GRADE 10 MATHEMATICS LEARNERS' ERRORS AND MISCONCEPTIONS WHEN SIMPLIFYING ALGEBRAIC FRACTION

BY

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" The most useful piece of learning for the uses of life is to unlearn what is untrue"

Antisthenes

DECLARATION

I declare that this report is my own work and that no portion of it has been copied from other sources (unless quoted). All phrases and sentences taken directly from other sources have been cited and the references listed in full in the reference list.

Signature: ....

Date: ....

DEDICATION

I dedicate this dissertation to my dearest family, especially Joel and Madam Comfort Efua Onyienyiwa Nyamekye, and even to you Jeacy and Mr. Onyipanka.

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I am grateful!

ABSTRACT

This study presents findings on algebraic fractions that was conducted among grade 10 learners in the Eastern Cape Province, South Africa. The study analysed the learners' performance in algebraic fractions and explored the mistakes made by the learners as they engaged in the tasks of simplifying algebraic fractions. The study required learners to respond to algebraic fraction tasks. A group of 136 grade 10 high school learners were purposefully sampled. They responded to twelve grade 10 mathematics questions that consisted of two knowledge questions, four routine procedural questions, four complex procedures questions and two problem solving questions which were the cognitive levels guide suggested in the TIMSS 1999 study. Descriptor code keys were used to code, categorise and thematised the findings. When the learners' overall performances were analysed for all twelve questions, the results showed that the majority of the learners did not comprehend what simplify algebraic fractions meant, and their conceptual knowledge of basic algebraic expression and simplification of such expressions was quite weak. The study revealed learners' conceptual errors as well as procedural errors as being most prominent. These errors hinder learners' ability to solve other mathematical problems that require prior knowledge to simplifying algebraic fractions.

Keywords: Algebraic fractions; misconceptions and errors.

LIST OF ABBREVIATIONS

CAPS : Curriculum and Assessment Policy Statement

FET : Further Education and Training

MKT : Mathematical Knowledge for Teaching

MQI : Mathematical Quality Instruction

GET : General Education and Training

DBE : Department of Basic Education

NCTM: National Council of Teachers of Mathematics

PCK : Pedagogical Content Knowledge

CK : Content Knowledge

LTSM: Learner Teacher Support Material

SMCK: Subject Matter Content Knowledge

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CHAPTER 1: INTRODUCTION

1.1 INTRODUCTION

The idea that learners develop ‘misconceptions’ lies at the heart of much of the empirical research on learning and teaching of mathematics and science nowadays. Chapter 1 of this study presents the insights into as well as the background to the study which aimed at investigating learners’ performance in problems involving algebraic fractions. It also includes the statement of the problem, the purpose of the study, the aim and objectives of the study, research questions and the significance of the research, definitions of operational terms of the study, the organisation of the study and the conclusion to the chapter.

1.2 BACKGROUND TO THE STUDY

Mathematics plays a vital role in our modern society especially in the scientific and technological development of any country (Park et al., 2016; Drew, 2015). It is becoming man’s survival kit. This may suggest that no country can grow scientifically and technologically above her mathematics status; this is a clear indication that mathematics is essential for science. It serves as a tool to sharpen our mind, reasoning ability and our personality, hence its immense contribution to individuals and the world in general.

Algebraic fractions are quotients of two algebraic expressions (Petit et al., 2015). Computations of these algebraic fractions usually involve co-ordination of other mathematical concepts such as equation, factorization, division, and exponents at Grade 10 level (Kaput, 1999). The challenge for most teachers is to acknowledge the fact that understanding of algebraic fraction concepts forms an important base for mathematical learning in general, and adopt this as a foundation for teaching mathematics to learners.

All of us, including our learners, make mistakes from time to time. Some individuals suggest that if you do not make mistakes you are not working hard enough. Truly, it is not only learners that make mistakes with computations - everyone does. However, there is a difference between the careless mistakes we all make, and misconceptions about mathematical ideas and procedures. Learners learn concepts, and sometimes they also learn misconceptions in spite of what teachers try to teach them. Error analysis often reveals misconceptions learners have (Bush & Karp, 2013; Idris, 2011; Luneta & Makonye, 2010 & Olivier, 1989). Kaplan (2007) also argues that learners find algebraic fractions difficult to learn, probably because they demand understanding of other concepts such as exponents, factorization and division. For instance, a grade 10 learner (21) answered

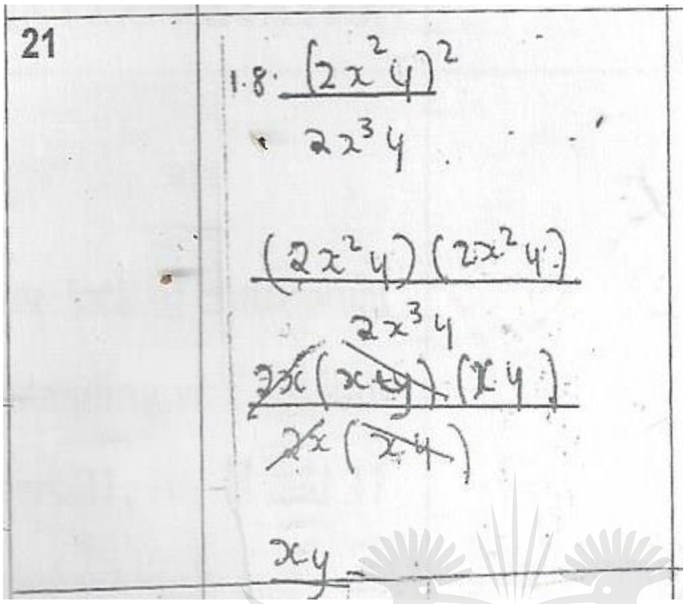
	CORRECT
	$\frac{(2x^2y)^2}{2x^3y}$ $\frac{4x^4y^2}{2x^3y}$ $2x^{4-3}y^{2-1}$ 2xy

Figure 1: Algebraic fraction depicting possible misunderstanding of mathematical concept. Source: Author

The learner (21) express the numerator as $(2x^2y)(2x^2y)$ and wrongly factorize 2x as the common factor from both the numerator and the denominator. The principle of factorization with algebraic fraction applies when there is operation sign plus or minus in the expression. This is a practice based reason for the study. The results of the study can be used to assist teachers to address learners' misconceptions and errors in algebraic fractions and provide instructional approaches to address mathematical errors.

Algebraic fractions have been one of the topics that most learners fear to learn (Mhakure et al., 2014 & Egodawatte, 2011). As to why it is difficult for some learners, Warren (2003) propounds that learners' difficulties with algebraic fractions stem from their lack of understanding of the structure of arithmetics. Siemon et al. (2013: 540) and Figueras et al. (2008) also observed that learners' difficulty with algebraic fractions is due to the nature of it – a fraction with numbers and variables. The algebraic fraction concept is an integral part of both the General Education and Training (GET) and the Further Education and Training (FET) curricula. The Curriculum and Assessment Policy Statements (CAPS) for mathematics emphasises that learners should be able to simplify algebraic fractions with confidence, yet algebraic fractions is one of the most challenging concepts for learners to grasp. It is essential to note also that knowledge of algebraic fractions helps with simplification of several mathematical topics, for example, simplification of functions, calculus, trigonometry, financial mathematics, etc. (Siemon et al., 2013). As a result, a good knowledge of algebraic fractions is crucial for success in mathematics education (Brown & Quinn, 2006; Mhakure et al., 2014). This affirms that learners in high school have difficulties with simplification of algebraic fractions among other topics such as circle geometry and trigonometry. This contributes to under-performance in terms of the matric mathematics pass rate in the country (Karim et al., 2010; Aliberti, 1981; Luneta, 2015). This study will suggest possible strategies to minimize misconceptions in learning of mathematics.

Following the introduction and the research background is the underlying problem warranting the study. The significance and the aims of this study follow. The formulated primary research questions and sub-questions in tandem with the specific objectives of the study are then presented. Next, a brief literature review of the key concepts of this research study is provided by highlighting the main topics discussed in the literature chapter. The research instrument and outline for this dissertation are then described.

1.3 STATEMENT OF THE PROBLEM

The National Diagnostic Report (2013:136) points out a concern that the matriculants perform badly in simplification of algebraic fractions among other things. The Chief Examiner's report in both mathematics first and second papers also outlined that matriculants consistently have difficulties with simplification of fractions, algebraic fractions and equations (DBE, 2013). According to the National Diagnostic Report (2013:136) although algebra and equations cover 30% in both grade 10 and 11, and 17% of the grade 12 curricula, learners find simplification of algebraic fractions difficult. The report added that many of the candidates were "discredited" because of their inability to simplify algebraic fractions.

Very often learners find simplification of algebraic fractions confusing and display misconceptions learners develop when dealing with this section (Figueras et al., 2008; Mhakure, et al., 2014; Siemon et al. 2013; Brown & Quinn, 2006; Groff, 2006:552). The study therefore sought to identify possible learners' misconceptions when dealing with algebraic fractions and what could be the root causes of these misconceptions.

1.4 SIGNIFICANCE OF THE RESEARCH

The report resulting from this research on mathematical misconceptions will be useful in devising approaches that will help with identification and minimization of misconceptions and errors. Similar to other research papers on the same topic, it will assist in giving directions on mathematics teaching strategies which will remedy such misconception occurrences.

Teachers who read this study report will be aware of some of the common misconceptions that develop amongst learners in algebraic fractions lessons. They will therefore be better equipped and prepared for the lessons. The study will also enlighten all education stakeholders on how misconceptions in simplification of algebraic fractions and other mathematics concepts can be identified and rectified.

1.5 AIM OF THE STUDY

The study is aimed at:

- Identify displayed errors by learners resulting from misconceptions.
- Categorise the errors.
- Suggest ways in which the misconceptions discovered in the study can be minimized.

1.6 OBJECTIVES OF THE STUDY

According to Cooney, Davis and Henderson (1975), objectives should be stated in terms of observable learners' behaviour. The study should meet the following objectives:

- To identify possible reasons for learners' misconceptions in simplifying algebraic fractions.
- To identify errors that result from the learners' misconceptions and how they can be categorized.
- To suggest ways in which these misconceptions can be minimized.

1.7 RESEARCH QUESTIONS

1.1 What are the misconceptions that learners display through the errors they make when solving algebraic fractions?

1.2 What are the categories of errors that learners display when solving algebraic fractions?

1.3 How can these misconceptions be minimized?

1.8 MOTIVATION FOR THE STUDY

Long (2005:15) attributed learners' learning difficulties to concept learning: "challenges in learning stem from a failure to comprehend concepts on which procedures are based". In other words, misconceptions are neither inborn nor are they instantaneous. Rather, learners have acquired those misconceptions during their learning process for inconclusive reasons. Whatever the reasons may be, problems of this nature are particularly worthy of investigation as there is still a lack of robust research in identifying learners' misconceptions for more than one conceptual area collectively. In order to explore such issues, the algebraic fractions concept has been chosen in this study. Learners' misconceptions about mathematics particularly affect their further studies. In trigonometry, an example like:

Prove that $\frac{sin2x}{1+cos2x} = tanx$

Solution

$= \frac{2sinxcosx}{1+(2cos^2x-1)}$
$= \frac{2\text{sinx.cosx}}{2cos^2}$
$= \frac{sinx}{cosx}$
= tanx

Similarly, the use of first principles to find the derivative of

f(x) $=\frac{1}{2}x^2$

Solution

$f'(x) = \lim_{h\to o}\frac{f(x+h)-f(x)}{h}$
$f'(x) = \lim_{h\to o}\frac{\frac{1}{2}(x+h)^2-\frac{1}{2}x^2}{h}$
$f'(x) = \lim_{h\to o}\frac{xh-\frac{1}{2}h^2}{h}$
$f'(x) = \lim_{h\to o}(x+\frac{1}{2}h)$
$\therefore\ f'(x) = x$

The above examples on trigonometry and differential calculus depict how algebraic fractions form part of solutions of other sections of high school mathematics. Hence, lack of understanding of algebraic fractions will hinder the comprehension of other concepts. This study sought to identify and categorize errors as a result of misconceptions and suggest possible strategies to curb this learning barrier.

1.9 INSTRUMENTS

Tasks: learners were given a test containing questions on algebraic fractions. The questions were designed such that their solutions to the problems will require both procedural and conceptual knowledge. The context of the questions was not designed only to focus on the learners' ability to perform, but also to assess their competence to interpret questions correctly and formulate correct mathematical representations.

Questionnaires: A questionnaire was given to each of the respondents. The open-ended questions were used to grant the respondents space to share their experiences regarding solutions of the algebraic fraction questions. These first-hand feedbacks from the respondents reinforced the researcher's analysis of the collated data.

1.10 DEFINITIONS AND TERMS

1.10.1 Algebraic fraction

Algebraic fractions are expressions thought of as generalizations of fractions or rational numbers. Algebraic fractions build on the properties learnt about fractions and algebra. Computation of algebraic fractions is challenging to learners, possibly because they require the understanding of mathematical concepts like division, variable, equation, perfect squares, exponent, factorization and rational numbers. Inadequacy in the understanding of these concepts leads to difficulties in solving algebraic fractions (Kaput, 1999; Mhakure et al., 2014). So algebraic fractions is not a singular idea but is composed of different forms of thought and an understanding of symbolism. It is a vital strand of the curriculum but it should be imbedded in all areas of mathematics. This may enable learners to think mathematically in all the phases.

1.10.2 Misconceptions and errors

There are unnecessarily mistakes and challenges that learners display in their responses to mathematics tasks. Luneta and Makonye (2010a) pointed out that errors and misconceptions may relate but the two constructs are different and they identify two types of errors: systematic and unsystematic errors. According to Lukhele, Murray and Olivier (1999) unsystematic errors are exhibited without the intention of learners; learners may not repeat such errors and learners can correct them independently upon second look. According to Watanabe (1991) some learners use short cuts to solve mathematical problems which result in such errors. Systematic errors may be repeated, systematically constructed or reconstructed over a period due to a grasp of incorrect conceptions of solving a particular problem (Idris, 2011). Nesher (1987) described misconceptions as a display of an already acquired system of concepts and algorithms that have been wrongly applied.

1.10.3 Misconceptions

Misconceptions may occur when an idea is not incorporated into an appropriate schema (Skemp, 1978:25). Skemp further described this phenomenon as instrumental understanding, thus ideas that are isolated and without meaning. Drew et al. (2008:15) also define a misconception as a misapplication of a rule, an over- or under-generalization or an alternative conception of the situation. In this study, ‘misconceptions’ shall be viewed as an alternative conception idea which differs from expert understanding.

1.10.4 Errors

Misconceptions generate errors. This is because errors result from learners' misconceptions of new ideas. Errors are wrong answers or mistakes which are regular, planned and repeated again and again. Luneta (2008:35) defined errors as a mistake, slip, blunder or inaccuracy and deviation from accuracy. These errors are the indicators or symptoms of some misconceptions. For the example in Vignette A, the learner had conceived that quotients expressions are the same as pairs of linear or quadratic expressions. Hence, the answer given by the learner was wrong. But this error did not just happen; rather it is as a result of misconception.

1.11 RESEARCH ORGANISATION

The dissertation is organised as follows.

Chapter 1 provides a brief background to the study, the statement of the research problem, the significance of the study, the aims and the objectives of the study, the research questions, motivation for the study, and the definitions of some key terms used in the study. As a theoretical basis to the study and to examine the previous work done in this area, a literature review was carried out as the next step.

Chapter 2 is a literature review chapter that is organized into several areas starting from an introduction. Different views of studying cognition and different notions of constructivism are discussed next. The nature of mathematics, fraction concepts, the nature of algebra, algebraic fractions and their associated misconceptions are also discussed. Chapter 2 ends with a classification of various error types in the literature pertaining to the topics under investigation in the research. Possible effective strategies to deal with misconceptions are also discussed.

Chapter 3 is devoted to discussion on the methodological constructs of the research. It explains the qualitative approach that was employed for this study. A mathematics test on algebraic fractions was used as the main research instrument and tested for its reliability and validity. Interviews were used as part of the case study method. The chapter ends with a brief discussion on the sample, data analysis methods, and ethical considerations.

Chapter 4 addresses the main findings pertaining to the categorization of errors. A rubric containing error types was designed for each conceptual area considered in the study. **Chapter 5** summaries, recommends and concludes with suggestions of strategies that can be used to minimize misconceptions and errors in algebraic fractions.

CHAPTER 2: LITERATURE REVIEW

2.1 INTRODUCTION

According to Hart (1998), a literature review is the effective evaluation of selected documents on a research topic. Creswell (2010) and Babbie (2008) also observed that a good literature review leads logically to the research question. In line with that, this chapter reviews literature on learners' misconceptions in the Further Education and Training (FET) phase as well as the theoretical frameworks which underpin the study. It first considers two of Shulman's knowledge domains: Content knowledge (CK) and Pedagogical content knowledge (PCK) and then constructivism as the basis for a theoretical framework for the study. It then goes on to explore from a constructivist viewpoint the nature of mathematics, algebra and algebraic fractions before ending the discourse with discussions on misconceptions and errors regarding the algebraic fraction concept.

2.2 THEORETICAL FRAMEWORK

Shulman's (1986) knowledge bases in teaching, grounded with the constructivist's views of teaching and learning, are outlined below. Shulman identified seven knowledge domains pertinent to teacher knowledge and these were:

1. Content knowledge;
2. General pedagogical knowledge;
3. Curriculum knowledge;
4. Pedagogical-content knowledge;
5. Knowledge of learners and their characteristics;
6. Knowledge of educational contexts; and
7. Knowledge of educational ends, purposes and values the philosophical grounds.

Years later, Collison (1996) also proposed a theoretical model for becoming an exemplary teacher, which encompasses Shulman's seven knowledge domains within a triad of knowledge: professional knowledge (i.e. content knowledge, curricular, and pedagogical knowledge), interpersonal knowledge (i.e. relationships with students, the educational community, and the local community), and intrapersonal knowledge (i.e. reflection, ethics and dispositions). Collison further revealed that a teacher's understanding of what it means to be a teacher involves developing and integrating professional, interpersonal and intrapersonal knowledge in ways that allow learners to construct the physical, social and intellectual environment of their classrooms (1996). The learners are also equipped for life beyond the classroom.

Among the knowledge bases of Shulman, content knowledge (CK) and pedagogical content knowledge (PCK) were discussed in detail for this study due to their relevancy. Whereas CK focuses on knowing the key content ideas to be taught (Luneta et al., 2013), the PCK focuses on ways of representing and formulating the subject to make it understandable to learners as well as providing an understanding of what makes learning specific themes easy or difficult to comprehend (Shulman, 1987).

To educate learners nowadays, teachers need to understand subject matter deeply and flexibly so that they can assist learners create useful concept images, relate one idea to another, and address misconceptions. Teachers need to see how ideas connect across fields and to everyday life (Hill et al., 2008). Kilpatrick (2001) also identified five different strands (conceptual understanding; procedural fluency; strategic competence; adaptive reasoning and productive disposition) that teachers should equip learners with in order to be critical thinkers and problem solvers. This kind of understanding creates a concrete foundation for pedagogical content knowledge that enables teachers to make ideas accessible to learners (Shulman, 1987 & McNeil et al., 2016). Teaching is not a matter of knowing how to answer some questions**.** It is even more than mere transmitting of concepts and ideas to learners. Rather, it involves bringing out the accumulated ideas and experiences those learners come to class with and working on those ideas and experiences together with the learners by way of refining, reorganizing, co-constructing and repairing these ideas and experiences into meaningful and compressible form for learners to assimilate (Shulman, 2000). This may be the foundation on which teaching of mathematics through problem solving focuses. According to Shulman (2000), teaching is about making the internal and external capabilities of an individual and can only be achieved if teachers engage learners actively. Hill et al. (2008) advocates that teachers can engage learners actively if their mathematical knowledge for teaching (MKT) and mathematical quality instructions (MQI) are up to scratch. It is obvious that both subject matter content knowledge and pedagogical content knowledge play critical roles for effective teaching and learning.

2.2.1 Subject matter content knowledge (SMCK)

Shulman (1986) defined subject matter content knowledge as the amount and organization of knowledge intrinsically in the mind of the teacher. He argues that teachers' subject matter content knowledge should not be limited to knowledge of facts and procedures; but also be an understanding of both the substantive and syntactic structures of the subject matter.

The substantive structures are the various ways in which the basic concepts and principles of the discipline are organized to incorporate its facts. Teachers will therefore be able to use appropriate teaching materials to teach mathematics well only when they comprehend the network of fundamental concepts and principles of problem solving in a holistic manner (Shulman, 1986). The syntactic structure of a discipline is also the set of ways in which truth or falsehood, validity or invalidity, are established (Shulman, 1986). The syntactic structure is used to establish the most appropriate claims about a particular phenomenon. Teachers' knowledge must therefore go beyond mere definitions of accepted truths in the subject matter domain.

In summary, to provide for effective teaching and learning of mathematics, mathematics teachers' content knowledge of concepts cannot be underplayed (Brophy and Alleman, 1991). One may ask how can a teacher unpack subject matter with ease if the content knowledge of the teacher is weak? The researcher believes that teachers' knowledge of mathematics is essential to their ability to teach effectively. Where the teachers' knowledge is rich and rational, better connected and more integrated, they will tend to teach the subject more effectively, represent it in more varied ways, and motivate and respond fully to learners' questions promptly (Walle, 2007). Where their knowledge is limited, they will likely tend to depend on the textbook for content and in general portray the subject as a collection of static, factual knowledge. This suggests that the teacher uses mainly non-problem solving questions, and often selects only what he/she thinks he/she can teach.

2.2.2 Pedagogical content knowledge (PCK)

Shulman defined pedagogical knowledge as the know-how of teaching that the teacher needs to have in order to effectively teach the content. This knowledge alone, though important, may not suffice for teaching purposes. The researcher particularly considered pedagogical content knowledge which is essential for this study. Lee and Luft (2008:1345) defined PCK as experiential knowledge and skills acquired through classroom experience. PCK includes knowledge of how learners learn specific content, the misconceptions and errors associated with certain concepts as well as the assessment tasks, remedial activities and enrichment tasks needed to stretch the mathematics learners (Luneta, 2015). According to Shulman (1986), pedagogical content knowledge is knowledge about how to combine pedagogy and content effectively. It involves knowing what approaches fit the content, and knowing how to plan content to suit better teaching. It also involves knowledge of teaching strategies that incorporate appropriate conceptual representations to address learner difficulties and misconceptions and to foster meaningful understanding, knowledge of what the learners bring to the classroom and knowledge that might be either appropriate or dysfunctional for the particular learning task at hand. Shulman (1986) lays emphasis on the pedagogical content knowledge as the combination of the most regular taught topics, the most useful forms of representations of those ideas, the most powerful analogies, examples, illustrations, explanations and demonstrations in the art of teaching. Pedagogical content knowledge also includes the ways of representing the subject matter that makes it comprehensible to learners with diverse views and understandings. In teaching mathematics from an activity oriented base and through problem solving techniques, teachers need to design and present the lesson using appropriate learner teacher support materials (LTSM) that can enable the learners to construct their own knowledge of the concept. Teachers need to know the pedagogical strategies and techniques most appropriate for reorganizing the understanding of learners who might appear before them as blank slates (Shulman, 2000).

Much of the current research has demonstrated that knowledge is a powerful force in learning and instruction, and it is also pervasive, individualistic, and modifiable (Alexander, 1996). Stengel (1997) maintains that to get at pedagogical knowing conceptually as well as substantively requires focusing on teachers. It is in teaching that knowing resides, and it is revealed through it. Since effective teaching is in large part defined in relation to learners' learning, one cannot see knowledge until the learning is apparent. Thus, in order to define, situate, and understand pedagogical content knowledge, teachers and their learners must be observed to determine if learning is apparent through how teachers teach. Hill et al. (2008:431) describes how teachers' knowledge identifies mathematical knowledge for teaching (MKT) and mathematical quality instruction (MQI) to impact on instruction. The authors explained MKT as something mathematics teachers need more than an individual working in diverse professions. They also need the subject matter knowledge that supports teaching, for example, why and how specific mathematical procedures work, how best to define a mathematical term for a particular grade level, and the types of errors learners are likely to make with a particular content. The MQI is also explained as a composite of several dimensions that characterize the rigor and richness of the mathematics of the lesson, including the presence or absence of mathematical errors, mathematical explanation and justification, mathematical representation, and related observables. Speer et al. (2015) assert that teachers with affordance differ in the richness of the mathematics available to their learners. This may imply that a mathematics teacher may have a strong academic background and yet struggle to unpack content with ease to his/her learners. Rather, a good MKT and MQI will enable the teacher to make an impact on the learners. Hill et al. (2008:457) emphasized that teachers with good MKT provide better instruction by avoiding mathematical errors and missteps. They deploy their mathematical knowledge to support more rigorous explanations and reasoning, better analysis and use of learner mathematical ideas and simply more mathematics overall.

In view of above, it is likely that teachers might be very knowledgeable about their content area such as mathematics but may somehow lack MQI to assist learners to comprehend. This is the very aspect Shulman considers critical to teachers' knowledge, where one moves from personal comprehension to preparing for the comprehension of others. According to Ball (2000) a teacher cannot hope to explain a mathematical concept if he does not have full comprehension of that mathematical concept. Only when the mathematics teacher understands something well enough is he able to teach others. He needs to overcome the various obstacles that might otherwise deny his learners access to knowledge. Studies have also shown that novice teachers often struggle to represent concepts in an understandable manner to their learners because they have little or no MKT at their disposal (Kagan, 1992; & Reynolds, 1992).

Having discussed in some detail the essence of subject matter content knowledge and pedagogical content knowledge that mathematics teachers need to equip themselves with to have an impact on their learners, the next section dwells on learning theories and how best teachers can use MKT and MQI for effective teaching and learning.

2.3 LEARNING THEORY: CONSTRUCTIVISM

A theory is a scientifically acceptable set of principles offered to explain a phenomenon (Schunk, 2012). Theories provide frameworks for interpreting environmental observations and serve as bridges between research and education (Suppes, 1974). Although learning theorists, researchers and practitioners do not have one universally acceptable definition for learning (Shuell, 1986), most of these experts accept the definition below because it captures the criteria central to learning. According to Schunk (2012:16), learning is:

"an enduring change in behaviour, or in the capacity to behave in a given fashion, which results from practice or other forms of experience".

Learning theories have evolved over the years, from learning as the attainment of learned behaviour by way of conditioning (Watson, 1997), through learning as a cognitive process that goes beyond environmental influences (Baroody & Ginbury, 1990), to learning as the construction of knowledge through active involvement and social interaction and based upon past experiences and current knowledge (Ernest, 2010). Several learning theories exist; examples include behaviourism, cognitivism, constructivism and humanism. This study used constructivism learning theory to explain how learners conceive and misconceive concepts. Constructivism was adapted for this study because it has been empirically validated in several studies to explain misconceptions (Luneta, 2008; Olivier, 1989; Ojose, 2015; Egodawatte, 2011; Smith et al., 1994; Drews, 2008). I believe in constructivism because I view an individual as a knowledge constructor. Constructivists believe that mathematics does not grow through a number of unquestionable established theorems, but through the constant improvement of guesses by speculation and criticism (Fletcher, 2005). Considering constructivism's solid theoretically grounds and the consistent empirical support presented in early research studies, constructivism learning theory is suitable for this study.

A constructivist perspective on learning (Piaget, 1970; Skemp, 1978) assumes that concepts are not taken from experience as suggested by behaviourists, rather learners construct knowledge with their own life experiences. Constructivists hold the view that all learning involves the interpretation of phenomena, situations and events, including classroom instruction from the perspective of the learner's existing knowledge. Constructivism emphasizes the roles of prior knowledge in learning. Therefore, knowledge does not simply rise from experience. Rather, it rises from the interaction between experience and our current knowledge structures. Olivier (1989: 2) affirms this notion by stating that instruction affects what learners learn, but it does not determine it, because the learner is an active participant in the construction of her own knowledge. This construction activity involves the interaction of a learner's existing ideas and new ideas. Learners do not only interpret knowledge, but they also organize and structure this knowledge into units of interrelated concepts. Piaget (1970) described such a unit of interrelated ideas in the mind of a learner as schema. Learners make use of these schemas by retrieving and utilizing acquired ideas. Learning then basically involves the interaction between a learner's schemas and new ideas. The interaction involves two interrelated processes:

- Assimilation: If some new, but recognizably familiar, idea is encountered, this new idea can be incorporated directly into an existing schema that is very much like the new idea. In this process the new idea contributes to our schemas by expanding existing concepts, and by forming new distinctions through differentiation.
- Accommodation: Sometimes a new idea may be quite different from existing schemas; we may have a schema which is relevant, but not adequate to assimilate the new idea. Then it is necessary to re-construct and re-organize our schema. The re-construction leaves previous knowledge intact, as part or subset or special case of the new modified schema.

Thus, conception takes place when an idea is incorporated into an appropriate existing schema. However, sometimes some new ideas may be so different from any available schema to the extent that it makes it impossible to link them to any existing schema. For instance, a grade 2 learner given algebraic fraction problems to solve will likely remain in a state of disequilibrium only briefly realizing very soon that he or she hasn't got the foggiest idea of what these symbols mean. Assimilation or accommodation becomes impossible under such circumstances. In such cases, the learner memorizes the idea. This is rote learning because it is not linked to any previous knowledge. It is not understood; it is isolated knowledge therefore difficult to remember. Such rote learning is probably the cause of many mistakes in mathematics as learners try to recall partially remembered and distorted rules (Walle, 2007).

To the constructivist, learning leads to changes in learners' schemas. From a constructivist perspective, misconceptions are vital to learning and teaching, because misconceptions form part of learners' conceptual structure that will interact with new concepts and influence new learning mostly in a negative way because misconceptions generate errors (Ada & Kurtulus, 2010; Idris, 2011; Li, 2006).

There are three types of constructivism that are applicable to mathematics education. They are discussed below.

2.3.1 Radical constructivism

The term 'radical constructivism' was proposed by Von Glasersfeld. In his view, radical constructivism is a theory of knowing rather than a theory of knowledge because it avoids the usual connection between knowledge and the real world (Von Glasersfeld, 1998:95). Constructivism emphasizes the value of knowledge that is not universal but individual, personal and subjective. Hence, radical constructivists posit that reality resides in the mind of each person (Raskin, 2002; Von Glasersfeld, 1998). An individual makes sense of events according to his or her own experiences, beliefs, and knowledge (ibid). In other words, learning takes place when individuals make use of their existing knowledge and experience to make sense of new material. Learning materials should therefore be structured around problems, questions and situations that may not have one correct answer. This is in affirmation with the radical constructivist view that says that knowledge is always a result of a constructive activity. It cannot be transferred to a passive respondent. It should be actively constructed by each individual learner.

2.3.2 Social constructivism

Social constructivists assert that knowledge is individually constructed and socially mediated. Thus, by participating in a broad range of activities with others, learners tend to appropriate (internalize) the outcomes produced by working together: these outcomes can include new strategies and knowledge (Kim, 2001; Efran et al, 2014; Duit & Treagust, 1998).

Social constructivism is based on three specific assumptions reality, knowledge, and learning. To understand and apply models of instruction that are rooted in the social constructivists, it is important to know the assumptions that underlie them.

Reality: Social constructivists believe that reality is constructed through human activity. Members of a society together invent the properties of the world (Kukla, 2000). For the social constructivist, reality cannot be discovered: it does not exist prior to its social invention. **Knowledge:** To social constructivists, knowledge is also a human product, and is socially and culturally constructed (Ernest, 1999). Individuals create meaning through their interactions with each other and with the environment they live in.

Learning: Social constructivists view learning as a social process. It does not take place only within an individual, nor is it a passive development of behaviours that are shaped by external forces (McMahon, 1997). Meaningful learning occurs when individuals are engaged in social activities.

Group activities given to learners encourage interaction through communication and help to extend each learner's understanding of the concept concerned. The nature of the learner's social interaction with knowledgeable members of the society is important. Without the social interaction with more knowledgeable others it is impossible to acquire social meaning of important symbol systems and learn how to use them. Learners develop their thinking abilities by interacting with more experienced individuals.

Reality is constructed through human activity. Members of a society together invent the properties of the world (Kim, 2001). People create meaning through their interactions with each other and the objects in the environment. Learning is a social process. It occurs when people are engaged in social activities (1999).

A group of learners are given some algebraic fraction problems to work through. By using the different perspectives, they have gained from their different backgrounds, they can help each other solve the problems more effectively than if they had worked alone.

2.3.3 Sociocultural learning theory

Vygotsky's sociocultural learning theory is an extension of social constructivism. It describes learning as a social process and the origination of human intelligence in society or culture (Warschauer, 1997; Lantolf, 2000; Vygotsky, 1978). Sociocultural theory emphasizes the role in the development of co-operative dialogues between learners and more knowledgeable members of society. Learners learn the culture of the community (ways of thinking and behaving) through these interactions (Vygotsky, 1978).

Darling-Hammond et al. (2001) and Luneta et al. (2013) posit that teaching strategies and materials organized by teachers to enable learning as well as promoting equity learning are informed by the teachers' conceptual framework on learning theories. When it comes to how learners acquire knowledge, learning theorists suggest different theories. Early theorists assert that knowledge acquisition is based on cognitive processes and is independent from the environment. Conversely, contemporary learning theory such as sociocultural learning advocates that learning cannot be achieved independently from the social and cultural aspects that influence the learners, as well as the sociocultural interactions which are vital to learning (Vygotsky, 1978; Lantolf et al., 2000). This is necessary because learners use semiotic, cultural and psychological tools to learn and manipulate activities.

