Bose-Einstein condensates in a disordered potential: Anderson localization

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Abstract: We have simulated a Bose-Einstein condensate in a disordered potential, whose main characteristics such as the frequency of its maxima and its amplitude can be controlled through two parameters: the correlation length and the disorder strength. Studying the 1D case, we have encountered Anderson localization, which is the localization of individual particles or waves in a disordered energetic landscape, under specific parameters for both the interacting and the non-interacting regime. We have not been able to unambiguously observe the same kind of localization for 2D systems, although we provide an explanation for this result in terms of the topology of the 2D disordered potential, and interactions.

I. INTRODUCTION

Bose-Einstein condensation is a quantum phenomenon that was first predicted in 1924 by Satyendra Nath Bose and Albert Einstein, and consists in a macroscopic occupation of a single-particle state. This is an effect that occurs below a certain critical temperature (of the order of hundreds of nK), at which the thermal de Broglie wave length of the particles becomes comparable to the size of the system. In these conditions, the system can be described by means of a macroscopic wave function that corresponds to the condensate wave function. Bose-Einstein condensation was merely theoretical until the development of cooling techniques, such as laser cooling or evaporative cooling [1]. It was in 1995, when Boulder and MIT experimental groups succeeded in creating an atomic Bose-Einstein condensate (BEC) of Na and Rb, respectively [2].

When the diluteness condition is fulfilled ($\rho|a|^3 \ll 1$), the mean-field framework is a suitable theory to describe the behaviour of BECs. The static properties and the dynamical evolution of the condensate wave function can be analysed by means of the Gross-Pitaevskii (GP) equation which is a Schrödinger-like equation with a non-linear term that accounts for contact interactions [3].

0 from the interaction term, there is a non-interacting single-particle hamiltonian in the GP equation that contains an external potential whose form can widely vary depending on the object of to study. In this work, we want to see the effect of disorder in BECs.

Disorder is commonly present in nature and its effect can modify the nature of physical systems, even when the disorder is weak. One of the most important phenomena under the presence of disorder is the localization of individual particles or waves in a disordered energetic landscape, also known as Anderson localization [4]. In 1958, P. W. Anderson proved that for random potential barriers, the difference of their amplitudes and phases can suppress the propagation amplitude of an electronic matter wave. It is indeed possible to observe Anderson localization also in atomic matter waves, such as atomic BECs.

A disordered potential can be created with speckle patterns. They are produced when a laser light is reflected by a surface, which is created with randomly-distributed impurities that cause the light to scatter. These coherent light waves produce a complex distribution called speckle pattern, producing a disorder potential proportional to the local laser intensity [5]. Another emergent way of creating any kind of external potential is using *Digital Micromirror Devices*, consisting of an array of individual pixels that can modulate the light determinating its deflected direction [6].

For Anderson localization to happen, we need to control multiple variables which are going to be explained in detail in the results section. Consequently, it was difficult to achieve an unambiguous observation of the Anderson localization experimentally for BEC. It was in 2010 when two experiments, one in Paris using ^{87}Rb atoms and another one in Florence using ^{39}K atoms, studied this phenomenon and succeeded to observe the localized states [7].

Our aim is to numerically simulate these systems by solving the GP equation describing a BEC in a disordered potential, and observing Anderson localization. For this work we have used the Trotter-Suzuki *Python* package [8], which is a massively parallelized algorithm that efficiently solves the GP equation both in the time-independent and the time-dependent version in 1D and 2D.

Section II introduces the GP equation and the expression for the disordered potential. The Sect. III shows the numerical results obtained from the solution of the GP equation in the 1D case, and discusses the problems and perspectives in the 2D case. The conclusions of our work are analysed in Sect. IV.

