Closed-Form SER Expression for APSK Based on the Kite Structure

Alain Richard Ndjiongue, Hendrik C. Ferreira & Telex M. N. Ngatched

***Abstract*—In communication systems, amplitude phase shift keying (APSK) arises as a potential solution to mitigate performance degradations due to channel non-linearities. Since it combines the characteristics of phase shift keying (PSK) and those of amplitude shift keying (ASK), APSK is also likely to be robust in systems influenced by phase noise. It is therefore solicited in cascaded systems involving visible light communications (VLC) as second communication technology. This paper proposes the design of an APSK constellation, based on the kite structure, suitable for such cascaded systems. The constellation design is detailed and the symbol error probability (SEP) is derived in closed-form. The bit error rate (BER) performance is compared to those of QAM and PSK. Findings reveal that for $2\eta = 4$ (η being the common number of symbols per ring), the proposed APSK scheme has similar performance with *Q*PSK and *4*QAM, and, for $2\eta > 8$, the APSK scheme outperforms the corresponding PSK, making it a good candidate for cascaded VLC systems. Even though the proposed scheme is outperformed by QAM, it gains up to about 5 dB over the PSK for an error rate of 10^{-8}.**

***Index Terms*—APSK scheme, kite structure, closed-form SER, closed-form BER.**

I. Introduction

In communication engineering, when the transmitted signal is characterized by a high peak-to-average power ratio (PAPR), the considered scheme suffers from serious degradation of its performance when the signal passes through a non-linear channel [1]. Amplitude phase shift keying (APSK), which presents a lower PAPR when compared to the conventional quadrature amplitude modulation (QAM) [2], is a good modulation scheme to be used in non-linear channels [2, 3]. An APSK constellation is constructed by a distribution of symbols on concentric rings [3]. The idea was originally proposed in [4], and since then, APSK has witnessed a lot of research interests [5]. APSK have been proposed for radio frequency (RF) [3, 5], for satellite [6], and for optical [7] communications.

On the other hand, the avalanche of connectivity needs and the ever growing demand for higher bandwidth and faster data transmission, have saturated the RF spectrum. One of the solutions to this dilemma is to consider optical wireless communications (OWC) technologies. Among them, visible light communications (VLC), is emerging as a class of OWC that, in a near future, will successfully relieve the RF spectrum in many applications. VLC technology is motivated by the rapid growth of light technologies that have considered light emitting diodes (LEDs) as a primary light source due to its low electrical power consumption. This provides VLC with the advantage of exploiting the LED to couple the communication signal to the channel while it is used for illumination. Suitable modulation techniques are available to be used in VLC, exploiting both single color LEDs or red-green-blue LEDs (RGB-LEDs). RGB-LEDs provide high data rate compared with single color LED. This is due to the fact that RDB-LEDs divides the visible light spectrum into three separate communication channels. Color-shift keying (CSK) is a specific modulation scheme designed for RGB LEDs [8]. In addition to the increased data throughput, other advantages of CSK include intensity flicker free lighting and relaxed in-rush current requirements that may apply when modulating a large array of LEDs using other techniques such as on-off keying or variable pulse position modulation (VPPM).

Since VLC systems rely upon visible radiations to convey information through the wireless environment, the transmission distance barely reaches a couple of hundred meters, conferring to VLC a short range property [8]. Therefore, in many applications, VLC is to be fed by a pivot network to form a cascaded system. To achieve a correct combination between the pivot network and VLC, two main topologies are proposed, namely, decode-and-forward (DF) and amplify-and-forward (AF). When compared to DF, the AF scenario is cost effective and provides the system with less complexity. Solutions have been proposed to combine RF with VLC [9], fiber optics (FO) with VLC [10], and power line communications (PLC) with VLC [11].

A solution for direct mapping between the pivot network and VLC, suitable for the AF scenario have been proposed recently in [12] using phase shift keying (PSK) and CSK. PSK was used to modulate the data over the pivot network while CSK was exploited in VLC. The combination obtained, *M*PSK-CSK, exploited both the magnitude and the phase of the PSK complex symbol to achieve a conversion between PSK and CSK. This *M*PSK-CSK system, however, has a drawback related to the weakness of PSK against phase noise. APSK, being a scheme blending advantages of both PSK and amplitude shift keying (ASK), is more robust against phase noise. It is therefore interesting to investigate its combination with CSK. However, most APSK constellations proposed in the literature, i.e., [13], are not suitable for a direct mapping with CSK as they contain some symbols which have the same argument.

