \cong \mathbb{C}P^2 \times \mathbb{C}P^1$, and this space is described by the coordinates $\{y_1, \bar{y_1}, y_2, \bar{y_2}, z\}$.

3. Quiver gauge theory on $X_{1,1}$

Quiver diagrams are a powerful tool in representation theory, and this motivates their appearance in gauge theories, where the field content can be described by these directed graphs. In this section we will demonstrate the basic features of quiver gauge theories by considering them on the spaces $X_{1,1}$ and Q_3. We start the survey with a brief review of how quiver diagrams arise in the context of gauge theories[4] on reductive homogeneous spaces G/H.

3.1. *Preliminaries of quiver gauge theory*

The condition generating the quiver diagrams, which we will usually refer to as *equivariance condition*, can be understood from two points of views: On the one hand, one could consider equivariant vector bundles in a rigorous algebraic fashion as it is done in [8,9], purely based on the representation theory of the Lie algebras involved. On the other hand, the equivariance

[3] Considering the metric cone is tantamount to studying the conformally equivalent cylinder for the discussion in this article. One can obtain an orthonormal basis by rescaling the forms $\tilde{e}^\mu := r\, \mathrm{e}^\mu$. We will mainly use the cylinder for the description here.

[4] Note that for us the term *quiver gauge theory* always refers to the structures arising from the bundle equivariance. Thus our definition is not directly related to other forms of quiver gauge theories in the literature, e.g. [37], which are based on brane physics.

condition occurs quite naturally in the context of instanton studies, e.g. [38–42], as invariance condition on gauge connections on reductive homogeneous spaces G/H.

Equivariant vector bundles We sketch the basics of equivariant vector bundles and their relation to quiver gauge theories, following roughly [8,10]. For the application of this approach we refer also to the examples in [16,23]. Let G/H be a Riemannian coset space modeling the internal degrees of freedom, M^d a d-dimensional Riemannian manifold, and let $\pi : \mathcal{E} \to M^d \times G/H$ be a Hermitian vector bundle[5] of rank k, i.e. a vector bundle with structure group $\mathrm{U}(k)$. Suppose that the Lie group G acts trivially on M^d and in the usual way on the coset space. Then the bundle is called G-*equivariant* if the action of G on the base space and on the total space, respectively, commutes with the projection map π and induces isomorphisms among the fibers $\mathcal{E}_x \simeq \mathbb{C}^k$. By restriction and induction of bundles, $\mathcal{E} = G \times_H E$, G-equivariant bundles $\mathcal{E} \to M^d \times G/H$ are in one-to-one correspondence with H-equivariant bundles $E \to M^d$ [8].

Since the action of the closed subgroup H on the base space is trivial, the equivariance of the bundle implies that the fibers must carry representations of H. We assume that these H-representations stem from the restriction of an irreducible[6] G-representation $\mathcal{D}$ which decomposes under restriction to H as follows

$$\mathcal{D}_{|H} = \bigoplus_{i=0}^{m} \rho_i , \tag{3.1}$$

where the ρ_i are irreducible H-representations. This yields an isotopical decomposition[7] of the vector bundle E as a Whitney sum in the very same way

$$E = \bigoplus_{i} E_i \quad \text{with} \quad (E_i)_x \cong \mathbb{C}^{k_i} \quad \text{carrying} \quad \rho_i \tag{3.2}$$

and induces a breaking of the generic structure group $\mathrm{U}(k)$ of the bundle to

$$\mathrm{U}(k) \longrightarrow \prod_{i=0}^{m} \mathrm{U}(k_i) \quad \text{with} \quad \sum_{i=0}^{m} k_i = k. \tag{3.3}$$

The action of the entire group G on the decomposition (3.2) connects different representations ρ_i, i.e. it leads to homomorphisms from $\mathrm{Hom}\left(\mathbb{C}^{k_i}, \mathbb{C}^{k_j}\right)$. In this way, the fibres of the G-equivariant bundle are representations of a *quiver*[8] $(\mathcal{Q}_0, \mathcal{Q}_1)$, where $\mathcal{Q}_0$ denotes the set of vertices and $\mathcal{Q}_1$ the set of arrows. Each vertex $v_i \in \mathcal{Q}_0$ carries a vector space isomorphic to $\mathbb{C}^{k_i}$ with an H-representation, and the arrows are represented by linear maps among these spaces. The entire G-equivariant bundle thus carries a representation of the quiver, and this construction is called

5 One should keep in mind that the fundamental objects of a gauge theory are *principal bundles* $(P, p, X; K)$ with total space P, base space X, projection map p, and gauge group K although we will work completely in terms of vector bundles in this article. They can be thought of as associated to the relevant principal bundle P.

6 This assumption is not mandatory for the approach, but the restriction to irreducible representations makes it clearer because reducible representations would lead to results involving those of smaller, irreducible representations.

7 In general, one can split the summands further into $E_i = \tilde{E}_i \otimes \underline{V_i}$, where $\underline{V_i}$ is an irreducible H-representation and the subgroup H acts trivially on $\tilde{E}_i$ [15]. Since we consider an abelian subgroup H, the irreducible representations are 1-dimensional, so that H acts as multiple of the identity on the entire space E_i. For an example of a non-abelian subgroup H, consider for instance [23].

8 For details on representations of quiver diagrams, see for example [43,44].

a *quiver bundle*. Since the allowed arrows of the quiver diagram arise from the commutation relations of the generators with the elements of the subalgebra $\mathfrak{h}$, this approach is entirely based on the representation theory of $\mathfrak{h}$ and $\mathfrak{g}$, and it can be realized using (parts of) the weight diagram of the Lie algebra $\mathfrak{g}$.

Invariant gauge connections The equivariance condition leading to the quiver diagrams also occurs naturally when studying instanton solutions of invariant gauge connections on reductive homogeneous spaces, e.g. in [30,41,42]. Let G/H be a reductive homogeneous space with the $\mathrm{Ad}(H)$-invariant splitting

$$\mathrm{span}\langle I_\mu\rangle := \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} =: \mathrm{span}\langle I_j\rangle \oplus \mathrm{span}\langle I_a\rangle, \tag{3.4}$$

where the generators satisfy

$$\left[I_j, I_k\right] = C^l_{jk} I_l, \quad \left[I_j, I_a\right] = C^b_{ja} I_b, \quad \text{and} \quad [I_a, I_b] = C^c_{ab} I_c + C^j_{ab} I_j; \tag{3.5}$$

the space $\mathfrak{m}$ can be identified with the tangent space of G/H. Let e^μ be the 1-forms dual to the generators I_μ, which obey the structure equation

$$\mathrm{d}e^\mu = -\frac{1}{2} C^\mu_{\rho\sigma} e^{\rho\sigma} = -\Gamma^\mu_\nu \wedge e^\nu + T^\mu, \tag{3.6}$$

where Γ^μ_ν are the connection 1-forms describing a (metric) connection Γ on the homogeneous space, and T^μ is its torsion. Due to a known result from differential geometry [45] and following the approach used for example in [41], we can express a G-invariant connection $\mathcal{A}$ on the homogeneous space as

$$\mathcal{A} = I_j \otimes e^j + X_a \otimes e^a, \tag{3.7}$$

where the skew-hermitian matrices X_a, the *Higgs fields*, describe the endomorphism part. The connection[9] $\Gamma := I_j \otimes e^j$ takes values entirely in the vertical component $\mathfrak{h}$ and is obtained by declaring the torsion to be $T(X, Y) := -[X, Y]_{\mathfrak{m}}$ for $X, Y \in T_e(G/H)$. The curvature $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ of (3.7) is then given by

$$\mathcal{F} = \mathcal{F}_\Gamma + \left(\left[I_j, X_a\right] - C^b_{ja} X_b\right) e^j \wedge e^a + \frac{1}{2}\left([X_a, X_b] - C^c_{ab} X_c\right) e^{ab} + \mathrm{d}X_a \wedge e^a. \tag{3.8}$$

For the connection to be G-invariant, terms containing the mixed 2-forms $\mathrm{e}^j \wedge e^a$ must not occur, so that one obtains – assuming that the last term in (3.8) does not yield incompatible contributions[10] – the *equivariance condition* [41,45]

$$\left[I_j, X_a\right] = C^b_{ja} X_b. \tag{3.9}$$

Thus the equivariance forces the endomorphisms X_a to act (with respect to the adjoint action) on the fibres of the bundle as the generators I_a in (3.5) do.

Construction procedure Based on the outline above, we can construct an equivariant gauge connection and the corresponding quiver bundle for $X_{1,1} = \mathrm{SU}(3)/\mathrm{U}(1)_{1,1}$ in the following way. Let

[9] In principle one could also use different connections Γ as starting point in the ansatz (3.7). See the comments in Section 4.

[10] This holds true e.g. for constant matrices or those with $X_a = X_a(r)$, as we will consider on the metric cone $C(G/H)$ in Section 4.

$$\mathbb{C}^k = \left(\mathbb{C}^{k_0}, \mathbb{C}^{k_1}, \dots, \mathbb{C}^{k_m}\right)^{\mathrm{T}} \tag{3.10}$$

be a decomposition of the representations on the fibres in $(m+1)$ terms, which yields the breaking of the structure group (3.3) and the isotopical decomposition as in (3.2). Since the irreducible representations ρ_i of the abelian subgroup $H = \mathrm{U}(1)_{1,1}$ are 1-dimensional, the group H acts as

$$\left(\zeta_0\, \mathbf{1}_{k_0}, \zeta_1\, \mathbf{1}_{k_1}, \dots, \zeta_m\, \mathbf{1}_{k_m}\right) \tag{3.11}$$

on the vectors (3.10). The constants ζ_i can be obtained from an irreducible representation of the $\mathrm{U}(1)_{1,1}$-generator on an $(m+1)$-dimensional vector space. This fact and the way how the quiver diagrams arise motivate to consider the gauge connection as a block matrix of size $(m+1)^2$, whose structure is determined by the $(m+1)$-dimensional G-representation in which the entries are (implicitly) replaced by homomorphisms. By construction and due to the equivariance condition (3.9), the quiver diagram is then based on (parts of) the underlying weight diagram of the chosen G-representation. If the subgroup H is a maximal torus, the quiver coincides with the weight diagram because all Cartan generators occur as operators I_j in (3.9). For smaller subgroups there might be degeneracies as double arrows in the diagram, while larger groups require a collapsing of vertices in the weight diagram along the action of the ladder operators of $\mathfrak{h}$ as it is done, for instance, in [11,16]. We will clarify this procedure for the abelian subgroup $H = \mathrm{U}(1)_{1,1}$ in the following.