II. THE GROSS-PITAEVSKII EQUATION AND THE DISORDER POTENTIAL

Let us first start by introducing the many-body Hamiltonian written in second quantization, with the field operators $\hat{\Psi}^\dagger(\vec{r})$ and $\hat{\Psi}(\vec{r})$:

$$\hat{\mathcal{H}} = \int d\vec{r}\,\hat{\Psi}^\dagger(\vec{r})\left[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V_{\text{ext}}(\vec{r})\right]\hat{\Psi}(\vec{r}) + \frac{1}{2}\int d\vec{r}\,d\vec{r}\,'\Psi^\dagger(\vec{r})\Psi^\dagger(\vec{r}\,')V(|\vec{r}-\vec{r}\,'|)\hat{\Psi}(\vec{r})\hat{\Psi}(\vec{r}\,'). \quad (1)$$

Using the Heisenberg equation of motion:

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi} = \left[\hat{\Psi}(\vec{r}), \hat{\mathcal{H}}\right]. \quad (2)$$

Replacing the operator fields with their mean value since the non condensed fraction of the BEC can be neglected: $\varphi(\vec{r}) = \langle\hat{\Psi}(\vec{r})\rangle$, we get:

$$i\hbar\frac{\partial}{\partial t}\varphi(\vec{r},t) = \left[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V_{\text{ext}}(\vec{r})\right]\varphi(\vec{r},t) + \int d\vec{r}\,'\varphi^*(\vec{r}\,',t)V(|\vec{r}-\vec{r}\,'|)\varphi(\vec{r}\,',t)\varphi(\vec{r},t). \quad (3)$$

Where $\varphi(\vec{r})$ is normalized to the total number of particles N, V_{ext} represents an external potential and $V(|\vec{r}-\vec{r}\,'|)$ is the interaction potential.

For a dilute system (so that $\rho|a|^3 \ll 1$, being ρ the density and a the scattering length), there will be only interaction if there is direct contact between particles, thus we can replace the interaction potential, $V(|\vec{r}-\vec{r}\,'|)$ with $g\delta(\vec{r}-\vec{r}\,')$, $g = 4\pi\hbar^2 a/m$ being the coupling constant. If we replace this term in Eq. (3) we obtain the Gross-Pitaevskii [9] (GP) equation:

$$i\hbar\frac{\partial}{\partial t}\varphi(\vec{r},t) = \left[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V_{\text{ext}}(\vec{r})\right]\varphi(\vec{r},t) + g|\varphi(\vec{r},t)|^2\varphi(\vec{r},t). \quad (4)$$

This GP is a non-linear Schrödinger-like equation that in general has to be solved using numerical methods. In this work we are going to solve the GP equation with a disordered potential using the Trotter-Suzuki algorithm.

There are mainly two ways of creating numerical disorder. There is a rough method that consists in overlapping sinusoidal functions with incommensurable frequencies: $V_{\text{des}}(\vec{r}) = \sum_{i=1}^{N} A_i \sin(\vec{k_i}\cdot\vec{r})$. The second method is to compute an inverse Fourier transform of gaussianly-distributed disorder:

$$V_{\text{des}}(\vec{r}) = \text{Re}\left[\mathcal{F}^{-1}\left[2\pi^2 l_c^{d/2} u_0 \exp\left(i\varphi_{\vec{k}}\right)\exp\left(-\frac{\vec{k}^2 l_c^2}{16}\right)\right]\right], \quad (5)$$

where $\mathcal{F}^{-1}$ represents the inverse Fourier transform, l_c is the correlation length, u_0 the strength of the disorder, d is the dimension, $\varphi_{\vec{k}}$ is a random matrix with values uniformly between 0 and 2π (which is going to be the cause of noise) [10]. This expression allows us to control the parameters of the disorder thanks to the correlation length l_c[12], which is related to the average distance between maxima and minima, and the disorder strength u_0: the amplitude of the potential. The 2D disorder potential form is shown in Fig. 1. Panel a) being a color map of the disorder potential. The potential along the x and y-axis are represented in panels b) and c), respectively. The panel d) shows the histogram of the disorder.