In this paper, the design of an APSK constellation based on the kite structure and suitable for a combination of RF/PLC/FO and VLC in an AF scenario is proposed. To reduce the complexity of the color and lighting compensator module necessary on the VLC side, the proposed constellation is made of only two rings. The kite structure is obtained by adopting

Table I: Selected Notation Used in the Text

Symbol	Definition
"i", "o"	Inner and outer ring indexes
θ_i, θ_o	Angle delimiting two symbols on inner and outer rings
η	Common number of symbols on both rings
r_i, r_o	Radius of inner and outer rings
β	Ratio between r_i and r_o ($r_i = \beta r_o$)

a uniform distribution of the symbols over the same ring with each signal point defined with a different phase. Its closed-form symbol error rate (SER) expression is derived and its performance is compared to that of the corresponding PSK and QAM.

II. APSK Based on the Kite Structure

Selected notation that will be used in the text is summarized in Table I and Fig. 1 depicts the general constellation diagram of the APSK based on the kite structure. The design is motivated by two main constraints: (*i*) Since the optical power related to each symbol in an RF/PLC/FO-VLC AF scenario depends on the modulus of the APSK complex symbol used to produce the color, the number of rings are chosen in such a way that the mismatch between optical powers due to rings is easily compensated. Hence, two rings are used to reduce the compensator complexity. (*ii*) To better detect colors (after the direct APSK-CSK mapping), it is required to keep the same angle between symbols on the same ring, i.e., both inner and outer rings, and keep the same number of symbols on both rings ($\eta_i = \eta_o = \eta$). Also, since the colors are regularly organized on the color wheel, it is required to keep the same angular spacing between symbols on both rings. The symbols are then equally distributed over the rings in the way to always form a kite (for example, $B - \mathbf{s}_i - \mathbf{s}_{o-1} - \mathbf{s}_{i-1} - B$ in Fig. 1 is a kite). Moreover, a linear combination is kept between θ_i and θ_o, $\theta_o = \theta_i + \pi/\eta$. Hence, both rings have the same number of symbols. Therefore, the signal constellation in this scheme is given by

$$\mathbf{z}_i = r_i e^{j\left(\frac{2\pi}{\eta} i + \theta_i\right)} \quad \text{i} = 0, \ldots, \eta - 1, \tag{1}$$

for the inner ring, and

$$\mathbf{z}_o = r_o e^{j\left(\frac{2\pi}{\eta} o + \theta_o\right)} \quad \text{o} = 0, \ldots, \eta - 1, \tag{2}$$

for the outer ring. The kite structure makes it easy to compute the distance $d = d_{s_i\_s_{o-1}}$. This is obtained by computing θ_i, $r_i = B\mathbf{s}_i$, $r_o = B\mathbf{s}_{o-1}$ and $BB' = r_i \cos(\theta_i/2)$, and, finding $b = r_o - BB'$, where B is the common center of both rings and B' is the intersection between $B\mathbf{s}_{o-1}$ and $\mathbf{s}_i\mathbf{s}_{i-1}$. Finally, utilizing $a = r_i \sin(\theta_i/2)$, the characteristic distance, d, of the proposed APSK is given by

$$d = \left[E_s\left(\beta^2 + 1 - 2\beta\cos\left(\frac{\theta_i}{2}\right)\right)\right]^{1/2}, \tag{3}$$

where β is the ratio of the inner ring radius by the outer ring one ($r_i = \beta r_o$). Based on the allowed set of values of β ($0 \leq \beta \leq 1$), d is bounded as

$$\sqrt{E_s} \leq d \leq \sqrt{2E_s(1 - \cos\left(\frac{\theta_i}{2}\right)}, \quad \theta_i = \pi, \tag{4}$$

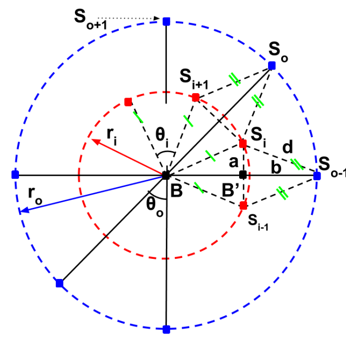

Fig. 1: Constellation design for APSK based on the kite structure.