3.2. Equivariance condition and quiver diagrams of $X_{1,1}$

The aforementioned approach is now applied to the space $X_{1,1}$. Following the outline above and according to (3.7), we write an SU(3)-invariant connection $\mathcal{A}$ on $M^d \times X_{1,1}$ as [30]

$$\mathcal{A} = A + I_8 \otimes e^8 + \sum_{a=1}^{7} X_a \otimes e^a =: A + I_8 \otimes e^8 + \sum_{\alpha=1}^{3} \left(Y_\alpha \otimes \Theta^\alpha + Y_{\bar\alpha} \otimes \Theta^{\bar\alpha}\right) + X_7 \otimes e^7, \tag{3.12}$$

where A is a connection on M^d. Moreover, we have defined complex endomorphisms

$$Y_1 := \frac{1}{2}\left(X_1 + \mathrm{i}\, X_2\right), \quad Y_2 := \frac{1}{2}\left(X_3 + \mathrm{i}\, X_4\right), \quad Y_3 := \frac{1}{2}\left(X_5 + \mathrm{i}\, X_6\right) \quad \text{with} \quad Y_{\bar\alpha} := -Y_\alpha^\dagger . \tag{3.13}$$

In terms of the structure constants (A.3) the field strength of the connection $\mathcal{A}$ is given by [30]

$$\begin{aligned}
\mathcal{F} = {}& \mathrm{d}A + A \wedge A + (\mathrm{d}Y_\alpha + [A, Y_\alpha]) \wedge \Theta^\alpha \\
&+ (\mathrm{d}Y_{\bar\alpha} + [A, Y_{\bar\alpha}]) \wedge \Theta^{\bar\alpha} + (\mathrm{d}X_7 + [A, X_7]) \wedge e^7 \\
&+ \frac{1}{2}\left(\left[Y_\alpha, Y_\beta\right] - C^{\gamma}_{\alpha\beta} Y_\gamma\right) \Theta^{\alpha\beta} \\
&+ \left(\left[Y_\alpha, Y_{\bar\beta}\right] - C^{\gamma}_{\alpha\bar\beta} Y_\gamma - C^{\bar\gamma}_{\alpha\bar\beta} Y_{\bar\gamma} + \mathrm{i}\, C^{7}_{\alpha\bar\beta} X_7 + \mathrm{i}\, C^{8}_{\alpha\bar\beta} I_8\right) \Theta^{\alpha\bar\beta} \\
&+ \frac{1}{2}\left(\left[Y_{\bar\alpha}, Y_{\bar\beta}\right] - C^{\bar\gamma}_{\bar\alpha\bar\beta} Y_{\bar\gamma}\right) \Theta^{\bar\alpha\bar\beta} + \left([X_7, Y_\alpha] - \mathrm{i}\, C^{\beta}_{7\alpha} Y_\beta\right) e^7 \wedge \Theta^\alpha \\
&+ \left([X_7, Y_{\bar\alpha}] - \mathrm{i}\, C^{\bar\beta}_{7\bar\alpha} Y_{\bar\beta}\right) e^7 \wedge \Theta^{\bar\alpha} + \left([I_8, Y_\alpha] - \mathrm{i}\, C^{\beta}_{8\alpha} Y_\beta\right) e^8 \wedge \Theta^\alpha \\
&+ \left([I_8, Y_{\bar\alpha}] - \mathrm{i}\, C^{\bar\beta}_{8\bar\alpha} Y_{\bar\beta}\right) e^8 \wedge \Theta^{\bar\alpha} + [I_8, X_7]\, e^{87}.
\end{aligned} \tag{3.14}$$

Following some notation in the literature, e.g. in [16], we call

$$\phi^{(\alpha)} := Y_{\bar{\alpha}} \text{ for } \alpha = 1, 2, 3, \quad \text{and} \quad X_7 \tag{3.15}$$

the *Higgs fields* and set[11]

$$\hat{I}_8 := -\sqrt{3}\,\mathrm{i}\, I_8 = \mathrm{diag}\,(2, -1, -1) \qquad \text{and} \qquad \hat{I}_7 := -\,\mathrm{i}\, I_7 = \mathrm{diag}\,(0, -1, 1)\,. \tag{3.16}$$

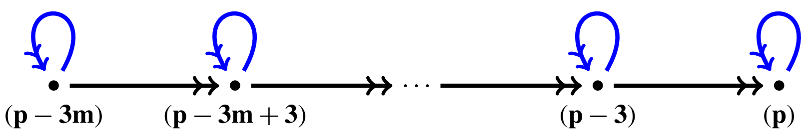
The equivariance condition (3.9), equivalent to the vanishing of the last three terms in (3.14), then reads

$$\left[\hat{I}_8, \phi^{(1)}\right] = 3\phi^{(1)}, \quad \left[\hat{I}_8, \phi^{(2)}\right] = -3\phi^{(2)}, \text{ and } \left[\hat{I}_8, \phi^{(3)}\right] = 0 = \left[\hat{I}_8, X_7\right]. \tag{3.17}$$

Consequently, the endomorphisms $\phi^{(1)}$ and $\phi^{(2)\dagger}$ will have the same block form and the form of $\phi^{(3)}$ coincides with that of X_7, but their entries are still arbitrary and not related to each other. The commutation relations (3.17) provide the action of the Higgs fields on the quantum numbers (ν_7, ν_8) associated to the two Cartan generators $\hat{I}_7$ and $\hat{I}_8$ of SU(3)

$$\begin{aligned} \phi^{(1)} &: (\nu_7, \nu_8) \longmapsto (*, \nu_8 + 3), \\ \phi^{(2)} &: (\nu_7, \nu_8) \longmapsto (*, \nu_8 - 3), \\ \phi^{(3)} &: (\nu_7, \nu_8) \longmapsto (*, \nu_8), \\ X_7 &: (\nu_7, \nu_8) \longmapsto (*, \nu_8). \end{aligned} \tag{3.18}$$

Since the quantum number ν_7 does not enter the equivariance condition, it is reasonable[12] to label the vertices in the quiver diagram only by the number ν_8, so that one obtains effectively a modified version of the holomorphic chain [9]: a diagram consisting of double arrows between adjacent vertices and double loops at each vertex,



(3.19)

where the black two headed arrows denote the contributions by $\phi^{(1)}$ and $\phi^{(2)\dagger}$, while the endomorphisms $\phi^{(3)}$ and X_7 are represented by the blue two headed loops.[13] Here, the integer p denotes the highest weight (with respect to ν_8) of the representation $\mathcal{D}$. The endomorphism part of the invariant connection associated to this modified holomorphic chain of length $m + 1$ is then given by

[11] As mentioned above, we implicitly interpret the numbers in the Cartan generators as numbers times identity operators.

[12] This corresponds to the isotopical decomposition (3.2): The representation $\mathcal{D}$ is restricted under the subgroup $\mathrm{U}(1)_{1,1}$ rather than under a maximal torus $\mathrm{U}(1) \times \mathrm{U}(1)$.

[13] Using one arrow with two heads as symbol for two arrows improves the readability of the more complicated diagrams like Fig. 3 significantly.

$$X_a e^a = \begin{pmatrix} \Psi_p & \Phi_{p-3} & 0 & \dots & 0 \\ -\Phi^\dagger_{p-3} & \Psi_{p-3} & \Phi_{p-6} & \dots & \vdots \\ 0 & -\Phi^\dagger_{p-6} & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \Psi_{p-3} & \Phi_{p-3m} \\ 0 & \dots & 0 & -\Phi^\dagger_{p-3m} & \Psi_{p-3m} \end{pmatrix}, \tag{3.20}$$

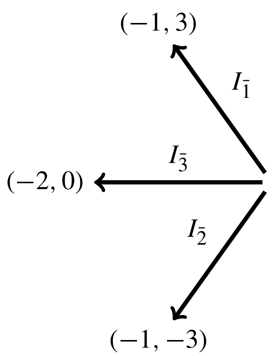
where we have defined the abbreviations

$$\Phi_{p-3j} := \phi^{(1)}_{p-3j} \otimes \Theta^{\bar{1}} - \phi^{(2)\,\dagger}_{p-3j} \otimes \Theta^2 \quad \text{and} \quad \Psi_{p-3j} := \phi^{(3)}_{p-3j} \otimes \Theta^{\bar{3}} + (X_7)_{p-3j} \otimes e^7 \tag{3.21}$$

for $j = 0, 1, \dots, m$, and the indices label the tail of the arrow. The remaining contribution to the invariant connection (3.12) is given by the diagonal parts

$$\Gamma + A = \operatorname{diag}\left(\mathbf{1}_{k_l} \otimes \frac{3l-p}{\sqrt{3}}\,\mathrm{i}\,e^8 + A_{p-3l}\right)_{l=0,\dots,m}, \tag{3.22}$$

where A_{p-3l} is a component – according to the isotopical decomposition (3.2) of the fibres – of a connection on the bundle $E \to M^d$. Equations (3.20) and (3.22) describe the general solution. For comparisons with gauge theories of similar geometric structures like Q_3, it is advantageous to consider not only the decomposition under the subgroup H, i.e. labeling the vertices only by ν_8 as we did, but to study the equivariance conditions in the entire weight diagram of G. Since the weight diagrams[14] of the relevant Lie algebras are well-known, one can quickly construct the invariant connection by implementing the rules (3.18) and can then project to the relevant quantum numbers. In the following, we will consider the triangular/hexagonal weight diagram of SU(3), spanned by the roots



(3.23)

and the conjugated operators I_α.

3.2.1. Examples

We consider three explicit examples of SU(3) representations and the quiver diagrams associated to them.

[14] For representation theory of $\mathfrak{su}(3)$ see e.g. [46].

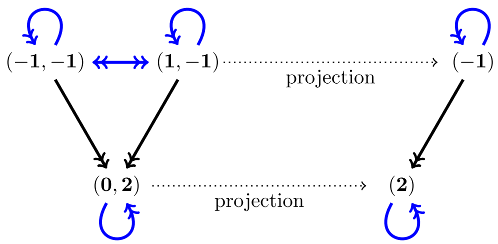


Fig. 1. Quiver diagram of $X_{1,1}$ for the fundamental representation $\underline{\mathbf{3}}$ of SU(3): The left diagram stems from the implementation of the equivariance condition in the weight diagram of SU(3) and the right one is the holomorphic chain with the loop modification and the double arrows, obtained from the projection by forgetting about the second quantum number ν_7, i.e. identifying points along horizontal lines.