Efforts to move learners from A to Z to enable them acquire new skills and knowledge are informed by several learning theories (Darling-Hammond et al., 2001). Jean Piaget, a psychologist, proposed cognitive constructivism theory, which states that learning is a developmental cognitive process in that learning becomes learner-centred and therefore learners are exposed to construct their own knowledge. Lev Vygotsky (1978) extended this theory and stated rather that learning involves social interaction. Thus if "more knowledgeable other "work with a struggling learner at a particular point in time, this struggling learner will be able work alone without any help. Kozulin, one of the followers of Vygotsky, also described sociocultural learning as a cognitive development which includes high order learning rooted in social interactions and mediated by psychological tools (Kozulin, 1998).

Current trends in education support sociocultural learning theory as a framework for teaching and learning, most especially when it comes to mathematics education. Discussion on the tenets underpinning sociocultural theory such as mediation, higher mental order, zone of proximal development (ZPD), psychological tools and others will be more valuable in this discourse.

According to Vygotsky (1998), the use of psychological tools in teaching and learning promote higher mental functioning. It takes the form of social interaction through conversation thus making use of language. Language is an example of a psychological tool. Other semiotic tools used for meaning making include problem solving tools, signs, symbols, mental models, etc. The use of these tools during dialogue or conversation helps learners to make sense of concepts and internalize knowledge (Kozulin, 1998; Warschauer, 1997).

Another key principle of sociocultural theory is development of higher functions as stipulated in the definition of this construct. Vygotsky assert that learners possess lower mental functions such as elementary perception, attention and will which they use to do new things. As these children interact with their environment and make use of psychological tools, gradually they develop higher mental functions and abilities such as abstract reasoning, logical memory, planning, decision making, etc. (Kozulin, 1998).

Zone of proximal development (ZPD) is a well-known tenet of sociocultural theory. Vygotsky observed that there is a distance between a person's ability to solve problems independently and the ability to solve problems with the help of others, which he referred as ZPD (Darling-Hammond et al., 2001). This Vygotsky elaborated as a necessity in learning because sometimes learners get stuck and need help from others. This could be the teacher, peer or the parent. Such interactions help develop the person’s mental abilities leading to an independent learner (de Valenzuela, 2006).

Scaffolding learners to be autonomous is also a principle that sociocultural theorists’ advocates. This process occurs within the ZPD. Here the experienced or the one who had mastered the concept already supports the learner or breaks the task into simpler parts and provides clues such as gestures, eye gazes, unfinished solutions, etc. and encourage the learner to finish it alone. This support is then gradually withdrawn until the learner becomes autonomous (de Valenzuela, 2006). Lantolf (2000) associated the concept of scaffolding in learning with structures/platforms used when building houses - these structures aid the builders to build. As the building nears completion, gradually these supporting boards are taken away.

Vygotsky also asserts that all forms of learning are mediated (Kozulin, 1998). This higher mental order is only achieved when learning has been mediated. According to Kozulin mediated learning is when a teacher or a peer or textbook or language is placed between the learner and what is to be learnt (concept). The mediator’s role is to enable the learner to learn. As the learner develops higher mental functions, he is motivated and uses acquired conceptual knowledge in diverse ways. The goal of mediated learning is to enable learners to become motivated, effective and independent learners.

Sociocultural theorists posit also that communication and language play vital roles in learning. This is because teachers use communications and language as a mediation tool to unpack and justify a concept. According to Hiebert and Lefevre (1986), “knowledge is concept understood”. Walle (2007) also referred to this as knowledge rich in relationship and understanding. Hiebert et al. (1996) assert that the major ingredient in developing understanding is communication. “Communication involves talking, listening, writing, this involves oneself in social interaction by sharing thoughts and listening to other views” (Hiebert, 1996; Vygotsky, 1978). This helps in building meaning of ideas. Hence, sociocultural theorists view language and communication as central to learning. (Vygotsky, 1978; Luneta, 2015).

Dynamic assessment according to Vygotsky’s sociocultural theory, teaching and assessment form an integral part of overall learning experience (de Valenzuela, 2006; Warschauer, 1997). Thus, teachers should not use summative assessment only as a means of judging achievement but also include formative assessment such as group work activities, homework, etc. Vygotsky further explain that what a learner can do with the help of others - teacher or peer - is indicative of the learner’s potential for achievement without assistance. Vygotsky disbands the idea of assessing learners' actual development by means of summative assessment being used to decide on the learners' potential development and argued that responsiveness to help is an indication of cognitive ability or learners' potential development. Sociocultural theories also advocate use of dynamic assessment as opposed to traditional assessment methods (Shepard, 2000; Kozulin, 2003; Lantolf et al. 2000). Vygotsky (1998) defined dynamic assessment as assessment that identifies cognitive processes of learners that are fully developed and those that are in the state of being developed at the time of the assessment. This model of assessment consists of pre-test, mediation, a brief period of revision at home and post-test (Shepard, 2000). Kozulin (1998), argued that dynamic assessment enables the teacher to design optimal educational intervention for each learner, which also leads to improvement in the learning attainment of learners. For instance, to teach a mathematical concept such as algebraic fractions, structured questions regarding the real world which require use of algebraic fractions are given to the learners to determine their prior knowledge on the topic. The second aspect of dynamic assessment deals with mediation. This involves looking for clues from the pre-test for how best the concept should be taught, how the learner responded to the context, the language and appropriateness of the activities. This enables the teacher to give an informed lesson, sometimes with homework activities to complement the lesson. The last stage is the post-test; the context of the post-test activities should not be different from the pre-test. This form of assessment enables learners to know their improvement rate, since they are not competing against others but against themselves. Hence, sociocultural theorists advocate that dynamic assessment gives a better form of assessment than traditional forms of assessment (Kozulin, 1998).

Sociocultural Theory And Mathematics Education

Figure 2: Sociocultural theory and mathematics education. Source: Author

2.3.4 Criticism of constructivism learning theory

Other schools of thought argue that if everyone had a different experiential world, no one could agree on any knowledge. They argue that constructivism is a stance that denies reality (Kilpatrick, 1987). In contrast, constructivists argue that agreement on social and scientific issues does not prove that what we experience has objective reality. The models that we construct about something are our own constructs that are accessible to us (von Glaserfeld, 1991). A similar criticism is that if everyone is able to construct their own knowledge, then everyone's constructs must be equally valid. Constructivists view these notions differently and posit that the constructive process does not happen in isolation, but is subject to social influences. The constructs of knowledge are both social and individual (Kitchener, 1986). Finally, both individual as well as social constructions of knowledge are important if we are to think of a combined and complete notion of constructivism (Shepard, 2000; Kozulin, 2003; Lantolf et al., 2000).

2.3.5 Classroom implications of constructivism to be teacher

In constructivism, teachers and learners are viewed as active meaning makers who continually give contextually based meanings to each other's words and actions as they interact (Kozulin, 2003; Schunk, 2012; Raskin, 2002). From this perspective, mathematical structures are not perceived, intuited or taken in but are constructed by reflectively abstracting from and re-organising sensorimotor and conceptual activity (Piaget, 1970). Thus, the mathematical structures that the teacher 'sees' are considered to be the product of his or her own conceptual activity and could be different from those of the learners (Von Glaserfeld, 1989). Consequently, the teacher cannot be said to be a transmitter of such structures nor can he or she build any structures for learners. The teacher's role here is viewed as that of a facilitator in the learning process. Indeed, if learners are to be empowered and given greater control over their own lives, then, as Fletcher (1997) points out, they should be encouraged to choose their own areas of study in mathematics and should also be encouraged to work in groups to solve mathematical problems. It means encouraging learners to use active techniques to create more knowledge and then to reflect on and talk about what they are doing and how their understanding is changing. The teacher makes sure he/she understands the learners' prior conceptions, and guides the activity to address them and then build on them (Kozulin, 2003; Lantolf et al., 2000). The constructivist teacher encourages learners to often assess how the class in the classroom, the teacher's view of learning must point to a number of different teaching practices. activities are helping them gain understanding. By questioning themselves and their strategies, learners in the constructivist classroom ideally become "expert learners" as described by Piaget. This gives them ever broadening tools to keep learning. With a well-planned classroom environment, the learners learn how to learn. When they continuously reflect on their experiences, learners find their ideas gaining in complexity and power, and they develop increasingly strong abilities to integrate new information. The teacher's main role is to facilitate and encourage these learning and reflection processes (Kozulin, 2003).

In summary, constructivism is an epistemology, or philosophical explanation about the nature of learning (Terhart, 2003). Constructivists believe that knowledge is not imposed from outside people but rather formed inside them (Kivinen & Ristelä, 2003). Constructivist theories differ from those that claim complete self-construction through to those that hypothesize socially mediated constructions, to those that argue that constructions match reality. Constructivism requires that we structure teaching and learning experiences to challenge learners' thinking so that they will be able to construct new knowledge. A core premise is that cognitive processes are situated (located) within physical and social contexts (Resnick, 1996; Lave & Wenger, 1991). Vygotsky's sociocultural theory emphasizes the social environment as a facilitator of development and learning. The social environment influences cognition through its tools - cultural objects, language, symbols, and social institutions. Cognitive change results from using these tools in social interactions and from internalizing and transforming these interactions. A key concept is the zone of proximal development (ZPD), which represents the amount of learning possible by a learner given proper instructional conditions.

Vygotsky's theory contends that learning is a socially mediated process. Learners learn many concepts during social interactions with others. Structuring learning environments to promote these interactions facilitates learning. Vygotsky believed that language and the zone of proximal development are critical for the development of self-regulation. A key is the internalization of self-regulatory processes.

Constructivists believe that teachers convey their expectations to learners in many ways. Teachers' expectations influence teacher–learner interactions, and some research shows that, under certain conditions, expectations may affect learner achievement. Teachers should expect all learners to succeed and provide support (scaffolding) for them to do so.

The goal of constructivist learning environments is to provide rich experiences that encourage learners to learn. Constructivist classrooms teach big concepts using much learner activity, social interaction, and authentic assessments. Learners' ideas are enthusiastically sought and, compared with traditional classes, there is less emphasis on superficial learning and more emphasis on deeper understanding.

Some instructional methods that fit well with constructivism are discovery learning, inquiry teaching, peer-assisted learning, discussions and debates, and reflective teaching.

2.4 THE NATURE OF MATHEMATICS

Mathematics is a key subject in South Africa and other countries as stated earlier. In the national Curriculum Statement (NCS R-12) document, often referred to as the CAPS document, mathematics is defined as language that makes use of symbols and notations for describing numerical, geometric and graphical relationships. It is a human activity that involves observing, representing and investigating patterns and qualitative relationships in physical and social phenomena and between mathematical objects themselves. It helps to develop mental processes that enhance logical and critical thinking, accuracy and problem solving that will contribute in decision-making (DBE, 2011).

The CAPS document for FET Mathematics describes mathematics as very inclusive and highlights ten important mathematical contents such as Functions; Number patterns, sequences and series; Finance, growth and decay; Algebra; Differential calculus; Probability trigonometry; Euclidean geometry; Analytical geometry and statistics (Education, 2015).

All the ten topics afore-mentioned reveal hidden patterns that help us understand the world around us (Parker, 2006). The understanding of the nature of mathematics knowledge contributes to a deep understanding of learning difficulties of mathematics. It is not only a list of facts and techniques which learners memorise but is made up of a number of processes which together form a way of thinking (Booker et al., 2014; Gynnild et al., 2007). Views held on the nature of mathematics according to Mereku (2004) can be described in terms of the elements or constituents of knowledge embodied in the subject. Mereku expounded that the constituents of mathematical knowledge or things that must be learned to possess mathematical knowledge are usually expressed by rules, definitions, methods and conventions (Parker, 2006). These constituents also have theoretical, communicative and methodological implications (Gynnild et al., 2007). Learners find it difficult to understand abstract concepts. They arise from the nature of mathematics, its symbolism and language (Siemon et al., 2013:109; Booker et al., 2014; Mereku, 2004). The mathematical concepts are very many and are represented using mathematical symbols, which by their nature are very abstract right from functions to statistics. Even the mathematical concepts learners are taught in the primary school are far removed from reality (Siemon et al., 2013). Similarly, mathematical symbols are seldom experienced in real life situations that have meaning to learners. If they experience them at all, they have no real value to them until they start with symbolic work in school; hence this abstract nature and structure of mathematics make abstraction, generalisation, deduction and recall of concepts and principles difficult for learners to comprehend (Li, 2006; Barcellos, 2005; Siemon et al., 2013).

Furthermore, mathematics offers distinctive modes of thought which are both a versatile and powerful survival tool which includes modelling, abstraction, optimization, logical analysis, inference from data and use of symbols. Our exposure and experiences with mathematical modes of thought enable us to create a "mathematical reservoir" - a capacity that enables us to read critically, identify misconceptions, assess risk and even suggest alternative solutions (Mereku, 2004). Mathematics empowers us to be critical thinkers and understand better the information-laden world in which we live now (Savery, 2015).

Algebraic fractions is part of algebra taught in high school. Algebra, trigonometry and probability are some of high school mathematics topics taught. These topics provide a firm knowledge base to the teachers in training in such a way that they can contextualize their teachings to a young audience's cognitive level and consolidate pre-existing mathematical constructs into solid cognitive structures (conceptual knowledge).

2.4.1 Fraction concepts

A fraction can be defined as part of a whole. It is made up of the top part referred to as the numerator and the bottom part referred to as the denominator. Fractions are taught in the early stages of learning and form an integral part of college or university mathematical course. They are applied in a range of topics in mathematics including algebra. Many examination questions involve direct or indirect computation using fractions. Fractions is one of the mathematics topics that many learners often struggle with (Brown & Quinn, 2006; Idris, 2011; Lamon, 1999; Themane, 2014; Ball, 1993). The word 'fraction' is taken from the Latin word '*Frangere*' which means '*to break*' (Downes & Paling, 1965). This suggests that a fraction may be described as a part of a whole where the whole could be a unit or a set of objects. Idris (2011) pointed out the importance to realise that the pairs of numbers and the phrases *'one fourth', 'two thirds',* etc. are not fractions but merely symbols and words representing the concept of fractions. He further stated that to understand what a fraction is we must first look at how they arise. 'A *half*' is what we get when we share something equally into two parts. He noted that what '*a half*' is subject to what we started with. This suggests that '*a half*' of an apple may not be the same as '*a half*' of a banana. It is not possible therefore to show any single object and say '*this is what a half* *is*'. Figure 4 illustrates diagrams that represents a half.

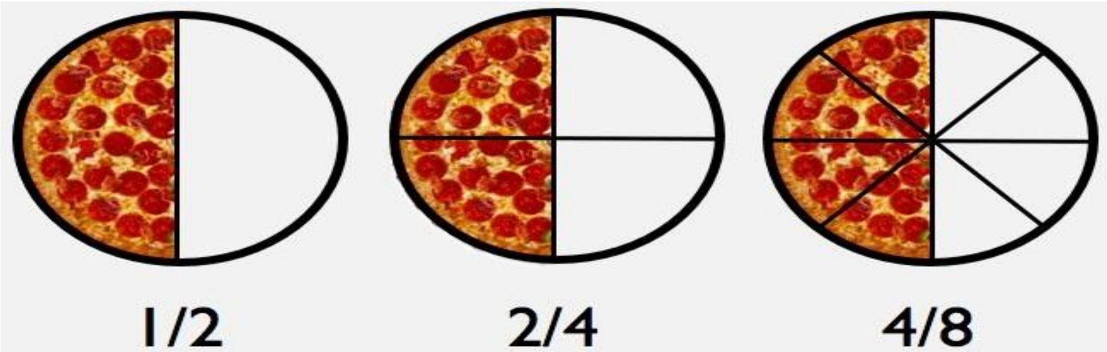


Figure 3: Graphical representation of fractions. Source: Author

From the above discussions, fractions are not objects but actions (Charalambous & Pitta-Pantazi, 2007). It is only when we learn to represent these actions that these symbols can be treated as objects. However, Hilton and Pedersen (1983) said the words 'a half', 'a quarter', 'a third', and 'three-quarter' are used frequently in everyday speech and their meaning is clear to the reader, suggesting that one can say 'a fourth' instead of 'a quarter'. The word 'half' can be used in trickily different ways. For instance, would you like a piece of cake? We may reply 'please just a half'. The host or hostess may then cut the piece of cake into two equal pieces and give him one of those pieces. He has received 'a half' of the original piece of cake. On the other hand, an estate agent showing us two possible lots for purchase may say 'plot X is more attractively situated but there is only half as much land as on plot Y'. This does not imply that plot X has been created by cutting up lot Y; the estate agent means only that the amount of land on X is the same as the amount of land we would get by taking half of lot Y. This also means that when we say that 10 cents is a tenth of R1.00, we certainly do not mean that 10 cents is obtained by cutting or breaking R1.00 into ten equal pieces and taking one of the pieces; rather, we mean that 10 cents is worth a tenth of R1.00, that we would require ten 10 cents to purchase what we can purchase for R1.00.

Fractions, decimals and percentages are number ideas that are not whole numbers. These three concepts are closely related to each other for the fact that one can move from one domain to the other (Walle et al., 2012). In line with this point, the teacher must help the learner to see these relations and how to move from one form to the other (Usiskin, 2007). In developing the concept of fractions, the teacher must be able to utilise real life situation activities (Walle, 2007:317).

According to Charalambous and Pitta-Pantazi (2007) and Ross and Bruce (2009), research in mathematics education has identified the multifaceted nature of fractions as a contributing factor to the core difficulties experienced by teachers and learners during teaching and learning of fractions and related concepts. This multifaceted construct is made up of five interrelated sub-constructs as follows: part/whole (which explains the idea of partitioning an object or set into smaller sections); ratio (explains procedures associated with finding the equivalent fractions); operator (explains approaches to develop multiplication operations of fractions); measure (also refers to idea that fractions are identified by their size) and finally quotient (which also explains that any fraction can be represented as a division). Learning of fractions is difficult probably because it requires a deep understanding of all the above constructs.

Van de Walle (2007) also asserts that fraction concepts are well identified as an area of difficulty for many learners. There are two possible obstacles to understanding of fraction concepts:

- Fractions cannot be thought of as separate, independent entities. They have meaning only in relation to the whole to which they apply. To recognise a fraction of anything, you need a concept of the whole. It is relatively easy to imagine the whole apple of which you have a quarter, but it is not easy to imagine the whole kilogram of which you have a quarter, or the whole hour of which a quarter has passed.
- Complicated notations by which fractions are symbolized. The numeral at the bottom of a fraction (denominator) has an entirely different function from the numeral at the top (numerator). For instance, the denominator of the fraction tells us that the 'whole' has been divided into three equal shares. The numerator tells us that two of those shares are under consideration. The word 'denominator' means 'the thing that names'. The denominator of the fraction gives the fraction its name 'third'. The word 'numerator' means 'the thing that numbers'. Hence the numerator of the fraction tells us the number of thirds to be considered. The denominator and numerator for fractions also make it possible to denote the same fraction in infinitely many ways, for instance the concept of equivalent fraction. This idea takes a long time to sink in, and can prove another obstacle to understanding. To overcome the first obstacle, we should always in the early stages refer to the whole to which any fraction applies. We should not talk about a 'quarter' but 'a quarter of an apple' or 'a quarter of a metre' or 'a quarter of twelve', etc.

2.4.2 Addition and subtraction of fractions

Dolan (2000) observes that apart from whole number computations, no topic in elementary mathematics curriculum demands more time than the study of fractions. According to Dolan, for learners to understand, the teaching about fractions and their operations must be grounded in concrete models, for instance, the use of set, length and area models to build a solid foundation of fraction concepts (Van de Walle, 2007).

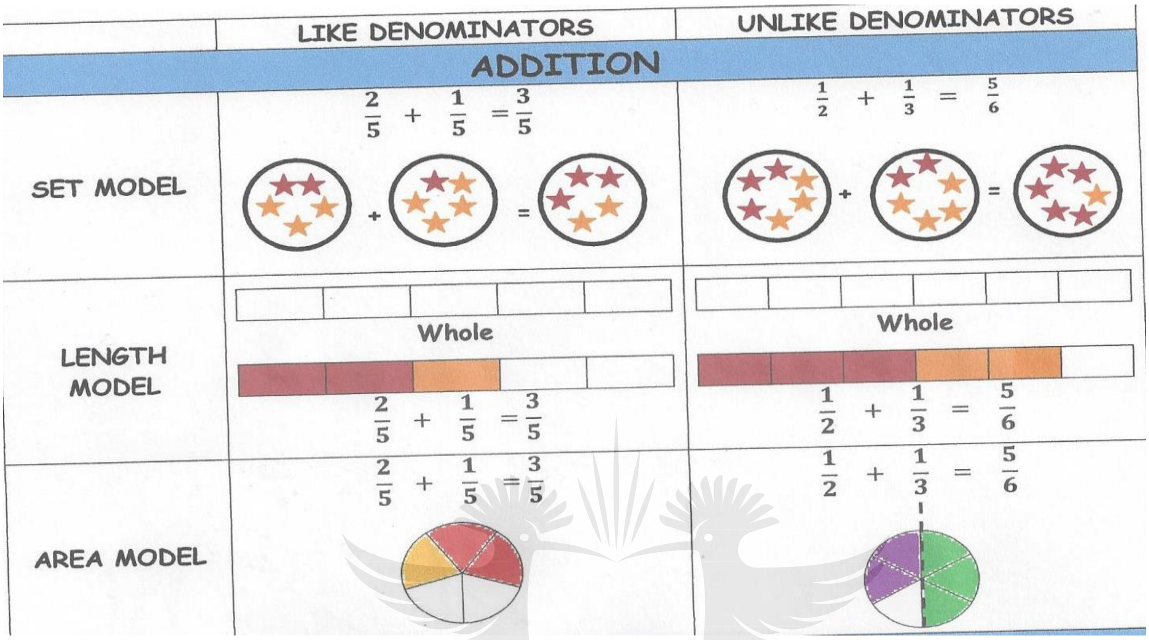


Figure 4. Set, length and area models. Source: Walle (2007:295)

A firm foundation for number sense involving fractions and a deeper understanding of the algorithms for operations must be developed before formal work with fractions. According to Brown and Quinn (2006) and Idris (2011), before learners are introduced to addition and subtraction of fractions, they must be able to rename fractions using their equivalence to confirm their readiness for operations on rational numbers. However, learners often think that whenever two fractions are added, the result is less than 1 (Owusu & Manu, 2007). This is because their exposure to addition of fractions is always less than 1. This means that they need early exposure to problems where the sum is greater than 1 to erase such misconception.

Teaching addition and subtraction of fractions for better understanding, it is expected that we use concrete materials (Van de Walle, 2007:317). However, usually the first step is to learn to add and subtract fractions with the same denominator which is straightforward and activities using concrete materials are easy to devise (Van de Walle, 2007). Van de Walle also suggested that paper folding and shading, number line and Cuisenaire rods could be used to teach addition and subtraction of like fractions.

2.4.3 Multiplication and division of fractions

The product of two fractions is a fraction with the following properties: the numerator is the product of the numerators of the given fractions. The numerator is also the product of the denominators of the given fractions. In general, for any a, b, c and d, when c ≠ 0 and d ≠ 0

$$\frac{a}{c} \times \frac{b}{d} = \frac{ab}{cd}$$

The operation of division may be defined in terms of the multiplicative inverse, the reciprocal. A quotient can be expressed as the dividend times the reciprocal of the divisor. In general, for any numbers a, b, c, and d, when c ≠ 0 and d ≠ 0

$$\frac{a}{c} \div \frac{d}{b} = \frac{a}{c} \times \frac{b}{d} = \frac{ab}{cd}$$

Armstrong and Bezuk (1995) assert that, typically, understanding of these two constructs is procedural in nature; however, Hill et al. (2008) posit that teachers with high MKT and MQI will unpack this concept with ease for their learners to comprehend. Hiebert and Behr (1998) also observed teachers teach fractions often using instruction approaches that do not encourage learners to construct their own knowledge, and instruction rarely provides learners with structured learning experiences to help them acquire conceptual and procedural knowledge.

Hill et al. (2008) opined that teachers with high MKT and MQI will use instructional activities that support the process of linking symbols with concrete materials and verbal interaction. In this vein, Armstrong and Bezuk (1995:86) recommend instruction on operations on fractions be built on actions on objects rather than being based solely on manipulation of symbols according to a set of rules and procedures. This is because it does not connect learners to the real world, thus resulting in difficulty to comprehend concepts.

In the next section, historical background on algebra and a discussion on algebraic fractions are presented.

2.4.4 Historical background of algebra

It is important to learn about algebra before looking at the errors that occur in algebraic fractions. Algebra has been stated to be one of the broadest concepts in mathematics which is more than what people define as a collection of symbols, rules and procedures (French, 2004). It has been defined as a symbolic system (can be identified by symbols), a calculus (can be used to compute numerical solutions) and a representational system (Ablon, 1981).

Algebra is taught in schools to equip the learner with skills used to solve equations. Knowledge learnt in algebra classes can be applied in solving real problems. Algebra is an important branch of mathematics; today it is studied not only in high schools and universities but also in lower phases. Algebra is useful like the other branches of mathematics. Algebra by nature has different types and since the study is about school algebra it will be appropriate to distinguish school algebra from the other types. The knowledge of algebra is useful in other mathematical topics like calculus and geometry. For example, in geometry, algebraic methods are used when solving geometric problems, especially when synthetic techniques (from known to unknown) become cumbersome, e.g., the synthetic proof of the Pythagoras theorem which an average geometry learner usually finds somewhat difficult to follow. Similarly, deductive reasoning of vertical, adjacent, complementary, supplementary angles in geometry also makes use of algebra, likewise, deriving sum of angles, interior and exterior angles. In analytical geometry, algebraic concepts are also applied. Again, in school algebra substitution plays an important role in problems involving calculus, sequences and series and nature of roots. Factorisation of algebraic expressions is also of importance in school algebra such as in solving of quadratic equations. In school algebra, the knowledge and understanding of algebra contributes a lot to problem solving and drawing of graphs (Bell, 1996).

Algebra is one of the oldest branches of mathematics. There is historical evidence that the Babylonians were versed in its methods 4000 years ago. In 2000 B.C. the Babylonians used algebraic methods in solving problems. However, they used no mathematical symbols other than primitive numerals. This lack of symbolism in algebra continued for many centuries. Gradually, some of the more common words used in mathematics were abbreviated, which led to a syncopated algebra. Symbolic algebra, however, did not begin to emerge until 1500 A.D. One person who can be credited with the early development of symbolic algebra is the French mathematician Viete (1540-1603) in about 1600 A.D. (Van Reeuwijk, 1995:144). Classical algebra was introduced in about 830 A.D. by al-Khwarizmi in the Middle East. The name algebra comes from the Arabic al-jabr, which was the title of an algebra text written by al-Khwarizmi. It was presented as a list of rules and procedures needed to solve specific linear and quadratic equations. Until the end of the eighteenth century, algebra could roughly be described as the branch of mathematics which dealt with the solution of equations. The nineteenth century marks the beginning of modern algebra. Modern algebra, in addition to its concern with solving equations, supplies the language and patterns of reasoning used in other branches of mathematics.

Problems involving quadratic equations were solved more than 3000 years ago. Such problems and their solutions have been interpreted from ancient Egyptian and Babylonian tablets. Although a general method is not given, the solutions involved completion of squares. Wheeler (1996) accounts for the long period of time it took to develop algebra. He states that the full development took at least 1000 years. Algebra was seen as a completion of arithmetic. Arithmetic, he notes, appears to need the real numbers for its trouble-free functioning, and these could not be fully developed without the aid of algebra. Bell (1996), however states, that there is a multiplicity of algebras, not just one. This confirms that algebra is not restricted to the world of numbers and may therefore not be inextricably tied to arithmetic.

An understanding of the fundamental concepts of algebra and of how these concepts may be applied is necessary in most technical careers. In the 18th century there existed two substantially different, but mutually supplementary concepts of algebra. One of these considered algebras to be a science of equations and their solutions; the other a science of quantities in general. The latter concept is the "calculation with letters". There are different types of algebra, namely, modern (abstract) algebra, Boolean algebra, linear algebra and school algebra. This research focuses on high school algebraic fractions.

Modern algebra developed between 1770-1870 and deals with the theory of groups and fields. The concepts of modern algebra have been found to be very useful in other branches of mathematics and social sciences. A chemist may use modern algebra in a study of the structure of crystals; a physicist may use modern algebra in the design of electronic computers, and a mathematician may use modern algebra in the study of logic. Boolean algebra is a branch of algebra named after George Boole (1815-64). It combines algebraic methods and logic. The basic principles of Boolean algebra relate to logic. Knowledge of Boolean algebra is very useful in fields requiring the application of mathematics and logic. Electronic computer programming and the construction of electronic circuits are examples of such fields (Gillian, 2016).

Linear algebra is the branch of mathematics concerned with the study of vectors, vector space or (linear space), linear transformation, and surface linear equations. Linear algebra is the earlier of the two mathematical disciplines devoted to the study of that broad and useful notion called linearity: the study of lines and planes in analytic geometry and the system of linear algebraic equations. Linear algebra has a concrete representation in analytic geometry. The history of modern linear algebra dates back to the years 1843 and 1844. Most mathematical problems encountered in scientific or industrial applications involve solving a linear system. For example, systems of linear equations and reduction of matrices to standard forms are applications that belong to everyone. Linear systems arise in applications to areas like business, economics, sociology, ecology, genetics, electronics, engineering and physics.

2.4.5 What is algebra in high school?

The word 'algebra', as explained by Mason (1996:73), is "derived from the problems of *al-jabr* (literally, adding or multiplying both sides of an equation by the same thing in order to eliminate negative/fractional terms), which were paralleled by problems of *al-muqabala* (subtracting the same thing from or dividing the same thing into both sides)". From this explanation, it can be realized that the definition of algebra reflected a limited view of the subject, restricted only to the process of solving equations. It is acknowledged that the meaning of algebra has developed and broadened from algebra as a process to object (an algebra). In an attempt to define algebra, Wheeler (1996:319) describes algebra as a symbolic system (its presence is recognized by symbols), a calculus (its use in computing numerical solutions to problems), and also as a representational system (it plays a major role in the mathematization of situations and experiences).

While to some people algebra is merely a collection of symbols, rules and procedures, to mathematicians it is much more than that. According to Kieran (1992:391), algebra is conceived as a branch of mathematics that deals with symbolizing and generalizing numerical relationships and mathematical structures, and with operating within those structures. Algebra is conceived, by some as a study of a language and its syntax, a study of procedures for solving certain classes of problems. In the latter, algebra is not only seen as a tool for problem-solving but also as a tool for expressing generalizations. It is also viewed as the study of regularities governing numerical relations, a conception that centres on generalizations that can be widened by including the components of proof and validation (Bednarz, Kieran & Lee, 1996:4).

Algebra is about identifying patterns and generalizing those patterns. Generalizing involves seeing a pattern, expressing it clearly in verbal terms, and then using the symbols to express the pattern in general terms. Sfard, quoted by Bednarz et al. (1996:103), affirms that most authors unanimously agree to the early origins of algebra because they "… spot algebraic thinking wherever an attempt is made to treat computational processes in a somehow general way." One of the most salient features that distinguish algebra from arithmetic is its generality. According to Mason (1996:74), "… generalization is the life-blood, the heart of Mathematics."

2.4.6 Why algebra is taught in schools

Several authors (Collins & Dacey, 2011; Ablon, 1981) have advanced different, though not contradicting views, regarding the goals of algebra instruction. These are cited in Thorpe (1989), and some indicate that algebra is taught so as to develop learners' skills in the solution of equations, that is, finding numbers that meet specific conditions (Siemon et al., 2013:234). Schoenfeld sees the purpose of algebra teaching as "… to teach learners to use symbols to help solve real problems, such as mixture problems, rate problems and so forth" (Schoenfeld, 1986). Algebra is taught in schools to equip learners with knowledge and skills that would enable them to become sufficiently at ease when working with algebraic formulas and so that they can read scientific literature more intelligently (Siemon et al., 2013). According to Flanders (1987) the goal of algebra teaching is "… to prepare learners to follow derivatives in other subjects, for example, in physics and engineering". In addition, French (2002: 3), holds the view that algebra "provides a valuable training in thinking skills and a respect for rigorous argument", and also gives insight into the explanations of a wide range of phenomena in the world.

As indicated earlier, mathematics is a human activity that deals with qualitative and quantitative relationships of space and time. Algebraic concepts, principles and methods provide powerful intellectual tools for representing the quantitative information and reasoning about this information. The main topics covered in algebra include variables, relations, function, equations and inequations (inequalities), and graphs. It is imperative, therefore, that pupils should be able to handle and manipulate the symbolic mathematical language - algebra. Competency in algebra will also enable pupils to cope with learning more advanced mathematics.

According to the CAPS document (2011), typical topics in algebra in the FET phase are:

1. properties of real numbers;
2. algebraic representation and formulae;
3. solution of linear equations and inequalities;
4. indices;

5. algebra of matrices;
6. coordinates, graphs, relations and function notation;
7. sets.

Since this study was concerned with problems that are encountered in algebraic expressions, only those aspects that relate to algebraic expressions were considered. The specific objectives for teaching algebraic expressions are that learners should be able to multiple a binomial by a trinomial, factorising trinomials and factoring by groups in pairs; simplifying, adding and subtracting algebraic fractions with monomial denominators(DBE, 2012:9). The next section will therefore discuss algebraic fractions and the skills that are involved in learning this concept.