Table II: Zeros of Eq. (6) for Normalized Values of θ_i.

APSK	*2-2*	*4-4*	*8-8*	*16-16*	*32-32*	*64-64*
θ_i	π	$\pi/2$	$\pi/4$	$\pi/8$	$\pi/16$	$\pi/32$
β_1	~~-0.5774~~	~~-1.9319~~	~~3.8306~~	~~1.5555~~	~~1.2115~~	~~1.0943~~
β_2	0.5774	0.5176	0.6302	0.7583	0.8584	0.9227

and

$$\sqrt{2E_s(1 - \cos\left(\frac{\theta_i}{2}\right)} \leq d \leq \sqrt{E_s}, \quad \theta_i \leq \pi/2. \tag{5}$$

Note that for the same value of β, d changes with the constellation size. The analysis of d and a leads to the following design axioms:

- (*i*) *Axiom 1*:
  $\forall\theta_i$, $0 < \theta_i \leq \pi$, it always exists a value of β such that the APSK of size 2η, having two rings, can be constructed.
- (*ii*) *Axiom 2*:
  $\forall\theta_i > \pi$, $\theta_o = \theta_i + \pi/\eta$ does not hold.
- (*iii*) *Axiom 3*:
  $\forall\theta_i$, $0 \leq \theta_i \leq 2\pi$, if $\beta = 1$ the APSK is a PSK of size 2η and if $\beta = 0$ the APSK is a PSK of size η.

The distances d and $2a$ (see Fig. 1) are crucial in this scheme as they considerably influence its performance. The characteristic equation of the scheme is obtained by equating Eq. (3) and $2a$, and is given as follows

$$\beta^2\left(1 - 4\sin^2\left(\frac{\theta_i}{2}\right)\right) - 2\beta\cos\left(\frac{\theta_i}{2}\right) + 1 = 0. \tag{6}$$

The solutions of Eq. (6) are

$$\beta_{1,2} = \frac{\cos\left(\frac{\theta_i}{2}\right) \pm \sqrt{3}\sin\left(\frac{\theta_i}{2}\right)}{1 - 4\sin^2\left(\frac{\theta_i}{2}\right)}. \tag{7}$$

Eq. (6) defines the critical point where the symbol error probability (SEP) changes its expression since, depending on the value of β, the minimum distance of the scheme is either d or $2a$. A graphical analysis of Eq. (6) is depicted in Fig. 2, for selected values of θ_i. It shows that as the constellation size increases, the accepted zero of Eq. (6) is closed to 1. These are grouped in Table II, where unaccepted solutions are strikethrough. These are not accepted because they are negative or greater than 1. As stated earlier, the distances d and $2a$ dictate the performance of the proposed scheme and, depending on the value of β, three cases arise ($d < 2a$, $d = 2a$,

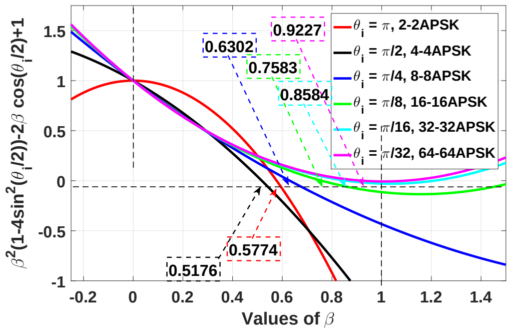

Fig. 2: Roots of Eq. (6), for selected values of θ_i and β.