Fundamental representation Applying the prescription (3.18) to the single triangle of the weight diagram of the defining representation $\underline{\mathbf{3}}$ provides the quiver diagram in Fig. 1. Of course, this diagram could be also obtained by direct evaluation of the commutator of $\hat{I}_8 = \mathrm{diag}(2, -1, -1)$ with an arbitrary 3×3-matrix $(\bullet)$

$$\left[\hat{I}_8, (\bullet)\right] = \begin{pmatrix} 0\bullet & 3\bullet & 3\bullet \\ -3\bullet & 0\bullet & 0\bullet \\ -3\bullet & 0\bullet & 0\bullet \end{pmatrix}. \tag{3.24}$$

Then the equivariance condition requires the Higgs fields to be of the form

$$\phi^{(1)} = \begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \phi^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, \quad \phi^{(3)} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \quad X_7 = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \tag{3.25}$$

which again yields the quiver diagram Fig. 1. This translates into the invariant gauge connection

$$\mathcal{A}_{\underline{\mathbf{3}}} = \begin{pmatrix} \frac{2}{\sqrt{3}}\,\mathrm{i}\,e^8 \otimes \mathbf{1} + \Psi_{0,2;0,2} & \Phi_{-1,-1;0,2} & \Phi_{1,-1;0,2} \\ -\Phi^\dagger_{-1,-1;0,2} & -\frac{1}{\sqrt{3}}\,\mathrm{i}\,e^8 \otimes \mathbf{1} + \Psi_{-1,-1;-1,-1} & +\Psi_{1,-1;-1,-1} \\ -\Phi^\dagger_{1,-1;0,2} & -\Psi^\dagger_{1,-1;-1,-1} & -\frac{1}{\sqrt{3}}\,\mathrm{i}\,e^8 \otimes \mathbf{1} + \Psi_{1,-1;1,-1} \end{pmatrix}, \tag{3.26}$$

where we have defined

$$\begin{aligned} \Phi_{i,j;k,l} &:= (\phi^{(1)})_{i,j;k,l} \otimes \Theta^{\bar{1}} - (\phi^{(2)\dagger})_{i,j;k,l} \otimes \Theta^2, \\ \Psi_{i,j;k,l} &:= (\phi^{(3)})_{i,j;k,l} \otimes \Theta^{\bar{3}} + (X_7)_{i,j;k,l} \otimes e^7; \end{aligned} \tag{3.27}$$

the $\mathrm{U}(1) \times \mathrm{U}(1)$-charges (i, j) denote the tail of the arrow, and (k, l) its head. Going back to the effective quiver diagram, i.e. the modified holomorphic chain, yields

$$\mathcal{A}_{\mathbf{3}} = \begin{pmatrix} \frac{2}{\sqrt{3}}\,\mathrm{i}\,e^8 \otimes \mathbf{1} + \Psi_{2;2} & \tilde{\Phi}_{-1;2} \\ -\tilde{\Phi}^\dagger_{-1;2} & -\frac{1}{\sqrt{3}}\,\mathrm{i}\,e^8 \otimes \mathbf{1} + \tilde{\Psi}_{-1;-1} \end{pmatrix}, \tag{3.28}$$

which agrees with the general result (3.20). The anti-fundamental representation $\underline{\bar{\mathbf{3}}}$, of course, leads to an analogous diagram and connection.

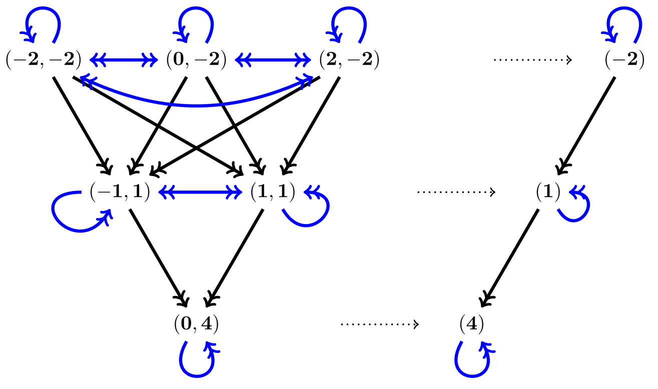


Fig. 2. Quiver diagram for the representation $\underline{\mathbf{6}}$ with the same notation as before.

Representation $\underline{\mathbf{6}}$ The six-dimensional representation $\underline{\mathbf{6}}$ of SU(3) causes the more complicated quiver diagram depicted in Fig. 2, which could be also obtained by the direct evaluation of the commutation relation with the $\mathrm{U}(1)_{1,1}$-generator $\hat{I}_8 = \operatorname{diag}(4, 1, 1, -2, -2, -2)$. The resulting fields are of the form

$$\phi^{(1)} \text{ and } \phi^{(2)\dagger} = \begin{pmatrix} 0 & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \phi^{(3)} \text{ and } X_7 = \begin{pmatrix} * & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \end{pmatrix}. \tag{3.29}$$

We skip the explicit index structure of the invariant gauge connection which can be read from the quiver diagram, Fig. 2, and provide only the result for the modified holomorphic chain

$$\mathcal{A}_{\underline{\mathbf{6}}} = \begin{pmatrix} \frac{4}{\sqrt{3}}\,\mathrm{i}\,e^8 \otimes \mathbf{1} + \Psi_{4;4} & \tilde{\Phi}_{1;4} & 0 \\ -\tilde{\Phi}^\dagger_{1;4} & \frac{1}{\sqrt{3}}\,\mathrm{i}\,e^8 \otimes \tilde{\mathbf{1}} + \tilde{\Psi}_{1;1} & \tilde{\Phi}_{-2;1} \\ 0 & -\tilde{\Phi}^\dagger_{-2;1} & -\frac{2}{\sqrt{3}}\,\mathrm{i}\,e^8 \otimes \tilde{\mathbf{1}} + \tilde{\Psi}_{-2;-2} \end{pmatrix}. \tag{3.30}$$

It is interesting to compare this block matrix of size 3×3 with that of the adjoint representation in the following example, which – on the level of the modified holomorphic chain – only differs in the occurring quantum numbers and, thus, the connection Γ.

Adjoint representation $\underline{\mathbf{8}}$ The $\mathrm{U}(1)_{1,1}$-generator in the adjoint representation is given by $\hat{I}_8 = \operatorname{diag}(3, 3, 0, 0, 0, 0, -3, -3)$ and the weight diagram is a hexagon with two degenerated points at the origin.[15] The Higgs fields must thus have the shape

[15] The representation of the other Cartan generator reads $\mathrm{ad}(-\mathrm{i}\, I_7) = \operatorname{diag}(-1, 1, -2, 0, 0, 2, -1, 1)$, which causes the degeneracy at $(0, 0)$.

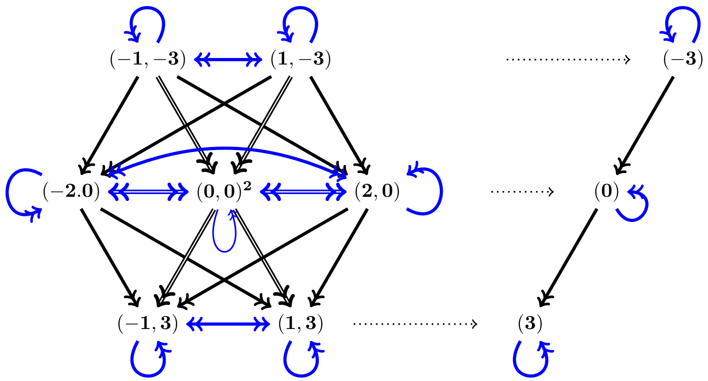


Fig. 3. Quiver diagram for the adjoint representation $\underline{\mathbf{8}}$ of SU(3). Note that due to the degeneracy of $(0, 0)$ each arrow involving the origin must be counted twice (depicted as arrows consisting of two lines), i.e. there are, for instance, *four* arrows between $(0, 0)$ and $(1, -3)$ etc.

$$\phi^{(1)} \text{ and } \phi^{(2)\dagger} = \begin{pmatrix} 0 & 0 & * & * & * & * & 0 & 0 \\ 0 & 0 & * & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\phi^{(3)} \text{ and } X_7 = \begin{pmatrix} * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & * & * & 0 & 0 \\ 0 & 0 & * & * & * & * & 0 & 0 \\ 0 & 0 & * & * & * & * & 0 & 0 \\ 0 & 0 & * & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \end{pmatrix}, \tag{3.31}$$

and the quiver diagram in Fig. 3 contains a large number of arrows. The identification leading to the modified holomorphic chain yields as connection

$$\mathcal{A}_{\underline{6}} = \begin{pmatrix} \sqrt{3}\,\mathrm{i}\, e^8 \otimes \mathbf{1} + \Psi_{3;3} & \tilde{\Phi}_{0;3} & 0 \\ -\tilde{\Phi}_{0;3}^{\dagger} & \tilde{\Psi}_{0;0} & \tilde{\Phi}_{-3;0} \\ 0 & -\tilde{\Phi}_{-3;0}^{\dagger} & -\sqrt{3}\,\mathrm{i}\, e^8 \otimes \tilde{\mathbf{1}} + \tilde{\Psi}_{-3;-3} \end{pmatrix}. \tag{3.32}$$

As mentioned before, this modified holomorphic chain of length 3 is different from that of the six-dimensional representation, (3.30), only due to the quantum numbers that appear.

The huge number of arrows in the last two examples have shown that it is advantageous to use only the relevant quantum number ν_8 rather than the entire weight diagram of G, but for comparisons with Q_3 the latter description is also useful. The occurrence of degeneracies in the entire weight diagram of SU(3) due to the weaker equivariance condition is similar [15,24] to the case of the five-dimensional Sasaki–Einstein manifold $T^{1,1} := (\mathrm{SU}(2) \times \mathrm{SU}(2))\,/\mathrm{U}(1)$ in comparison with its underlying manifold $\mathbb{C}P^1 \times \mathbb{C}P^1$.