2.4.7 Algebraic fractions and operations

Algebraic fractions are expressions that may be thought of as extensions of fractions or rational numbers. It is a mathematical concept often communicated through symbolic mathematical language (Graham & Thomas, 2000). This language uses numbers, letters and other conventional symbols. Kaplan (2007) asserts that algebraic fractions are challenging to learners, probably because they demand understanding of several mathematical concepts such as exponent, factorization, division, variables, equations, perfect squares and rational numbers. Inadequacy or lack of any of the above concepts hinders understanding of algebraic fractions. A good understanding of algebraic fractions at the grade 10 stage should enable the learner to have success in learning of mathematics (Fennell, 2007).

2.4.7.1 Basic rules of algebra

The operations performed on algebraic fractions are similar to those performed on numeral fractions in arithmetic, which are addition, subtraction, multiplication and division.

A. Commutative and associative properties of addition and multiplication

Basically learners should understand these properties in order to work well with algebraic fractions. These properties say that, for a list of items that one wants to add or multiply, it doesn’t matter how you order the list (communicative property) or which additions or multiplications you perform first (associative property). Therefore, learners should feel at ease when making the following re-arrangements (emphasis on the parentheses).

$x + y + p = (x + y) + p = x + (y + p) = y + (x + p)$ etc. $xyp = (xy)p = x(py) = y(xp)$ etc.

B. Distributive property

.

$x(y+p) = xy + xp$

This rule or property is one of the essential reasons for using parentheses when mathematical expressions are written. To write x times the sum of y+p, one needs to write it as x(y+p). Some learners may write xy+p, an error which often occurs as a result of misconception irrespective of semiotic tool use for teaching and learning of the concept (Vermeulen, 1995).

The Distributive Property of Multiplication over Addition is a good way to simplify calculations (Vermeulen, et al., 1996). It combines two different operations in a shortcut to make solving variables easier. The Distributive Property of Multiplication over Addition: The general rule for this property is for any numbers y, p, and x, $px + yx = (p + y)x$. Imagine one rectangle with area px and add it to another rectangle with area yx. The length of one side would equal x and the length of the other side would equal p + y. To make the problem less abstract, numbers could be substituted. The variable p could equal 4, y could also equal 3, and the variable x could equal 2. Does 4(2) + 3(2) equal (4 +3)2? It does, because 8 + 6 = 14. The Distributive Property of Multiplication over Subtraction: The general rule for this property is for any numbers y, p, and x, $px – yx = (p-y)x$. This can also be illustrated with numbers substituting for the letters, as in the example above. Does 4(2) – 3(2) equal (4-3)2? It does, because 8-6 =2.

C. **Summary on laws of exponents**

Laws of exponents are inherently vital knowledge that learners use when simplifying algebraic fractions. As simple as laws of exponents may look in mathematics, a very high percentage of learners find it difficult to deal with them (McNeil & Alibali, 2005). The laws of exponents (Miller et al., 2001) are as follows:

- $x^p\, x^q \;= x^{p+q}$
- $x^p\, x^{-q} \;=\; \frac{x^p}{x^q} = x^{p-q}$
- $(x^p)^q \;= x^{pq}$
- $(xy)^p = x^p y^q$
- $(\frac{x}{y})^p = \frac{x^p}{y^p} = (\frac{y}{x})^{-p}$
- $(x)^0 = 1$

Where x, y, p and q are any quantity or algebraic expression

A. FACTORISING, ADDITION AND SUBTRACTION OF ALGEBRAIC FRACTIONS

Computations using algebraic fractions are similar to calculations involving fractions as alluded to earlier. Hence, when adding or subtracting fractions with different denominators, we must first find the lowest common multiple. Two examples below:

1. $\frac{x-y}{3} + \frac{x}{4}$	2. $\frac{x}{4y} - \frac{2+x}{y}$
$\frac{4(x-y}{3(4)} + \frac{x(3)}{4(3)}$	$\frac{xy}{4y} - \frac{4y(2+x)}{4y}$
$\frac{4x-4y}{12} + \frac{3x}{12}$	$\frac{xy-8y-4xy}{4y}$
$\frac{4x-4y+3x}{12}$	$\frac{-3xy-8y}{4y}$
$\frac{x-4y}{12}$	$\frac{y(-3x-8)}{y(4)}$
	$\frac{-3x-8}{4}$

Sometimes it is difficult to find a simple expression that is a multiple of two algebraic expressions. When that happens, it is perfectly acceptable to multiple the two expressions together even though this will not necessarily form the smallest common multiple. It is advisable to check at the end of the calculation in the final fraction that there are no common factors in the numerator and the denominator. If there are, factorize and cancel them to give an equivalent but simpler fraction.

B. MULTIPLICATION AND DIVISION OF ALGEBRAIC FRACTIONS

Just as in numerical fractions, the trick with simplifying the multiplication and division of algebraic fractions is to look for common factors during the calculations. Once the common factors are identified and cancelled out, a simplified equivalent fraction is formed. It is important to realise that dividing by a fraction is the same operation as multiplying by the reciprocal.

$\frac{1}{\frac{1}{x}} = 1 \div \frac{1}{x} = 1 \times \frac{x}{1} = x$
$\frac{x}{3} \div \frac{x}{6} = \frac{x}{3} \times \frac{6}{x} = 2$

Quotient divisions by definition are fractions but rarely do learners view fraction in the form $\frac{a}{b}$ as a division. According to Oksuz and Middleton (2007) most of the misconceptions learners have about algebraic fractions is because learners interpret quotients as pairs of whole numbers. For example, the learner viewed

$\frac{9X^2Y^2}{3X^2Y}$ as pair of $9x^2y^2$ and $3x^2y$

and subtracted like terms. Therefore, $\frac{9X^2Y^2}{3X^2Y} = 6y$ Hence, learners see nothing wrong with subtracting the denominator expression from the numerator. Walle (2007) recommended that getting learners to draw diagrams area models about divided quantities and applying this idea accurately with diagrams is one ideal approach to resolve this type of misconception. For instance,

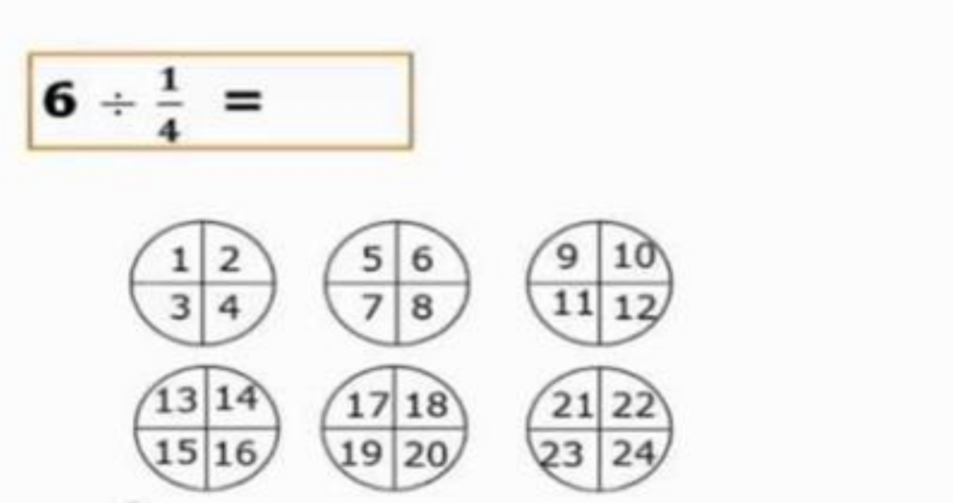


Figure 5. Area Model. Source: Van de Walle, (2007: 296)

Above is a simple area model example of division of fractions. Six divide by one-fourth-area model depicting 24 pieces when six items are divide by one-fourth. Algorithmically this is also correct. Van de Walle (2007), however, failed to categorize the errors and even examine possible reasons for these misconceptions.

2.5 THE ROLE OF LANGUAGE IN MATHEMATICS EDUCATION

Learners' understanding of mathematics is evidenced by the quality of the teachers' explanations (Bahr et al., 2010; Luneta, 2015c). Hill et al. (2008: 432) also found that teachers' quality instruction has a link with learner achievement. This implies that teachers not only need to possess conceptual knowledge, skills and strategies that underpin "the mathematics they teach but also need deep knowledge of the links between these ideas, what makes them difficult, and how they are best communicated" (Siemon et al., 2011:15). According to Lampert and Cobb (2003:237) communication and language are "the primary means by which mathematics is taught and learned".

Over the past two decades, mathematics education reformers have been increasingly concerned with what goes on in mathematics teaching and learning situations, especially in the classroom. However, the role of instructional language in learning mathematics has remained out of focus in mathematics education research. The manner of use of instructional language during the teaching by mathematics teachers as a factor in quality of learning of mathematics needs to be examined. Language is vital to the processing of any concept, whether mathematical or not

(Setati, 2005; Luneta, 2015). The significance of language as a semiotic tool in the classroom is considered important in all activities associated with effective teaching and learning of school subjects of which mathematics is a part. This shows that the use of appropriate language both written and oral cannot be avoided in effective teaching and learning of mathematics.

In Beal et al. (2010) and Setati's (2005) view, the capability to talk about mathematics is of importance for all teachers and learners of the subject. Language and communication are essential elements of teaching and learning mathematics as it facilitates the transmission of mathematical knowledge and allows for teacher learner interactions. They explain further that language permits mathematics learners to work out meanings, to convey their understanding, help develop their thinking further and to express their answers with others. This is because mathematical learners' language is in two-fold in that they are required to have competence in the language of instruction and in the language of mathematics.

To emphasise this issue, Setati (2005) opines that some borrowed words and ambiguous terms from everyday English are a key issue that causes significant problems for learners in mathematics. They give details that these words tend to be ambiguous due to multiple meanings they might have in the mathematics register, vis-a-vis its everyday use. The non-mathematical meanings of these terms can influence mathematical understanding, as well as being a source of confusion. Words such as: angle, average, base, common, complete, degree, difference, differentiation, divide, figure, form, grid, high, improper, integration, product, proper, property, remainder, right, volume, etc. are examples of words that may cause confusion. It is clear from the above that if learners are not given the competence in using mathematical vocabulary to explain mathematical task to others, to ask or answer questions, and when working in groups, it is going to create linguistic difficulty in the study of the subject, in that, when teachers do not use mathematical language effortlessly, their learners are unable to describe mathematical ideas and concepts using appropriate language.

According to Kahn (2005), teachers' interactions with learners can result in consensus and common understandings of issues brought about during practical work. Everywhere in education, there is an urgent need to ensure that the language of instruction issues receive adequate attention. In Vygotsky's (1978) view, teachers' use of instructional language during teaching is based on the recognized role of language in concept formation and development. It also shows its vital importance to learners' learning of mathematical concepts. The U.S. National Centre for Educational Statistics – NCTM (1981) also comments that an increase in the number of learners' ages 5 to 14 years from the level of 2.4 million to 3.4 million by the year 2000 raised major concern on mathematics learning of these ages. An inadequate grasp of the language of instruction is a major source of under achievement in schools. Ohta (2001) found that teacher-learner communication encourages learners to increase their knowledge in classroom. In support of this, Adler and Setati (2001) indicate that when teachers and learners work together to create intellectual and practical activities, it shapes both the form and the content of the target language as well as the processes and outcomes of individual development. Shellard & Moyer (2002) also opine that there are three critical components to effective mathematics instruction; teaching for conceptual understanding; developing children's procedural fluency and promoting strategic competence through meaningful problem-solving investigations. Thus, teachers' instructions in the GET phase should build on learners' emerging capabilities for increasingly abstract reasoning, including: thinking hypothetically, comprehending cause and effect and reasoning in both concrete and abstract terms (Protheroe, 2007). In South Africa, the language policy of education makes clear that schools should adopt two languages as mediums of instruction at both GET and FET phases, for instance English and IsiXhosa. The learners are expected to be able to read and use numbers correctly, reason logically, solve problems and communicate mathematical ideas effectively using English. The learners' mathematical knowledge, skills and competency at this stage enable them to make more meaning of the world around them and also develop interest in mathematics. However, teachers' use of instructional language in mathematics at the GET and FET phases has been a catalyst in the creation of great linguistic problems in terms of comprehension (Setati, 2005).

In summary, mathematics is a unique form of communication, and it is important to understand the way the world can be viewed and interpreted using mathematics. In fact, teaching mathematics requires knowledge of concepts and skills, and the ability to translate between general and specific situations. On the part of the learners, they must have knowledge and understanding of language of mathematics and develop skills in its application. Symbolic records allow them to represent their knowledge, while written and oral communication encourage clarification and justification of ideas.

2.6 MISUNDERSTANDING PERPETUATED BY MATHEMATICS TEACHERS

Many research studies have shown that some teachers harbour some misunderstanding, which is eventually passed on to the learners they teach (Setati, 2005; Luneta, 2015b; Beal et al., 2010). These are mainly as a result of language (Setati, 2015). Language can either help learners to understand a concept or hinder their understanding (Luneta, 2015). Arzarello (1998) reports that many learners do not understand algebraic language correctly and, as a result, their thinking and performance are badly affected. Representation in algebra and symbols or the language of algebra, in general, are likely to be a major factor affecting misunderstanding. Inappropriate information or lack of it which leads to misconceptions by teachers is likely to be some of the major contributing factors affecting misunderstanding of algebraic fractions in schools.

MISCONCEPTIONS AND ERRORS IN ALGEBRAIC FRACTIONS

Misconceptions

According to constructivists, misconceptions happen because the new idea has no link with existing knowledge; as a result assimilation or accommodation becomes impossible (Olivier, 1989). Misconception leads to serious learning difficulties in mathematics, since learners try to make use of their previous inadequate teaching informal thinking or poor remembrance (Resnick, 1983). Drews et al. (2005:18) defined a misconception as a misapplication of a rule, an over or under-generalization or an alternative conception of the situation. Research on misconceptions suggests that repeating a lesson to emphasise a point does not help learners who have acquired alternative conceptions or misconceived (Champagne, Klopfer & Gunstone, 1982; McDermott, 1984; Resnick, 1983). Learners tend to be attached to their misconceptions because they actively constructed them and gives them smart solutions (Naseer & Hassan, 2014). Identifying misconceptions and employing diverse and effective strategies to re-educate the learner in order to correct this learning barrier is probably ideal.

Resnick (1983) suggested that changing the conceptual framework of learners is one ideal way of repairing mathematical and science misconceptions. He expounds that it is not usually successful to merely explain the errors and misconceptions to the learners. This is because misconceptions have to be changed internally partly through the learners' belief systems and partly through their own cognition. Mestre (1987) also affirmed the constructivist view that learners do not come to the classroom blank; rather they come with informal theories constructed from everyday experiences.

According to dictionary.reference.com/browse/misconception (11/01/2016) a misconception is an erroneous conception, mistaken notion. Due to the subjective nature of being human, it can be assumed that everyone has some kind of misconception. If a concept cannot be proven to be either true or false, then it cannot be claimed that disbelievers have a misconception of the concept by believers no matter how much the believers want a concept to be true and vice versa. Misrepresentation of a concept is not a misconception but may produce a misconception. According to Li (2006) and Luneta (2015) learner errors are the symptoms of misconceptions. Generally, misconceptions manifest through errors. The challenging issue concerning misconceptions is that many people have difficulty correcting misconceptions because the false concepts may be deeply ingrained in the mental map of the individual. Some people do not like to be proven wrong and will continue clinging to a misconception even if they have been proven wrong (McDonald, 2010).

This view is consistent with that of Hammer (1996:99) who thought students' misconceptions:

1. Are strongly held, stable cognitive structures: This means that the learners formed misconceptions are solidified cognitively and become intricately difficult to eradicate.
2. Differ from expert understanding; when evaluated by someone with a strong knowledge base, they find that the learner's perceptions of mathematical constructs are incomprehensible due to weakened concept images.

   Affect in a fundamental sense how students understand natural phenomena and scientific explanations.

4. Mathematical cognition is a complex mental process that takes time to build with early childhood forming the prime to form the base mathematical constructs. If the learner acquires weak concept images at this stage, then his/her ability for abstract thinking later in the sciences and mathematics disciplines is heavily impacted.
5. Must be overcome, avoided, or eliminated for students to achieve expert understanding. Due to the negative consequences of weak mathematical representative foundations, it is imperative that all teachers are aware of how their teaching in early childhood may affect the child later in life.

Figure 2.3 depicts Luneta's (2015) view of how alternative conceptions are formed if teaching of mathematical concepts is not unpacked effectively.

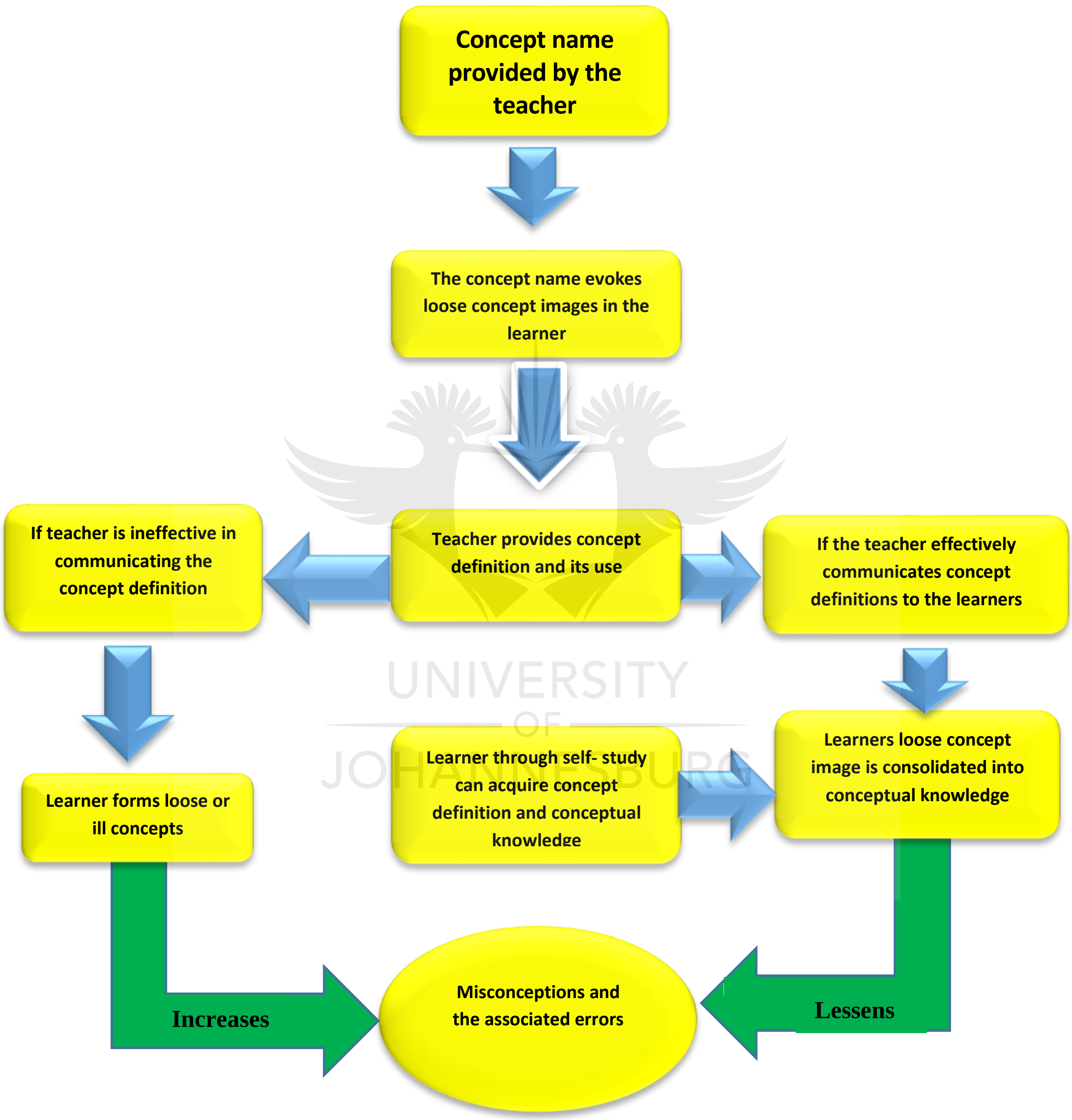


Figure 6: Conceptual knowledge and communication in mathematics. Source: Luneta, (2015)

In unpacking mathematical concepts the teacher provides a concept name followed by a concept definition and the use of the concept in mathematics (Tall and Vinner, 1981). This information evokes a weak concept image in the learner. If the teaching was effective and the teachers provided the appropriate definition and illustrations contextualised to the learner's experiences and cognitive level, the learner's concept image is consolidated into conceptual knowledge. When the teacher's knowledge base is weak, it results in facilitation that weakens further the learner's concept image that might lead to misconceptions and errors. Figure 2.3 illustrates how mathematical concepts are communicated from the teacher to the learner.

Research on learners' misconceptions confirms that learners learn by their mistakes as long as those mistakes and misconceptions are identified, addressed and discussed in teaching (Luneta, 2008; Luneta, 2015; Li, 2006; Merthens,1996). Merthens (1996) found that misconceptions are common among learners and one way to deal with them is to allow the learners to go through them before addressing them. According to Swan (2001:154) a misconception is not wrong thinking but is a concept in embryo or a local generalization that the learner has made. It may in fact be a natural stage of development. Although the teacher should employ effective teaching strategies to unpack concepts, misconceptions cannot simply be avoided (Swan 2001:150). Therefore, it is important to have strategies for remedying as well as for avoiding misconceptions.

To align myself with the views above, misconceptions in this study are referred as an alternative conception of an idea which differs from expert understanding.

2.7.2 Errors

As stated earlier, misconceptions generate errors. This is because errors result from learners' misconceptions of new ideas. Errors might be referred to as wrong answers or mistakes which are regular, planned and repeated again and again. Luneta and Makonye (2010: 35) defined errors as a mistake, slip, blunder or inaccuracy and deviation from accuracy. These errors are the indicators or symptoms of some misconceptions. Swan (2001:150) also asserts that errors could be the result of 'careless or misinterpretation of symbols or text'. Every error is an indication of a misconception and these misconceptions are solid concepts for the learners. Hence, diagnosing and analyzing learners' work critically has become an essential activity for teachers. One of the main methods used to analyse learners' errors is to classify them into certain categorizations based on an analysis of students' behaviours.

Orton (1983) also classified errors into three categories as:

1) Structural error: is an error which arises from some failure to appreciate the relation involved in the problem or to grasp some principle essential to its solution.

2) Arbitrary error: is that error in which the subject behaved arbitrarily and failed to take into account the constraints laid down in what was given.
3) Executive error: is that error where the learner fails to carry out manipulations, though the principles involved may have been understood.

Radatz (1979) classified the errors in terms of (1) language difficulties. Mathematics is like a "foreign language" for learners who need to know and understand mathematical concepts, symbols, and vocabulary. Misunderstanding the semantics of mathematics language may cause learners' errors at the beginning of problem solving; (2) difficulties in processing iconic and visual representation of mathematical knowledge; (3) deficiency in requisite skills, facts, and concepts, for example, students may forget or be unable to recall related information in solving problems; (4) incorrect associations or rigidity, that is, negative transfer caused by decoding and encoding information; and (5) application of irrelevant rules or strategies. Other researchers (Newman, 1977; Watson, 1980) have also used the classifying method but based theirs on the model of problem solving. Watson used Newman's (1977) model of the sequence of steps in problem solving: reading and comprehension, transformation, process skills, and encoding, to identify learners' possible errors. He thought that learners' errors may be due to deficiency in one or several of the above steps. In order to verify those hypotheses about students' errors, Watson designed both word and computation problems to compare errors made by two groups of learners, with lesser and greater abilities. He found that most initial errors made by the more able group were at the stage of reading and comprehension. However, the less able group learners made many more errors when applying and selecting mathematics processes. The above classification method was simply used to describe learners' errors, but lacked detailed analysis of why learners were unable to perform well in some steps. For example, why did learners not select correct mathematics processes or operations? What strategies effectively helped learners make correct decisions? Why did learners have special difficulty in understanding mathematics language?

Naseer and Hassan (2014) also categorized learners' errors as:
Detachment error: When learners lack an aspect of structure sense, they often make detachment errors (in particular detachment of the negative sign): $x^2 - 5x = 25$ & $5x$ -x^2=25. An instructional suggestion would be to "use order of operations to develop an understanding of transformations that can keep the value of an expression equal" (Banerjee & Subramaniam, 2012).

Errors due to lack of technical vocabulary: Analysis of answer scripts revealed that many learners lack technical vocabulary. For example, learners were unable to differentiate between "factorize", "solve", and "complete the square". Some learners had factorized when they were asked to complete the square for the expression $x^2 - 6x$. It was also observed that when learners were asked to factorize the expression $x^2 - 10x + 21$, they gave solutions to the equation $x^2 - 10x + 21 = 0$. Teachers could directly explain what these terms mean and use these technical terms as often as possible. According to Narayan (2009) traditional teaching techniques such as rote learning play an important role in building a solid foundation in correct concept formation.

Misconceptions with operational symbols: These misconceptions could be due to their earlier learning experiences. For example, a plus sign is a signal to conjoin two terms together like 1 + 1 = 1 1. However, in algebra $x + 1 \neq 1\ x$
According to Welder (2012:260) these misconceptions can be prevented by exposing the underlying structure of algebra to learners while working with arithmetic prior to learning formal algebra.

Misconceptions with letter usage: The main reason for this type of misconception is the use of misleading teaching materials. One way of addressing this issue is by carefully distinguishing variables and abbreviations (Welder, 2012). According to Watson (cited in Welder, 2012) variables should be introduced once learners learn how to recognise and record pattern and write pattern rules in words. Warren and Cooper (cited in Welder, 2012) stressed the importance of exposing elementary students to recognise and write growing patterns as a way of preparing them for algebra.

Mathematical language errors: When analysing the data, the researcher adopted Nolting and Hodes' definitions of the various errors for this study because their definitions of errors are most complete and encompass the ideas of others. The researcher, however, added mathematical language error identified by Radatz (1978) to the three error types identified by Nolting and Hodes. The description of language error by Radatz (1978) seemed to be similar to that of Naseer and Hassan's (2014) three error types: error due to technical vocabulary; misconception with operational symbols error and misconception with letter usage. Furthermore, literature reviews consistently highlighted mathematical language error as a source of error. Some studies carried out on errors and misconceptions attributes learners' lack of understanding with use of mathematical symbols as a source. (Resnick, 1983; Luneta & Makonye, 2010:35; Li, 2006; Luneta, 2015; Swan, 2001). In this study language errors were referred to as the misconceptions associated with mathematical symbols and vocabulary.

A further elaboration on the Nolting and Hodes' identified errors and their applicability to this study types are presented in Table 1.

Error categorization by different researchers				
Orton (1983)	Radatz (1978)	Naseer & Nassan (2014)	Nolting & Hodes (1998)	Applicability to this study
Structural error	Language difficulty	Detachment error	Careless error	Careless error
Arbitrary error	Difficulties in processing iconic and visual representation	Technical vocabulary error	Procedural error	Mathematical language error
Executive error	Deficiency in requisite skills, facts, and concepts	Misconception with operational symbols	Concept error	Procedural error
	Incorrect associations or rigidity	Misconception with letter usage	Application error	Concept error
	Application of irrelevant rules or strategies			Application error

Table 1. Error categorization by different Researchers. Source: Author

Careless errors

Hodes and Nolting (1998) described careless or unintended errors as errors that occur even when the learner has the required knowledge but they are due to distractions. Careless mistakes can often be corrected easily upon a second look. For example, when a learner was asked to explain his solution of the algebraic fraction expression below, he quickly realized that the exponent for the **a** should be 4, hence the answer should be - $4a^4b^2$ instead of- $4a^3b^2$. This is careless/ unintended error. In this study, an error which occurs when the learner has the required knowledge but slips up as a result of action is also classified as a careless error.

Figure 7. Example of a careless error. Source: Author

Procedural errors

Procedural errors, on the other hand, are mix up of rules or formulas as a result of lack of relational understanding of what they are doing (Skemp, 1978 as cited in Walle, 2007). For example, multiplication of decimal numbers: 0.5 x 0.6. A learner may give 3.0 as an answer and explain the answer as multiplication of two numbers as always a bigger number. Another learner may answer and justify an algebraic fraction as follows:

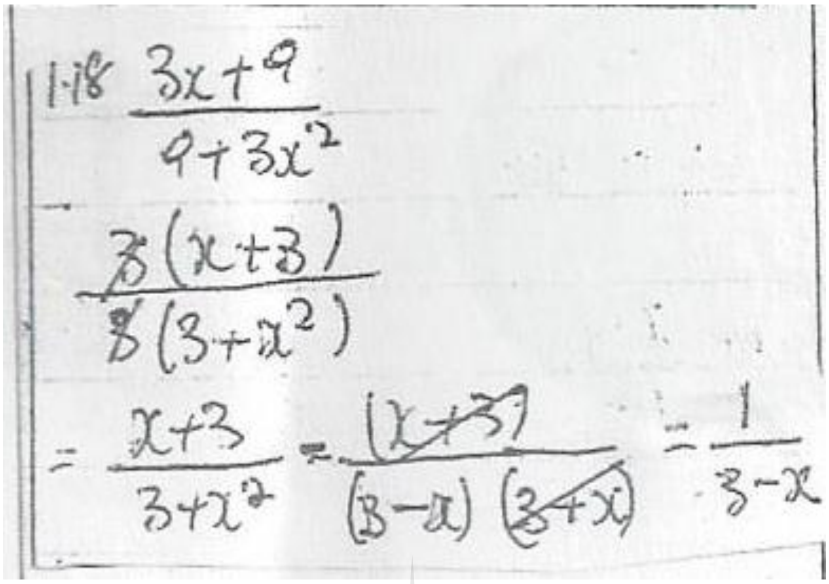


Figure 8. Example of a procedural error. Source: Author

The above solution to an algebraic fraction indicates a procedural error and lack of conceptual understanding. Radatz (1978) also seems to have classified procedural error as a deficiency in requisite skills, facts and concepts as a source of error. Orton’s (1983) description of arbitrary error also tallies with Nolting and Hodes’ procedural error. In this study, errors as a result of lack of understanding of the rules needed to simplify an activity shall be referred as procedural errors.

Concept errors

Concept errors also occur when learners do not understand the properties or principles underpinning the topic concerned because learners attach their own meaning. For example, to find the volume of an object, the learner may think of the volume of a sound on his/her TV. The solution below also indicates a learner who lacks conceptual understanding to solve the problem. Hence, the learner has created an alternative concept which differs from expert understanding. Errors as a result of lack of conceptual understanding underpinning the topic concerned are termed concept errors for this study. The example below shows a concept error typically perpetuate by learners.

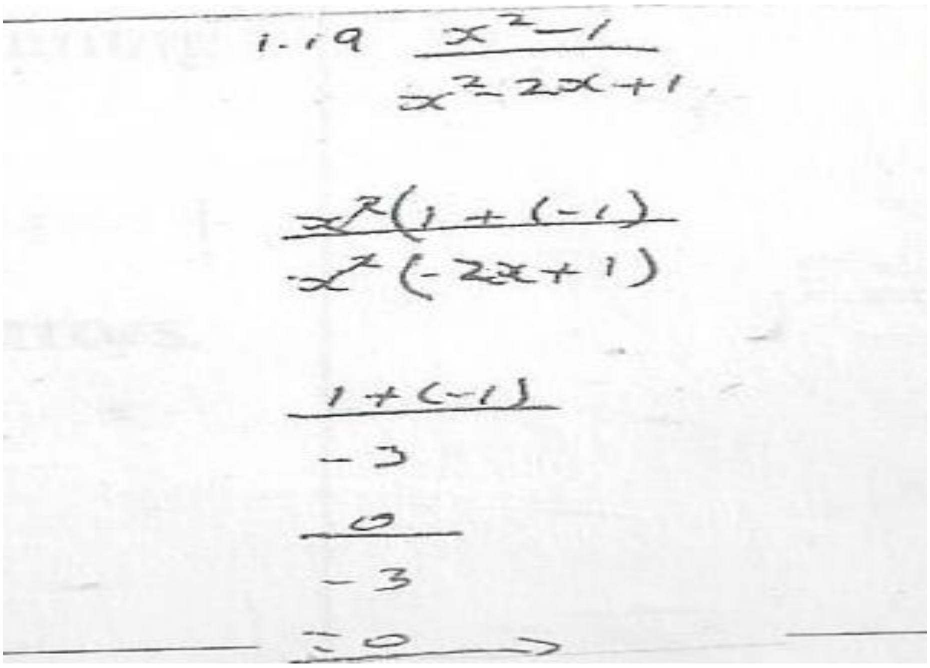


Figure 9. Example of a concept error. Source: Author

Application error

According to Nolting and Hodes (1998) application errors also occur when learners know the concept but cannot apply it to specific situations or problems. Radatz (1978) also classified this as application of irrelevant rules or strategy. In this case, usually the learner lacks the conceptual understanding to apply the learnt concepts and skills in an unfamiliar context. A further interrogation with the learner on the question below depicts that the learner knows how to work with factorization, division, exponents and perfect square. However, he/she lacks the understanding as to when to apply them. If an error is due to the learner's inability to apply a known concept appropriately to specific or different problems, it shall be referred as an application error in this study.