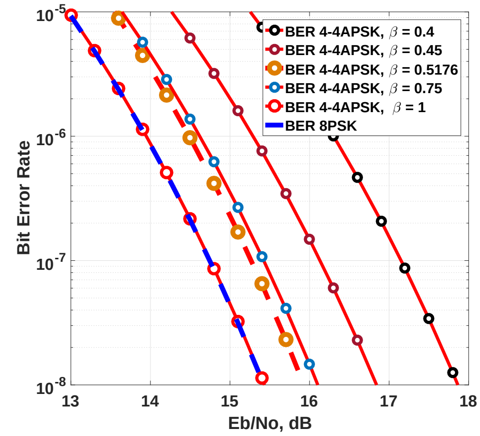

Fig. 3: Performance of a *4-4*APSK based on the kite structure.

or $d > 2a$). Knowing that symbols on the same ring will be transmitted with the same SEP, the maximum likelihood detection (MLD) rule is used to determine the SER expression per ring. The final SER is obtained by averaging the SERs from both rings. Due to the kite structure, each symbol on the inner ring is surrounded by four symbols. Hence, symbols on the inner ring are transmitted with a SEP given by

$$\begin{aligned} p_{ei} &= 1 - \Big(1 - 2Q\Big(\frac{d}{2\sigma}\Big)\Big)\Big(1 - 2Q\Big(\frac{a}{\sigma}\Big)\Big) \\ &= 2\Big[Q\Big(\frac{a}{\sigma}\Big) + Q\Big(\frac{d}{2\sigma}\Big)\Big] + 4Q\Big(\frac{a}{\sigma}\Big)Q\Big(\frac{d}{2\sigma}\Big), \end{aligned} \tag{8}$$

where $Q(.)$ refers to the Q-function. Also, as result of the kite structure, those on the outer ring have two nearest neighbors. Hence, they are transmitted with a SEP expressed as

$$p_{eo} = 1 - \Big(1 - 2Q\Big(\frac{d}{2\sigma}\Big)\Big) = 2Q\Big(\frac{d}{2\sigma}\Big). \tag{9}$$

By averaging p_{ei} and p_{eo}, the final SEP for the proposed APSK is given by

$$p = Q\Big(\frac{a}{\sigma}\Big) + 2Q\Big(\frac{d}{2\sigma}\Big) + 2Q\Big(\frac{a}{\sigma}\Big)Q\Big(\frac{d}{2\sigma}\Big). \tag{10}$$

Assuming a transmission affected by an additive white Gaussian noise (AWGN) of variance $\sigma^2 = N_0/2$, defining the quantities $\psi_1 = \beta^2 - 2\beta\cos(\theta_i/2)$ and $\psi_2 = \beta\sin(\theta_i/2)$, and substituting d and a, Eq. (10) becomes

$$\begin{aligned} p &= Q\Bigg(\psi_2\sqrt{2}\sqrt{\frac{E_s}{N_0}}\Bigg) + 2Q\Bigg(\sqrt{\frac{\psi_1}{2}}\sqrt{\frac{E_s}{N_0}}\Bigg) \\ &+ 2Q\Bigg(\psi_2\sqrt{2}\sqrt{\frac{E_s}{N_0}}\Bigg)Q\Bigg(\sqrt{\frac{\psi_1}{2}}\sqrt{\frac{E_s}{N_0}}\Bigg). \end{aligned} \tag{11}$$

III. NUMERICAL RESULTS

The general SEP expression proposed in Eq. (11) is valid for $2\eta \geq 8$ since it is derived from constellations where symbols on the inner ring have four nearest neighbors. This is not the case for $2\eta = 4$ (*2-2*APSK) as symbols on the inner ring are surrounded by three nearest neighbors. Nevertheless, the analysis of the *2-2*APSK is not interesting as its best performance will correspond to β = 1, which is similar to that of *4*QAM or *Q*PSK.