3.3. Dimensional reduction of the Yang–Mills action

In the previous section we have completely characterized the form of a G-invariant gauge connection by applying the rules (3.18) in the weight diagram and in terms of the results (3.20) and (3.22). Given such a gauge connection $\mathcal{A}$ on $M^d \times X_{1,1}$ with field strength $\mathcal{F}$, we now determine its standard Yang–Mills action

$$S_{\mathrm{YM}} = -\frac{1}{4} \int\limits_{M^d \times X_{1,1}} \operatorname{tr} \mathcal{F} \wedge *\mathcal{F}, \tag{3.33}$$

yielding the usual Yang–Mills Lagrangian[16]

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4}\sqrt{\hat{g}}\, \operatorname{tr} \mathcal{F}_{\hat{\mu}\hat{\nu}}\mathcal{F}^{\hat{\mu}\hat{\nu}}, \tag{3.34}$$

where we denote $\hat{g} := \det g_{X_{1,1}} \det g_{M^d}$. Using the Sasaki–Einstein metric (2.10),

$$\left(g_{X_{1,1}}\right)_{\alpha\bar{\beta}} = \frac{1}{2}\delta_{\alpha\beta} \quad \text{and} \quad \left(g_{X_{1,1}}\right)_{77} = 1, \tag{3.35}$$

and the field strength components from (3.14), one obtains as Lagrangian

$$\begin{aligned}
\mathcal{L}_{\mathrm{YM}} = \sqrt{\hat{g}}\, \operatorname{tr}_k \Bigg\{ & \frac{1}{4} F_{\mu\nu} \left(F^{\mu\nu}\right)^\dagger + 2 \sum_{\alpha=1}^{3} \left| D_\mu \phi^{(\alpha)} \right|^2 + \frac{1}{2} \left| D_\mu X_7 \right|^2 \\
& + 2 \left| \left[\phi^{(1)}, \phi^{(1)\dagger}\right] - \mathrm{i}\, X_7 + \sqrt{3}\,\mathrm{i}\, I_8 \right|^2 + 2 \left| \left[\phi^{(2)}, \phi^{(2)\dagger}\right] - \mathrm{i}\, X_7 - \sqrt{3}\,\mathrm{i}\, I_8 \right|^2 \\
& + 2 \left| \left[\phi^{(3)}, \phi^{(3)\dagger}\right] - \mathrm{i}\, X_7 \right|^2 + 4 \left| \left[\phi^{(1)}, \phi^{(2)}\right] - 2\phi^{(3)} \right|^2 + 4 \left| \left[\phi^{(1)}, \phi^{(3)}\right] \right|^2 \\
& + 4 \left| \left[\phi^{(2)}, \phi^{(3)}\right] \right|^2 + 4 \left| \left[\phi^{(1)}, \phi^{(2)\dagger}\right] \right|^2 + 4 \left| \left[\phi^{(1)}, \phi^{(3)\dagger}\right] + \phi^{(2)\dagger} \right|^2 \\
& + 4 \left| \left[\phi^{(2)}, \phi^{(3)\dagger}\right] - \phi^{(1)\dagger} \right|^2 + 2 \left| \left[\phi^{(1)}, X_7\right] - \mathrm{i}\,\phi^{(1)} \right|^2 + 2 \left| \left[\phi^{(2)}, X_7\right] - \mathrm{i}\,\phi^{(2)} \right|^2 \\
& + 2 \left| \left[\phi^{(3)}, X_7\right] - 2\,\mathrm{i}\,\phi^{(3)} \right|^2 \Bigg\}.
\end{aligned} \tag{3.36}$$

Here, we have defined the covariant derivatives $D_\mu \phi^{(\alpha)} := \left(\mathrm{d}\phi^{(\alpha)} + \left[A, \phi^{(\alpha)}\right]\right)_\mu$ for $\alpha = 1, 2, 3$ and $D_\mu X_7 := (\mathrm{d}X_7 + [A, X_7])_\mu$, the field strength $F_{\mu\nu} := (\mathrm{d}A + A \wedge A)_{\mu\nu}$ and we write $|X|^2 : = X X^\dagger$. Since the fields $\phi^{(\alpha)}$ and X_7 are assumed to be independent from internal coordinates of $X_{1,1}$ (due to equivariance), the additional dimensions can be integrated out easily, which yields only a prefactor $\operatorname{vol}\left(X_{1,1}\right)$ for the dimensional reduction of the Lagrangian. In this way, one obtains from a pure Yang–Mills theory on $M^d \times X_{1,1}$ a Yang–Mills–Higgs action on M^d, where the endomorphisms $\phi^{(a)}$ and X_7 constitute a non-trivial potential provided by the internal geometry of $X_{1,1}$.

16 We use the set of indices $\{\hat{\mu}\} = \{\mu, \alpha, \bar{\alpha}, 7\}$ with μ referring to M^d.

3.4. Reduction to quiver gauge theory on Q_3

The equivariance condition and the examples of the quiver diagrams in the previous section have shown that the quiver gauge theory on $X_{1,1}$ depends on only one of the two quantum numbers of SU(3). This yields effectively a modified holomorphic chain as quiver diagram or, considered in the original weight diagram of SU(3), a diagram with multiple arrows and degeneracies. As mentioned in the discussion of the Sasaki–Einstein structure on $X_{1,1}$ in Section 2.2, the space is a U(1)-bundle over the (Kähler) space Q_3, so that it is natural to consider the reduction from the gauge theory on $X_{1,1}$ to that on Q_3 by removing the contact direction as a degree of freedom. Since we then divide by a Cartan subalgebra, the quiver diagram is simply the weight diagram of SU(3) without the degeneracies which have been caused by the weaker conditions on $X_{1,1}$. This reduction can be performed by setting the terms containing $e^7 \wedge \Theta^\alpha$ or $e^7 \wedge \Theta^{\bar\alpha}$ in the field strength (3.14) to zero. This provides the additional equivariance conditions

$$\left[X_7, \phi^{(1)}\right] = -\mathrm{i}\,\phi^{(1)}, \quad \left[X_7, \phi^{(2)}\right] = -\mathrm{i}\,\phi^{(2)}, \quad \text{and} \quad \left[X_7, \phi^{(3)}\right] = -2\,\mathrm{i}\,\phi^{(3)}. \tag{3.37}$$

For the reduction to Q_3, the field X_7 must be proportional to I_7 and setting $X_7 = I_7$ fixes the action of the Higgs fields to be

$$\begin{aligned}
&\phi^{(1)} : (\nu_7, \nu_8) \longmapsto (\nu_7 - 1, \nu_8 + 3) \\
&\phi^{(2)} : (\nu_7, \nu_8) \longmapsto (\nu_7 - 1, \nu_8 - 3) \\
&\phi^{(3)} : (\nu_7, \nu_8) \longmapsto (\nu_7 - 2, \nu_8).
\end{aligned} \tag{3.38}$$

This, indeed, requires the quiver diagrams in Fig. 4 to coincide with the weight diagrams of the chosen representations and yields the results[17] from [16,21]. The endomorphism part of the gauge connection, e.g. for the fundamental representation, reads

$$\mathcal{A}_{\underline{3}} = \begin{pmatrix} \mathbf{1} \otimes \frac{2}{\sqrt{3}}\,\mathrm{i}\,e^8 & -\Phi^{(2)\dagger}_{0,2;-1,-1} & \Phi^{(1)}_{1,-1;0,2} \\ \Phi^{(2)}_{0,2;-1,-1} & \mathbf{1} \otimes \left(-\frac{1}{\sqrt{3}}\,\mathrm{i}\,e^8 - \mathrm{i}\,e^7\right) & \Phi^{(3)}_{1,-1;-1,-1} \\ -\Phi^{(1)\dagger}_{1,-1;0,2} & -\Phi^{(3)\dagger}_{1,-1;-1,-1} & \mathbf{1} \otimes \left(-\frac{1}{\sqrt{3}}\,\mathrm{i}\,e^8 + \mathrm{i}\,e^7\right) \end{pmatrix} \tag{3.39}$$

with $\Phi^{(\alpha)} := \phi^{(\alpha)} \otimes \Theta^{\bar\alpha}$. Since the quiver diagram is the weight diagram of SU(3), the Higgs fields have the block shape of the generators (A.1) and the central idea of quiver gauge theory becomes evident: One modifies the bundle (2.8) by inserting compatible endomorphisms $\phi^{(\alpha)}$ as entries in the block matrices describing the gauge connection.

The Lagrangian of the gauge theory on $M^d \times Q_3$ is then given by that on $M^d \times X_{1,1}$ without the terms containing commutators with X_7,

$$\begin{aligned}
\mathcal{L}_{Q_3} = \sqrt{\hat g}\,\mathrm{tr}_k \Bigg\{ &\frac{1}{4} F_{\mu\nu} \left(F^{\mu\nu}\right)^\dagger + 2 \sum_{\alpha=1}^{3} \left| D_\mu \phi^{(\alpha)} \right|^2 + 2 \left| \left[\phi^{(1)}, \phi^{(1)\dagger}\right] - \mathrm{i}\,I_7 + \sqrt{3}\,\mathrm{i}\,I_8 \right|^2 \\
&+ 2 \left| \left[\phi^{(2)}, \phi^{(2)\dagger}\right] - \mathrm{i}\,I_7 - \sqrt{3}\,\mathrm{i}\,I_8 \right|^2 + 2 \left| \left[\phi^{(3)}, \phi^{(3)\dagger}\right] - \mathrm{i}\,I_7 \right|^2
\end{aligned}$$

[17] Note that the orientation of the Higgs fields depends on the chosen convention of the holomorphic structure; we denote as Higgs fields ϕ^α the endomorphisms accompanying the anti-holomorphic forms $\Theta^{\bar\alpha}$.

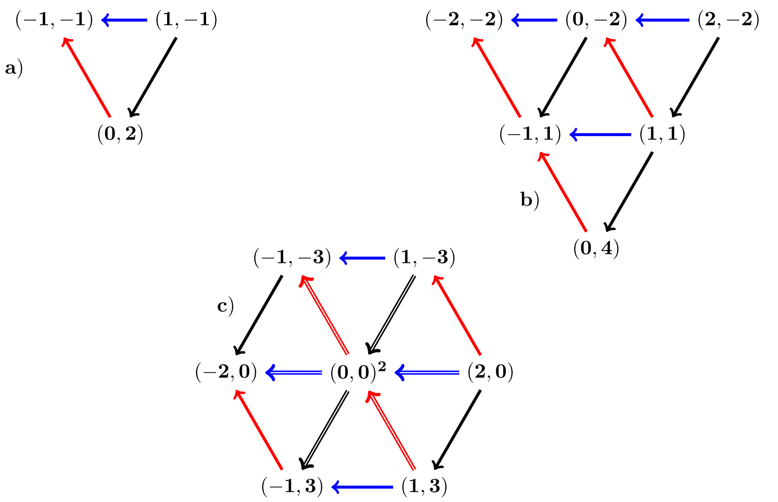


Fig. 4. Quiver diagrams of Q_3 for a) fundamental representation $\underline{\mathbf{3}}$, b) representation $\underline{\mathbf{6}}$, and c) adjoint representation $\underline{\mathbf{8}}$ (with the degenerated origin) of SU(3). The arrows denote the Higgs fields $\phi^{(1)}$ (black), $\phi^{(2)}$ (red), and $\phi^{(3)}$ (blue), according to the condition (3.38). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$
\begin{aligned}
&+4\left|\left[\phi^{(1)},\phi^{(2)}\right]-2\phi^{(3)}\right|^2+4\left|\left[\phi^{(1)},\phi^{(3)}\right]\right|^2+4\left|\left[\phi^{(2)},\phi^{(3)}\right]\right|^2\\
&+4\left|\left[\phi^{(1)},\phi^{(2)\dagger}\right]\right|^2+4\left|\left[\phi^{(1)},\phi^{(3)\dagger}\right]+\phi^{(2)\dagger}\right|^2+4\left|\left[\phi^{(2)},\phi^{(3)\dagger}\right]-\phi^{(1)\dagger}\right|^2\Bigg\},
\end{aligned}
\tag{3.40}
$$

because the vanishing of them is subject to the further equivariance conditions (3.37).