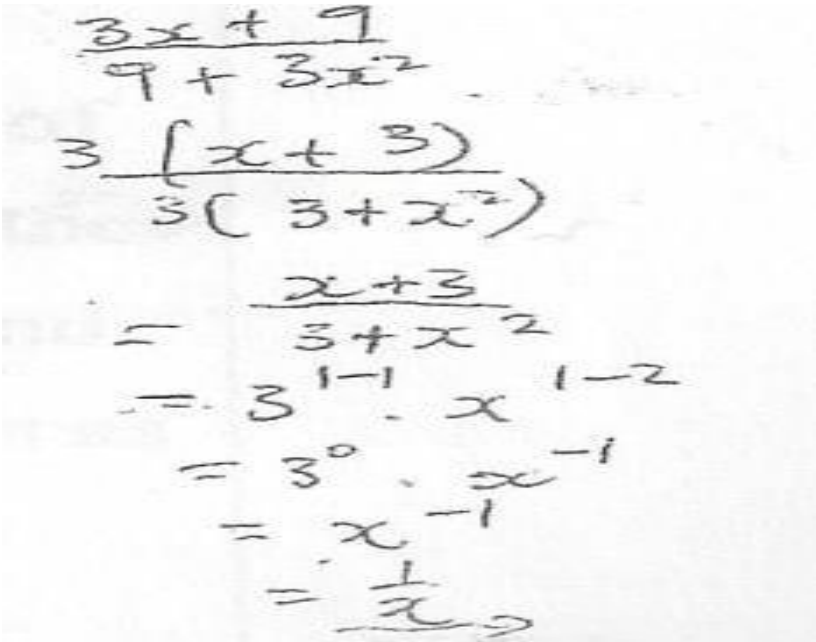


Figure 10. Example of an application error. Source: Author

In view of the above, we can assert that misconceptions and errors provide teachers with an in-depth knowledge of how they should handle certain mathematical concepts.

Hodes and Nolting (1998) also proposed four types of errors and explained them as follows:

- Careless errors: mistakes made which can be caught automatically upon reviewing one's own work.
- Conceptual errors: mistakes made when the learner does not understand the properties or principles covered in the text and lecture.
- Procedural errors: these errors occur when learners skip directions or misunderstand directions, but answer the question or the problem anyway.
- Application errors: mistakes that learners make when they know the concept but cannot apply it to a specific situation.

Mathematical knowledge is interrelated and misconceptions in one branch of mathematics may be carried into other areas of mathematics. A poor mastery of basic concepts may limit a learner to pursue other areas of study. Over and above all algebraic fractions are a core content at the FET phase in south Africa.

2.7.3 Possible sources of misconceptions and errors

To analyse these errors and design effective instructions, teachers need to have an in-depth knowledge of possible reasons for these errors. Numerous reasons could be cited for them. Some of them are inadequate prior knowledge of the topic; distractions during instructions; language barriers; lessons not suitable for most of the learners' learning styles; level of cognitive thinking below the concept been taught and refusal on the part of the learners to spend ample time to study in order to understand and acquire the skills needed (Luneta & Makonye, 2011; Van de Walle, 2007; Setati &Adler, 2001; Hodes & Nolting, 1998).

Adequate prior knowledge is unequivocally essential to understand new concepts. Lack of this prior knowledge means that learners are 'lost in the classroom' as a result of which they attach their own meaning to concepts which leads to errors. Similarly, distractions during instruction time has been identified as a source of error making.

Another debatable cause of error is the language of instruction as affirmed by Setati and others. Usually language of instruction and assessment is different from learners' mother tongue; hence, some of the learners find it difficult in terms of understanding the questions but some critics of language of instruction believe otherwise. The instructional approach of the teacher might be to develop learners from procedural to conceptual knowledge or the other way round. In either case, acquisition of skills is vital. Learners therefore need to spend enough time with their books for practice; failure to do this usually leads to errors. It is necessary for teachers to do error analysis and design appropriate strategies for effective teaching. Possible strategies for error analysis may include the following but most important is to code the type of error such as reading of the learners. The teacher could, for example, ask the learner to read a question. If the learner could not read or recognise some key words or symbols, then the teacher should code that as a reading barrier.

If the learner can read but cannot explain the question, then we have a comprehension problem at hand. Another way of identifying an error might be to ask learners to show how they got their answers. If they cannot explain, it means you have a processing skills problem to deal with. Careless errors which is as a result of oversight, but can be corrected on the second attempt is also another way of analysing error (Hodes & Nolting, 1998).

Inappropriate instruction method can also be coded as teaching. To find out about this, learners could be asked to assess the topic as being interesting or boring or easy or difficult. If they respond 'boring' or 'difficult', it might indicate possible error-making in their activities (Li, 2006).

If error analysis will enable teachers to design effective instructions, then it should be embraced since it might help to identify the root causes of the types of errors discussed above and how best to remedy these learning barriers to effective learning and teaching. Learners will benefit from error analysis by knowing the correct concept and teachers will also know the right path to effective teaching. For example, error analysis indicating careless errors implies the need for more examples on the main steps. If learners have inadequate prior knowledge on the topic concerned, teachers should be prepared to go down and bring them up (Themane, 2014).

Display posters in the classroom that have content information will also go a long way to help learners who easily forget what has been taught (Van de Walle et al., 2015).

Teachers can also use different approaches if learners do not understand what has been taught. Misconceptions are very difficult to detect; however, if problem solving questions are used, learners will develop the conceptual understanding necessary for mathematics education. Cooperative group work could also be used to reduce error making since it enables learners to reflect on their own ideas and even that of others.

In conclusion, it is essential for the teacher to do error analysis on learner activities to look for patterns of errors or mistakes in learners' activities as well as the root causes of such errors so that effective targeted instructions are designed to remedy the errors. Some of the reasons identified as causes of errors are careless errors, language errors, procedural errors, and concept and application errors. In this research, identified errors from the learner-participants were coded using these error types, except careless error. This is because, although careless error is an error type, it is not as a result of misconceptions that learners have. As alluded to earlier, it can be rectified by the learner upon second look. Also emphasizing the main steps in the case of careless errors should enable learners to overcome their mistakes. More activities and problem solving activities can also be used to develop both procedural and conceptual understanding before learners misconceive or misuse formulas. For mathematics teachers to be effective, error analysis should be part of their daily activities. Error analysis will enable teachers to unpack contents considering both conceptual and procedural understanding in their teaching.

2.7.4 Conceptual and procedural knowledge in mathematics

The common objective of teaching is to enable learners to understand concepts or acquire particular knowledge. Thus, teachers deploy different teaching pedagogies to enable learners to acquire knowledge. In mathematics education, teaching strategies used enable learners to acquire two interrelated types of knowledge: conceptual and procedural knowledge. According to Hiebert and Lefevre (1986) conceptual knowledge is knowledge that is understood. Skemp (1978) also referred to this construct as knowledge rich in relationship and understanding. Procedural knowledge is also a type of knowledge that enables learners to apply a rule to solve problems without understanding how it works, usually as a result of rote learning. Hence, knowledge of these two constructs by mathematics teachers should enable learners to deliver. Critics of procedural knowledge argue that procedural knowledge acquisition should not be entertained, while some scholars also assert that the combination of the two constructs holds the key to high attainments in mathematics.

Another school of thought has it that the two constructs intertwine somehow in that the two constructs do not develop independently. One's conceptual understanding influences the procedures they use (Gelman & Williams, 1997 as cited in Rittle-Johnson & Alibali, 1999). For instance, if explanation is done verbally to learners before practice, it implies acquisition of conceptual understanding rather than procedural understanding. The other way round is also feasible, for example, the use of trial and error to solve factorization problems. Thereafter, reflecting on the activities implies movement from procedural to conceptual understanding.

If both knowledge concepts are necessary in teaching of mathematics, then which teaching pedagogy should maths teachers use? Is it by developing conceptual knowledge first, which should enable learners to acquire knowledge with understanding and then develop procedural skills thereafter, or start with the procedural understanding instructional approach before developing conceptual knowledge or blend the two constructs together? Incorporating the two constructs appears to be the best approach for effective mathematics teaching. Research has shown that there is a strong relationship between children's understanding of mathematical concepts and procedure skills, and that conceptual understanding comes before procedural understanding, as, for example, with addition of algebraic fractions (Codding, et al., 2016; Rittle-Johnson & Alibali, 1999).

If these two constructs hold the key to understanding of mathematics, then mathematics educators should use these two teaching pedagogies to unpack mathematics content. However, if the focus is on conceptual understanding, learners find it very difficult with the understanding. On the other hand, procedural understanding incapacitates learners because they are not able to apply acquired skills under different context. Incorporating the two constructs is hailed as an effective teaching and learning strategy (Codding et al., 2016). However, which of them should precede the other poses a question. Use of these constructs to unpack mathematics contents should depend mainly on the mathematical development stage of the learners.

2.7.5 Effective teaching and learning strategies to deal with misconceptions: Algebraic fractions

Teaching and learning of algebraic fractions poses difficulties to learners due to its abstract nature as well as the pedagogies used by teachers in presenting this very concept which usually focuses on procedures and algorithms instead of concentrating on deep conceptual understanding. Brown and Quinn (2006:125) asset that," If algebraic fractions are for everyone, then a bridge must be built to span the gap between arithmetic and algebraic fractions. The building materials being conceptual understanding and the ability to perform arithmetic manipulations of algebraic fractions".

There are a number of principles that appear in literature on effective teaching and learning of mathematical concepts such as algebraic fractions. According to Lappan and Briars (1995), among these principles is a problem-oriented curriculum that focuses on ideas before skills. Teacher actions that are effective include deriving concepts, using cooperative group work, encouraging frequent mathematical communication, and using multiple representations and multiple strategies.

The National Council of Teachers of Mathematics (2000) report on problem solving standard states that high schools' maths learners should be able to "build new mathematical knowledge through problem solving; solve problems that arise in mathematics and elsewhere; monitor and reflect on the process of mathematical problem solving" (NCTM as cited in Walle, 2007:15). Procedural understanding or memorization of procedure or formula only helps the learner to some extent. The learner struggles when the problem becomes complex and unfamiliar. The NCTM advocates that, "a major goal of high school mathematics is to equip learners with knowledge and tools that will enable them to formulate, approach, and solve problems beyond those which they have learnt". This implies that learners need opportunities created by teachers to develop problem solving skills, for instance, giving tasks that will enable learners to know the applicability of algebraic fractions in real life. Swan (2001:74) affirms this by his assertion that "this will be possible if learners are given real-life problems involving algebraic fractions". It is also true that real-life problem activities will also enable the learners to construct their own understanding thus helping them to acquire conceptual understanding.

Just following procedures to solve mathematics problems and doing or proofing mathematics concepts with understanding are two different dimensions in mathematics. Mathematical ideals are formed through a process of analysing problems, trying a number of strategies to solve them, evaluating the strategies' effectiveness, looking for new strategies and verifying that a particular strategy is valid (Rittle-Johnson and Siegler, 1998). Thus, if learners go through this process of learning a concept for themselves, they will acquire appropriate conceptual understanding of the concept. Hence, effective teaching and learning of algebraic fractions should imply that teachers use problems that highlight algebraic fractions and allow the learners to figure out the solutions themselves. Of course, some of the algebraic fractions may be complex but then scaffolding becomes necessary.

Co-operative learning involves groups of learners working to complete a common task (Siegel, 2005). This is typically done with a smaller group of learners. Research has shown that learners perform better when this learning strategy is used than when learners work alone. Co-operative learning gives learners the chance to analyse and evaluate the mathematical thinking and strategies, a key concept NCTM promotes. The interactions with other learners can help deepen the level of understanding for all learners involved. Viable group work requires a lot of preparation; this is because all the learners in a group must be engaged for the group work to be effective. This also means that the level of the activities must be challenging and developmentally appropriate for the group.

Discussion and writing about mathematics helps learners to reflect on their own thinking and redefine their ideas. The ability to communicate mathematically can only be developed through practice, hence mathematics teachers can provide learners with activities that will help them in that regard. Lappan and Briars (1995) assert that one effective strategy that teachers can use is by encouraging frequent mathematical communication. They can restate the learners' ambiguous or unclear statements in a better mathematical way and allow the learner to agree or disagree with the rephrasing of his/her original thought. This helps the learner to claim ownership of their ideas and construct meanings of the concepts. Furthermore, effective mathematics teachers may also unpack concepts using more than one approach to enable all the learners to get the most out of their instruction. This is necessary because learners have different strengths, weaknesses and learning styles, hence the use of different teaching strategies helps (Swan, 2001).

2.8 CONCLUSION

In the above discussions, learners' misconceptions were reviewed. Theoretical explanations of why some misconceptions are inevitable are provided. There is also the need on the part of teachers to identify learners' alternative conceptions or misconceptions to help them learn mathematics effectively and efficiently. Ignoring learners' misconceptions may have negative effects on learners' new learning and will also reinforce more misconceptions. It is imperative that learners are encouraged to developed mathematical mental habits and acquire the necessary skills by allowing them to derive concepts themselves. This can be achieved if the mathematics teachers acquire high MKT and MQI. Co-operative group work, frequent mathematical communication, use of diverse teaching strategies and use of real-life problems when teaching topics such as algebraic fractions will enable learners to acquire conceptual understanding, thus minimizing errors as a result of misconceptions.

This chapter is relevant to the problem as an empowering mathematics teachers with MKT and MQI will assist to minimize misconceptions with learning and teaching of algebraic fractions and mathematics in general. The next chapter discusses the appropriate research design and methodology to assess misconceptions when learners are computing algebraic fractions.

Chapter 3: RESEARCH DESIGN AND METHODOLOGY

3.1 INTRODUCTION

A research design is "concerned with the research methodology, approach and data collection methods and the subsequent analysis of the data" (Robert-Holmes, 2014:70). According to Creswell (2013), the success of any given research depends on the appropriateness of the method to the research and its design. Hofstee (2009:109) also reiterates that "careful thought about your method can easily end up saving you an enormous amount of time, effort and frustration". The methodology of any given research provides the different methods that are used in the research (Easterby-Smith et al., 2002). The research methodology therefore offers an opportunity for the researcher to provide the methods that were employed to arrive at conclusions in relation to an identified problem. Additionally, it offers an avenue for the researcher to state the reasons for choosing a particular method (Saunders, Lewis, & Thornhill, 2007).

This chapter begins by presenting the paradigms underlying research, and then the research design concept. This is then followed by the research methodological approach employed in the study. The data collection instruments used in the study are then presented with justification for the choice of each instrument. What follows this is the sample identification and the selection process and, finally, the approach to data analysis and measures to ensure the trustworthiness of the study is presented.

3.2 RESEARCH OBJECTIVES

The primary objectives of the study were to identify, categorize and suggest ways to minimize learners' misconceptions in algebraic fractions. These were achieved through the following secondary objectives:

In order to achieve these objectives, there was the need to identify appropriate method(s). The next sections therefore discuss the appropriate research methodologies adopted for the study.

3.3 RESEARCH PARADIGM

A paradigm, sometimes called a philosophy, according to Taylor, Kermode and Roberts (2007:5), is "a broad view or perspective of something". In other words, a worldview or a set of assumptions about how things work. In addition, Weaver and Olson's (2006:460) explanation of paradigm indicates the effect of paradigm on research by concluding, "paradigms are patterns of beliefs and practices that regulate inquiry within a discipline by providing lenses, frames and processes through which investigation is accomplished". Consequently, for a better structure of inquiry and methodological adoptions, there is the need for a probe of the paradigm used for this study before any discussion of particular methodologies adopted for the study.

Denzin and Lincoln (2005) gave three reasons why it is useful to understand the paradigm of a research. Firstly, an understanding of a research paradigm can help clarify research design issues, like the type of evidence required and the ways to gather and interpret after collection. Secondly, it will be beneficial to researchers as it can help them recognize the best designs for a particular research. This helped in identify the limitations of particular approaches. Thirdly, knowledge of research paradigm can allow the researcher to identify and develop designs that may be outweighing past experiences. Also this understanding can help suggest how to adapt research designs in relation to constraints of different subject or knowledge structures. This is, therefore, a clear indication that a researcher should be aware of and have a better understanding of the differences among research paradigms.

Koenig et al. (2011) classified and categorized research paradigms into three: positivism, interpretivism and critical postmodernism. Interpretive researchers believe that our knowledge of reality is gained only as a result of social constructions, such as language, consciousness, shared meanings, documents, tools, and other artefacts. Thus, interpretivists assume that reality consists of people's subjective experiences of the external world, and reality is given or socially constructed (Riemer, 2008; Seale, 2000; Shenton, 2004; Silverman, 2000). Positivism, on the other hand, is associated with quantitative research. It involves hypothesis testing to obtain 'objective' truth. It is also used to predict what may happen at a future date. Critical realism is a subtype of positivism that incorporates some value assumptions on the part of the researcher. It involves looking at power in society. Researchers primarily rely on quantitative data to do this.
The positivists assume the social world exists externally and its properties should be measured through objective methods, instead of being inferred subjectively through sensation, reflection or intuition, and are independent of the observer (researcher) and his or her instruments (Punch, 2005). The critical postmodernism paradigm is a mixture of two different worldviews, i.e. critical theory and postmodern scholarship (Gephart, 1999). Critical researchers assume that social reality is historically constituted and that it is produced and reproduced by people (Myers, 2009).

Although people can intentionally act to change their social and economic circumstances, critical researchers believe that their ability to do so is inhibited by various forms of social, cultural and political domination (Myers, 1997). Gephart (1999) also asserts that critical postmodernism seeks to attain social transformation to displace the existing structures of power and domination and making available opportunities for social participation among people originally excluded and dominated. Critical postmodernism enquiry has been to deconstruct discourse to expose hidden structures of domination, particularly dichotomies, and then reconstructed to present alternative, less exploitive social activities (Boje, 2001).

Pragmatism assumes "the worth of a proposition or theory is to be judged by the consequences of accepting the proposition or theory" (Kelder, Marshall, & Andrew, 2005:4). As a result, theories and ideas are considered beneficial tools for the purpose of increasing our ability to expound and utilize phenomena (Levy & Hirschheim, 2012). The pragmatists see truth through inquiry into the relevance of propositions, models and theories "with the aim of helping people to better cope with the world" or develop better situations for themselves (Murphy in Johnson & Onwuegbuzie, 2004; Mulhall, 2003). A pragmatist is of the view that what is practically relevant and useful must be asserted through dialogue and argument, and not just through claims to experience. The pragmatist researcher tries to avoid the concept of what is truth and reality, and rather focuses on using different ways they believe appropriate and uses the result in ways that have positive consequences within their value system. To the pragmatist researcher, reality is too complex and the many different standpoints in the specific cultural and/or social setting of research interest must be accommodated (Kelder et al., 2005). Pragmatism assumes knowledge is provisional, socially created and situated in history (Kelder et al., 2005). Hence, theory is only deemed to be true after it has been proved to be useful and then only in the context and the period within which it is established to be useful (Kelder et al., 2005; Levy & Hirschheim, 2012).

Even though these paradigms have clear philosophical viewpoints, it is sometimes difficult for some researchers to identify themselves with a particular tradition because, in practice, it is possible they may accept foundational ideas of one tradition but prefer collecting data and generalizing the findings using other traditions. Moreover, the viewpoint adopted might be inconsistent in a particular individual researcher. In some instances, a researcher from one philosophical standpoint may develop ideas which fit perfectly with those of the other view (Easterby-Smith et al., 2002).

3.3.1 Paradigm adopted for this research and its relevance

The study was grounded in the interpretivism and pragmatism paradigms for a number of reasons. The study aimed at identifying, thematizing and suggesting ways of minimizing learners' misconceptions when dealing with algebraic fractions and used the directed content analysis approach and, since pragmatism focuses on practical applications, consequences, relevance and usefulness, it resonates well with the aims of the directed content analysis approach. Additionally, the assertion that theory and decisions are only relevant to a particular context and period of time permits the on-going enquiry and continuity in the improvement of the developed strategies (Johnson & Onwuegbuzie, 2004; Kelder et al., 2005). Pragmatism adopts a pluralistic approach by rejecting an in-compatibilist method and presents researchers with an opportunity to select a mix of methodologies to better solve their research problem (Johnson & Onwuegbuzie, 2004; Levy & Hirschheim, 2012). A number of factors are of significance in the development context where this study will be done. Firstly, pragmatism views research as value-oriented (Johnson & Onwuegbuzie, 2004). Secondly, it seeks to develop visible strategies and validate the purpose of the research (Kelder et al., 2005). Finally, there is the need for evaluating the ethics of decisions, actions and the research process (Kelder et al., 2005). This is to ensure the research is done the right way and makes a difference. Additionally, research based on qualitative surveys via interviews is identified by Kelder et al. (2005) as a good research strategy which is suitable under the pragmatic philosophy. The aim of the study is to suggest a viable strategy (or framework) to assist learners and teachers to minimize misconceptions when simplifying algebraic fractions. From the pragmatist viewpoint, the framework isn't proven but constructed from the researchers' and participants' experiences during the study (Kelder et al., 2005). These conditions fit very well with the objectives of this study. The use of multiple research paradigms is gradually becoming popular (Mingers, 2001; Babbie and Babbie, 2001; Berg, 2001), as this leads to some richer and more reliable research outcomes as well as creativity (Mingers, 2003; Saunders, 2007).

In view of the latter statement, the study employed another paradigm, interpretivism, to the data collected from the interview. The interview focused on gaining understanding of learners' misconceptions when dealing with algebraic fractions. Interpretivism was used to interpret the findings and results to gain an understanding of how misconceptions can be minimized to enable effective teaching and learning of algebraic fractions.

3.4 RESEARCH METHODOLOGY

Research methodology is the approach to the research process and includes a number of data collecting methods (Collis & Hussey, 2009). The choice of method reflects the aims of the study, availability of time and resources, philosophical assumptions and approach (Saunders et al., 2009). As a result, research can be divided into five main categories (Saunders et al., 2009):

- ***Evaluation research:*** Seeks to give a systematic assessment to provide accountability in situations where changes were expected to be made on some object.
- ***Theoretical and pure research:*** It is a research design where scholars and researchers expand knowledge in a specific field with the aim of helping solve existing problems.
- ***Applied and policy research:*** It is meant to help policymakers to make informed decisions. This form of research is both theoretical and practical.
- ***Critical and feminist research:*** This form of research critiques basic hypotheses and conventional research strategies previously ignored. Feminists support research that contributes towards advancement of women.
- ***Quantitative and qualitative research:*** There is the use of numerical data in quantitative research and qualitative research involves the use of experiences or verbal data.

This study adopted the qualitative research approach because it helped answering the research questions. This is therefore further discussed in the sub-section below.

3.4.1 Qualitative research approach

A qualitative research approach involves the collection of a variety of materials that describe routine and problematic instances and meanings in individuals' lives. It employs the use of descriptive and interpretive methods of gathering and analyzing data (Marshall & Rossman, 2010; Creswell, 2003). Qualitative research attempts to make meaning or interpret issues through the meanings people assign to them (Denzin & Lincoln, 2005). Thus, it explores and discovers issues about the problem because very little is known about the problem (Creswell, 2015). Research that develops knowledge through social constructions such as consciousness, language, shared meanings, symbols, stories, documents, interviews, and other interactions is considered interpretive research (Klein & Myers, 1999; Yin, 2015).

3.4.1.1 Rational for using the qualitative research approach in this study

The philosophical foundation in interpretive research is that there is the need for interpretation in order to understand a phenomenon (Creswell, 2003). Interpretive methods usually aim at “producing an understanding of the context of the information system, and the process whereby the information system influences and is influenced by the context” (Walsham, 1993:4). In recent times, the use of interpretive research has become a popular alternative to the more traditional positivist and critical theory strategies (Klein & Myers, 1999; Yin, 2015). This is because it is very difficult to model or understand a phenomenon using traditional positivist approaches (Klein & Myers, 1999; Yin, 2015), especially due to the complex relationship that exists between people, technology, politics and other organizational factors within the research domain. This is because positivist approaches try to develop fixed, predictive affiliations and elements (Creswell, 2003.)

As a result of the above reasons, the use of the qualitative research approach was appropriate for this study. Moreover, Creswell (2007) concurs with the use of qualitative methodologies for research involving learners’ behaviours. Hence, this is the best approach for studying learners’ misconceptions.

3.4.2 Research design

The research design links the theory and argument that necessitated the research and the empirical data collected (Frankfort-Nachmias & Nachmias, 2008). Research design is the act of organizing research activity, including the collection of data in ways that are most likely to achieve the research aims (Thorpe et al., 2002). According to MacMillan and Schumacher (2001:166), research design is a plan for choosing subjects, research sites, and procedures for collecting data to answer the research question(s). It is therefore imperative that the researcher chooses a design that is likely to achieve the research aims and objectives of the study. According to Tharenou et al.,(2007), it is ideal that the researcher starts with the question and follows it with the appropriate research design (Tharenou et al., 2007). The design allows the researcher, among other things, to come out with a general outline for the data collection and analysis of a study (Iacobucci & Churchill, 2009). The current study seeks to suggest possible strategies to minimize learners’ misconceptions and errors when learning algebraic fractions and mathematics in general. In order to answer the problem, the researcher used the inductive content analysis approach and followed the specific guidelines of this strategy throughout. The next section gives a brief description of different research designs before a detail account of the adopted research design, the content analysis design, is provided.

Some of the designs that can be employed in a qualitative research process are:

- ***Grounded theory:*** The empirical data is usually analyzed to allow for the construction or verification of a theory from the data (Creswell, 2003).

- ***Content analysis:*** In this approach, there is the identification of themes, patterns, or biases from the content of sources like books, music, television, human interaction and transcripts of conversion (Leedy & Ormrod, 2009).
- ***Ethnography:*** This approach involves the use of observation to gather information about the behaviour of a particular social group for a long period of time (Creswell, 2003).
- ***Action research:*** This is a situation where a researcher or group of researchers undertakes a research with the aim of improving the quality of the organization (Thomas, 2004).
- ***Phenomenological:*** In this approach, a researcher detects a phenomenon through how it is perceived by a subject in the study (Leedy & Ormrod, 2009).
- ***Case studies:*** This research approach explores in depth an individual, group, institution, organization or community in its natural settings (Creswell, 2003).

For the purpose of this study, a content analysis approach was adopted because it allowed for the assessment of errors which were as a result of alternative conceptions of a concept. The content analysis strategy is further discussed in the following sub-section.

3.4.2.1 Content analysis

Content analysis is a method of analyzing written, verbal or visual communication messages (Anfara & Mertz, 2006). Content analysis has also been defined as a systematic, replicable technique for compressing many words of text into fewer content categories based on explicit rules of coding (Bryan, 2004; Denzin & Lincoln, 2008; Denzin & Lincoln, 2005; Kawulich, 2005; Morse et al., 2002). Content analysis as a research method is a systematic and objective means of describing and quantifying phenomena (Kawulich, 2005; Mulhall, 2003; Punch, 2005). Content analysis allows the researcher to test theoretical issues to enhance understanding of the data. Through content analysis, it is possible to distil words into fewer themes. It is assumed that when classified into the same categories, it is more useful (Morse et al., 2002).

Content analysis is a research method for making replicable and valid inferences from data to their context, with the purpose of providing knowledge, new insights, a representation of facts and a practical guide to action (Kawulich, 2005). The aim is to attain a condensed and broad description of the phenomenon, and the outcome of the analysis is new constructs or concepts. Usually the purpose of those concepts or categories is to build up a model. The researcher makes a choice between the terms 'concept' and' category' and uses one or the other (Kynga¨ s & Vanhanen, 1999). For example, if the purpose of the study is to develop a theory, it is recommended that the term ‘concept’ be used as a substitute for ‘category’. However, in this study, when describing the analysis process, we use the term ‘category’.

Critics of the content analysis method argue that it is not sufficiently qualitative in nature and uses less detailed statistical analysis and is too simple to use (Seale, 2000). In the early days, the differentiation of content analysis was limited to classifying it primarily as a qualitative vs quantitative research method (Hsieh & Shannon, 2005:89). Silverman (2010:101) views things differently and asserts that, “it is possible to attain simplistic results by using any method whatsoever if skills of analysis are lacking. The truth is that this method is as easy or as difficult as the researcher determines it to be”.

Irrespective of criticism, content analysis has an established position in educational research and offers researchers several major benefits. One of these is that it is a content-sensitive method (Berg, 2001), and another is its flexibility in terms of research design (Harwood & Garry 2003). It is also much more than a naive technique that results in a simplistic description of data (Riemer, 2008).

Content analysis is a method that may be used with either qualitative or quantitative data; furthermore, it may be used in an inductive or deductive way. Which of these is used is determined by the purpose of the study if there is not enough former knowledge about the phenomenon or if this knowledge is isolated, the inductive approach is recommended (Lauri &Kynga, 2005). The categories are derived from the data in inductive content analysis. Deductive content analysis is also used when the structure of analysis is operationalized on the basis of previous knowledge and the purpose of the study is theory testing (Kynga¨ s & Vanhanen, 1999). An approach based on inductive data moves from the specific to the general, so that particular instances are observed and then combined into a general statement (Chinn & Kramer, 1999). A deductive approach is based on an earlier theory or model and therefore it moves from the general to the specific (Burns & Grove, 2005). This study is deductive in nature since models or categories developed by Nolting and Hodes (1998) as well as Radatz (1978) were used for this study. These inductive and deductive approaches have similar preparation stages.

Both inductive and deductive analysis processes are represented as three main stages: preparation, organizing and reporting. Despite this, there are no systematic or definite rules for analyzing data; the key feature of all content analysis is that the many words of the text are classified into much smaller content categories (Creswell, 2014). The preparation stages start with selecting the unit of analysis (McCain, 1988; Cavanagh, 1997; Guthrie et al., 2004). This can be a word or a theme (Polit & Beck, 2004). Deciding on what to analyze in what detail and sampling considerations are important factors before selecting the unit of analysis (Bryman, 2012). The sample must be representative of the universe from which it is drawn (de Vos et al., 2012). Probability or judgment sampling is necessary when a document is too large to be analyzed in its entirety (Bryan, 2004). A unit of meaning can consist of more than one sentence and contain several meanings. On that account, using it as a unit of analysis makes the analysis process difficult and challenging (Bryan, 2004; Graneheim & Lundman, 2004). On the other hand, an analysis unit that is too narrow, for example one word, may result in fragmentation (Graneheim & Lundman, 2004). Depending on the research question, the unit of analysis can also be a letter, word, sentence, portion of pages or words, the number of participants in discussion or the time used for discussion (Shenton, 2004; Polit & Beck, 2004). In this study the unit of analysis was conception, which learners exhibited through errors they make when computing mathematical problems. Graneheim and Lundman (2004) pointed out that the most suitable unit of analysis is whole interviews or observational protocols that are large enough to be considered as a whole and small enough to be kept in mind as a context for meaning unity during the analysis process. According to Robson (1993), researchers are guided by the aim and research question of the study in choosing the contents they analyse.

Next in the analytic process, the researcher strives to make sense of the data and to learn 'what is going on' (Morse & Field, 1995:37) and obtain a sense of the whole (Mulhall, 2003; Punch, 2005). According to Dey (1993), when reading the data these questions should be considered:

Who is telling?
Where is this happening?
When did it happen?
What is happening?
Why?

The aim is to become immersed in the data, which is why the written material is read through several times (Polit & Beck, 2004). No insights or theories can spring forth from the data without the researcher becoming completely familiar with them (de Vos et al., 2012). After making sense of the data, analysis is conducted using an inductive or deductive approach (Babbie, 2007).Researchers opting for inductive content analysis start with organization of the qualitative data. This process includes open coding, creating categories and abstraction (Morse et al., 2002).

Open coding means that notes and headings are written in the text while reading it. The written material is read through again, and as many headings as necessary are written down in the margins to describe all aspects of the content (Hsieh & Shannon, 2005). The headings are collected from the margins on to coding sheets (Anfara & Mertz, 2006; Dey, 1993) and categories are freely generated at this stage (Burnard, 1991). After this open coding, the lists of categories are grouped under higher order headings (Kawulich, 2005; Morse et al., 2002).

The aim of grouping data was to reduce the number of categories to similar categories (Dey, 1993). However, Dey (1993) points out that creating categories is not simply bringing together observations that are similar or related; instead, data are being classified as 'belonging' to a particular group and this implies a comparison between these data and other observations that do not belong to the same category. The purpose of creating categories is to provide a means of describing the phenomenon, to increase understanding and to generate knowledge (Berg, 2001). When formulating categories by inductive content analysis, the researcher comes to a decision, through interpretation, as to which things to put in the same category (Dey, 1993; Creswell et al., 2004).

Abstraction means formulating a general description of the research topic through generating categories (Polit & Beck, 2004). Each category is named using content-characteristic words. Sub-categories with similar events and incidents are grouped together as categories and categories are grouped as main categories (Dey, 1993; Kynga¨ s & Vanhanen, 1999). The abstraction process continues as far as is reasonable and before developing a model or possible categories. Figure 13 below shows the inductive and deductive content analysis processes.