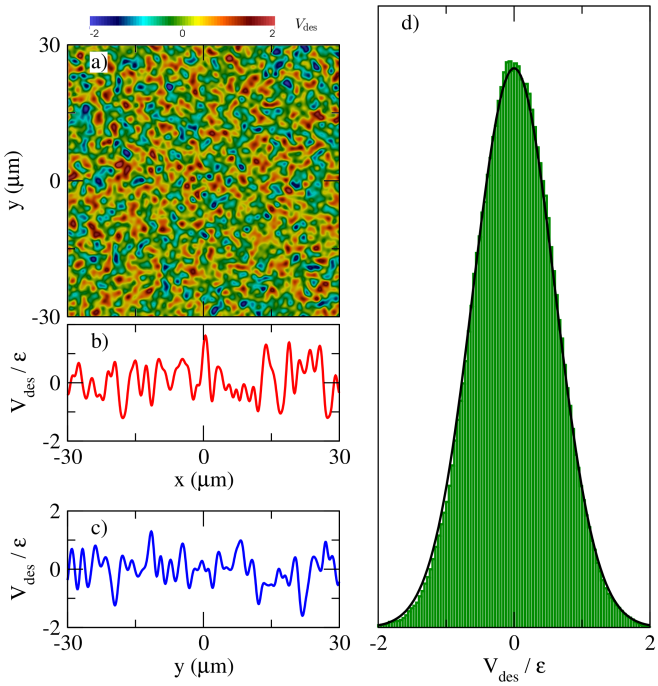

FIG. 1: The form of the disorder potential used (eq 5). a) represents the disorder form in a 2D grid, with its maxima (red) and minima (blue), b) is the x-axis view of the same disorder potential showed above, with the units expressed in μm, c) is the y-axis view, again for the same disorder showed above and lastly d) is the histogram of the points of our grid where the potential is acting and shows the Gaussian behaviour of the width of the potential peaks, with the amplitude of this histogram being of the order of u_0. The potential is written in terms of $\varepsilon = 0.476$ peV. We have used for this case $u_0 = 1.5\varepsilon$ and $l_c = 4.5$ μm.

III. RESULTS

To observe Anderson localization, initially, we create a Bose-Einstein condensate with a gaussian wave function in a disorder potential, and we let the BEC to expand until it reaches localization after a certain time. When the BEC has surpassed this localization time, it remains localized, and we can then state that Anderson localization has taken place.

If we take into account the atom-atom interaction, the repulsion between them screens the disorder [7] and therefore weakens Anderson localization. However, during the expansion of the BEC, the interaction energy reduces, and the repulsion effect becomes less important.

As we mentioned, our aim is to numerically find An-

derson localization by using the Trotter-Suzuki package [8] to simulate a BEC under the disorder potential introduced in the previous section. We are going to have a simulation controlled by six parameters. The first one is the grid length, which is going to be centered to 0 so the grid length is 2 times the maximum value $x_{\max}$ in the x-axis, and correspondingly for the y-axis. The second one is the grid step Δx and Δy for x and y-axis, respectively. The number of points in the lattice n_x for the x-axis, are related with the previous parameters in such a way that $2x_{\max} = n_x \Delta x$, and correspondingly for the y-axis. The last four parameters are the disorder strength u_0, the correlation length l_c, the coupling constant g (or equivalently, the scattering length a) and lastly the initial amplitude of the gaussian-state condensate σ.

For the observation of Anderson localization, one has to take into account some restrictions when choosing values for the different parameters. A sufficiently large grid is necessary to prevent the condensate from localizing after reaching the end of the lattice, since this kind of localization is not physical. Nevertheless, large grid lengths require very large computational times. Therefore, one has to be careful with these dimensions. If the grid length is changed and the number of points remains the same, we have to take into account that the grid step is modified as well. To keep the same step we have therefore to change the number of points for each axis too. In addition, the amplitude of the initial gaussian-state condensate σ needs to be at least 2 or 3 times the correlation length to be able to see the effect of the disorder. Once the initial expansion is finished, we expect the amplitude to remain relatively constant, assuming that it has become localized.