4. Instantons on the metric cone $C\left(X_{1,1}\right)$

The implementation of the equivariance condition (3.17) has determined the general form of the gauge connection, expressed in the associated quiver diagram, and the action functional, but has not restricted the entries of the endomorphisms. Further conditions and relations among the endomorphisms can be imposed by studying vacua of the gauge theory, i.e. by minimizing the action functional (3.33). To this end, we will evaluate the *Hermitian Yang–Mills equations* – a certain form of generalized self-duality equations – on the metric cone $C(X_{1,1})$, as it has been done in similar setups, e.g. [23,24], and describe their moduli space, following [33–35].

4.1. Generalized self-duality equation

A very useful tool for obtaining minima of a Yang–Mills functional in gauge theory is to evaluate a first-order equation implying the second-order Yang–Mills equations [2–4]. Given a connection $\mathcal{A}$ on an n-dimensional manifold whose curvature $\mathcal{F}$ satisfies the generalized self-duality equation

$$
*\mathcal{F}=-*Q\wedge\mathcal{F}
\tag{4.1}
$$

for a 4-form Q, one obtains by taking the differential [5]

$$\nabla^{\mathcal{A}} \wedge *\mathcal{F} + (\mathrm{d} * Q) \wedge \mathcal{F} = 0 \quad \text{with} \quad \nabla^{\mathcal{A}} \wedge *\mathcal{F} := \mathrm{d} * \mathcal{F} + \mathcal{A} \wedge *\mathcal{F} + (-1)^{n-1} * \mathcal{F} \wedge \mathcal{A}, \tag{4.2}$$

which is the usual Yang–Mills equation with torsion term $(\mathrm{d} * Q) \wedge \mathcal{F}$. Explicit formulae for the choice of the form Q, in dependence of the geometry of the manifold, such that the torsion term vanishes even if the form Q is not co-closed have been given in [5]. Their construction is based on the existence of (real) Killing spinors, and thus also applies to Sasaki–Einstein structures. A connection $\mathcal{A}$ whose curvature satisfies (4.1) for the form Q given by [5] is called a *(generalized) instanton*. For a Sasaki–Einstein manifold the form Q reads [5]

$$Q = \frac{1}{2}\omega \wedge \omega, \tag{4.3}$$

such that we have

$$Q = -\frac{1}{4}\left(\Theta^{1\bar{1}2\bar{2}} + \Theta^{1\bar{1}3\bar{3}} + \Theta^{2\bar{2}3\bar{3}}\right) = e^{1234} + e^{1256} + e^{3456}. \tag{4.4}$$

The corresponding instanton equation (4.1) on $X_{1,1}$ is satisfied for the connection $\Gamma = I_8 \otimes e^8$, which we used for expressing the G-invariant connection in (3.12); see Appendix A.2. The form Q_Z occurring in the instanton equation on the cylinder, which is conformally equivalent to the metric cone, over a Sasaki–Einstein manifold reads [5][18]

$$Q_Z = \mathrm{d}\tau \wedge P + Q \qquad \text{with} \qquad P = \eta \wedge \omega \tag{4.5}$$

and one thus obtains

$$Q_Z = \frac{1}{2}\Omega \wedge \Omega, \tag{4.6}$$

where Ω is the Kähler form of the Calabi–Yau cone and the cylinder, respectively. Since the Calabi–Yau manifold is of complex dimension 4 and as we have chosen the standard form of the Kähler form, the 4-form Q_Z is self-dual, such that $\mathrm{d} * Q_Z = \mathrm{d}Q_Z = 0$, and the Yang–Mills equation without torsion follows from the instanton equation (4.1). We evaluate the instanton equation (4.1) with the form Q_Z by imposing the – equivalent – *Hermitian Yang–Mills equations* (HYM) [42,47,48]

$$\mathcal{F}^{(2,0)} = 0 = \mathcal{F}^{(0,2)} \qquad \text{and} \qquad \Omega \lrcorner \mathcal{F} := *(\Omega \wedge *\mathcal{F}) = 0, \tag{4.7}$$

where $\mathcal{F}^{(2,0)}$ refers to the $(2,0)$-part with respect to the complex structure J. The first equation is a holomorphicity condition and the second one can (sometimes) be considered as a stability condition on vector bundles; they are also known as *Donaldson–Uhlenbeck–Yau equations*.

4.2. *Hermitian Yang–Mills instantons on $C\left(X_{1,1}\right)$*

We consider the same ansatz (3.7) [30], now including also the additional form $e^{\tau} := \mathrm{d}\tau := \frac{\mathrm{d}r}{r}$ on the cylinder,

[18] They provide the form for a whole family of compatible metrics and we consider one special value here.

$$\begin{aligned}\mathcal{A} &= I_8 e^8 + Y_\alpha \Theta^\alpha + Y_{\bar\alpha}\Theta^{\bar\alpha} + X_7 e^7 + X_\tau e^\tau \\ &= I_8 e^8 + Y_\alpha \Theta^\alpha + Y_{\bar\alpha}\Theta^{\bar\alpha} + Y_4\Theta^4 + Y_{\bar 4}\Theta^{\bar 4},\end{aligned} \tag{4.8}$$

where we set[19]

$$Y_4 := \frac{1}{2}\left(X_\tau + \mathrm{i}\, X_7\right). \tag{4.9}$$

Due to the equivariance, the endomorphisms are "spherically symmetric", i.e. they can only depend on the radial coordinate, $X_a = X_a\,(r)$. After the implementation of the same equivariance conditions as before,

$$\left[I_8, Y_{\bar 1}\right] = -\sqrt{3}\,\mathrm{i}\, Y_{\bar 1}, \quad \left[I_8, Y_{\bar 2}\right] = \sqrt{3}\,\mathrm{i}\, Y_{\bar 2}, \quad \text{and} \quad \left[I_8, Y_{\bar 3}\right] = 0 = \left[I_8, Y_{\bar 4}\right], \tag{4.10}$$

the non-vanishing components of the field strength read (see [30])

$$\begin{aligned}&\mathcal{F}_{\alpha\beta} = \left[Y_\alpha, Y_\beta\right] - C^\gamma_{\alpha\beta} Y_\gamma, \quad \mathcal{F}_{\bar\alpha\bar\beta} = \left[Y_{\bar\alpha}, Y_{\bar\beta}\right] - C^{\bar\gamma}_{\bar\alpha\bar\beta} Y_{\bar\gamma}, \\ &\mathcal{F}_{\alpha\bar\beta} = \left[Y_\alpha, Y_{\bar\beta}\right] - C^\gamma_{\alpha\bar\beta} Y_\gamma - C^{\bar\gamma}_{\alpha\bar\beta} Y_{\bar\gamma} + C^7_{\alpha\bar\beta} Y_4 - C^7_{\alpha\bar\beta} Y_{\bar 4} + \mathrm{i}\, C^8_{\alpha\bar\beta} I_8, \\ &\mathcal{F}_{\alpha 4} = \left[Y_\alpha, Y_4\right] - \frac{1}{2} r \dot Y_\alpha - \frac{1}{2} C^\beta_{7\alpha} Y_\beta, \quad \mathcal{F}_{\alpha\bar 4} = \left[Y_\alpha, Y_{\bar 4}\right] - \frac{1}{2} r \dot Y_\alpha + \frac{1}{2} C^\beta_{7\alpha} Y_\beta, \\ &\mathcal{F}_{\bar\alpha 4} = \left[Y_{\bar\alpha}, Y_4\right] - \frac{1}{2} r \dot Y_{\bar\alpha} - \frac{1}{2} C^{\bar\beta}_{7\bar\alpha} Y_{\bar\beta}, \quad \mathcal{F}_{\bar\alpha\bar 4} = \left[Y_{\bar\alpha}, Y_{\bar 4}\right] - \frac{1}{2} r \dot Y_{\bar\alpha} + \frac{1}{2} C^{\bar\beta}_{7\bar\alpha} Y_{\bar\beta}, \\ &\mathcal{F}_{4\bar 4} = \left[Y_4, Y_{\bar 4}\right] - \frac{1}{2} r\left(\dot Y_4 - \dot Y_{\bar 4}\right).\end{aligned} \tag{4.11}$$

Evaluating the condition $\mathcal{F}_{\bar\alpha\bar\beta} = 0$ leads to

$$\left[Y_{\bar 1}, Y_{\bar 2}\right] = 2Y_{\bar 3}, \quad \left[Y_{\bar 1}, Y_{\bar 3}\right] = 0 = \left[Y_{\bar 2}, Y_{\bar 3}\right] \tag{4.12}$$

(together with their complex conjugates from $\mathcal{F}_{\alpha\beta} = 0$). Thus, this part of the holomorphicity condition imposes algebraic relations on the quiver. In contrast, from $\mathcal{F}_{\bar\alpha 4} = 0$ we obtain the following flow equations

$$r\dot Y_{\bar 1} = -Y_{\bar 1} + 2\left[Y_{\bar 1}, Y_{\bar 4}\right], \quad r\dot Y_{\bar 2} = -Y_{\bar 2} + 2\left[Y_{\bar 2}, Y_{\bar 4}\right], \quad r\dot Y_{\bar 3} = -2Y_{\bar 3} + 2\left[Y_{\bar 3}, Y_{\bar 4}\right]. \tag{4.13}$$

The remaining equation $\Omega \lrcorner \mathcal{F} = 0$ requires

$$r\left(\dot Y_4 - \dot Y_{\bar 4}\right) = 2\left[Y_1, Y_{\bar 1}\right] + 2\left[Y_2, Y_{\bar 2}\right] + 2\left[Y_3, Y_{\bar 3}\right] + 2\left[Y_4, Y_{\bar 4}\right] - 6\left(Y_4 - Y_{\bar 4}\right). \tag{4.14}$$

Constant endomorphisms: For the special case of constant matrices X_a, the situation corresponds to that of the underlying Sasaki–Einstein manifold $X_{1,1}$ with the parameter τ (or r, respectively) just as a label of the foliation along the preferred direction of the cone. Gauging the field X_τ to zero, one recovers then from (4.13) exactly the additional equivariance conditions (3.37), which appeared in the discussion of the gauge theory on Q_3. Thus the equivariant gauge theory on Q_3 can be considered as a special instanton solution[20] of the more general setup on the metric cone $C\left(X_{1,1}\right)$.

19 The field X_τ associated to the radial direction could be gauged to zero [30].

20 The vanishing of the contributions stemming from the form e^7 is obvious from the Yang–Mills action (3.33) and the instanton condition (4.1). Due to $*_7 Q \propto e^7$ those terms do not contribute to the action for instanton solutions, and this is equivalent to the further equivariance conditions (3.37).