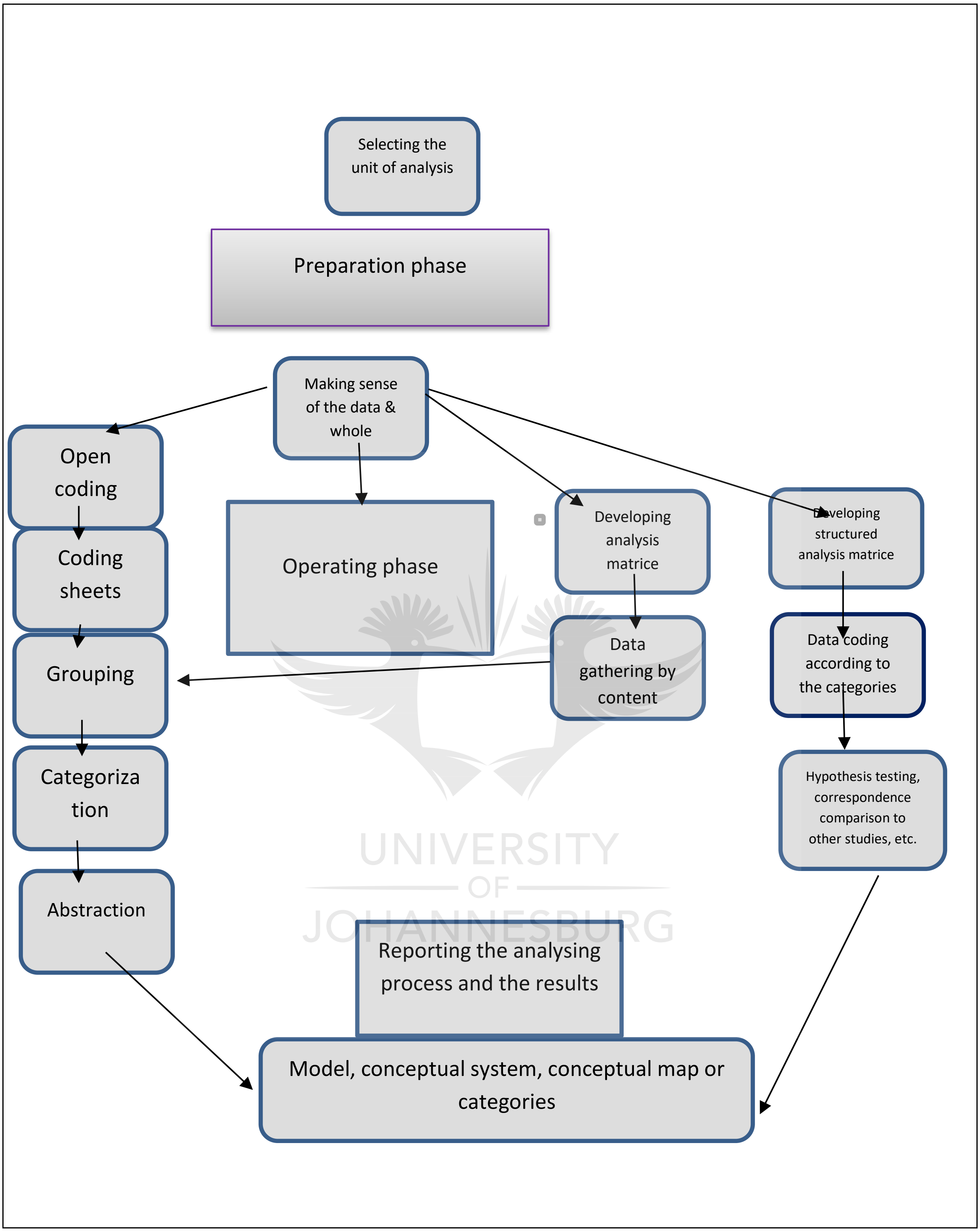


Figure 11. Preparation, organizing and resulting stages in the content analysis process. Source: Elo & Kynga (2008)

3.4.2.1.2 Deductive content analysis

This study used deductive content analysis since four categories of errors were identified with the assistance of the literature review. In this case the categories were pre-determined. Deductive content analysis is often used in cases where the researcher wishes to retest existing data in a new context (Polit & Beck, 2004). This may also involve testing categories, concepts, models or hypotheses (Marshall & Rossman, 1995). If a deductive content analysis is chosen, the next step is to develop a categorization matrix (Table 2) and to code the data according to the categories. In deductive content analysis, a structured matrix of analysis can be used, depending on the aim of the study (Kynga¨ s & Vanhanen, 1999). It is generally based on earlier work such as theories, models, mind maps and literature reviews (Polit & Beck, 2004, Hsieh & Shannon, 2005). After a categorization matrix has been developed, all the data are reviewed for content and coded for correspondence with or exemplification of the identified categories (Polit & Beck, 2004).

If the matrix is structured, only aspects that fit the matrix of analysis are chosen from the data (Berg, 2001). In this study four identified themes: language, procedural, concept and application errors were used. This can also be called testing categories, concepts, models or hypotheses (Marshall & Rossman, 1995). When using a structured matrix of analysis, it is possible to choose either only the aspects from the data that fit the categorization frame or, alternatively, to choose those that do not. In this way, aspects that do not fit the categorization frame can be used to create their own concepts, based on the principles of inductive content analysis. The choice of method depends on the aim of the study (De Vos et al., 2011).

The analysis process and the results should be described in sufficient detail so that readers have a clear understanding of how the analysis was carried out and its strengths and limitations (Babbie, 2007). This means a detail explanation of the analysis process and the validity of results. Elements of validity in content analysis are universal to any qualitative research design. There are additional factors to take into consideration when reporting the process of analysis and the results. The results are described contents of the categories such as the meanings of the categories. The content of the categories is described through sub-categories (Marshall & Rossman, 1995). Creating categories is both an empirical and a conceptual challenge, as categories must be conceptually and empirically grounded (Dey, 1993). Successful content analysis requires that the researcher can analyze and simplify the data and form categories that reflect the subject of study in a reliable manner (Kynga¨ s & Vanhanen, 1999). Credibility of research findings also deals with how well the categories cover the data (Graneheim & Lundman, 2004). It is important to make defensible inferences based on the collection of valid and reliable data (Bryman, 2012). To increase the reliability of the study, it is necessary to demonstrate a link between the results and the data (Polit & Beck, 2004). This is why the researcher must aim at describing the analyzing process in as much detail as possible when reporting the results. Appendices and tables were used to demonstrate links between the data and results. To facilitate transferability, the researcher gave a clear description of the context, selection and characteristics of participants, data collection and process of analysis. Demonstration is needed of the reliability of the findings and interpretations to enable someone else to follow the process and procedures of the inquiry. Authentic citations could be used to increase the trustworthiness of the research and to point out to readers from where or from what kinds of original data categories are formulated (Patton, 1990; Sandelowski, 1993). Table 2 shows the error analysis coding template (rubric) that was used:

Table 2. Error due to misconceptions categorization(rubric)

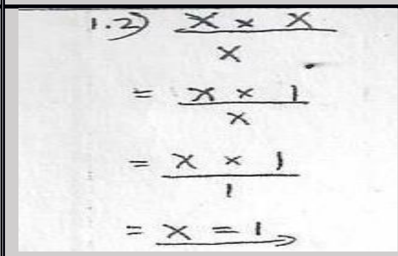
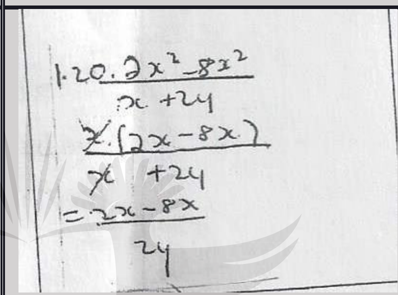
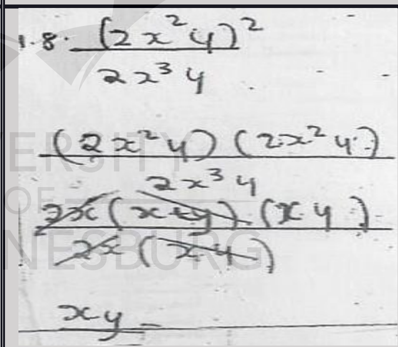
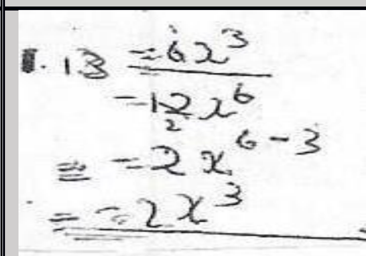
Categories	Adopted definitions	Example	Code
Mathematical language error	Mathematical language error is an error associated with mathematical symbols and vocabulary.		**LE**
Procedural error	Errors as a result of lack of understanding of the rules needed to simplify an activity.		**PE**
Concept error	Errors as a result of lack of principles or properties for conceptual understanding underpinning the topic concerned.		**CE**
Application error	Errors due to the learners' inability to apply known concept appropriately to specific or different problems		**AE**

Table 2. Error due to misconceptions Categorization(Rubric). Source: Author

There are various opinions about seeking agreement (Graneheim & Lundman, 2004), because each researcher interprets the data according to his/her subjective perspective and co-researchers could come up with an alternative interpretation (Berg, 2001). Content validation requires the use of a panel of experts to support concept production or coding issues. Graneheim and Lundman (2004) defend the value of dialogue among co-researchers to find agreement on the way in which the data are labelled.

In brief, content analysis is extremely well-suited to analyzing the multifaceted, sensitive phenomena characteristics of social issues. An advantage of the method is that large volumes of textual data and different textual sources can be dealt with and used in corroborating evidence. Especially in educational research, content analysis has been an important way of providing evidence for a phenomenon where the qualitative approach used to be the only way to do this, particularly for constructs such as misconceptions. The disadvantage of content analysis relates to research questions that are ambiguous or too extensive. In addition, excessive interpretation on the part of the researcher poses a threat to successful content analysis. However, this applies to all qualitative methods of analysis.

The next section discusses the data collection instruments adopted for this study and the reasons for using them.

3.5 DATA COLLECTION INSTRUMENTS

Several instruments for data collection exist but the best instrument must be the type that will be able to address the various research questions that have been framed (Kothari, 2004). This study employed more than one instrument for collection of data. Using more than one instrument in a research allows the researcher to view the situation from different perspectives (Oates, 2008:37). Additionally, the use of more than one instrument enables the outcomes to be substantiated or questioned through comparison with other available methods. To be able to answer the formulated research questions, the study collected data from primary sources thus from a work activity and followed up with face-to-face interview to probe for possible reasons for identified misconceptions. Figure 3.3 shows the data collection process used for this study.

Data Collection Process

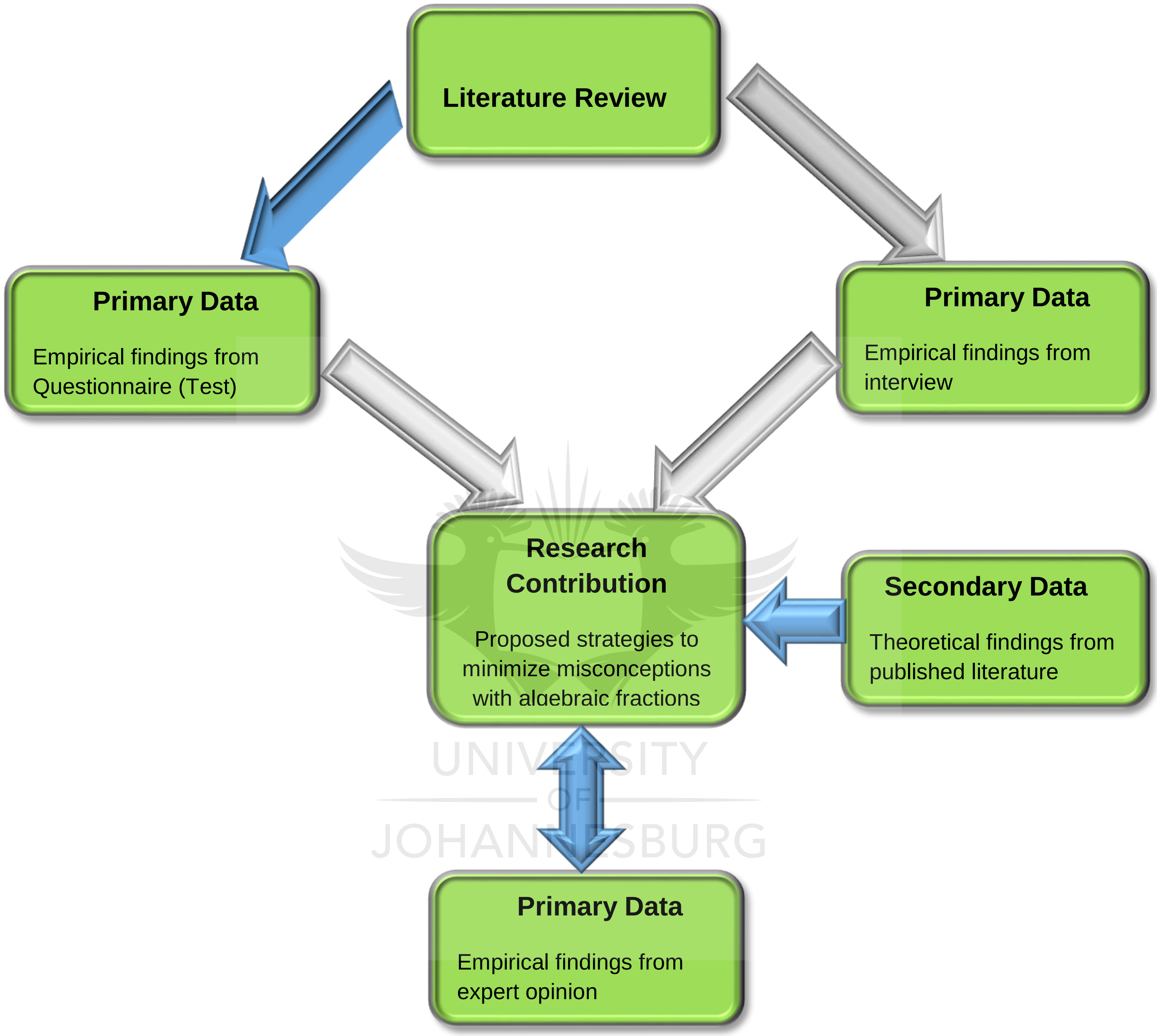


Figure 11. Data collection process for the study. Source: Author

From figure 14, the reviewed literature was used as the theoretical basis for the study. The development of the initial research framework, interview and the test were influenced by this theoretical base. The interview and test were used to collect the empirical data.

3.5.1 Primary data collection instruments

The primary sources of data for this study were a questionnaire, interviews and expert opinion. These are discussed in the next section.

3.5.1.1 Questionnaires

A questionnaire is a form or set of forms containing a number of questions in a definite order (Kothari, 2004), and are designed to address a statistically significant number of subjects. Questionnaires have the advantage of reaching a wider audience and de Vos et al. (2011) argue that a questionnaire has the ability to cover a wider geographical area and gives a higher degree of freedom.

The use of a questionnaire as a data collection instrument is characterised by risk of non-response, bias and wrongful interpretation of questions (Collis & Hussey, 2009). Additionally, there is the reluctance of respondents to answer as a result of questionnaire fatigue, thereby affecting the effectiveness of this data collection instrument (Babbie, 2007). It is therefore imperative that questions developed incorporate the use of a relevant response format. As a result, the questionnaire was pre-tested to ensure its suitability. Questions were selected from National past papers and its content was guided by the programme of assessment policy document (DBE, 2012).

3.5.1.2 Programme of assessment

The four cognitive levels used to guide all assessment tasks were based on those suggested in the TIMSS study of 1999. Descriptors for each level and the appropriate percentages of tasks, tests and examinations which should be at each level are given below in Table 3.

Table 3. Programme of assessment. Source: DBE (2012)

Cognitive levels	Description of skills to be demonstrated	Examples	Example as per the study
Knowledge 20%	1.Straight recall 2.Identification of correct formula on the information	1. Write down the domain of	$\left(\frac{x}{2}-\frac{y}{3}\right)\left(\frac{x}{2}+\frac{y}{3}\right)$

	sheet (no changing of the subject) Use of mathematical facts 3.Appropriate use of mathematical vocabulary	the function y=f(x)=$\frac{3}{x}+2$ (Grade 10) 2. The angle AÔB subtended by arc AB at the centre O of a circle	
Routine procedure 35%	• Estimation and appropriate rounding of numbers • Proofs of prescribed theorems and derivation of formulae • Identification and direct use of correct formula on the information sheet(no changing of the subject) • Perform well known procedures	1. Solve for x: $x^2-5x=14$ (Grade 10) 2. Determine the general solution of the equation $2\sin(2x-30^0)+1=0$ (Grade 11) 3. Prove that the angle AÔB subtends by arc AB at the centre O of a circle is double the size of the	$\left(\frac{2x^2y^2}{3\,y^{-2}}\right)^2$

	• Simple applications and calculations which might involve many steps • Derivation from given information may be involved • Identification and use (after changing the subject) of correct formula • Generally similar to those encountered in class	angle $A\hat{C}B$ which the same arc subtends at the circle. (Grade 11)	
Complex procedures 30%	• Problems involve complex calculations and/or higher order reasoning • There is often not an obvious route to the solution • Problems need to be based on a	1. What is the average speed covered on a round trip to and from a destination if the average speed going to the destination	$\frac{(2^{x+1})^3}{\sqrt{64^x}}$

	real world context • Could involve making significant connections between different representations • Require conceptual understanding	is 100km/h and the average speed for the return journey is 80km/h? (Grade 11) 2. Differentiate $\frac{(x+2)^2}{\sqrt{x}}$ with respect to x (Grade 12)	
Problem solving 15%	• Non-routine problems (which are not necessarily difficult) • Higher order understanding and processes are often involved. • Might require the ability to break the problem down into its constituent parts	Suppose a piece of wire could be tied tightly around the earth at the equator. Image that this wire is then lengthened by exactly one meter and held so that it is still around the earth at the equator. Would a mouse be able to crawl between the wire and earth? Why or why not? (Any grade)	$\frac{x^2-1}{3} X \frac{1}{x-1} - \frac{1}{2}$

Table 3. Programme of assessment. Source: DBE(2012)

The questionnaire includes twelve questions (see Appendix A): questions 1 and 2 were cognitive level 1 questions; questions 3, 4, 5 and 6 were cognitive level 2 questions; questions 7, 8, 10 and 11 were cognitive level 3 questions and questions 9 and 12 are cognitive level 4 problems. The responses gathered from the learners' scripts were analysed for themes and categorized.

The findings of the questionnaire are carefully described in the next chapter.

3.5.1.2 Interviews

Interviews are methods of soliciting information through the use of oral or verbal conversation between a researcher and the sample (Yin, 2009). Interviews can help researchers pursue specific problems that may result in a focussed and constructive suggestion (Shneiderman & Plaisant, 2005). According to Silverman (2016) interviews are the best way to confront a face-to-face situation. Personal attendance brings about a more confidential atmosphere and allows for a more natural way of interrogation. It also makes it easier to detect from mimics and gesticulations where follow-up questions can offer the opportunity to go in-depth of certain issues of concern and where the interview object feels uncomfortable. However, interviews have the disadvantage of usually being time consuming and resource demanding as the amount of data collected is very high and usually unstructured.

Based on the need and research design, interviews can either be focus-group, structured, semi-structured or unstructured interviews. The study adopted semi-structured interview.

- **Focus-Group Interviews**

  This type of interview is the least structured as a result of the difficulty in bringing structure in a group. However, this style of interview results in a rich sample of data. Individuals in a group are able to develop and express ideas they would not have thought of on their own (Creswell, 2010). This type of interview follows different sets of individual interviews and seeks to further explore the general nature of the observations made from the different individual interviews (Cohen et al., 2011). Focus-group interviews were not used for this study because the researcher tried to avoid situations whereby the respondents will be influencing each other.

  Structured Interviews

  With this type of interview, the interviewer adopts a set of predetermined questions which are mostly short and clearly worded. These questions are usually closed-ended and therefore demand precise answers in the form of set options read out or supplied on paper. A structured interview is easy to organize, and easy to standardize due to the fact that the same questions are asked to all subjects. This type of interview is mostly employed when the goals of the study are clearly understood and specific questions can be identified (Preece et al., 2002). Respondents were further probed to clarify their responses, hence the use of structured interviews alone was deemed not sufficient for this study.

- **Unstructured Interviews**

  This type of interview allows the interviewer to ask open-ended questions with the subject having the opportunity to freely offer his/her opinion. It requires the careful attention of the interview participants since it is like a discussion on an identified issue. The scope of the interview is not predetermined but rather determined by the interview participants. This makes it difficult to standardize the interview since each interview takes on its own format (Preece et al., 2002). It is time consuming to conduct such an interview and difficult to analyze the gathered data. The study did not use strictly unstructured interviews because there are chances to get diverted from the entire interview. In some cases, interview questions in an unstructured interview have no judgement about the answer. The interviewer or the respondents tend to divert from the topic and deviate totally away from the purpose of the study. Hence, only experienced people in unstructured interviews have to be co-opted or else the real purpose of this method of interview might go to waste (Yin, 2015). Because the respondents of this study were learners, a strictly unstructured interview was deemed not suitable for the study.

- **Semi-structured Interviews**

  This type of interview involves some aspect of structured and unstructured interviews and, as a result, employs both open-ended and closed-ended questions during interview sessions. The interview begins as unstructured by stating the core question, and the session is subsequently controlled by asking certain probing questions that require the subject to elaborate or provide more information. According to Creswell (2007) the interview style offers a framework that guides the researcher, and allows for the acquisition of additional information and other avenues explored. The use of semi-structured interviews offers an opportunity for the subjects to expand on responses and allow for further probing by the researcher. The study adopted this method because, after the test scripts of the learner-participants were reviewed, open-ended questions were asked to explore the possible source of the misconception.

For the purpose of this study, data was collected from the 60 learner-participants using the semi-structured interview approach (See Appendix B). The interviews were conducted to ascertain possible misconceptions.

3.5.1.3 Expert opinion

Expert opinion is a technique use to seek views of experts in functional areas of the outcome. Expert groups are used to evaluate the research outcome through criticisms (Molich & Jeffries, 2003). For example, experts recommend that early childhood mathematics teachers have a strong knowledge base and a strong understanding of the relationship between concepts. Therefore, the expert group gives comments and suggestions on the presented material, which is then incorporated into the findings.

Appropriate experts must be selected to ensure the appropriateness of their comments on the presented material. Experts selected for an expert review process for a study should meet four criteria, namely: knowledge and experience relevant to the research; capacity and willingness to participate; sufficient time to participate; and effective communication skills (Skulmoski, Hartman, & Krahn, 2007). For the current study, relevant experts (colleague mathematics teachers) who met these criteria were selected. The 'expert teachers' were tasked with offering comments on the various stages of selecting of questions and the findings. The expert reviews also validated the proposed strategies to minimize misconceptions when learners dealt with algebraic fractions.

3.5.3 Pilot study

After designing the questionnaire and interview guide a pilot study (see Appendix C) was conducted with some learners. The objective of the pilot study is to ensure that the questionnaire was an adequate and refined research instrument to be used to gather information from respondents (Hofstee, 2006). The pilot study was critical in refining the questionnaire to ensure that the most appropriate and relevant responses were collected using the questionnaire. Some suggestions were made by respondents and the expert teachers regarding the length, ambiguity, form, content, and phrasing of some questions. Modifications and refinements were therefore made to the questionnaire in line with these suggestions and feedback obtained from the pilot study.

The next section describes the steps taken to obtain the research sample.

3.6 ENTRY INTO THE RESEARCH SITE

The study area, Lady Frere, is a settlement in the Chris Hani District Municipality in the Eastern Cape Province of South Africa. It is located on the Cacadu River, 51km north-east of Queenstown and 53km south-west of Cala. With a population of 4,024 in 2011, Lady Frere covers an area of 22.1km (8.5sq metres). Figure 15 shows a map of the Lady Frere District.

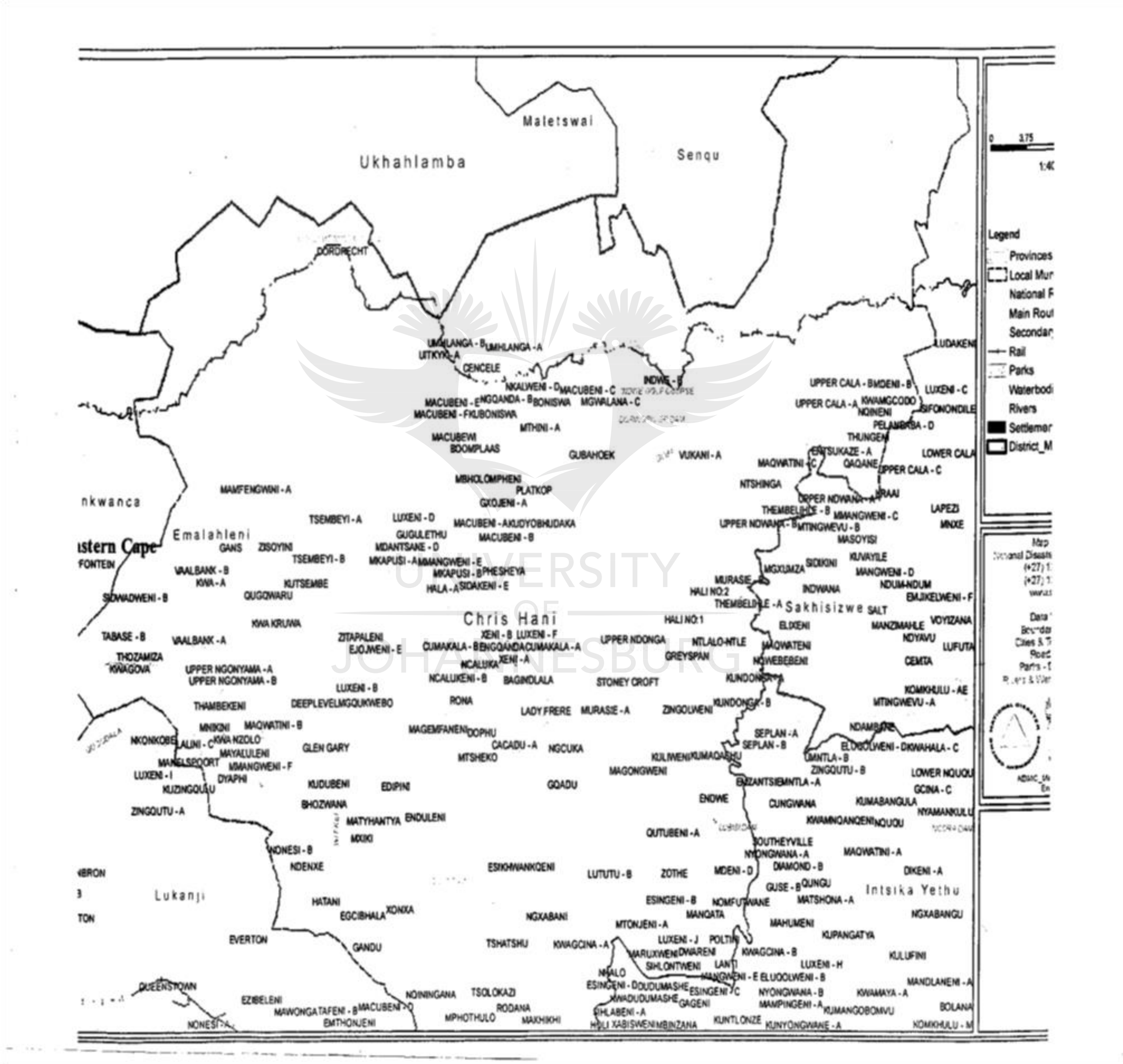


Figure 12. Map of Lady Frere District. Source: Chris Hani Municipality

3.6.1 Overview of the study area

Lady Frere District has 26 secondary schools of which two are boarding schools. Sunshine Secondary School is one of the two boarding schools. With a population of approximately 1000, the school caters for learners from grades 8 to 12. There are 256 (34% of grade 10 learners), 240 (32% of grade 11 learners), and 160 (21% of grade 12 learners) studying mathematics in the school. The high percentage of learners in grade 10 doing mathematics makes it ideal for the current study.

3.7 SAMPLE IDENTIFICATION AND SELECTION PROCESS

Babbie (2008) defines sample as a sub-group of the target population that the researcher plans to study for generalization about the target population. A target population is a group of individuals with some common defining characteristic that the researcher can identify and study. In this study, the target population was the grade 10 mathematics learners. The researcher used 120 learners for the test and the interviews as well. All the interviews and questionnaires were administered in English. The duration of the test was 30 minutes and they were conducted during study time, thus between 16:00 and 18:00. Interviews were also conducted after a week.

Sampling is the process of gathering information concerning an entire population by targeting only a portion of it (Kothari, 2004). It is therefore critical to employ a sampling strategy that is suitable for the study.

3.7.1 Sampling strategy

Sampling strategy is the process adopted in order to obtain a sample from a population under study (Kothari, 2004). According to Marshall and Rossman (2010), poor sampling may lead to research of low credibility and trustworthiness. The sample is selected from a part of the population and conclusions may be drawn about the population (Babbie, 2008). There is, therefore, the need for the sample of the research to be representative of the population of interest (Creswell, 2007). The appropriateness of a chosen sample will ensure a valid and reliable conclusion.

For the purpose of the current study, a non-probability sampling strategy was preferred. This is because the study sought to provide sufficient description about the context of the sample. There are many types of non-probability sampling strategies. A brief description of these is provided below (Babbie, 2008; Marshall & Rossman, 2010):

- **Quota sampling:** In this method of sampling there is the adoption of pre-defined characteristics in the selection of units of the sample. As a result, the entire sample will have the same distribution of characteristics as the population under study.
- **Snowball sampling:** In this method of sampling subjects for the inclusion of the sample in the study may use their social networks to refer potential subjects. This method is also known as 'chain referral sampling'.
- **Purposive sampling:** This method involves the employment of the researcher's personal experience, skills, or previous research findings to choose the most appropriate sample to answer the research questions. This method is also known as 'judgmental sampling'.
- **Convenience sampling:** This method of sampling involves the selection of the most accessible subjects for use in the study. Although this strategy of sampling may be least demanding in terms of effort, money and time, it is likely to produce poor quality data and lacks intellectual integrity if care is not taken.

For this study, purposive sampling was used to select the learners that best fit the purpose of the study. This sampling technique was used because it allows for the access of basic data and trends regarding the study. Additionally, it gives an indication of a particular phenomenon occurring within the sample and relationships among different phenomena. The next sub-section explains how the sample size was calculated.

3.7.2 Sample size

In any research, it is ideal to select and test the whole population, but in most cases the population is too large to include all individuals of the population (Silverman, 2016). The sample strategy and sample size used in research can therefore be affected by the availability of resources (Saunders et al., 2009), and, in this regard, the current research study's limitations concern financial support and time availability for data collection and analysis. It was thus impractical to collect data from the entire population and hence the usage of a sample strategy was essential.

The purposive sampling method was used to identify the participants. This means that participants were selected intentionally based on their experience with the topic under study (Creswell, 2003:125). The selected secondary school is one of the boarding schools in Lady Frere District, Eastern Cape. The population of the school is approximately 1000 learners from grade 8 to 12. There are 160 learners studying mathematics in the school in grade 10. These 160 learners at the school who study mathematics in grade 10 form 15.4% of the total grade 10 learners in the district. Hence, selecting targeted secondary school grade 10 mathematics learners as a sample school for the study is laudable with reference to the sample size. The study used grade 10 learners who had been taught algebra in the first four weeks of first term. Learners were interviewed to ascertain identified errors and misconceptions. 135 sampled learners were stratified into three groups of high achievers, middle achievers and low achievers. In this study, learners who scored above 60% in their previous mathematics tests were classified as high achievers; those who scored between 40% and 59% were classified as middle achievers and those with scores between 0% and 39% were low achievers. This ensured that I considered the mixed ability in the class.

3.8 DATA COLLECTION

The purpose of data collection for a chosen research using the most relevant and applicable method (qualitative and quantitative) is to be able to address a problem scientifically and appropriately to achieve the research goals (Boaduo, 2010:109). The researcher administered a questionnaire, conducted interviews and used notes as data collection tools (Mouton, 2005). According to Creswell (2014), field notes are the researcher's record of what has been observed in the field, descriptions of individual responses, the setting and what happened during the recording of a conversation. In this study, the researcher administered the questionnaires personally by hand and conducted face-to-face interviews to ascertain possible learner misconceptions on algebraic fractions.

Creswell (2014) defined an interview as a conversation between two or more people and it consisted of three elements, namely, the interviewer, the interviewee and the context of the interview including issues or questions raised. The interview technique is flexible and adaptable and it can be used with many different problems (McMillan & Schumacher, 2010:150). The interviewer realized the verbal and non-verbal behaviours of the interviewee and was able to motivate and persuade the participant to provide the required information. The researcher established positive relationships with the interviewees and understood the context of their responses. On the other hand, Babbie (2007) opines that a questionnaire is economical; it has standardized questions and can be written for specific purposes. By using questionnaires the participants were assured of anonymity and they could respond to the questions in their own time without any form of pressure.

3.9 DATA ANALYSIS TECHNIQUES

Data analysis is a critical stage in any research process. According to Yin (2015), data analysis involves breaking down data into manageable themes, patterns, trends and relationships. Singh (2006:223) also describes data analysis as breaking down existing complex factors into parts, putting parts together for the purpose of interpretation. It begins with the design of the study, and is followed with the data collection process after which the analysis becomes the focus and ends with the writing of the report. According to Yin (2009) data analysis includes processes such as thematic and content analysis. The categories with the same labels were compared, and further notes were made to increase the number of findings.

The data collected from participants through the questionnaires were interpreted and analyzed using thick descriptions and some descriptive statistics tables and statistical graphs. The responses from the interviews were tabulated, recorded and later transcribed to give an accurate interpretation of what was revealed in the data. Patterns in the data were considered when interpreting the data in order to give precise interpretation of the information revealed by the collected data. In this study, data patterns were coded with the four identified error categories. The next subsections explain how data collected was analyzed.

3.9.1 Primary data analysis

The analysis of the qualitative data involved two major steps: data preparation and descriptive statistics. The process of data preparation begins with editing, followed by coding and finally data is transformed into a database structure. Descriptive statistics is then employed to explain the basic features of the data collected to help show a summarized form of the data. This stage of the analysis gave the researcher an insight into the wording of the questions and the respondents' understanding of the questions.

The qualitative data was analyzed using content analysis. This approach enabled the researcher to report the experiences of the participants gathered during the interview process. Content analysis is regarded as a useful analytic approach to analyze qualitative data and present rich, detailed, and complex accounts of data (Fereday & Muir-Cochrane, 2006; Braun & Clarke, 2006). The approach involves "identifying, analyzing and reporting patterns (themes) within data" (Braun & Clarke, 2006:79). It has proved to be flexible and an effective analysis strategy for qualitative data (Attride-Stirling, 2001; Tuckett, 2005; Braun & Clarke, 2006). Hence, the researcher was of the conclusion that its application in the current study would be suitable and beneficial.