Reducing the correlation length prevents the appearance of bound states in the minima of the disorder potential. This provides an energy scale, called mobility edge $E_m = h^2/2ml_c^2$, that has to be much larger than the disorder strength u_0, to forbid trivial localization in the minima of the potential. Moreover, if we assume that the coupling constant is very small but non zero, another energy scale is added in the system. It is known (and we have checked it in the present work), that Anderson localization can appear in non-interacting 1D BECs. However, it is still an ongoing research the study of the role of interactions in 2D and 3D systems, and we will briefly comment about the 2D case later on.

To conclude this previous analysis of the simulation, there is still another relevant factor to be mentioned: the time. It is known that for ^{87}Rb atoms, used as the reference for this simulation, the localization time occurs approximately after a second [11]. We have to perform a time evolution using a sufficiently small time step to control that the total energy remains approximately constant, but large enough for the program to be optimized and spend acceptable running times. We have periodically checked the density and energy of the condensate to control if localization is taking place or not and if the energy remains constant.

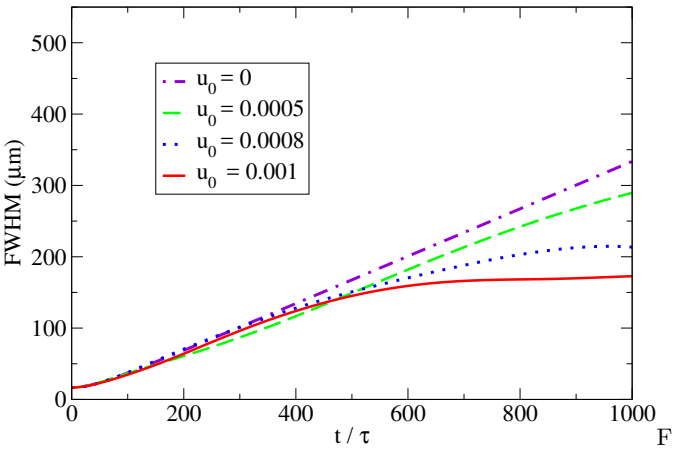

FIG. 2: Full width at half maximum (in μm) of the BEC gaussian-form density during a time evolution, for different values of u_0. This disorder strength u_0 has the same adimensional units as the potential in Fig. 1. The x-axis is adimensionally t/τ where $\tau = 2.76 \times 10^{-3}$ s.

A. 1D analysis

We have finally seen unambiguous Anderson localization with the following parameters: 3000 μm for the grid length with $n_x = 1000$ points, so that the grid step is $\Delta x = 3$ μm. The correlation length is $l_c = 10\mu$m. The initial full width at half maximum (FWHM) of the density in the gaussian-state BEC is 17 μm, which means that the amplitude of the wave function is 20 μm, which is 2 times the correlation length so that we can see the effect of the disorder. Once having fixed these parameters and for no interaction (so that $g = 0$), we proceed to check the evolution of the FWHM of the condensate as a function of time for different values of u_0, as seen in Fig. 2. The dot-dashed violet line, the dashed green line, the dotted blue line and solid red line corresponds to $u_0 = 0, 0.0005, 0.0008$ and 0.001 in units of ε.

For no disorder (dot-dashed violet line, $u_0 = 0$) the BEC expands at a constant rate indefinitely until it reaches the boundaries of the lattice. As the disorder strength increases, the BEC starts to localize after a certain localization time that decreases with u_0. For the particular case of $u_0 = 0.0005$, we have not been able to see localization. It can be a signature of two possible effects in the system. Since at weak disorder strength, the localization time is very large, maybe the simulation has not been long enough to capture the localization of the condensate in this potential. Nevertheless, if the simulation is reliable, it would mean that there exists a critical value for the disorder strength below which there is no Anderson localization. Unfortunately, our simulations are not able to distinguish between these two cases.