4.3. Moduli space of SU(3)*-equivariant instantons*

For a description of the moduli space of the equations (4.12), (4.13) and (4.14) (under the equivariance conditions (4.10)), it is advantageous to re-write them in a form similar to the Nahm equations. Then one can employ the techniques used by Donaldson [33] and Kronheimer [34] for the discussion thereof. We will briefly sketch the application of these methods to our system of flow equations, following [35], where the framed moduli space of solutions to the Hermitian Yang–Mills equations on metric cones over generic Sasaki–Einstein manifolds is discussed in this way. Note that the treatment [35] uses the canonical connection of [5] as starting point Γ for the gauge connection and that our connection $\Gamma = I_8 \otimes e^8$ in (4.8) differs from it (see Appendix A.2). This is why some modifications, in comparison with [35], will appear in our discussion.[21]

Changing the argument in the flow equations to $\tau = \ln(r)$ and setting[22]

$$Y_{\bar{\alpha}} =: \mathrm{e}^{-\tau} W_\alpha, \quad \text{for } \alpha = 1, 2, \quad Y_{\bar{3}} =: \mathrm{e}^{-2\tau} W_3, \quad \text{and} \quad Y_{\bar{4}} =: \mathrm{e}^{-6\tau} Z \tag{4.16}$$

eliminates the linear terms in (4.13) and (4.14). Defining $s := -\frac{1}{6}\mathrm{e}^{-6\tau} = -\frac{1}{6}r^{-6} \in (-\infty, 0]$ yields Nahm-type equations

$$\frac{\mathrm{d}W_1}{\mathrm{d}s} = 2[W_1, Z], \qquad \frac{\mathrm{d}W_2}{\mathrm{d}s} = 2[W_2, Z], \qquad \frac{\mathrm{d}W_3}{\mathrm{d}s} = 2[W_3, Z], \tag{4.17}$$

$$[W_1, W_2] = 2W_3 \quad \text{and} \quad [W_1, W_3] = 0 = [W_2, W_3] \tag{4.18}$$

(from $\mathcal{F}^{(2,0)} = 0$) and

$$\mu(W_\alpha, Z) := \frac{\mathrm{d}}{\mathrm{d}s}\left(Z + Z^\dagger\right) + 2\sum_{\alpha=1}^{3} \lambda^\alpha(s)\left[W_\alpha, W_\alpha^\dagger\right] + 2\left[Z, Z^\dagger\right] = 0 \tag{4.19}$$

(from $\Omega \lrcorner \mathcal{F} = 0$), with the non-negative functions

$$\lambda^1(s) = \lambda^2(s) := (-6s)^{-\frac{5}{3}} \quad \text{and} \quad \lambda^3(s) := (-6s)^{-\frac{4}{3}}. \tag{4.20}$$

The equation (4.19) shall be referred to as the *real equation* and the equations (4.17) and (4.18) as *complex equations*. The discussion of the moduli space is based on the invariance of the complex equations under the complexified gauge transformation [33]

$$W_\alpha \longmapsto W_\alpha^g := g W_\alpha g^{-1}, \quad \text{for } \alpha = 1, 2, 3 \text{ and } Z \longmapsto Z^g := g Z g^{-1} - \frac{1}{2}\left(\frac{\mathrm{d}g}{\mathrm{d}s}\right) g^{-1} \tag{4.21}$$

with $g \in \mathcal{C}((-\infty, 0], \mathrm{GL}(\mathbb{C}, k))$. A local solution of (4.17) can be attained by applying the gauge

$$Z^g = 0 \quad \Rightarrow \quad Z = \frac{1}{2} g^{-1} \frac{\mathrm{d}g}{\mathrm{d}s}, \tag{4.22}$$

[21] Of course, using the canonical connection of [5] yields the results of [35] also for $X_{1,1}$. However, for the discussion of the quiver diagrams in Section 3 the connection $\Gamma = I_8 \otimes e^8$ was more suitable because it is valued in the subalgebra $\mathfrak{h}$ and, thus, adapted to the setup of a homogeneous space. The canonical connection, in contrast, is adapted to the Sasaki–Einstein structure of $X_{1,1}$; see Appendix A.2.

[22] For the canonical connection (A.14) of a seven-dimensional Sasaki–Einstein manifold, the matrices scale as [35]

$$Y_{\bar{\alpha}} = \mathrm{e}^{-\frac{4}{3}\tau} W_\alpha \quad \text{for } \alpha = 1, 2, 3 \quad \text{and} \quad Y_{\bar{4}} = \mathrm{e}^{-6\tau} Z. \tag{4.15}$$

so that – due to the complex equations (4.17) – the gauge transformed matrices W_α^g must be constant,

$$W_\alpha = g^{-1} T_\alpha g. \tag{4.23}$$

To obtain solutions, one has to choose these constant matrices such that they satisfy (4.18). One special choice, for instance, could be to set $T_3 = 0$ and take for T_1 and T_2 elements of a Cartan subalgebra. Note that not only the scaling in (4.16) is different from that in [35], but also the conditions (4.18): There all three matrices have to commute with each other and, thus, also T_3 can be chosen as arbitrary element of a Cartan subalgebra. Adapting Donaldson's arguments [33,35], the real equation (4.19) can be – locally on an interval $\mathcal{I} \subset (-\infty, 0]$ – considered as the equation of motion (i.e. $\delta\mathcal{L} \propto \mu$) of the Lagrangian

$$\mathcal{L} = \frac{1}{2} \int\limits_{\mathcal{I}} \mathrm{d}s \left\{ 2|Z + Z^\dagger|^2 + 2\lambda^1(s)|W_1|^2 + 2\lambda^2(s)|W_2|^2 + 2\lambda^3(s)|W_3|^2 \right\}. \tag{4.24}$$

Employing (4.22) and (4.23), one can re-write this Lagrangian as [33,35]

$$\mathcal{L}(h) = \frac{1}{2} \int\limits_{\mathcal{I}} \mathrm{d}s \left\{ \frac{1}{4} \mathrm{tr} \left(h^{-1} \frac{\mathrm{d}h}{\mathrm{d}s} \right)^2 + 2 \sum_{\alpha=1}^{3} \lambda^\alpha \mathrm{tr} \left(h T_\alpha h^{-1} T_\alpha^\dagger \right) \right\} \quad \text{with} \quad h := g^\dagger g. \tag{4.25}$$

Since the potential term in this Lagrangian is non-negative, the existence of a solution to (4.19) as equation of motion follows from a variational problem [33]. One still has to ensure some technical aspects: the uniqueness of the solutions, the existence of the gauge transformation and the Lagrangian on the *entire* interval $(-\infty, 0]$, as well as the boundedness of μ. In the reference [35] these properties are proven, given that for framed instantons, i.e. those with $h = 1$ at the boundary of the interval $(-\infty, 0]$, the following condition

$$\exists g_0 \in \mathrm{U}(k): \lim_{s \to -\infty} W_\alpha = \mathrm{Ad}(g_0) T_\alpha \tag{4.26}$$

is satisfied for constant matrices obeying the conditions (4.18). For *their* constraints, i.e. mutually commuting matrices T_α, it is shown that the moduli space can be expressed as diagonal orbit in a product of coadjoint orbits [35]. In our case, however, due to the different constraints (4.12), the situation might be more involved. But we can at least conclude that (4.23) provides local solutions of the Nahm-type equations (4.17)–(4.19).

Moreover, it was shown in the references (see again [35]) that the real equation (4.19) can be considered as a moment map $\mu : \mathbb{A}^{1,1} \to \mathrm{Lie}(\mathcal{G}_0)$ from the space $\mathbb{A}^{1,1}$ of framed solutions to the complex equations into the Lie algebra of the framed gauge group $\mathcal{G}_0$. This result still holds here, despite the difference in the connections that are used. Hence the moduli space of equivariant Hermitian Yang–Mills instantons on metric cones over Sasaki–Einstein manifolds admits the description as Kähler quotient [35]

$$\mathcal{M} = \mu^{-1}(0) / \mathcal{G}_0. \tag{4.27}$$

5. Summary and conclusions

In this article we studied the SU(3)-equivariant dimensional reduction of gauge theories over the Sasaki–Einstein manifold $X_{1,1}$. We interpreted the condition of equivariance, which had already occurred in articles [30,31] on Spin(7)-instantons on cones over Aloff–Wallach spaces

$X_{k,l}$, in terms of quiver diagrams, and we discussed the general construction of the quiver bundles. This yielded a new class of Sasakian quiver gauge theories. The associated quiver diagram of this gauge theory is a *"doubled modified holomorphic chain"*, consisting of two arrows between adjacent vertices and two loops at each vertex, and three explicit examples thereof were considered in the article. For the comparison with the gauge theory on the underlying Kähler manifold Q_3 we studied the quivers also in the entire weight diagram of $G = \mathrm{SU}(3)$, which implied degeneracies of the arrows. This behavior is similar to the case [15,24] of the five-dimensional Sasaki–Einstein manifold $T^{1,1}$ over $\mathbb{C}P^1 \times \mathbb{C}P^1$. The reduction to the gauge theory on Q_3 led to the correct, expected result for the quiver diagram [16]: the weight diagram of SU(3).

For the investigation of the vacua described by this gauge theory we imposed the Hermitian Yang–Mills equations on the metric cone $C\left(X_{1,1}\right)$. The resulting flow equations have been re-written in a form similar to Nahm's equations, which allowed a discussion based on Kronheimer's [34] and Donaldson's [33] work and its generalized application to equivariant HYM instantons on Calabi–Yau cones [35]. Since we formulated the quiver gauge theory by using an instanton connection different from that of [5] in the gauge connection, some modifications appeared. While the real equation can be still interpreted as a moment map for framed instanton solutions, as in [35], and, thus, leads to a description of the moduli space as a Kähler quotient, the description based on coadjoint orbits is more involved: The HYM equations impose a non-trivial commutation relation on the gauge transformed matrices, in contrast to [35], where they have to commute with each other. Thus, the behavior is more complicated and further effort would be needed to study the consequences thereof in detail.

Acknowledgements

The author thanks Olaf Lechtenfeld, Alexander Popov, and Marcus Sperling for fruitful discussions and comments. This work was done within the project supported by the Deutsche Forschungsgemeinschaft (DFG, Germany) under the grant LE 838/13 and was supported by the Research Training Group RTG 1463 "Analysis, Geometry and String Theory" (DFG).