The performance of the first three constellation sizes *4-4*APSK, *8-8*APSK and *16-16*APSK, of the proposed scheme are shown in Figs. 3, 4 and 5, respectively. In each of these figures, the performance of the APSK is compared to that of a PSK of the same size. The *4-4*APSK represents the APSK based on the kite structure (for $2\eta \geq 8$) in which the best performance does not corresponds to the zeros of the characteristic equation, but matches that of the corresponding PSK. As shown in Fig. 3, the performance improves as β increases. At β = 0.5176, the scheme starts to degrade until β = 0.75, where it reverses and reaches its best performance at β = 1 (*8*PSK). From *8-8*APSK, as shown in Figs. 4 and 5, the best performance of the scheme is obtained exactly when d = $2a$. Thus, the best performance of the *8-8*APSK is obtained for β = 0.6302, that of the *16-16*APSK is obtained for β = 0.7583, that of a *32-32*APSK is obtained for β = 0.8584, and that of a *64-64*APSK at β = 0.9227. They all outperform the corresponding PSK (see Figs. 4, 5, and 6). In Fig. 6, the best performances of *4-4*APSK, *8-8*APSK, *16-16*APSK and *32-32*APSK are compared to corresponding QAM and PSK performances. At an error rate of 10^{-8}, a *4-4*APSK gains about 2.7 dB over the *8-8*APSK, while the *8-8*APSK gains about 3.3 dB over the *16-16*APSK, and the *16-16*APSK gains about 4.00 dB over the *32-32*APSK. It can also be observed from Fig. 6 that the APSK scheme outperforms the corresponding PSK scheme of the same constellation size. A *4-4*APSK has similar performance as an *8*PSK. However, *8-8*APSK, *16-16*APSK and *32-32*APSK gain respectively 1.8 dB, 3.5 dB and 5 dB over *16*PSK, *32*PSK and *64*PSK. All APSK of size $2\eta \geq 8$ are outperformed by the corresponding QAMs. *8*QAM, *16*QAM, *32*QAM and *64*QAM respectively gain about 1.5 dB, 2.2 dB, 3.0 dB and 5 dB over *4-4*APSK, *8-8*APSK, *16-16*APSK and *32-32*APSK. It should be noted that the comparison with QAM is just for illustrative purposes. QAM is very unlikely to be used in cascaded VLC systems involving a direct mapping with CSK modulation, for which the proposed APSK is designed. This is due to the fact that, since QAM is made of complex symbols, some of which might have the same modulus or argument, a direct mapping with CSK would result in poor performance in both lighting and communication.

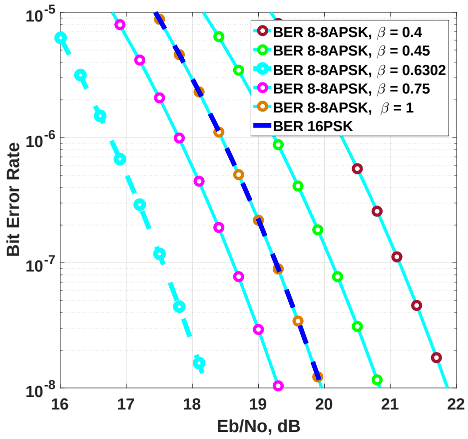

Fig. 4: Performance of a *8-8*APSK based on the kite structure.

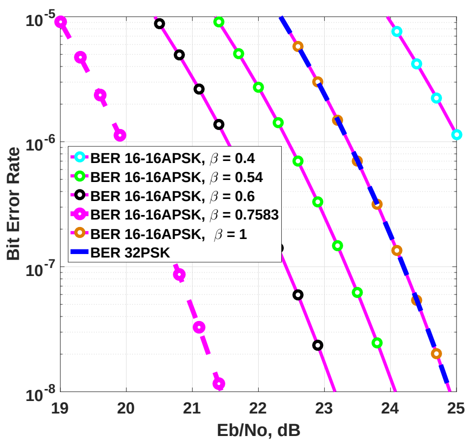

Fig. 5: Performance of a *16-16*APSK based on the kite structure.

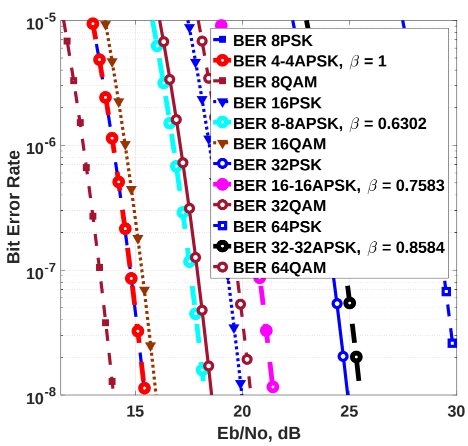

Fig. 6: Comparison between APSK based on the kite structure, QAM and PSK.