We have also studied the effect of the interaction in the expansion of the BEC. The results are displayed in Fig. 3. The green-dashed line represents the FWHM of the condensate as a function of time for $u_0 = 0.001\,\varepsilon$ and

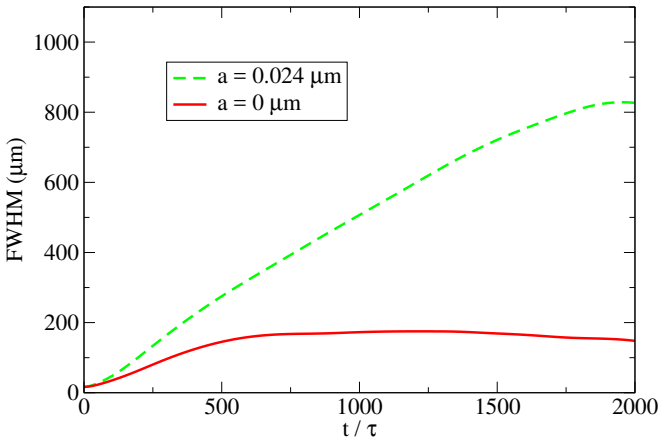

FIG. 3: Full width at half maximum (in μm) of the BEC gaussian-form density during a time evolution with and without interaction, using the scattering length parameter a in μm. Again, the x-axis is adimensionally t/τ where $\tau = 2.76 \times 10^{-3}$ s.

scattering length $a = 0.024$ μm, and the solid red line represents the FWHM for a non-interacting condensate and $u_0 = 0.001\,\varepsilon$.

As we can see, the localization time is much larger in the interacting case than in the non-interacting case. The reason of that is that interaction screens disorder, and effectively reduces the disorder strength. As we have discussed before, the localization time increases when the disorder strength decreases and thus the localization time also increases with interactions.

B. 2D analysis

We have also analysed Anderson localization in a 2D system. However, the large grid lengths and the long time evolutions made the simulation unattainable and we were not able to obtain conclusive results. Moreover, that was not the only problem that we had in the 2D case. The 2D disordered potential possesses a crucial difference with respect to its version in the 1D case, regarding the topology. Whereas in 1D there is always a maximum between two minima, in 2D one can trace a path continuously following the different minima of the lattice. It leads to two effects. One of them is the effective screening of the disorder strength due to the fact that particles can avoid the maxima of the potential, just become these points unexplored for the system. The second effect is that it is easier to find bound states localized in the minimum of the potential, effectively reducing as well the mobility edge.

It is possible that Anderson localization can not be found in non-interacting 2D Bose-Einstein condensates. The introduction of interaction can maybe solve this problem, but we have not been able to find unambiguous Anderson localization even in the interacting case, which is the worst situation for numerical calculations, since interactions increase the localization time, and therefore, the size of the condensate in the localized regime, thus requiring prohibitive grid lengths.

IV. CONCLUSIONS

In this work we have simulated a Bose-Einstein condensate and studied its behaviour under a disordered external potential, by numerically solving the time-dependent Gross-Pitaevskii equation with the Trotter-Suzuki package, to observe unambiguously Anderson localization in both 1D and 2D systems.

We first study how to get a proper expression for the disordered potential, checking that its proper form can be controlled through its parameters. We then have proceeded in creating the simulation program and analyzing the different parameters which have to be controlled to fulfil the observation of a localized state, such as the grid dimensions, the disorder parameters (l_c and u_0), the interaction, the initial amplitude of the condensate and the evolving time.

Finally, we have concluded that for 1D systems the Anderson localization occurs when properly controlling all the variables that will influence the simulation: the BEC initially expands at different rates depending on the strength of disorder applied, and if the disorder is sufficiently big, we have observed localization of the condensate. Localization in 1D occurs both for interacting and non-interacting systems. In contrast, in the 2D case, it has not been possible to observe Anderson localization. We have not been able to distinguish if this is due to the fact that we have not achieved the numerical conditions for localizing the initial BEC, or there are physical limitations, such as topology, or interactions.

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[12] This parameter has to be larger than the grid step, in order to be properly resolved, but smaller than the grid size, to allow the disorder to effectively play a role.