Appendix A

A.1. SU(3) *generators and structure constants*

The generators defined by the choice of the 1-forms in (2.8) read

$$I_1^- := \sqrt{2}\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}, \quad I_2^- := \sqrt{2}\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}, \quad I_3^- := \begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix},$$
$$I_7 := \mathrm{i}\begin{pmatrix}0&0&0\\0&-1&0\\0&0&1\end{pmatrix}, \quad I_1^+ := \sqrt{2}\begin{pmatrix}0&0&-1\\0&0&0\\0&0&0\end{pmatrix}, \quad I_2^+ := \sqrt{2}\begin{pmatrix}0&0&0\\-1&0&0\\0&0&0\end{pmatrix},$$
$$I_3^+ := \begin{pmatrix}0&0&0\\0&0&-1\\0&0&0\end{pmatrix}, \quad I_8 := \frac{\mathrm{i}}{\sqrt{3}}\begin{pmatrix}2&0&0\\0&-1&0\\0&0&-1\end{pmatrix}, \tag{A.1}$$

and we define the structure constants via the commutation relations

$$\left[-\mathrm{i}\,I_j, I_\alpha^-\right] = C_{j\alpha}^{\beta} I_\beta^-, \quad \left[-\mathrm{i}\,I_j, I_{\bar\alpha}^+\right] = C_{j\bar\alpha}^{\bar\beta} I_{\bar\beta}^+, \quad \left[I_\alpha^-, I_\beta^-\right] = C_{\alpha\beta}^{\gamma} I_\gamma^-,$$
$$\left[I_{\bar\alpha}^+, I_{\bar\beta}^+\right] = C_{\bar\alpha\bar\beta}^{\bar\gamma} I_{\bar\gamma}^+, \quad \left[I_\alpha^-, I_{\bar\beta}^+\right] = -\mathrm{i}\,C_{\alpha\bar\beta}^{j} I_j + C_{\alpha\bar\beta}^{\gamma} I_\gamma^- + C_{\alpha\bar\beta}^{\bar\gamma} I_{\bar\gamma}^+. \tag{A.2}$$

The non-vanishing structure constants are [30]

$$\begin{aligned}
&C^1_{3\bar2} = -C^2_{3\bar1} = -1 = -C^{\bar1}_{2\bar3} = C^{\bar2}_{1\bar3}, && C^3_{12} = 2 = C^{\bar3}_{\bar1\bar2},\\
&C^1_{71} = C^2_{72} = 1 = -C^{\bar1}_{7\bar1} = -C^{\bar2}_{7\bar2}, && C^3_{73} = 2 = -C^{\bar3}_{7\bar3},\\
&C^1_{81} = -C^2_{82} = -\sqrt{3} = -C^{\bar1}_{8\bar1} = C^{\bar2}_{8\bar2}, && C^3_{83} = 0 = C^{\bar3}_{8\bar3},\\
&C^7_{1\bar1} = C^7_{2\bar2} = C^7_{3\bar3} = -1, && C^8_{1\bar1} = -C^8_{2\bar2} = \sqrt{3}.
\end{aligned} \tag{A.3}$$

By the Maurer–Cartan equations,

$$\mathrm{d}\Theta^\alpha = -\mathrm{i}\,C^\alpha_{j\beta} - \frac{1}{2} C^\alpha_{\beta\gamma}\Theta^{\beta\gamma} - C^\alpha_{\beta\bar\gamma}\Theta^{\beta\bar\gamma}, \qquad \mathrm{d}e^j = \mathrm{i}\,C^j_{\beta\bar\gamma}\Theta^{\beta\bar\gamma}, \tag{A.4}$$

they yield again the structure equations (2.9). In terms of real forms

$$\Theta^1 =: e^1 - \mathrm{i}\,e^2, \quad \Theta^2 =: e^3 - \mathrm{i}\,e^4, \quad \text{and} \quad \Theta^3 =: e^5 - \mathrm{i}\,e^6, \tag{A.5}$$

the structure equations read

$$\begin{aligned}
&\mathrm{d}e^1 = \sqrt{3}e^{82} - e^{72} - e^{35} - e^{46}, && \mathrm{d}e^2 = -\sqrt{3}e^{81} + e^{71} - e^{36} + e^{45},\\
&\mathrm{d}e^3 = -\sqrt{3}e^{84} - e^{74} + e^{15} + e^{26}, && \mathrm{d}e^4 = \sqrt{3}e^{83} + e^{73} + e^{16} - e^{25},\\
&\mathrm{d}e^5 = -2e^{76} - 2e^{13} + 2e^{24}, && \mathrm{d}e^6 = 2e^{75} - 2e^{14} - 2e^{23},\\
&\mathrm{d}e^7 = 2e^{12} + 2e^{34} + 2e^{56}, && \mathrm{d}e^8 = -2\sqrt{3}e^{12} + 2\sqrt{3}e^{34}.
\end{aligned} \tag{A.6}$$

A.2. Connections and instanton equation

On the homogeneous space $X_{1,1} = G/H = \mathrm{SU}(3)/\mathrm{U}(1)_{1,1}$ we consider the connection with torsion

$$T(X, Y) := -[X, Y]_{\mathfrak{m}} \tag{A.7}$$

for vector fields X, Y on G/H, where $[\cdot,\cdot]_{\mathfrak{m}}$ denotes the projection of the commutator to the complement $\mathfrak{m}$; this yields the following torsion components

$$T^\mu_{\rho\sigma} = -C^\mu_{\rho\sigma} \qquad \text{for } \mu, \rho, \sigma = 1, \ldots, 7. \tag{A.8}$$

Using the structure equations and the Maurer–Cartan equation

$$\begin{aligned}
\mathrm{d}e^\mu &= -\frac{1}{2}C^\mu_{\rho\sigma}e^{\rho\sigma} = -C^\mu_{8\rho}e^8 \wedge e^\rho + \frac{1}{2}T^\mu_{\rho\sigma}e^{\rho\sigma}\\
&=: -\Gamma^\mu_\rho \wedge e^\rho + T^\mu,
\end{aligned} \tag{A.9}$$

one obtains the connection 1-forms

$$\Gamma^\mu_\rho = C^\mu_{8\rho}e^8 \qquad \Rightarrow \Gamma = I_8 \otimes e^8, \tag{A.10}$$

which is the U(1)-connection used in the ansatz for the gauge connection in (4.8). Its curvature

$$\mathcal{F}_\Gamma = \mathrm{d}\Gamma + \Gamma \wedge \Gamma = -2\sqrt{3} I_8 \otimes \left(e^{12} - e^{34}\right) \tag{A.11}$$

satisfies the instanton equation

$$*_7\mathcal{F}_\Gamma = -\left(e^{12}+e^{34}+e^{56}\right)\wedge e^7\wedge\mathcal{F}_\Gamma = -*_7\,Q\wedge\mathcal{F}_\Gamma. \tag{A.12}$$

for the 4-form $Q = e^{1234}+e^{1256}+e^{3456}$ from (4.4). Because of

$$(\mathrm{d}*_7 Q)\wedge\mathcal{F}_\Gamma \propto \left(e^{1234}+e^{1256}+e^{3456}\right)\wedge\left(e^{12}-e^{34}\right)=0 \tag{A.13}$$

the torsion term in (4.2) vanishes, so that the usual torsion-free Yang–Mills equation is obtained. This is the intention of using special geometric structures. Note, however, that our U(1)-connection does not coincide with what is defined as *canonical connection of a Sasaki–Einstein manifold* in [5]. Its torsion for a seven-dimensional Sasaki–Einstein manifold is defined via

$$T^a = \frac{2}{3}P_{a\mu\nu}e^{\mu\nu} \quad \text{for } a=1,\dots,6 \quad \text{and} \quad T^7 = P_{7\mu\nu}e^{\mu\nu} \quad \text{with} \quad P := \eta\wedge\omega = e^7\wedge\omega. \tag{A.14}$$

Since this definition does not require a homogeneous space, but only exploits the Sasaki–Einstein structure, it allows for general discussions of gauge theories on those spaces, as used for example in [35,38]. On $X_{1,1}$ this canonical connection is expressed by the connection matrix

$$\mathrm{d}\begin{pmatrix}\Theta^1\\ \Theta^2\\ \Theta^3\\ e^7\end{pmatrix} = \begin{pmatrix}\frac{1}{3}\mathrm{i}\,e^7+\sqrt{3}\,\mathrm{i}\,e^8 & 0 & -\Theta^{\bar{2}} & 0\\ 0 & \frac{1}{3}\mathrm{i}\,e^7-\sqrt{3}\,\mathrm{i}\,e^8 & \Theta^{\bar{1}} & 0\\ \Theta^2 & -\Theta^1 & -\frac{2}{3}\mathrm{i}\,e^7 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\wedge\begin{pmatrix}\Theta^1\\ \Theta^2\\ \Theta^3\\ e^7\end{pmatrix}+\vec{T}. \tag{A.15}$$

Thus, the canonical connection is adapted to the SU(3) structure of $X_{1,1}$.

On the metric cone (with the rescaled forms $\tilde{e}^\mu := re^\mu$) or on the conformally equivalent cylinder, respectively, the connection $\Gamma = I_8\otimes e^8$ is still an instanton for the form

$$\begin{aligned} Q_Z = \frac{1}{2}\Omega\wedge\Omega &= r^4\left(e^{1234}+e^{1256}+e^{12\tau 7}+e^{3456}+e^{34\tau 7}+e^{56\tau 7}\right) \\ &= \tilde{e}^{1234}+\tilde{e}^{1256}+\tilde{e}^{12\tau 7}+\tilde{e}^{3456}+\tilde{e}^{34\tau 7}+\tilde{e}^{56\tau 7} = *_8 Q_Z \end{aligned} \tag{A.16}$$

because we have

$$*_8\left(\tilde{e}^{12}-\tilde{e}^{34}\right) = -\left(\tilde{e}^{12}-\tilde{e}^{34}\right)\wedge\tilde{e}^{56\tau 7} = -Q_Z\wedge\left(\tilde{e}^{12}-\tilde{e}^{34}\right). \tag{A.17}$$

A.3. Details of the moduli space description

This section provides some technical aspects of the description in Section 4.3. For details, the reader should consult the references, in particular [35]. To show that the real equation follows (over some range) as equation of motion of the Lagrangian (4.24), one considers [33] the variation of the matrices W_a with respect to g close to the identity. Writing $g = 1+\delta g$, where δg is self-adjoint, one obtains from the gauge transformation (4.21)

$$\delta W_\alpha = (1+\delta g)\,W_\alpha\,(1+\delta g)^{-1} - W_\alpha = \left[\delta g, W_\alpha\right] \quad \text{for} \quad \alpha = 1,2,3 \tag{A.18}$$

and

$$\delta Z = (1+\delta g)\, Z\, (1+\delta g)^{-1} - Z - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}(1+\delta g)\,(1+\delta g)^{-1} = \left[\delta g, Z\right] - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\delta g. \tag{A.19}$$