Changes to the qualitative data obtained from expert opinion were made according to the feedback they provided. The outcome of the expert review helped to refine the proposed strategies to minimize misconceptions and errors in algebraic fractions.

Analysis tries to establish a better understanding of the data through an assessment of the relationships between concepts, constructs, or variables, and to identify the existence of trends and themes in data (Yin, 2009). As a result, patterns were thus drawn from the content analysis which was incorporated into the development of the proposed strategies.

3.10 VALIDITY AND TRUSTWORTHINESS OF THE STUDY

Trustworthiness of a research concerns the extent to which the data and its analysis are believable and trustworthy. According to Holloway and Wheeler (2002:256) a study is authentic when the strategies used are appropriate for the true reporting of the participants' ideas, when the study is fair, and when it helps participants and similar groups to understand their world and to improve it. Authenticity was achieved through the researcher's fairness to all subjects and getting their consent throughout the study.

The validity of a test instrument is equally important as its reliability. If a test does not serve its intended function well, then it is not valid. According to Cohen et al. (2011), there are four main types of validity: content, concurrent, predictive, and construct. Content validity addresses how well the content of the test samples the subject matter. Concurrent validity measures how well test scores correspond to already accepted measures of performance. Predictive validity deals with how well predictions made from the test are confirmed by subsequent evidence. This type of validity is not directly relevant to the current study. Construct validity is about what psychological qualities a test measures. This type of validity is primarily used when the other three types are insufficient. In this study, to preserve content validity, the content of the test (see Appendix A) was prepared by making use of national past mathematics exam papers. The content of the test was discussed with two mathematics teachers and their suggestions were included prior to the first administration of the test. Also, similar test construction procedures in the CAPS Assessment Statement Guidelines were consulted when preparing the test items.

Creswell (2013) points out that the trustworthiness in a research can be achieved by using four criteria: credibility, transferability, dependability and conformability. These are explained below:

- *Credibility:*

  Credibility is the level to which the data and data analysis are believable and trustworthy. Holloway and Wheeler (2002:255) point out participants must be able to understand the meaning they assign to situations and the truth of outcomes in their own social context. Credibility was ensured through the use of multiple data collection strategies and the use of expert review.

- *Transferability:*

  Transferability concerns the extent to which the outcomes of a study can be applied in other settings. This can be achieved by presenting a detailed description of the settings under study to give enough information for a good judgment of the applicability of the findings to other settings (Seale, 1999). This study achieved transferability as the proposed strategies to minimize misconceptions in algebraic fractions can be applied to other mathematical concepts.

- *Dependability:*

  According to Oates (2008:294), dependability concerns how well the research process is recorded and the data documented. There is the need for consistency and accuracy for a study to be considered dependable. In the case of the current study, dependability was achieved through the use of published literature and feedback from experts in the area of the study. The use of theories and models add dependability since they have been tested in several previous studies.

- *Conformability:*

  Conformability concerns the extent to which the findings of any research can be corroborated by others. That is, there is the need for the researcher to substantiate how constructs, themes and interpretations were achieved. This study used questionnaires and interviews undertaken to confirm the findings. In addition, the inclusion of feedback from experts led to the development of the proposed strategies.

These four criteria were applied to ensure the trustworthiness of this study.

The issue of ethics is of great concern in any research that involves human (Saunder et al., 2009). The next section, therefore, explains the ethical considerations that were put in place to tackle the ethical concerns for this study.

3.11 ETHICAL CONSIDERATIONS

In the planning of the research design and methodology for this research, the ethical implications which could adversely affect participants and the organizations were considered. According to Creswell (2008), practising ethics is a complex matter that involves more than just following a set of static guidelines such as those from a professional association. The research methodology adapted was therefore in agreement with suitable ethical norms. Sticking to ethical norms in research is of importance in any research (Kothari, 2004). Ethically conducting research requires researchers to actively interpret these principles for their individual projects, tailoring these ethical guidelines to suit the unique contexts of their research. McMillan and Schumacher (2006) affirm that in any educational research that focuses mainly on human beings the researcher is ethically responsible for protecting the rights and welfare of the participants which involves issues of physical and mental discomfort, harm and danger.

In these section ethical issues pertaining to the respondents in terms of permission, informed consent, and rights of participants, confidentiality and anonymity were discussed.

3.11.1 Permission

The researcher requested a recognition letter from the University of Johannesburg in the Faculty of Education. This was taken to the Lady Frere Education District Office where the school chosen for the study is situated. The researcher also wrote letters to the school to request permission to conduct the study (see Appendix D).

3.11.2 Informed consent

Learners who participate in a research project should give their informed consent to do so (Crowl, 1996:76). Informed consent is achieved by providing participants with an explanation of the opportunity to terminate their participation at any time with no penalty and full disclosure of any risks associated with the study (McMillan & Schumacher, 2006:143). The participants were informed about the purpose of the study and all the aspects that might influence their willingness to participate in the study. The researcher asked for informed consent forms from the University of Johannesburg (see Appendix E). The researcher distributed the informed consent forms to the targeted sample to complete.

3.11.3 Rights of participants

Bryman (2012) suggests that respondents in a research project should be allowed to exercise their right to be part of the research or not. Participants must be protected from physical and mental discomfort, harm and danger (McMillan & Schumacher, 1993:183). The participants were informed about all the possible risks involved in the study. They participated in the study and they were informed that they can withdraw any time from the study (see Appendix E).

3.11.4 Confidentiality

Babbie (2007) suggests that information obtained about the participants must be held confidential unless otherwise agreed on, in advance, through informed consent. This means that no one has access to the participants except the researcher. The researcher ensured that the data collected cannot be linked to the individual participants and assured the participants that the data collected from the interviews is only for academic purposes (see Annexure E). The settings and the participants should not be identifiable in print (de Vos et al., 2011). The respondents were assured that all information will be given anonymously to ensure their privacy.

3.12 CONCLUSION

The chapter presented the methodological approach adopted in the study and the justification for adopting a particular methodology. It first discussed available research paradigms underlying all research, namely positivism, interpretivism, critical postmodernism and pragmatism. This was followed by arguments to support the use of a particular paradigm relevant for this study. This study used two paradigms, pragmatism and interpretivism. Pragmatism was used because it focuses on the development of practical applications, consequences, relevance and usefulness. Interpretivists' beliefs were also adopted to help interpret the findings and results of the interviews to gain an understanding of possible reasons for learner misconceptions. The research design chosen for this study was the content analysis methodology which was also presented.

The chapter further discussed the instruments that were used to collect the empirical data, and how the collected data was analyzed. In this study, the instruments used to collect the primary data were a questionnaire (test), interviews, and expert opinion. The secondary data was sought through the review of relevant literature. The questionnaire was analyzed using descriptive statistics and the interviews through content analysis. Recommendations from experts were used to evaluate and refine the developed strategies. Moreover, the sample selected for the collection of the data is also outlined in this chapter.

Finally, measures taken to ensure the trustworthiness of this study were evaluated. All discussions on credibility, conformability, dependability and transferability will allow for a successful study, resulting in the development of the proposed strategies to minimize misconceptions and errors in algebraic fractions and mathematics in general. The next chapter (chapter 4) describes the findings from the questionnaire and interviews.

Chapter 4: DATA ANALYSIS, RESULTS AND DISCUSSIONS

4.1 INTRODUCTION

This report is about an in-depth study of the misconceptions and errors grade 10 learners encounter when tackling problems involving simplifying algebraic fractions. The purpose of this study was to identify the various reasons for learners' misconceptions when simplifying algebraic fractions, identifying the misconceptions the learner may encounter, group the learners' errors arising from the misconceptions and finally recommending ways in which the misconceptions identified can be minimized. The direct-content analysis was used in analysing the test scores.

The need to study this topic is due to the fact that misconceptions in learning are not inborn (Orton, 1983). They are acquired in the process of learning due to a range of reasons. Therefore, there was the need to evaluate those reasons. In this section, the levels of the cognitive demands of the questions used for the study were briefly discussed, a brief account was provided of how data coding was done before zooming in to error categorization and, finally, the findings were outlined and discussed.

4.2 DIFFERENT LEVELS OF COGNITIVE DEMANDS IN TASKS

An important practical step that every teacher takes daily in working towards assessment goals is the selection of tasks for learners to work on. Teachers do this with more or less thought on different occasions. The tasks that learners work on will influence their experiences of mathematics and are vital in their construction of knowledge and their mathematical development. It is important that mathematics teachers are able to choose tasks carefully and thoughtfully in order to achieve their goals for their learners' learning. This is particularly the case when working with new concepts of mathematics and learning.

Mathematical cognition is a relatively complex mental process involving various brain functions with the focus on quantity identification, comparison, and calculations. Mathematical cognition is necessary in understanding mathematical constructs as it relies heavily on the learner's ability to reason at abstract levels (Luneta, 2016). As the learner progresses through the mathematical curricula, he/she is required to think at higher and higher abstraction levels. A learner's mathematical cognition and understanding of mathematical constructs is highly dependent on his/her teacher's knowledge base and ability to disseminate that knowledge effectively to the students.

In ensuring reliability of the research study questions, the study adopted the Department of Education requirements for setting questions. The CAPS (DBE, 2015:16) assessment policy stipulates that assessment tasks have to follow the four cognitive levels suggested by the TIMSS study of 1999. These are knowledge (25%), routine procedure (35%), complex procedure (30%) and problem solving (15%). The four cognitive levels used in CAPS correspond directly with the subject Assessment Guidelines of 1999 TIMSS taxonomy of categories of mathematical demand (Stols, 2013:13). Column one of Table 4.1 depicts the four cognitive levels with their respected weighting considered for the study as shown below.

Cognitive levels	Questions	Number of questions
Knowledge 20%	1 & 2	2
Routine procedures 35%	3, 4, 5 & 6	4
Complex procedures 30%	8, 7, 10 & 11	4
Problem solving 15%	9 & 12	2

Table 4. Classification of test questions used for the study in relation with cognitive level assessment. Source: Author

Out of the twelve questions used for this study, questions 1 and 2 were considered knowledge questions because the two questions required learners to recall, identify and use correct formula as well as make use of appropriate mathematical vocabulary. For example, both questions 1 and 2 both require learners to recall the use of perfect square to simplify the problem.

Q1	$\left(\frac{x}{2}-\frac{y}{3}\right)\left(\frac{x}{2}+\frac{y}{3}\right)$ $\frac{x^2}{4}-\frac{y^2}{9}$	Q2	$36y^2$ - $\frac{x^2}{25}$ (6y - $\frac{x}{5}$)$(6y+\frac{x}{5})$

Four questions, question 3 to 6, were also considered routine procedure questions because these questions require learners to perform well known procedures derived from given information and do simple application calculations which might involve many steps. For instance, question 5, a routine procedure question, requires learner respondents to simplify the question as follows:

Q 5	$=\frac{10^{n-1}.12^{n+1}}{8^n.15^{n-1}}$ $=\frac{(2.5)^{n-1}.(2^2.3)^{n+1}}{(2^3)^n.(3.5)^{n-1}}$ $=\frac{2^{n-1}.5^{n-1}.2^{2n+2}.3^{n+1}}{2^{3n}.3^{n-1}.5^{n-1}}$ $= 2. 3^2$ $= 18$

Learner respondents were expected to make use of rules and perform well known procedures. For instance, changing of bases to their simplified form, before expanding by multiplying the exponents as a known routine procedure.

Four other questions, questions 7, 8, 10 and 11, were also considered complex procedures because they require learners to simplify questions that involve complex calculations and/or higher order reasoning. There is often not an obvious route to the solution of complex questions. Questions also need to be based on a real-world context and require conceptual understanding.

Questions 7, 8, 10 and 11 met the above criteria. For instance, question 11 requires learners to know the principles and properties underpinning the context of this cognitive question.

Q11	$$\frac{3}{x-4}-\frac{2}{x+3}-\frac{21}{x^2-x-12}$$ $$\frac{3}{x-4}-\frac{2}{x+3}-\frac{21}{(x-4)(x+3)}$$ $$\frac{3(x+3)+2(x-4)-21}{(x-4)(x+3)}$$ $$\frac{3x+9+2x-8-21}{(x-4)(x+3)}$$ $$\frac{5x-20}{(x-4)(x+3)}$$ $$\frac{5(x-4)}{(x-4)(x+3)}$$ $$\frac{5}{x+3}$$

The fourth cognitive level questions require learners to simplify questions that are non- routine problems (which are not necessarily difficult). Higher order understanding and processes are often involved. Based on the criteria stated above, questions 9 and 12 were considered appropriate for this cognitive level. For instance, question 12 is not necessarily difficult; however, requires some higher order thinking ability to factorize cubic expressions.

Q12	$\frac{x^3+1}{x^2-x+1} - \frac{4x^2-3x-1}{4x+1}$ $\frac{(x+1)(x^2-x+1)}{x^2-x+1}\frac{(4x+1)(x-1)}{4x+1}$ $x+1-(x-1)$ 2

The next section provides an account of how data coding was carried out.

4.3 DATA CODING

Ascertaining learners' misconceptions underlying the errors can be a challenging task to carry out. However, Li (2006:196) proposed two categories to aid with identification:

1. *The error type should be found consistently in different problems or contexts.*
2. *The type of error should appear consisitently in different questions or grades. Misconceptions of mathematics knowledge are held by many people over a long time. It is therefore expected that errors caused by misconceptions should not occur haphazardly.*

In this study data collection methods were employed in the bid to identify the misconceptions developed by learners in simplifying algebraic equations. Test questions (Appendix A) issued had twelve questions involving simplification of algebraic equations. The questions were free answer questions so as to monitor the consistency of learners' answers. The answer scripts were then marked paying close attention to the errors that the learners made. The errors discovered were analysed and coded with the rubric developed for this study (see table 2 below) (Kilpatrick, Swafford & Findell, 2001).

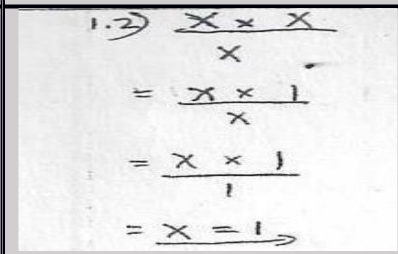
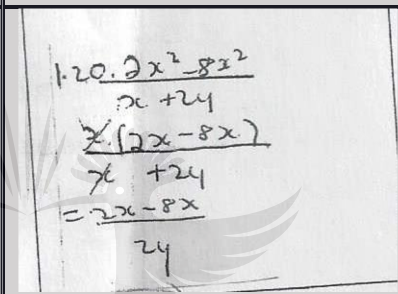
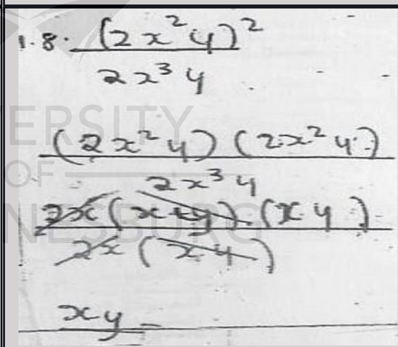
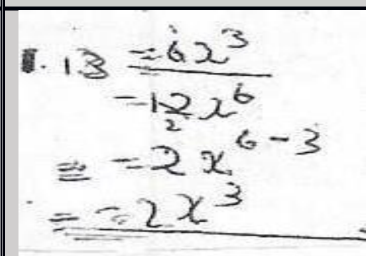
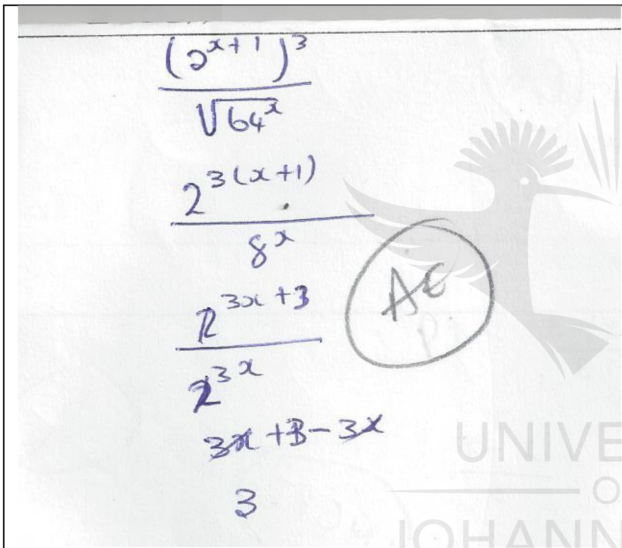
Categories	Adopted definitions	Example	Code
Mathematical language error	Mathematical language error is an error associated with mathematical symbols and vocabulary.		**LE**
Procedural error	Errors as a result of lack of understanding of the rules needed to simplify an activity.		**PE**
Concept error	Errors as a result of lack of principles or properties for conceptual understanding underpinning the topic concern.		**CE**
Application error	Errors due to the learners' inability to apply known concept appropriately to specific or different problems		**AE**

Table 2 : Errors due to misconceptions categorization. source: author

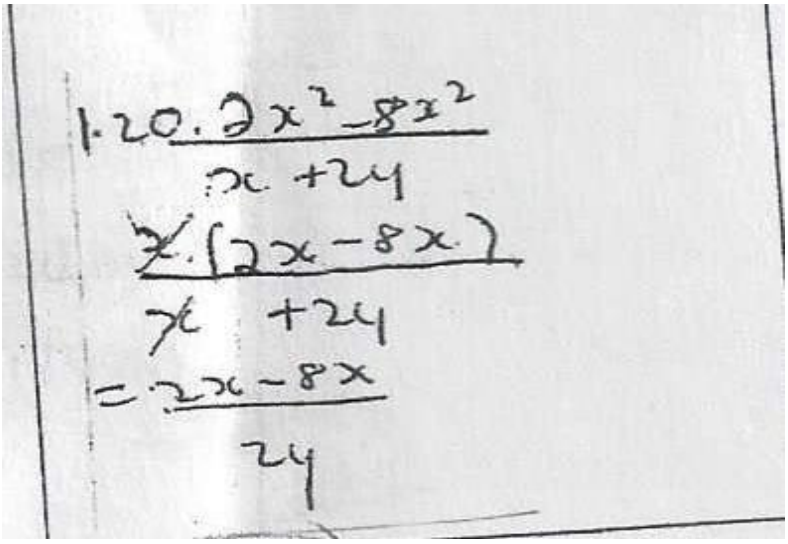
Learners' work analysed had some of these errors, learners swapped variables with numerals without substantiating them. A wide range of reasons have been discovered to be key contributors to errors in mathematics. Some of them include inadequate prior knowledge of algebraic fractions, language barriers, level of cognitive thinking being below the concept being taught.

In the table, above, the researcher has categorized the most common errors and their coding for analysis purposes. Application errors in mathematics are defined as errors that occur due to a learner's inability to apply already known concepts to one or different problems (Hodes & Nolting, 1998). For example, the first column depicts a learner's work and column 2 the supposed solution.

	$\frac{(2^{x+1})^3}{\sqrt{64^x}}$ $\frac{2^{3x+3}}{(8^{2x})^{1/2}}$ $\frac{2^{3x+3}}{(2^{3x}}$ $2^{3x+3-3x}$ 2^3 8

Vignette A

The learner was on course with the first two steps of the solution but misapplied the rules governing division of exponents. In this study, this type of error was coded **AE.** Concept errors in a mathematical context occur when the learner does not have the conceptual foundations underpinning the topic concerned. Errors involving the definition and application of mathematical concepts include the omission of essential attributes of a particular class or inclusion of non-essential elements in the definition. The concept errors were assigned the code **CE.** Procedural errors occur when a learner does not fully understand the



Vignette B

principles required to simplify a mathematical expression. In vignette B, for instance, the learner does not fully understand the rules involved in simplifying the algebraic expression ($2x^2$-$8x^2$) which leads to an incorrect solution to the problem. Additionally, the learner is unable to divide fractions correctly as is evidenced by ignoring the addition symbol in the denominator x+2y. Procedural errors in this case have been give the code **PE.** Mathematical language errors involve errors and misconceptions in the meanings associated with mathematical vocabulary and symbols. For example, the interchangeable usage of the terms ‘number’ and ‘digit’ may confuse learners. In this study, mathematical language errors were assigned the code **LE.** A sample of a learner’s script is provided in Appendix F.

To analyse learners’ misconceptions and errors - although time-consuming - all the 136 learner-participants were interviewed, usually after school hours. In qualitative studies, a focus group is a data collection method where a group of people gives their perceptions, beliefs, and opinions about a particular concept or topic (Krueger & Casey, 2014). Focus groups utilize a semi-structured interview approach where participants can bring new ideas to the discussion as a result of previous answers or the interviewer’s prompting. The researcher was the group moderator allowing for standardization of questions across groups of learners. For reliability check, a mathematics colleague of mine also independently selected 10% (14 participants) and interviewed them (Shernoff et al., 2016). Crusan et al. (2016) also employed a similar strategy to verify the reliability of the data collected for their study. Sometimes there was more than one error on a single answer. Based on our own hypotheses we compared the categories with each other to arrive at consensus. In an instance whereby an answer belongs to more than one category, we included them into the more likely category on a consensus bases (Ott & Longnecker, 2001).

The analysis of the data started with question by question analysis.

4.4 CATEGORIZATION OF ERRORS AS A RESULT OF POSSIBLE MISCONCEPTIONS

In this section, question by question were analysed with the four categories of the errors and associated misconceptions found in the 136 scripts, and how misconceptions manifested in the responses. The four error types are: Language error, Procedural error, Conceptual error and Application error.

Theme: Error analysis on Questions 1 & 2

According to the Curriculum and Assessment Policy Statement (CAPS) (DBE, 2011) document questions 1 and 2 of the questionnaire used were cognitive based *Knowledge cognitive level*. Learners were expected to make use of a formula sheet, do straight recall, and make appropriate use of mathematical vocabulary. Learners were to recall and make use of square identity understanding to simplify questions 1 and 2.

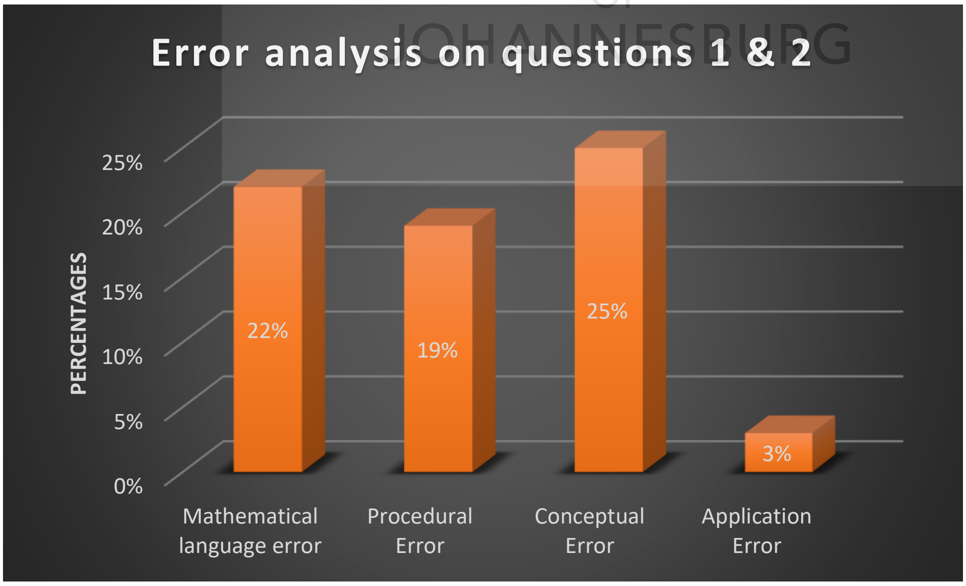
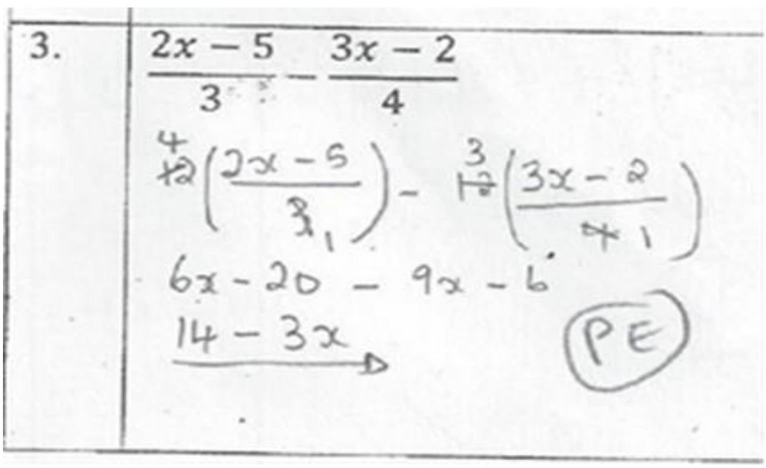


Figure 13. Error analysis on questions 1 & 2. Source: Author

Out of 295 items of questions 1 and 2 answered by the learner participants a quarter of them showed lack of the principle of square identity to simplify these questions. The graph depicts that language errors formed 22% of the errors displayed by the learners, whereas procedural errors formed 19%. This is an indication that one-fifth of the participants also lacked the understanding of the rules needed to simplify both questions 1 and 2. Almost 30% of the learners had questions 1 and 2 correctly answered, the highest success rate compared to any other question. This was expected because "knowledge cognitive level" based questions were not demanding compared to the other three cognitive levels. Learners at this cognitive level are expected to have learnt the basic mathematical foundations of addition, multiplication, subtraction, and division of figures whether whole numbers or fractional numbers. According to Egodawatte (2011) errors manifesting in the solving of these questions are probably due to insufficient explanations by the teacher which causes misunderstanding or incomplete comprehension among the children. This can be attributed to the fact that mathematics requires abstract thinking which may be problematic for early childhood teachers to construct abstract mathematical concepts in the natural language comprehensible to children.

Theme: Error analysis on questions 3, 4, 5 & 6

According to the policy document questions 3, 4, 5 and 6 are possible routine procedure cognitive level questions (DBE, 2015). Routine procedure questions demand that learners have the skill to estimate correctly, be able to proof theorems and derive formulas, identify and make use of correct formulas, perform well known procedures and be able to derive from given information. Test questions used in the study required learner participants to make use of these skills. Vignette C below is an example of a routine procedure cognitive level question.



Vignette C

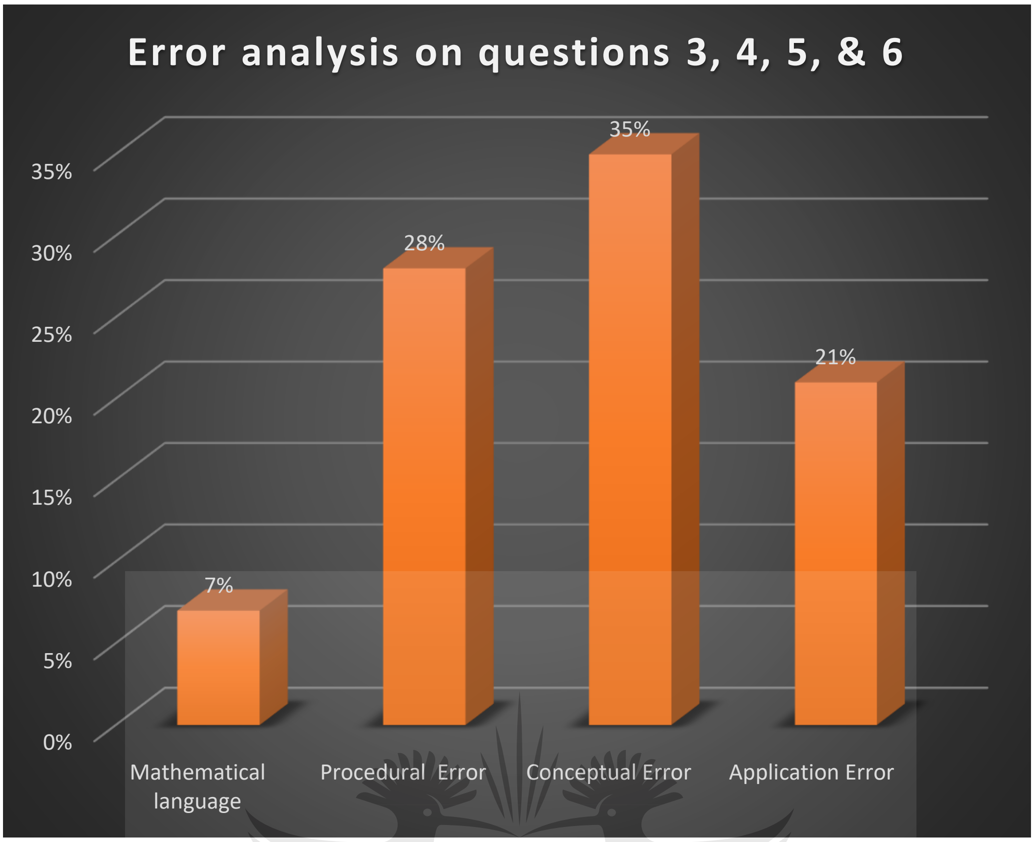


Figure 14. Error analysis on question 3,4,5 & 6. Source: Author

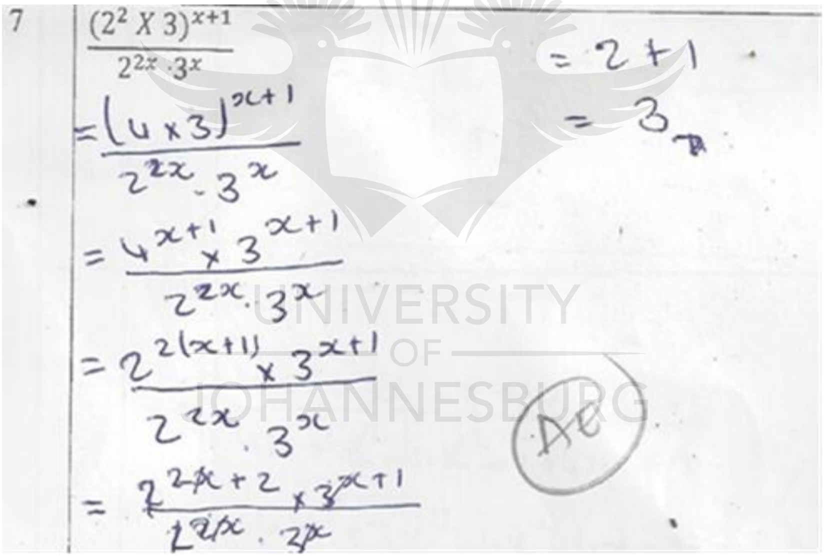
Reading from the graph, learner difficulties with questions 3, 4, 5 and 6 were basically conceptual, procedural and application errors. 35% of the errors from question 3 to 6 were due to conceptual error. This may suggest that the learners lack the principles or properties needed to solve questions 3 to 6. Question 3 for instance demands knowledge similar to addition and subtraction, and addition of fractions. Van de Walle et al. (2015) described such errors and their associated misconceptions as instrumental understanding - understanding which is not well connected (or rich). The difficulty that algebraic fractions poses to learners was also echoed by Brown and Quin (2006). Brown and Quinn (2006) assert that, "If algebraic fractions are for everyone, then a bridge must be built to span the gap between arithmetic and algebraic fractions. The building materials being conceptual understanding and the ability to perform arithmetic manipulations of algebraic fractions".

There are a number of principles that appear in literature on effective teaching and learning of mathematical concepts such as algebraic fractions. According to Lappan and Briars (1995), among these principles is a problem-oriented curriculum that focuses on ideas before skills. Teacher actions that are effective include deriving concepts, using cooperative group work, encouraging frequent mathematical communication, and using multiple representations and multiple strategies.

Theme: Error analysis on questions 10, 11, 8 &7

Based on the criteria for task cognitive levels, test questions 7, 8 10 and 11 used in this study were complex procedure questions. Complex procedure cognitive level questions demand that learners solve problems which involve complex calculations and/or higher order reasoning questions. Questions usually do not have obvious routes to the solution, and questions may involve making use of significant connections between different representations. Complex procedure and cognitive level questions require conceptual understanding.