IV. Conclusion

This paper presents the design and analysis of the SEP of an APSK scheme based on the kite structure, over a Gaussian channel. The proposed APSK scheme has two rings and is made of the same number of symbols regularly distributed over both rings. First, the ratio β between r_i and r_o was defined, which helped to find the sides of the kite. These sides of the kite were then used to define the characteristic equation of the scheme which is used in the derivation of the SEP. Second, the characteristic distance obtained was optimized for the best performance of the scheme. Third, a general SER expression was proposed. For $2\eta = 4$, the scheme has similar performance as *Q*PSK and *4*QAM. The scheme is analyzed for *4-4*APSK, *8-8*APSK, *16-16*APSK, and *32-32*APSK, and the obtained BER curves are compared to the performance of the corresponding PSK and QAM. The best of APSK outperforms the corresponding PSK for $2\eta > 8$, and, for $2\eta = 8$, *4-4*APSK has similar performance as the *8*PSK. The proposed APSK design is therefore a suitable candidate for RF/PLC/FO-VLC cascaded systems. The next step in this work is the analysis of the cascaded APSK-CSK scheme. This is currently being done and the results will be reported soon.

References

[1] Y. Rahmatallah and S. Mohan, "Peak-to-average power ratio reduction in OFDM systems: A survey and taxonomy," *IEEE Commun. Surveys Tuts.*, vol. 15, no. 4, pp. 1567–1592, Fourth 2013.

[2] Z. Liu, Q. Xie, K. Peng, and Z. Yang, "APSK constellation with Gray mapping," *IEEE Commun. Lett.*, vol. 15, no. 12, pp. 1271–1273, Dec. 2011.

[3] H. Méric, "Approaching the Gaussian channel capacity with APSK constellations," *IEEE Commun. Lett.*, vol. 19, no. 7, pp. 1125–1128, Jul. 2015.

[4] C. Thomas, M. Weidner, and S. Durrani, "Digital amplitude-phase keying with M-Ary alphabets," *IEEE Trans. Commun.*, vol. 22, no. 2, pp. 168–180, Feb. 1974.

[5] Q. Xie, Z. Wang, and Z. Yang, "Simplified soft demapper for APSK with product constellation labeling," *IEEE Trans. Wireless Commun.*, vol. 11, no. 7, pp. 2649–2657, Jul. 2012.

[6] R. D. Gaudenzi, A. G. i Fabregas, and A. Martinez, "Performance analysis of turbo-coded APSK modulations over nonlinear satellite channels," *IEEE Trans. Wireless Commun.*, vol. 5, no. 9, pp. 2396–2407, Sep. 2006.

[7] C. Hager, A. G. i Amat, A. Alvarado, and E. Agrell, "Design of APSK constellations for coherent optical channels with nonlinear phase noise," *IEEE Trans. Commun.*, vol. 61, no. 8, pp. 3362–3373, Aug. 2013.

[8] "IEEE standard for local and metropolitan area networks–part 15.7: Short-range wireless optical communication using visible light," *IEEE Std 802.15.7-2011*, pp. 1–309, Sep. 2011.

[9] X. Li, R. Zhang, and L. Hanzo, "Cooperative load balancing in hybrid visible light vommunications and WiFi," *IEEE Trans. Commun.*, vol. 63, no. 4, pp. 1319–1329, Apr. 2015.

[10] J. Latal, J. Vitasek, L. Hajek, A. Vanderka, O. Zboril, D. Pudis, P. Koudelka, and V. Vasinek, "New type of VLC communication transmitter based on optical fibres," in *Proc. 18th Int. Conf. Transparent Optical Netw. (ICTON)*, Trento, Italy, Jul. 10-14, 2016, pp. 1–4.

[11] J. Song, W. Ding, F. Yang, H. Yang, B. Yu, and H. Zhang, "An indoor broadband broadcasting system based on PLC and VLC," *IEEE Trans. Broadcast.*, vol. 61, no. 2, pp. 299–308, Jun. 2015.

[12] A. R. Ndjiongue, T. Shongwe, and H. C. Ferreira, "Closed-form BER expressions for HSV based MPSK-CSK systems," *IEEE Commun. Lett.*, vol. 21, no. 5, pp. 1–4, May 2017.

[13] Y. C. Chow, A. R. Nix, and J. P. McGeehan, "Analysis of 16-APSK modulation in AWGN and Rayleigh fading channel," *Electron. Lett.*, vol. 28, no. 17, pp. 1608–1610, Aug. 1992.