Using the results (A.18) and (A.19), one derives the following variations

$$\begin{aligned}\delta \int \mathrm{d}s\, |W_\alpha|^2 &:= \delta \int \mathrm{d}s\ \mathrm{tr}\ W_\alpha W_\alpha^\dagger = 2\mathrm{Re} \int \mathrm{d}s\ \mathrm{tr}\ \delta\left(W_\alpha\right) W_\alpha^\dagger = 2\mathrm{Re} \int \mathrm{d}s\ \mathrm{tr}\ \left[\delta g, W_\alpha\right] W_\alpha^\dagger \\ &= 2\mathrm{Re} \int \mathrm{d}s\ \mathrm{tr}\ \delta g \left[W_\alpha, W_\alpha^\dagger\right] \quad \text{for} \quad \alpha = 1,2,3\end{aligned} \tag{A.20}$$

and

$$\begin{aligned}\delta \int \mathrm{d}s\, |Z+Z^\dagger|^2 &= 2\mathrm{Re} \int \mathrm{d}s\ \mathrm{tr}\ \left(\left[\delta g, Z - Z^\dagger\right] - \frac{\mathrm{d}}{\mathrm{d}s}\delta g\right)\left(Z+Z^\dagger\right) \\ &= 2\mathrm{Re} \int \mathrm{d}s\ \mathrm{tr}\delta g \left(\frac{\mathrm{d}}{\mathrm{d}s}\left(Z+Z^\dagger\right) + 2\left[Z, Z^\dagger\right]\right).\end{aligned} \tag{A.21}$$

Putting the results from (A.20) and (A.21) together with the prefactors $\lambda^\alpha(s)$, (4.20) yields the Lagrangian (4.24) and shows that the real equation is the equation of motion thereof. That the Lagrangian can be defined for the entire range $s \in (-\infty, 0]$ and other technical issues can be found in [35]. The only quantitative difference is the concrete form of the factors $\lambda^\alpha(s)$ but this does not affect the general line of reasoning.

References

[1] M.F. Atiyah, N.J. Hitchin, I.M. Singer, Self-duality in four-dimensional Riemannian geometry, Proc. R. Soc. Lond. A 362 (1978) 425.

[2] E. Corrigan, C. Devchand, D.B. Fairlie, J. Nuyts, First-order equations for gauge fields in spaces of dimension greater than four, Nucl. Phys. B 214 (1983) 452.

[3] R.S. Ward, Completely solvable gauge-field equations in dimensions greater than four, Nucl. Phys. B 236 (1984) 381.

[4] C.M. Hull, Higher dimensional Yang–Mills theories and topological terms, Adv. Theor. Math. Phys. 2 (1998) 619, arXiv:hep-th/9710165v2.

[5] D. Harland, C. Nölle, Instantons and Killing spinors, J. High Energy Phys. 03 (2012) 082, arXiv:1109.3552.

[6] U. Gran, G. Papadopoulos, D. Roest, Supersymmetric heterotic string backgrounds, Phys. Lett. B 656 (2007) 119, arXiv:0706.4407.

[7] D. Kapetanakis, G. Zoupanos, Coset space dimensional reduction of gauge theories, Phys. Rep. 219 (1992) 4.

[8] L. Alvarez-Cónsul, O. García-Prada, Dimensional reduction and quiver bundles, J. Reine Angew. Math. 556 (2003) 1, arXiv:math-dg/0112160.

[9] L. Alvarez-Cónsul, O. García-Prada, Dimensional reduction, SL(2, $\mathbb{C}$)-equivariant bundles and stable holomorphic chains, Int. J. Math. 12 (2001) 159, arXiv:math-dg/0112159.

[10] B.P. Dolan, R.J. Szabo, Equivariant dimensional reduction and quiver gauge theories, Gen. Relativ. Gravit. 43 (2010) 2453, arXiv:1001.2429 [hep-th].

[11] O. Lechtenfeld, A.D. Popov, R.J. Szabo, Quiver gauge theory and noncommutative vortices, Prog. Theor. Phys. Suppl. 171 (2007) 258, arXiv:0706.0979.

[12] A.D. Popov, R.J. Szabo, Quiver gauge theory of nonabelian vortices and noncommutative instantons in higher dimensions, J. Math. Phys. 47 (2006) 012306, arXiv:hep-th/0504025.

[13] I. Biswas, Holomorphic Hermitian vector bundles over the Riemann sphere, Bull. Sci. Math. 132 (2008) 246.

[14] B.P. Dolan, R.J. Szabo, Dimensional reduction, monopoles and dynamical symmetry breaking, J. High Energy Phys. 03 (2009) 059, arXiv:0901.2491.

[15] O. Lechtenfeld, A.D. Popov, R.J. Szabo, Rank two quiver gauge theory, graded connections and noncommutative vortices, J. High Energy Phys. 09 (2006) 054, arXiv:hep-th/0603232.

[16] O. Lechtenfeld, A.D. Popov, R.J. Szabo, SU(3)-equivariant quiver gauge theories and nonabelian vortices, J. High Energy Phys. 08 (2008) 093, arXiv:0806.2791v2.

[17] C. Boyer, K. Galicki, Sasakian Geometry, Oxford University Press, 2008.
[18] J. Sparks, Sasaki–Einstein manifolds, Surv. Differ. Geom. 16 (2011) 265, arXiv:1004.2461.
[19] D. Joyce, Lectures on Calabi–Yau and special Lagrangian geometry, preprint, arXiv:math/0108088; M. Gross, D. Huybrechts, D. Joyce, Calabi–Yau Manifolds and Related Geometries, Springer, 2003.
[20] B.R. Greene, String theory on Calabi–Yau manifolds, preprint, arXiv:hep-th/9702155.
[21] A.D. Popov, R.J. Szabo, Double quiver gauge theory and nearly Kähler flux compactifications, J. High Energy Phys. 02 (2012) 033, arXiv:1009.3208v2.
[22] O. Lechtenfeld, A.D. Popov, R.J. Szabo, Sasakian quiver gauge theories and instantons on Calabi–Yau cones, preprint, arXiv:1412.4409.
[23] O. Lechtenfeld, A.D. Popov, M. Sperling, R.J. Szabo, Sasakian quiver gauge theories and instantons on cones over lens 5-spaces, Nucl. Phys. B 899 (2015) 848, arXiv:1506.02786.
[24] J.C. Geipel, O. Lechtenfeld, A.D. Popov, R.J. Szabo, Sasakian quiver gauge theories and instantons on the conifold, Nucl. Phys. B 907 (2016) 445, arXiv:1601.05719.
[25] J.P. Gauntlett, D. Martelli, J. Sparks, D. Waldram, Sasaki–Einstein metrics on $S^2 \times S^3$, Adv. Theor. Math. Phys. 8 (2004) 711, arXiv:hep-th/0403002.
[26] D. Martelli, J. Sparks, Toric geometry, Sasaki–Einstein manifolds and a new infinite class of AdS/CFT duals, Commun. Math. Phys. 262 (2006) 51, arXiv:hep-th/0411238.
[27] S. Aloff, N.R. Wallach, An infinite family of distinct 7-manifolds admitting positively curved Riemannian structures, Bull. Am. Math. Soc. 81 (1975) 93.
[28] F.M. Cabrera, M.D. Monar, A.F. Swann, Classification of G_2-Structures, J. Lond. Math. Soc. (2) 53 (1996) 407.
[29] T. Friedrich, I. Kath, A. Moroianu, U. Semmelmann, On nearly parallel G_2-structures, J. Geom. Phys. 23 (1997) 259.
[30] A.S. Haupt, T.A. Ivanova, O. Lechtenfeld, A.D. Popov, Chern–Simons flows on Aloff–Wallach spaces and Spin(7) instantons, Phys. Rev. D 83 (2011) 105028, arXiv:1104.5231v1.
[31] A.S. Haupt, Yang–Mills solutions and Spin(7)-instantons on cylinders over coset spaces with G_2-structure, J. High Energy Phys. 03 (2016) 038, arXiv:1512.07254v1.
[32] L. Castellani, L.J. Romans, N.P. Warner, A classification of compactifying solutions for $d = 11$ supergravity, Nucl. Phys. B 241 (1984) 429.
[33] S.K. Donaldson, Nahm's equations and the classification of monopoles, Commun. Math. Phys. 96 (1984) 387.
[34] P.B. Kronheimer, A hyper-Kählerian structure on coadjoint orbits of a semisimple complex group, J. Lond. Math. Soc. 42 (1990) 193.
[35] M. Sperling, Instantons on Calabi–Yau cones, Nucl. Phys. B 901 (2015) 354, arXiv:1505.01755.
[36] C.P. Boyer, K. Galicki, B.M. Mann, The geometry and topology of 3-Sasakian manifolds, J. Reine Angew. Math. 455 (1994) 183.
[37] B. Florea, S. Kachru, J. McGreevy, N. Saulina, Stringy instantons and quiver gauge theories, J. High Energy Phys. 05 (2007) 024, arXiv:hep-th/0610003v3.
[38] T.A. Ivanova, A.D. Popov, Instantons on special holonomy manifolds, Phys. Rev. D 85 (2012) 105012, arXiv:1203.2657.
[39] S. Bunk, O. Lechtenfeld, A.D. Popov, M. Sperling, Instantons on conical half-flat 6-manifolds, J. High Energy Phys. 01 (2015) 030, arXiv:1409.0030.
[40] S. Bunk, T.A. Ivanova, O. Lechtenfeld, A.D. Popov, M. Sperling, Instantons on sine-cones over Sasakian manifolds, Phys. Rev. D 90 (2014) 065028, arXiv:1407.2948.
[41] I. Bauer, T.A. Ivanova, O. Lechtenfeld, F. Lubbe, Yang–Mills instantons and dyons on homogeneous G_2-manifolds, J. High Energy Phys. 10 (2010) 044, arXiv:1006.2388.
[42] A.D. Popov, Hermitian Yang–Mills equations and pseudo-holomorphic bundles on nearly Kähler and nearly Calabi–Yau twistor 6-manifolds, Nucl. Phys. B 828 (2010) 594, arXiv:0907.0106.
[43] H. Derksen, J. Weyman, Quiver representations, Not. Am. Math. Soc. 52 (2) (2005) 200.
[44] R. Schiffler, Quiver Representations, Springer, 2014.
[45] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, vol. 1, Interscience Publishers, 1963.
[46] W. Fulton, J. Harris, Representation Theory, Springer, 1991.
[47] S.K. Donaldson, Anti-self dual Yang–Mills connections over complex algebraic surfaces and stable vector bundles, Proc. Lond. Math. Soc. 50 (1985) 1.
[48] K. Uhlenbeck, S.-T. Yau, On the existence of Hermitian Yang–Mills connections in stable vector bundles, Commun. Pure Appl. Math. 39 (1986) 257.