The graph below indicates that approximately 6 out 10 (57%) learners who answered those questions could not handle complex procedure cognitive questions**.** Vignette D is an example of a complex cognitive level question used for this study:



Vignette D

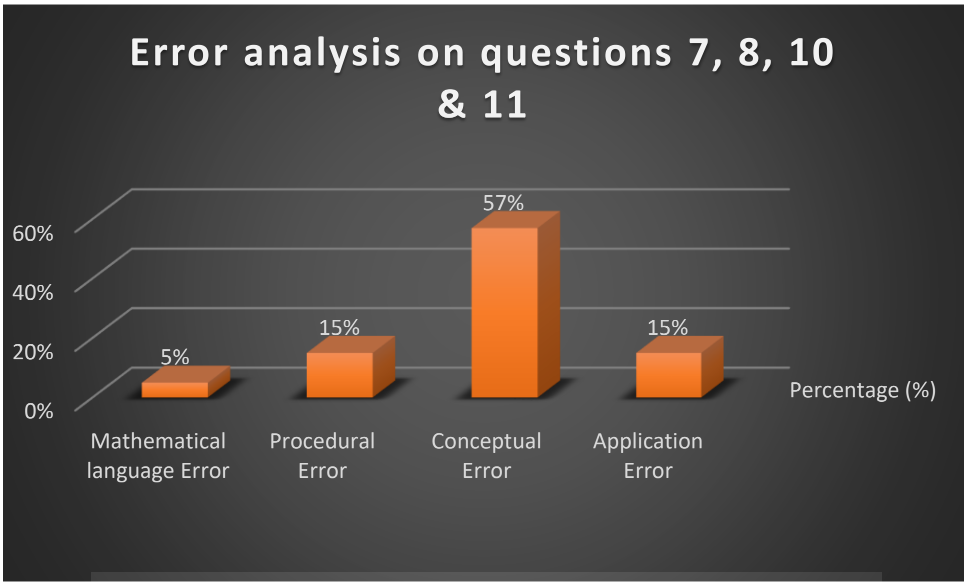


Figure 15. Error analysis on questions 7, 8, 10 & 11. Source: Author

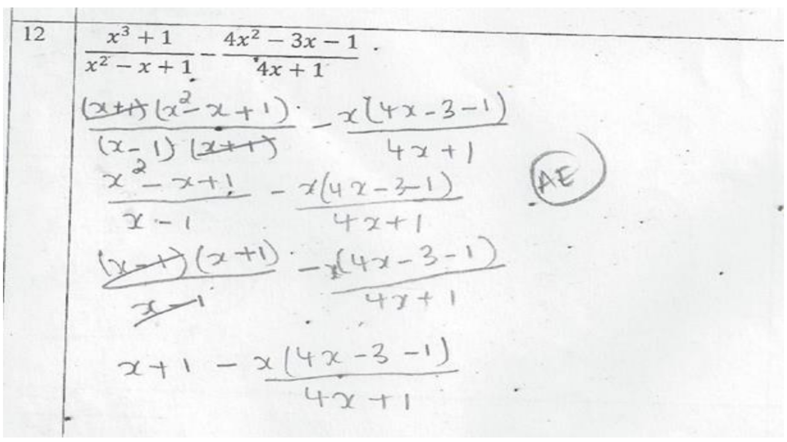
As the graph shows, at this stage, the errors that learners made were mainly conceptual where they did not understand the concepts that needed to be applied to the question or were unsure of the correct application of a certain concept to simplify the algebraic fractions. These errors mostly occur due to a lack of sufficient instruction by the teacher or weak mathematical constructs. Language errors, that is, errors caused by misunderstanding the instructions or mathematical language used, stood at 5% showing that the children understood the questions relatively easily while procedural errors and application errors accounted for 15% each.

The difficulties of students' learning of algebra have generated quite a number of studies. Welder (2012) approached the problem in terms of the cognitive gap between arithmetic and algebra. Van de Walle et al. (2015) also approached the problem in terms of a dialectic between procedural and relational thought. McDonald (2010) and Almeida (2010) centred their attention on the significance of errors made by learners learning algebra. Kieran (1992), Li (2006), Sfard and Themane (2014) highlighted the important sources of students' difficulties with the introduction to algebra. They revealed in these studies that the students often seem to have a limited view of algebraic expressions. Their notion of the solution of algebraic equations seems to be associated more with the ritual of the solution process rather than the numerical solution obtained, and they fail to grasp the meaning of the operations to be performed on the literal symbols, the algebraic expressions or the equations. It can be said that learner participants performed poorly with questions 7, 8, 10 and 11 because they failed to understand the principles and properties needed to simplify those questions. Themane (2014) gave special attention to the students' procedures of solving an equation prior to a formal instruction in algebra. Simeone et al. (2013) emphasised the importance of the acquisition of algebraic language and thought. Orton (1983) in his study also mentions the difficulties students experienced with elementary algebra, which appeared to obscure the fundamental ideas in calculus.

Hodes and Nolting (1998) also identified some of the root causes of students' difficulty in learning algebra as: The algebraic activity to perform, the nature of answers, the use of algebraic notations and conventions, and the meaning of letters and variables. The above difficulties could be as a result of teaching deficiencies, learning deficiencies and probably also the textbooks, the social background, the curriculum and examination influences. All these mainly centre on the teacher as the mediator and a guide. With adequate pedagogical content knowledge teachers are likely to present algebra in a manner which may enhance learning by learners and may serve as a solution to the learning difficulties mentioned above. To teach algebra with understanding mathematics teachers should be equipped with mathematical knowledge for teaching (MKT) and mathematics quality instruction (MQI) in order to deliver effective practice in the classroom where they teach (Hill et al., 2008). This should enable them to translate their knowledge into teaching to overcome the difficulties mentioned above.

Theme: Error analysis on questions 9 & 12

According to the criteria set in the Curriculum and Assessment Policy Statement (CAPS) (DBE, 2011) document, questions 9 and 12 of the test used were problem solving cognitive level questions.



Vignette E

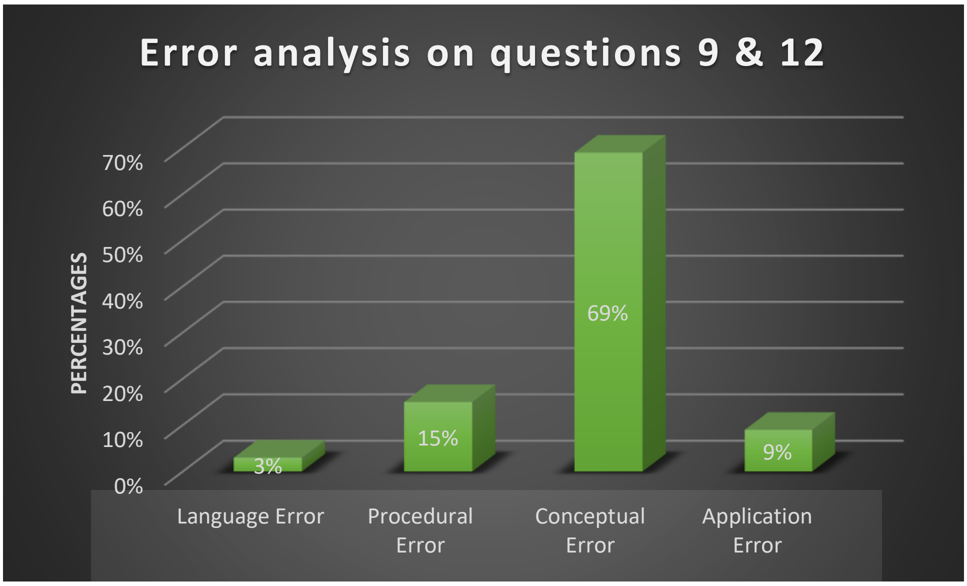


Figure 16. Error analysis on questions 9 & 12. Source: Author

Problem solving cognitive questions are non-routine problems that may not be necessarily difficult to solve but require higher order understanding. Nicholson (1992:67) explains problem solving as follows:

> *In problem solving, one finds the solution to a particular situation by a means which was not immediately obvious. A problem-solving task is one that engages the learners in thinking about and developing the important mathematics they need to learn.*

This can be contrasted with the traditional approach to teaching in which teachers explain a rule, provide an example, and then drill learners using similar examples. Many authors and researchers (Nicholson, 1992:66) have described problem solving as the essence of mathematics.

Figure 19 depicts that nearly 70% of the learner participants find problem solving questions difficult to deal with. An essential skill that Hiebert et al. (1997) believe that if learners are to understand mathematics, then it is more helpful to think of understanding as something that results from solving problems, rather than something we can teach directly.

Problem solving should be the cornerstone of the mathematics curriculum and instruction, fostering the development of mathematical knowledge and a chance to apply and connect previously constructed mathematical understanding.

This view of problem solving is presented in the *Revised National Curriculum Statement for Grades R-9 (schools)* (Department of Education, 2001:16-18). Problem solving should be a primary goal of all mathematics instruction and an integral part of all mathematical activity. Learners should use problem-solving approaches to investigate and understand mathematical content.

A significant proportion of human progress can be attributed to the unique ability of people to solve problems. Not only is problem solving a critical activity in human progress and even in survival itself, it is also an extremely interesting activity (Polya, 1957).

George Polya was well known for his book, *How to solve it*. He outlined four steps for solving problems, which are still used in many circles today: These are as follows:

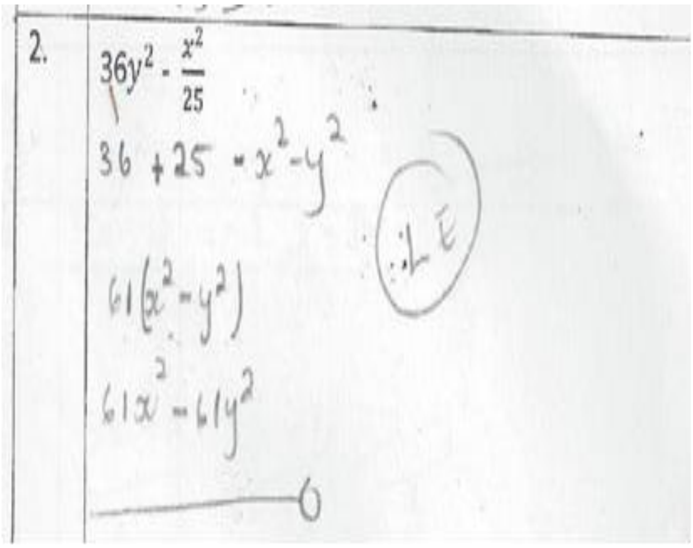
1. Understand the problem,
2. Devise a plan,
3. Carry out the plan, and
4. Look back.

4.5 DISCUSSIONS

There appeared to be some relationship between the learners' workings and their comments. Responses from the learners interviewed affirm the difficulties and challenges algebraic fractions pose to learners. Data collated and learners' responses were reported under four themes and statements from eight learners and are reported verbatim. In this section, the researcher discusses the research findings of the errors and possible misconceptions under each conceptual area with examples, carefully relating them to the various existing theories in literature.

Theme: Mathematical language error

Here is an example of a learner's error and response to the interview:



2. $36y^2 - \frac{x^2}{25}$

$36 + 25 - x^2 - y^2$

$61(x^2 - y^2)$

$61x^2 - 61y^2$

0

LE

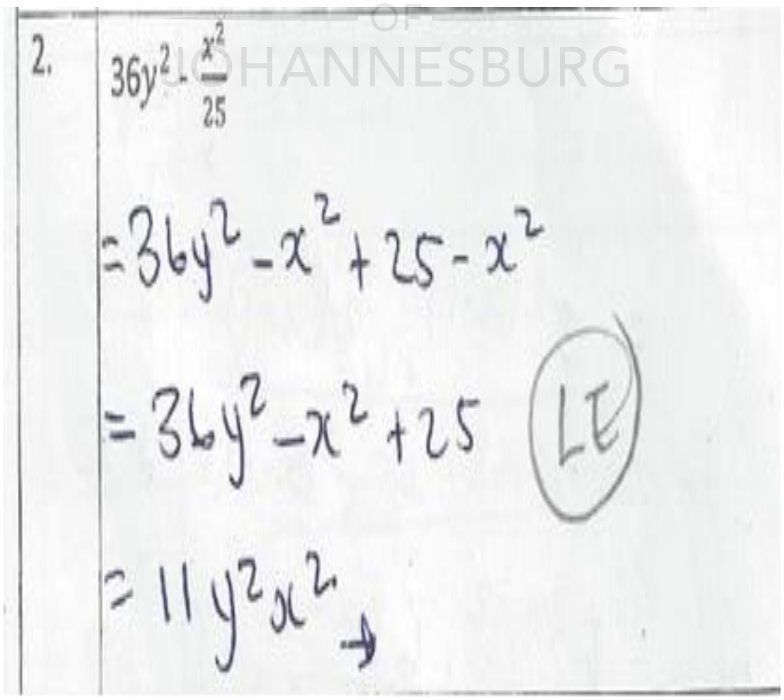
Vignette F1

Researcher: Can you explain your solution?

Learner 11: I wrote 36 and 25 on this side and then x^2 and minus y^2 on the other side. Then I add 25 and 36 to get 61 bracket x^2 minus y^2.

Researcher: There is a negative sign in front of x^2. What happen to it? (NB: learner did not answer)

Learner 11:



2. $36y^2 - \frac{x^2}{25}$

$= 36y^2 - x^2 + 25 - x^2$

$= 36y^2 - x^2 + 25$

$= 11y^2x^2$

LE

Vignette F2

Researcher: How did you go about this question?

Learner 120: Ok, I wrote the first the $36y^2$ minus x^2 plus this 25 minus x^2.

Researcher: Why minus x^2 plus 25 minus x^2.

Learner 120: Because of this minus (pointing to the minus sign) and the 25 is positive and then minus x^2.

Researcher: And then, how did you arrived at your answer?

Learner 120: 36 minus 25 is equal to 11 plus y^2x^2.

Radatz (1979) compared mathematical errors to a "foreign language" that learners need to know and to be able to understand mathematical concepts, symbols and vocabulary. Notably the misconception identified relating to operations and mathematical symbols is learners assuming that product and quotient expressions are the same as expressions of addition. For instance in Vignette F1 learner 11 wrote $36y^2$ - $\frac{x^2}{25}$ as $36 +25-x^2.$ - y^2 . Similarly in Vignette F2 learner 120 wrote $36y^2$ - $\frac{x^2}{25}$ as $36y^2$ - x^2 +25- $x^{2.}$. These misconceptions with mathematical operations and symbols were classified as mathematical language errors. This is because it indicates that the learner lacks understanding of mathematical symbols and operations.

Recent a study by Naseer and Hassan (2014:62) also highlighted the same sentiment, and categorised it as "detachment error". Thus, when learners lack an aspect of structure sense, that leads to detachment error. The authors cited an example as x^2 – x being equal to x- x^2. The possible misconception here is that the coefficient of each term in both vignettes were ignored by the learners. According the authors such learners lack the mathematical language and vocabulary to advance in the understanding of mathematics. Banerjee and Subramaniam (2012:73) offered an instructional suggestion to prevent and remedy this misconception which was to "use order of operations to develop an understanding of transformation that can keep the value of an expression equal".

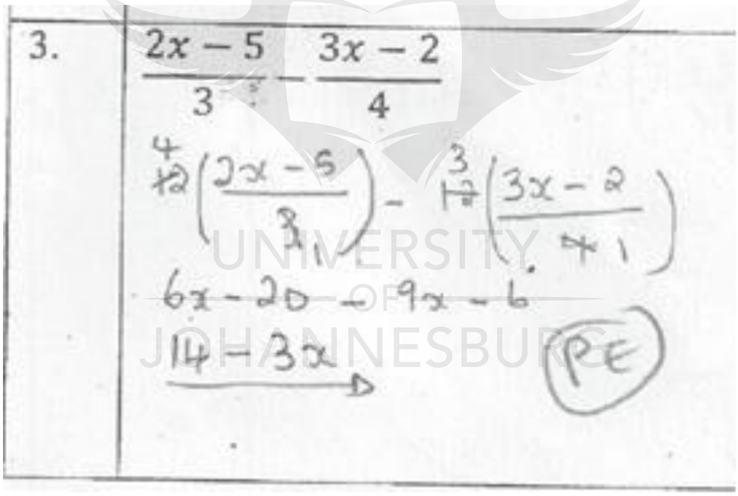
In answering question 2, a knowledge cognitive level question, the responses of both learners suggest language error as the two learners seemed to lack the knowledge of mathematical symbols and vocabulary to simplify the question. This is what researchers like Li (2006), Luneta

(2015), Resnick (1983) and Luneta & Makonye (2010) assert that knowledge of mathematical symbols and vocabulary are importance vehicles to comprehend mathematics concepts in general.

It is clear, as Setati (2005) put it, if learners are not given the competence in using mathematical vocabulary to explain mathematical tasks to others, to ask or answer questions, it is going to create linguistic difficulty in the study of the subject. When teachers do not use mathematical language effortlessly, their learners are unable to describe mathematical ideas and concepts using appropriate language. To overcome this learning barrier, Lappan and Briars (1995) suggest teacher actions that are effective should include deriving concepts, using cooperative group work, encouraging frequent mathematical communication, and using multiple representations and multiple strategies.

Theme: Procedural error

Here is an example of a learner's error and response to the interview:



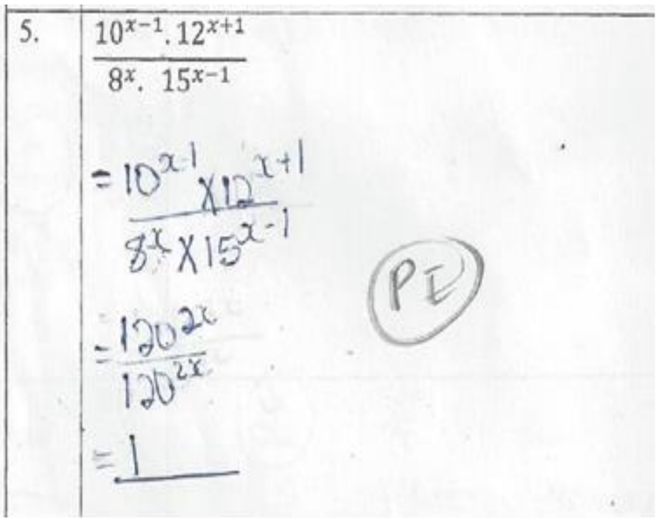
Vignette C

Researcher: Can you please explain your solution to me?

Learner 61: First I multiply 12 to each and then cancel the 12 with the denominators. After that I multiply 4 with 2x-5 and 3 with 3x-2 and then simply and got 14-3x.

Researcher: Why did you have to multiply each expression with 12?

Learner 61: Because I want to remove the denominators.



Vignette G

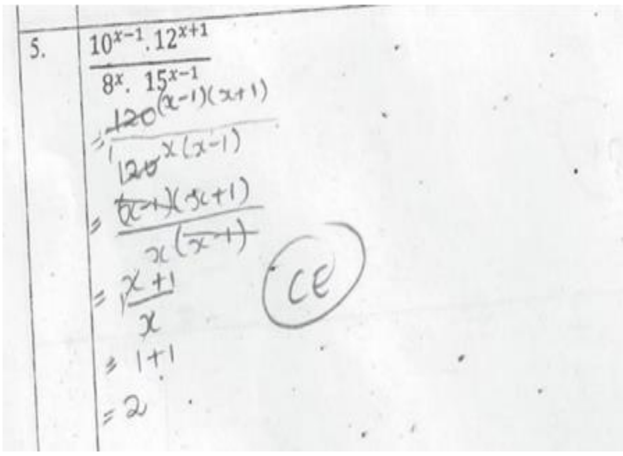
Researcher: Explain how you arrive at your answer.

Learner 92: I multiply the bases and add the exponents then cancel the numerator with the denominator to get 1, since they are the same.

From vignette F, the learners' work indicates an attempt to change the fraction expression to a pair of whole numbers. The error shown here is the use of the 12 to multiply both expressions instead of using the 12 as a LCD. According to Oksuz and Middleton (2007) and Mdaka (2011) most of the misconceptions learners have about algebraic fractions is because learners interpret quotients as pairs of whole numbers. The answer shows lack of knowledge of the rules needed to simplify this algebraic fraction question.

Similarly, vignette G also suggests that some learners could not easily simplify algebraic fractions. The learner multiplied the bases and added the exponents, an indication of lack of understanding of the necessary rules to solve the problem at hand. Approximately one-third of the learner respondents could not solve this routine procedure question (question 5). Based on the answers and the responses from the two learners, the closest descriptor was procedural error hence it was coded as such.

Theme: Conceptual error



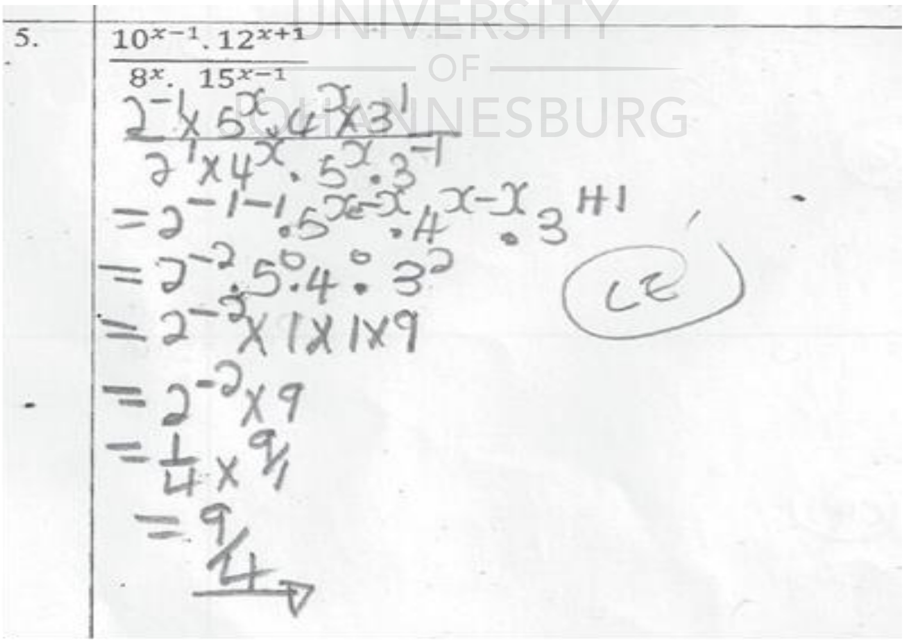
Vignette G

Researcher: Can you explain your solution?

Learner 100: First, I multiply the bases and the exponents. After cancelling the same bases, I was then left with the exponents $\frac{(x-1)(x+1)}{x(x-1)}$

(x-1) is a common factor so I cancel that to have $\frac{x+1}{x}$. X can cancel x to get 1. So x plus x equal to 2x.

Another example:



Vignette H

Researcher: How did you go about this question?

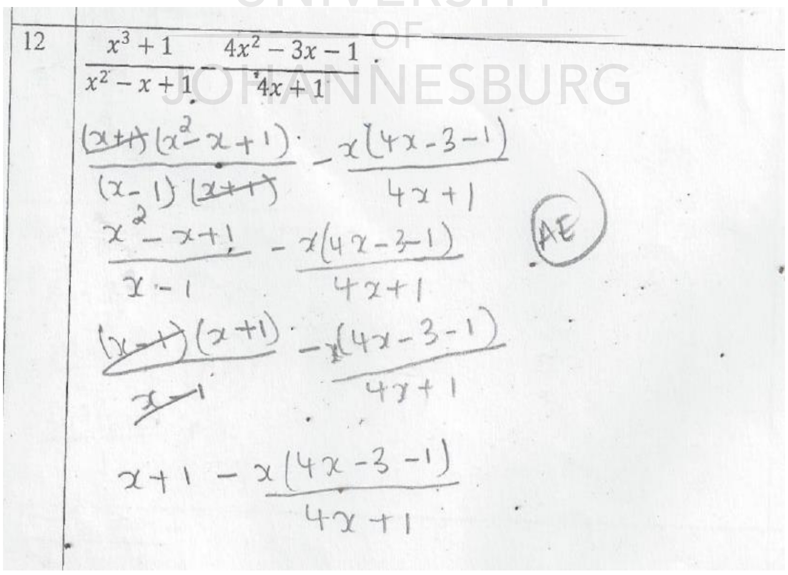
Learner 73: I reduce all the bases to their smallest bases. 10 can be express as 2 times 5 and so on. And then with the exponents each base gets one like this (pointing to her work).

Question 5 is also a routine procedural cognitive level question. 35% error frequency was recorded for conceptual errors. According to Nolting and Hodes (1998) and Luneta and Makonye (2013), when properties or principles underpinning a concept are not understood, learners commit conceptual errors.

From vignette G the learner multiplied both the bases and the exponents before cancelling out the bases. An indication of alternative conception in a previous learning situation. The learner further simplified this expression $\frac{x+1}{x}$ by cancelling out x from the numerator and denominator. This error displayed also indicates lack of principles needed to simplify this routine procedure cognitive question.

Although Vignette H shows that the first step of the learner was correct, that is, changing of the bases to their simplified form, the idea of attaching each base with an exponent points to possible misconception. According to the study rubric, these two misconceptions from vignette G and H were evaluated as concept errors because the learners' understanding shows lack of properties or principles required to simplify the question.

Theme: Application error

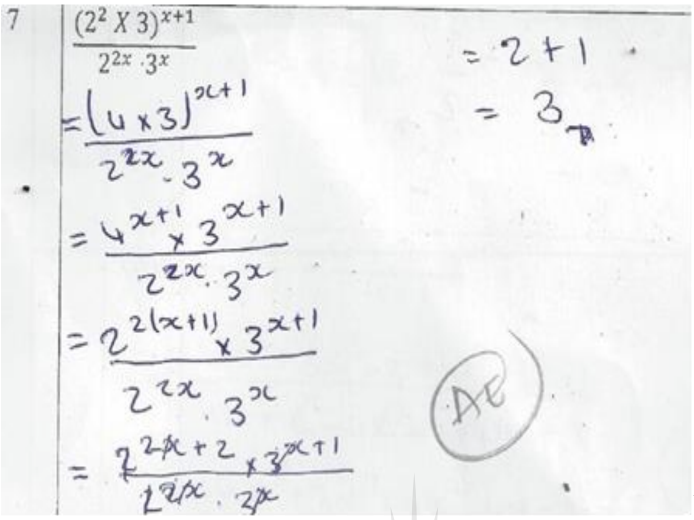


Vignette I

Researcher: How did you go about this question?

Learner 25: I factorized each expression except 4x+1. Then I cancelled the common factor (x+1) and got stuck.

Vignette J also illustrates the same theme:



Vignette J

Researcher: Can you explain your answer to me?

Learner 32: I change 2^2 to 4 before multiplying the exponents. Oh no! I don't need to change to the 2^2. So I change the 4 back to 2^2 . I then cancelled common factors 2^{2x} and 3^x remaining 2 +1 which is equal to 3.

Researcher: Why did you cancel 2^{2x} and 3^x

Learner 32: Because it is common and so you can cancel them out.

Question 9 in table 9 is an example of a problem solving cognitive level question. Summary results indicates that 69% of the learners who responded to questions 9 and 12 showed some misconceptions. Question 9 requires that learners factorised completely before cancelling:

$$\frac{x^3+1}{x^2-x+1} - \frac{4x^2-3x-1}{4x+1}$$

$$= \frac{(x^2-x+1)(x+1)}{x^2-x+1} - \frac{(4x+1)(x-1)}{4x+1}$$

$$= (x+1) - (x-1)$$

$$= x+1 - x+1$$

$$= 2$$

However, the learner managed to factorised $x^3 + 1$ expression but could not factorise

$4x^2 - 3x - 1$ and $x^2 - x + 1$. Subsequent interrogation revealed that at least the learner knows the concept but making use of it in a different context was the problem.

Similarly with Vignette F the learner struggled to apply the learnt concept. The error shown here is that, before the learner cancels out both 2^{2x} and 3^x, the expression should have been

$$\frac{2^{2x}.2^2.3^x.3^1}{2^{2x}.3^x}$$

Then the cancellation of the common factors will be ideal. The learner might have arrived at 4x3 being equal to 12 instead of 3. According to Nolting & Hodes (1998), Mdaka (2013) and Borasi (1996), application errors like this usually occur when learners apply rules or strategies wrongly. It results from the learner's lack of understanding of where to apply the concepts and skills learnt. Bosasi (1996) emphasized that this is due to misunderstanding of one or more of the required step(s).

To align myself with the above, the two vignettes from the sampled learners show misunderstanding of one or more steps wrongly used. Hence, the best descriptor code was application error.

Drawing from the vignettes I and J (application error), learner achievement in algebraic fractions seems to be mainly due to knowledge gaps in both fractions and algebra that forecloses substantial epistemic access to algebraic fractions. This could be that algebraic fractions are often communicated through symbolic mathematical language (Graham & Thomas, 2000). Hence, the difficulties or lack of understanding of algebraic fractions hinder the learning of them. Kaplan (2007) also noted this hiccup that learners encounter when simplifying algebraic fractions. A good understanding of algebraic fractions at an early stage, i.e. Grade 10, should enable the learners to have success in learning of mathematics (Fennell, 2007).

Understanding the errors and misconceptions learners make when simplifying algebraic fractions might be the beginning of the solution to low performance in mathematics since it might equip mathematics teachers to identify learning difficulties and challenges when unpacking algebraic fractions and other mathematical concepts.

Summary of the results of the error analysis

Figure 20 below gives an overview of the error analysis. 1632 items (total learner participants multiplied by total questions) were analysed. 2.2% of the items were incomplete, hence were not considered during the analysis stage. Approximately 10% of the items were answered correctly. As pointed out by Kaplan (2007) and the National Diagnostic Report (2013) algebraic fractions pose problems for learners, hence they contribute to learners' poor performance in mathematics.

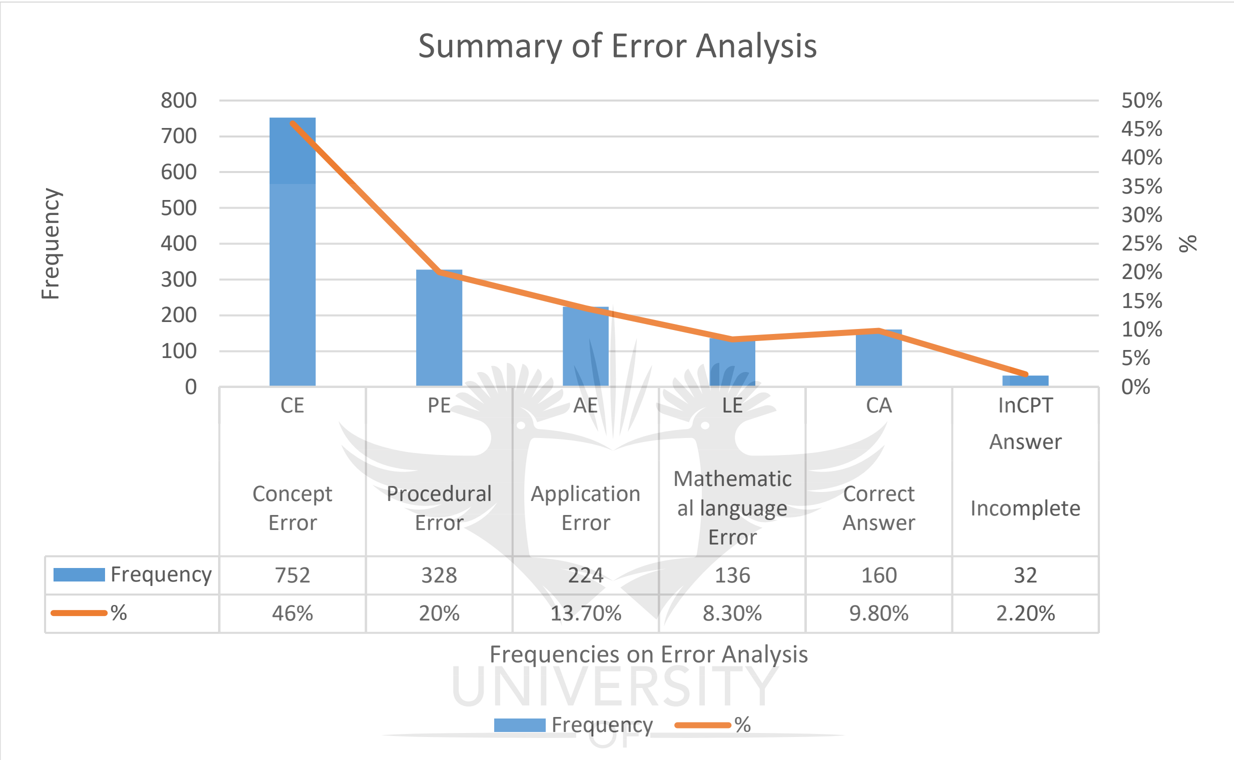


Figure 17. Frequencies of error analysis. Source: Author

Notably amongst the errors recorded were concept errors and procedural errors accounting for 46% and 20% respectively. Other studies on algebraic expressions also identified these two error constructs (Brown & Quinn, 2006; Egodawatte, 2011; Figueras, et al., 2008; Hodes & Nolting, 1998). Mathematical language error also accounted for about 8%. Although relatively low, Naseer and Hassan (2014) also identified this learning barrier.

4.6 RESEARCH FINDINGS

Based on data collated from this study and the discussions of the collated data, the following were noted.

Table 4.2 depicts a detailed summary of the analysed data. Column 1 shows the four categories of cognitive levels of thought for assessment activities. column 2 to 5 also harbours mathematical language error, procedural error, conceptual error and application error respectively. Correct simplifications of the learners were coded CA. Barely 160 out of 1632 items analysed were correct answers. Incomplete solutions were also assigned INCPT. The four error types identified constituted almost 90% when analysed versus the four cognitive levels for assessment activities.

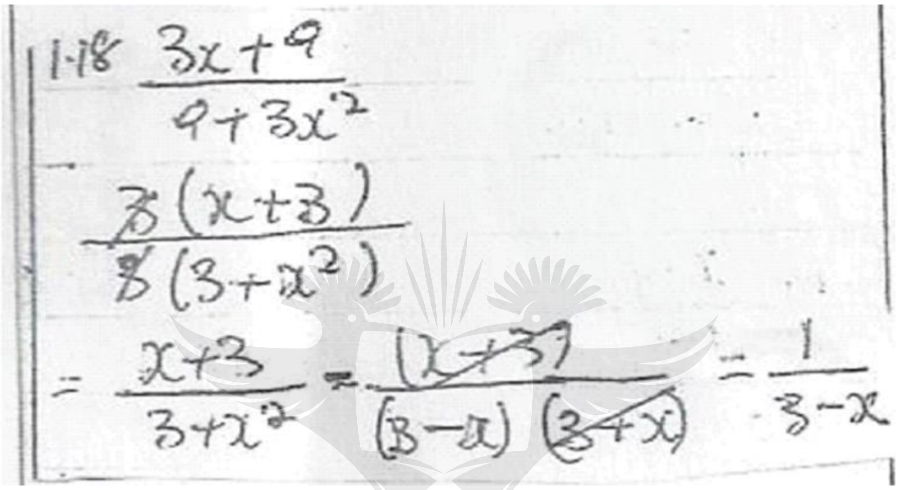
Table 4.2: Summary of cognitive levels per test question and the errors types

	LE	PE	CE	AE	CA	INCPT	TT
Q1 & 2 Knowledge	64	55	78	8	88	2	295
%	22%	19%	26%	<3%	30%	<1%	
Q 3,4,5&6 Routine Procedure	40	156	196	116	40	10	558
%	7%	28%	35%	21%	7%	2%	
Q7,8 &10 Complex Procedure	24	78	298	77	30	13	520
%	4.6%	15%	57%	15%	5.4%	3%	
Q 9 &12 Problem Solving	8	39	180	23	2	7	257
%	3%	15%	69%	9%	<1%	<3%	
TT	136	328	752	224	160	32	1632
Grand %	8.30%	20.00%	46.00%	13.70%	9.80%	2.20%	100.00%

Table 5.Overview of cognitive levels per Test questions and Error types of the study. Source: Author

Procedural errors

Procedural errors occur due to the mix up of rules and formulas during the computation of mathematical problems. Out of 1632 items analysed 328 (20%) showed that there was a mix up with the rules. This is typically a result of inadequacy in relational understanding of what the learner is doing. A common misconception in learners is that the product of two numbers is always a larger number. This is usually not the case in decimals, for example the product of 0.9 and 0.8 is 0.72 and not 7.2 as many learners will try to prove. Below is an illustration of an example of a procedural error in simplifying algebraic fractions.

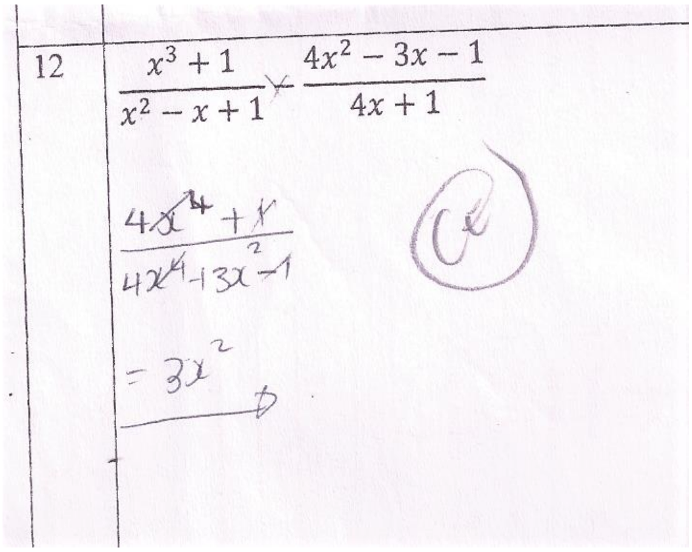


Vignette K

The working of the learner in the illustration above (Vignette K) manifests a procedural error and an inadequacy in conceptual understanding of simplification of algebraic fractions. The learner used wrong procedures to calculate the algebraic fraction question. The source of the procedural errors is likely deficiency in skill, facts and concepts required to tackle the problem. Orton (1983) described a similar phenomenon as ‘arbitrary error’. Orton described this as errors in which the learner behaved arbitrarily and failed to take into account the constraints laid down in what was given.

Concept errors

Some invalid use of principles or properties when computing algebraic fraction were categorised as concept error. "Concept errors occur when learners do not understand the principles or properties of the topic in question and attached their own understanding" (Booth, 1983). For example, when teaching the volume of solids, some learners may be stuck with the meaning of volume as the loudness of sound rather than the actual meaning of amount of space occupied by a solid. Concept errors manifest the deficiency of conceptual understanding in the topic in question. The Vignette L below illustrates a working of an learner showing concept errors.

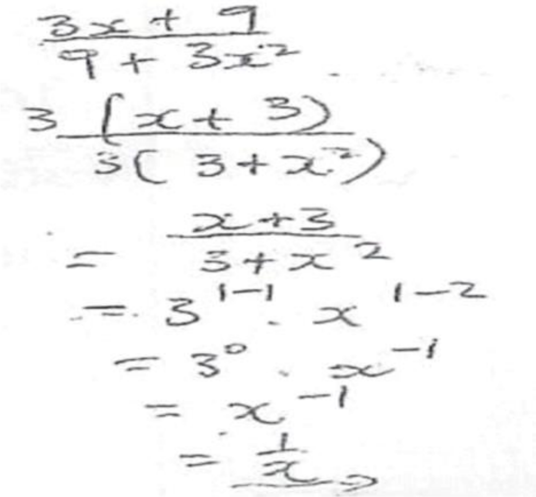


Vignette L

There are a number of learners who fail to use the required principles needed to solve the likes of vignette L shown above. Out of 1632 items (questions) analysed, 752 (46%) were errors made by the learners with insight of some concepts and showed some conceptual errors which were indications that these learners were yet to master the principles needed to simplify the question.

Application error

From table 4.2 13.7% of the 1632 items analysed were errors where learners wrongly applied a known concept. Application errors occur when learners know the concept of a topic but cannot apply it correctly in problems. It is the wrong application of correct rules or strategies. It results from the learner's lack of understanding of where to apply the concepts and skills learnt. The Vignette M below shows a case where a learner performs an application error.

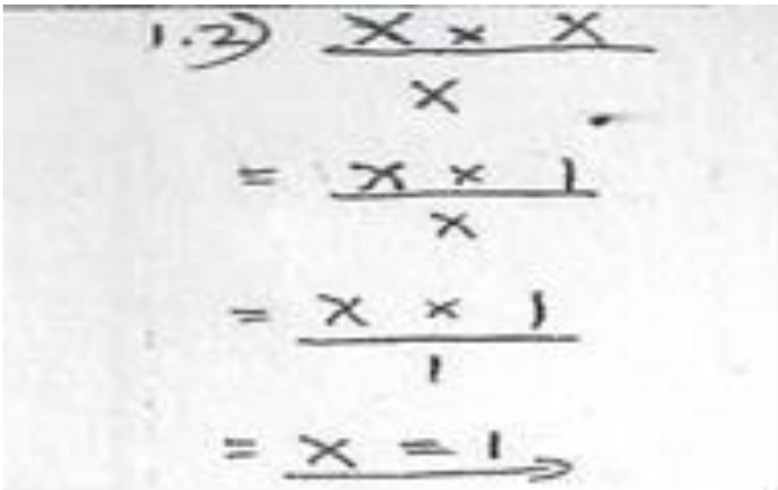


Vignette M

The learner doing the computation above can be thought of as knowing how to factorize, divide and how to work with exponents and perfect squares. However, it shows a lack of understanding of where and when to apply the concepts.

Mathematical language error

A mathematical language error was identified to be an error emanating from a learner lacking understanding of mathematical technical vocabulary, having a misconception when using operation symbols and from misconceptions with letter usage. It is a case where a learner lacks the technical vocabulary required to handle the problems. Some technical terms used in algebra fractions include factorize, simplify, perfect square and exponent among others. An error also occurs when a learner swaps a letter with a numeral without a valid reason. For instance, vignette N below shows a case where a learner performs such mathematical language error.



Vignette N

8.3% of the learners' work analysed had some of these errors, whereby learners swapped variables with numerals without substantiating them. To "come up with teaching strategies that will help reduce such occurrence of misconceptions among learners, teachers need to have a vivid knowledge about some of the possible reasons for the errors" (Lester et al., 2007). A wide range of reasons have been discovered to be key contributors to errors in mathematics. Some of them include inadequate prior knowledge of algebraic fractions, distraction during instruction and examinations, language barriers, level of cognitive thinking being below the concept being taught and learners' refusal to spend enough time to study and practise to master the skills learnt from lessons.

Inadequate prior knowledge

Lack of adequate prior knowledge, which is essential to understand the new concepts, results in the learner being confused in the classroom. In the long run, they attach their own misguided meanings to the new topics resulting in errors during tests. For example, when teaching simplification of algebraic fractions, if a learner is not conversant either with fractions or algebra they will end up lost during the lesson. They will develop their own way of solving algebraic fraction problems which are incorrect resulting in errors.

Language barrier

Usually, the language of instruction during classes and assessment is different from the learners' first language. This might lead to learners having difficulties understanding the question's requirements. In class, the learners will miss some concepts being discussed and end up erring in the tests. It is the teachers' role to ensure that the mode of delivery of their lessons is simple to ensure that each student understands. It is also the role of learners to spend extra time practicing the concepts learnt in class to improve their mastery.

Level of cognitive thinking being lower than the concept being taught

This occurs when "learners are taught a concept which is way beyond their level of understanding. This results to them grasping only chunks of the information delivered by the teachers" (Mistakes and Misconceptions in Mathematics, 2011). They will also understand the incorrect conception of the topic which is recognizable by their level of cognition. This results in errors in tests.

From table 4.2 about 99% of the problem-solving questions were wrongly solved. Thus non-routine problems, questions that were not necessarily difficult but demand higher order understanding, were a big challenge to the learners. This may suggest that the nature of the mathematical task or the cognitive level of the problem to be solved by the learner can cause errors in the learners' working and thus result in errors after computation. The error may be caused by a problem being difficult beyond the comprehension of the learner. Thus the learner will not understand what is required to solve it.

Also, the way an algebraic problem is presented can be a reason for errors. A task that in not presented appropriately may confuse the learners on what is required of them in the task despite maybe having an idea of how to solve problems of the same kind. Additionally, the error may be a result of translation complexity. Very complex problems or problem solving cognitive level activities may cause difficulties to the learner as they try to read and interpret the problem while figuring out what the task requires of them; however, it is imperative to use problem solving tasks to enable learners to comprehend mathematics. Misinterpretation of the task results in errors in the working.

Refusal of the learner to do extensive reading beyond class work

Each student, depending on his/her level of understanding, needs to spend some time after class work to practice the concepts learnt. The practice could be in the form of extensive reading and practice exercises or discussions with other learners. Failure to do this limits the learners' understanding to only what they learnt in class which may not be enough to ensure success in their tests. Bryant et al. (2016) show that while most teachers recommend that learners read beyond the given classwork and assignments, most students, especially in lower grade school, do not bother and, as a result, have lower mathematics scores as compared to students who read extensively. However, the lack of motivated self studying on the learner's part is mainly due to rigid class definitions that do not take into account the children's learning capabilities and cognitive development. As Bryant et al. (2016) shows, increased reading can improve the learner's scores dramatically but the motivation has to come from both the teacher and parents who will need to create enabling environments.

Algebraic fractions

Algebraic fraction take up the properties learnt about fractions and algebra. The computation of algebraic fractions is challenging to learners, possibly because they require the understanding of mathematical concepts like: division, variable, equation, perfect squares, exponent, factorization and rational numbers. Inadequacy in the understanding of these concepts leads to difficulties in solving algebraic fractions.

Misconceptions

"Constructivists state that misconceptions occur when a new idea has no connection with existing knowledge and therefore becomes impossible to understand" (McDonald, 2010: 34). This results in serious difficulties in learning as learners try to utilize their inadequate informal thinking and poor remembrance. Research has proved that repeating learning lessons does not assist learners who have misconceived the concepts. This is because learners tend to be attached to their misconceptions as they continuously construct with them providing them with smart solutions. A more ideal solution is to identify the misconceptions and apply different approaches to re-educate the learner to rectify his/her learning barrier. The study showed that 46% of the learners made conceptual errors where they do not fully understand a mathematical construct or concept, resulting in both conceptual and application errors. Misconceptions on the proper use of mathematical concepts is common among early childhood learners whose cognitive abilities may require more explanation or contextualizing as compared to the majority of other learners. Additionally, problems may be due to misconceptions that the learner already has when going to class and the teacher is simply unaware of the conceptual errors that the learner is apt to make.

The task of correcting misconceptions will involve changing the conceptual framework of the learner because misconceptions have to be changed partly through learners' belief systems and also through their cognition. The fact is that new learners do not come to class blank but with informal ideas generated from day-to-day activities. Research has proved that "learners can learn using their own misconceptions as long as they are identified, addressed and corrected in teaching" (Olivier, 2016). Errors in mathematics can be related to the learner, the teacher or the topic of study itself (in this case-simplifying algebraic fractions).

Teacher and mathematics errors

According Richmond et al. (2001) a teacher's attitude and confidence in mathematics is critical for efficient delivery of concepts to the learners. If teachers lack confidence or dislike algebra or mathematics, they end up teaching poorly and making many errors when teaching. These errors are propagated to the learners who will think they learnt the correct thing.

According to Ryan and Williams (2007) a teacher's lack of imagination and creativity in handling mathematics calculations may also be a factor leading to mathematical errors. A teacher who is creative happens to teach the mathematics concepts in a broader manner. They look for alternatives approaches to a calculation which reduces the chances of errors in learning.

The amount of knowledge a teacher has may be a factor in mathematical errors (Hill, et al., 2008). Too much knowledge in the subject may lead the teacher to not understanding the difficulties their learners are going through, thus their inefficiency in the learning process. On the other hand, too little knowledge of the concepts causes the teacher to teach them in a limited way (ibid).

A teacher's experience in the mathematics field is also a factor. Experience causes the teacher to know the possible misconceptions that can arise amongst their students. Thus experienced teachers are able to handle the misconceptions right in the learning process which is more efficient. Also their mode of teaching is better and more efficient for the learners.

It is also likely that a teacher might be very knowledgeable about his/her content area such as mathematics but may somehow lack mathematical quality instruction (MQI) to assist learners to comprehend. This is an area Shulman (2000:129-134) considers critical to a teacher’s knowledge, where one moves from personal comprehension to preparing for the comprehension of others. A teacher cannot hope to explain a mathematical concept if he/she does not have full comprehension of that mathematical concept. Only when the mathematics teacher understands something well enough is he able to teach others. Novice teachers often struggle to represent concepts in an understandable manner to their learners because they have little or no mathematical knowledge for teaching (MKT) at their disposal.

Misconceptions and errors

Misconceptions and errors are barriers to learning which need attention as they lead to low academic attainment. The results of the study showed that learners did not understand how to simplify algebraic fractions properly. The results of the study showed that conceptual, language, procedural and application errors hinder learners' learning of algebraic fractions. Notably among the four errors are conceptual and procedural errors; however, mathematical language errors relatively new to mathematics teachers also require attention.

4.7 CONCLUSION

In this study, learners' confusions were analysed. It was revealed that teachers need to distinguish learners' alternative conceptions or misconceptions to help them learn mathematics adequately and effectively. Overlooking learners' misconceptions may negatively affect learners' new learning and will likewise fortify more misinterpretations. It is basic that learners are urged to create mathematical mental propensities and obtain the fundamental abilities by permitting them to derive concepts themselves. Co-operative group work, frequent scientific communication, utilization of various teaching procedures and utilization of real life issues when explaining concepts such as algebraic fractions will empower learners to procure conceptual understanding and in this way minimize errors resulting from misconceptions.

Chapter 5: SUMMARY AND RECOMMENDATIONS OF THE STUDY

5.1 INTRODUCTION

This chapter presents a summary of the study, recommendations, and suggestions for further research. The main aim of this study was to identify and categorize learners' errors as well as suggest possible ways in which learners' errors due to misconceptions can be minimized when computing algebraic fractions.

Specifically, it sought to understand the rationale for learners' misconceptions when computing algebraic fractions. In addition, the study expounded on the implications of misconceptions and errors on learning of algebraic fractions and mathematics in general.

5.2 FOCUS OF THE STUDY

The study investigated misconceptions and errors that learners display when simplifying algebraic fractions. An analysis of the final examination results has pointed out a concern that most of the learners tested have performed poorly in algebra problems, especially the simplification of algebraic fractions (DoE, 2014). The study findings show that most of the learners tested found algebraic concepts problematic as the frequency of concept errors was 46% while that of procedural errors was 20%. These figures are alarming since the mathematical foundations formed during grade 10 school days are important in understanding more complex topics in mathematics and the sciences.

A multi-method approach involving the use of several research instruments, such as a comprehensive cognitive demand test, interviews, and document analysis for data collection, was adopted to carry out the investigation. Learners doing mathematics in grade 10 were identified, taking into account mixed abilities of the learners.

The research questions that guided the study were:

1. What are the misconceptions that learners display through the errors they make when solving algebraic fractions?
2. What are the categories of errors that learners display when solving algebraic fractions?
3. How can these misconceptions be minimized?

The data collected shows that in most cases learners are unable to simplify algebraic fractions due to conceptual errors, that is, errors related to understanding specific mathematical concepts and their application in different contexts. The conceptual errors accounted for almost 40% of all recorded errors in the tests while procedural errors accounted for an estimated 15%, especially in the tests requiring higher cognitive functions and proper procedures in simplifying the fractions. However, in the first two questions which are cognitive and relatively easy for the age group under study, most of the errors were application and conceptual errors where either the learners did not understand class content fully or, if they did, they were unsure of the application of concepts to different problems.

5.3 RECOMMENDATIONS

For a teacher to take corrective measures for errors in simplification of algebraic fractions they need to be able to identify their occurrence so as to help the learners. As most of the errors recorded in the study are conceptual errors, it means that learners have problems understanding the basic precepts of mathematics at an early age. However, the presence of misconceptions may not be entirely due to the teacher but it can also be misconceptions on the part of the learners where small gaps in comprehension may later result in further misconceptions thus creating an error avalanche. The teacher may not be aware of these alternative conceptions and their identification would help greatly in improving the quality of mathematics teaching.

In the event that the learner can read the question but can't explain the question, we have an understanding issue within reach. In this case, it becomes a language error problem which accounts for 8.3% in our study findings. These language errors can be eliminated by having the teacher contextualizing mathematical contexts in the language that learners can understand. This is especially important at an early age where the teacher has to be creative in relating mathematical constructs to the child's experiences and environment. Another method for recognizing errors may be to request that learners demonstrate how they got their answer. In the event that they can't, it implies they have a problem in processing the question.

As error analysis may empower teachers to plan successful instruction strategies, it ought to be grasped since it may enable teachers to recognize the underlying causes of the types of errors talked about above and how best to correct these learning barriers to effective learning and educating. Learners will profit from error examination by knowing the right concepts and teachers will likewise know the right way to minimize misconceptions. For instance, error examination demonstrating careless mistakes suggests using more examples on the principle steps. If the learners have insufficient prior information on the subject of concern, teachers ought to be ready to go through the required sections with the learners and help them understand. Showing illustrations in the classroom that have the topic content will likewise go far to help learners who forget easily what has been taught.

Teachers can likewise utilize diverse strategies if learners don't comprehend what has been taught. Misconceptions are extremely hard to recognize; however, if critical thinking questions are utilized learners will develop conceptual comprehension essential for mathematical education (Almeida, 2010). Group work could likewise be utilized to decrease mistake making since it empowers learners to think about their own thoughts and even that of others.

The research findings show that mathematics learners are prone to making errors when tackling problems. While teachers are aware that some of their learners may be making errors, the researcher found that most teachers are afraid of these errors. Beginning teachers are often afraid that they will be surprised by the number of learner errors and they may overlook them or remain unsure of how to handle learner errors. Other teachers tend to be annoyed at having to correct recurring errors among their learners which is especially true for misconceptions arising from a learner's previous knowledge. Taking everything into account, it is key for teachers to do error examination on learner exercises, i.e. search for examples of mistakes or slip-ups in learners' exercises and, in addition, the underlying causes of such errors so that they can effectively focus on instructions to correct the mistakes. Some of the reasons identified as reasons for errors are mathematical language errors, procedural errors, and concept and application errors. In this research, identified errors from the learners were grouped into these four error types as alluded to earlier.

5.4 SOME PROCEDURES THAT CAN BE APPLIED IN MANAGING MATHEMATICAL MISCONCEPTIONS

- Asking learners questions that oblige them to describe their beliefs and understandings (Clements & Sarama, 2014).
- When appropriate, asking questions intended to bring out and clarify the nature of the misconceptions (Silver, 2013).

- Presenting problems that learners cannot in any way solve using their incorrect understanding (Clements & Sarama, 2014).
- Helping the learners concentrate on concepts that will precisely describe the problem under thought (Sarwadi & Shahrill, (2014).
- Engaging learners in discussions about the advantages and disadvantages of the past versus the new understanding (Clements & Sarama, 2014).
- Pointing out contrasts between their misguided judgment and the better explanation (Booker, Bond, Sparrow & Swan, 2014).
- Convincingly demonstrating that an alternate understanding of events and concepts will empower them to comprehend things more beneficially than it would be possible if they proceeded with their past misconceptions (Webb et al., 2014).

Based on the findings of the study and the quest for quality mathematical education for our learners, the researcher recommends the hierarchy diagram (Figure 21) below that shows possible quality mathematical instruction which may lead to the acquisition of conceptual knowledge especially with mathematics education.

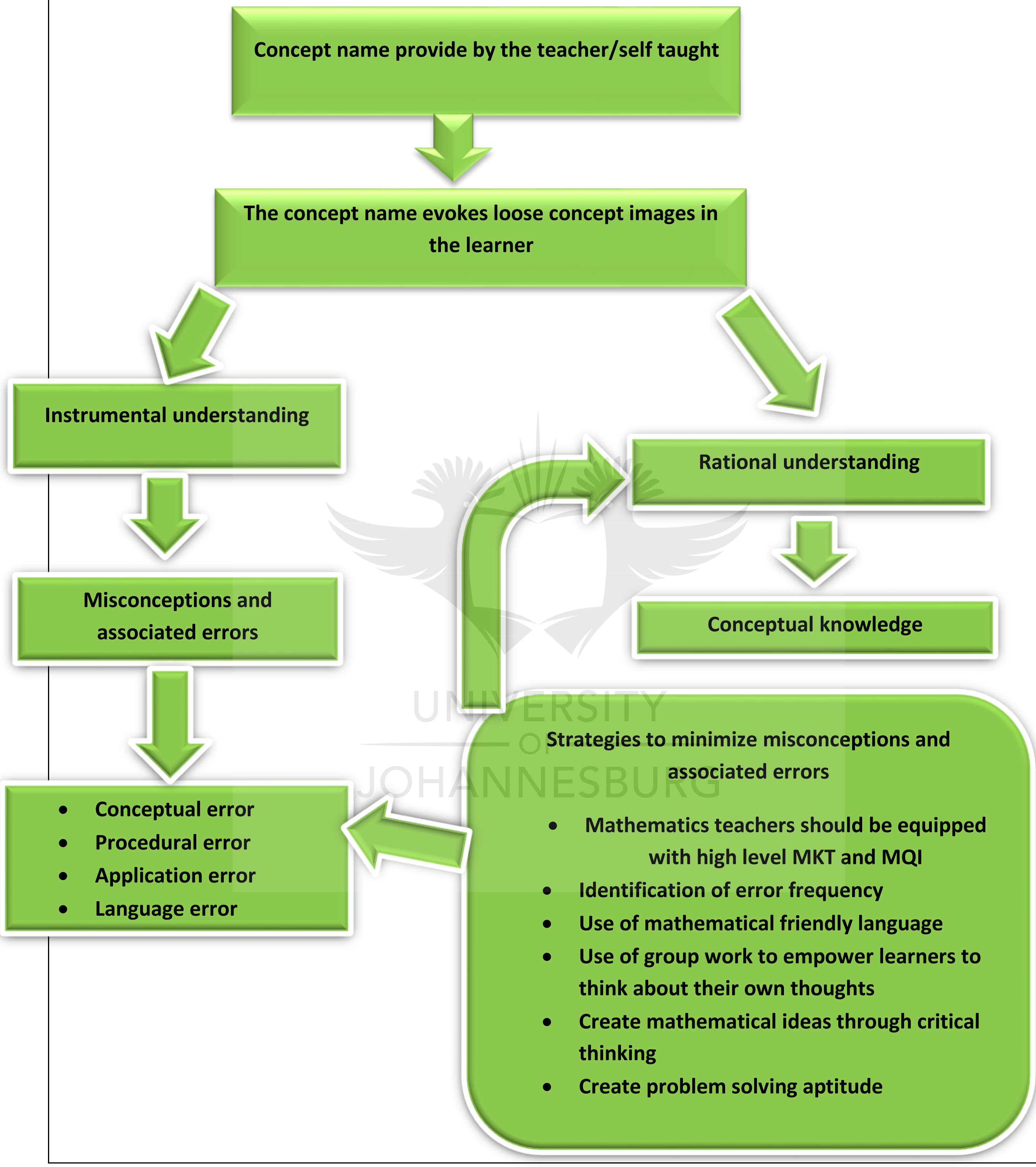


Figure 18. Misconceptions and errors and conceptual knowledge in mathematics. Source: Author

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APPENDIX A: QUESTIONNAIRE

NAME: ……………………………………………... DURATION: ONE HOUR

QUESTIONNAIRE ALGEBRAIC FRACTIONS

INSTRUCTIONS:

A. Simplify the following if possible and show all the workings.

B. Write neatly and legibly.

C. All answers will be treated with strictest confidence.

No	QUESTIONS AND LEARNERS' RESPONSES	YEAR
1.	$\left(\frac{x}{2}-\frac{y}{3}\right)\left(\frac{x}{2}+\frac{y}{3}\right)$	2009
2.	$36y^2$ - $\frac{x^2}{25}$	2008
3.	$\frac{2x-5}{3}-\frac{3x-2}{4}$	2010

4	$\left(\frac{2x^2y^2}{3\,y^{-2}}\right)^2$	2010
5.	$\frac{10^{x-1}.12^{x+1}}{8^x.\;\;15^{x-1}}$	2011

6.	$\frac{10^{2x+3}.\quad 4^{1-x}}{25^{2+x}}$	2015
7	$\frac{(2^2\ X\ 3)^{x+1}}{2^{2x}\cdot 3^x}$	2006

8.	$\frac{5(x-1)}{6} - \frac{x-1}{2} + \frac{x-3}{3}$	2011
9	$\frac{x^2-1}{3}\ X\ \frac{1}{x-1} - \frac{1}{2}$	2006

10	$\frac{(2^{x+1})^3}{\sqrt{64^x}}$	2007
11	$\frac{3}{x-4}-\frac{2}{x+3}-\frac{21}{x^2-x-12}$	2015
12	$\frac{x^3+1}{x^2-x+1}-\frac{4x^2-3x-1}{4x+1}$	2012

Accessed on 15/11/2015 from http://mathsandscience.com/tests-exams/gr-10-maths-exams/

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Acessed on 15/11/2015 http://www.thutong.doe.gov.za/ mathematics

APPENDIX B: INTERVIEW SCHEDULE

The following questions will be ask based on the identified errors in the learners' scripts. Identified errors will be coded. Open-ended questions will be used to allow learner respondents to explain their computations.

(Materials: Tape recorder, writing pads, pens)

Learners: x…………………………………………………..

Grade --------------   Date: ------------------------------

Time ----------------------   Duration of the interview: ----------------------

Interviewer: The researcher

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1. Learner : Can you explain your solution to question …. ? ………………………………

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2. You also answered question … , explain how you arrived at your answer:

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3. Can you explain your solution to question ……? ……………………………...............

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4. Learner X,…… Can you explain your solution to question ……? ……………………..

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5. Learner ……, Can you explain your solution to question……?……………………......

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APPENDIX C: PRETEST QUESTIONNAIRE

NAME: ………………………………………………... DURATION: ONE HOUR

QUESTIONNAIRE ALGEBRAIC FRACTIONS

A. Simplify the following if possible and show all the workings.

1.1	$\frac{x+x}{x}$	1.2	$\frac{x \times x}{x}$	1.3	$\frac{x}{x+\mathrm{y}}$	1.4	$\frac{x-\mathrm{y}}{x+\mathrm{y}}$
1.5	$\frac{x-\mathrm{y}}{x-\mathrm{y}}$	1.6	$\frac{x^2}{2x^2y}$	1.7	$\frac{2x^2y}{x}$	1.8	$\frac{(2x^2y)^2}{2x^3\mathrm{y}}$

1.9	$\frac{2x^3}{(2x^3\mathrm{y})^2}$	1.10	$\frac{2x^3\mathrm{y}^2 + 3x^3\mathrm{y}^2}{5x^3\mathrm{y}^2}$	1.11	$\frac{5x^2\mathrm{y}^3}{2x^3\mathrm{y}^3 + 3x^3\mathrm{y}^3}$	1.12	$\frac{(-2x^2\mathrm{y})^3}{2x^2\mathrm{y}}$ (Adler & Setati, 2016)

1.13	$\frac{-6x^3}{-12x^6}$	1.14	$\frac{x^2y^2 - xy}{x^3y^2 - x^2y}$	1.15	$\frac{4x^2 - 9}{6x - 9}$	1.16	$\frac{9x^2 - 1}{(3x - 1)^2}$
1.17	$\frac{2x^4 - 8x^3 + 6x^2}{-2x^2}$	1.18	$\frac{3x + 9}{9 + 3x^2}$	1.19	$\frac{x^2 - 1}{x^2 - 2x + 1}$	1.20	$\frac{2x^2 - 8x^2}{x + 2y}$

APPENDIX E: CONSENT LETTER

UNIVERSITY OF JOHANNESBURG

FACULTY OF EDUCATION

DEPARTMENT OF CHILDHOOD EDUCATION

INFORMED CONSENT FORM

Title:

GRADE 10 MATHEMATICS LEARNERS' ERRORS AND MISCONCEPTIONS WHEN SIMPLIFYING ALGEBRAIC FRACTIONS. LADY FRERE DISTRICT IN THE EASTERN CAPE PROVINCE OF SOUTH AFRICA

Name of the Researcher: J. Baidoo

Supervisor: Prof. K. Luneta

Purpose of the study/research: M Ed.

PARTICIPANT'S INFORMED CONSENT

The purpose of the study and the extent to which I will be involved was explained to me by the researcher in a language which I understood. I have understood the purpose of the study and the extent to which I will be involved in it. I unreservedly agree to take part in it voluntarily. I understand that I am free to withdraw from the study any time at any stage of my own will. I am aware that I may not directly benefit from this study. I am made aware that my responses will be recorded anonymously and that I may be audio or video taped for the purpose of this research.

Signed at (place)……………………….. on (date)……………………………………………..

By (full name) ………………………………………………………………………………

Witness(Full name)……………………………………date:…………………………………..

ENDORSEMENT BY THE HEAD OF PARTICIPANT'S INSTITUTION

Name:…………………………………..Signature:…………………………

OFFICE STAMP

APPENDIX D: PERMISSION LETTER

REQUEST LETTER

THE PRINCIPAL

LADY FRERE

5410

18/08/2015

Dear Sir

Re: REQUEST TO DO RESEARCH PROJECT IN YOUR SCHOOL

I, Joseph Baidoo, Master's student at the University of Johannesburg, Faculty of Education, as a requirement of my studies, am engaged in a research study titled:

"*GRADE 10 MATHEMATICS LEARNERS' ERRORS AND MISCONCEPTIONS WHEN SIMPLIFYING ALGEBRAIC FRACTIONS. LADY FRERE DISTRICT IN THE EASTERN CAPE PROVINCE OF SOUTH AFRICA"*

The study is supervised by Prof. K. Luneta.

The purpose of this research is to identify and categorize errors as a result of misconceptions of learners, as well as to suggest possible ways to curb these learning barriers. In order to examine learner errors and misconceptions, I wish to administer a test instrument to 136 learners in three grade 10 classrooms. Later, sixty learners will be selected for interviews based on their answers to the test. The test paper will take approximately one hour to answer and each interview will last within 20 to 30 minutes. Each interview will be tape-recorded for later transcription.

I would like to request the participation of your school in this study by allowing me to conduct the test and the interviews. The teachers will be given a summary of their interviews later. You will also be given an opportunity to receive a summary of the findings. I will not use learners' names or anything else that might identify them in the written work, oral presentations, or publications. The information remains confidential. They are free to change their minds at any time, and to withdraw even after they have consented to participate. They may decline to answer any specific questions. I will destroy the tape recording after the research has been presented and/or published which may take up to three years after the data has been collected. There are no known risks to you for assisting in this study.

If you would like more information, please contact me by phone at 0718644549 or by e-mail at baidoojoseph@gmail.com. Please contact me at your earliest convenience to discuss the work or to provide your consent to participate.

Thank you for your consideration.

Yours sincerely

Joseph Baidoo

APPENDIX F1 : SCRIPT AND OUTCOME OF INTERVIEW

APPENDIX B

INTERVIEW DAY

The following questions will be ask based on the identified errors in the learners' scripts. Identified errors will be coded. Open – ended questions will be use to allow learner- respondents to explain their computations.

(Materials: Tape recorder; writing pads; pens)

Learners: S

Grade ---------- Date: ----------

Time 15H00 Duration of the interview: ----------

Interviewer: The Researcher.

1. Learner S, Can you explain your solution to question 2. Qn 2

LEARNER S: I MULTIPLY 36 BY 36 TO GET 1296.
R: HOW DID YOU GET 0
L: I DIVIDE x^2 by 25 TO GET 0
R: WHERE IS THE Y VARIABLE
L: OH NO IT IS A MISTAKE, IT SHOULD BE 129by = 0 equal to 1296y

2. You also answered question 4, explain how you arrive by your answer:

I MULTIPLY 2 WITH 2 OF THE EXPONENT OF X TO GET 4.

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I MULIPLY 4 by 4 IS EQUAL 16 AND 16 TIMES 16 IS EQUAL TO 256xy divide BY $3y^{-2}$. I DIVIDE 256xy DIVIDE BY $3y^{-2}$ IS EQUAL TO $85,3xy^{-2}$.

3. Can you explain your solution to question 5

I DIVIDED 10^x BY 8^x TO GET $1,25^{x-1}$ AND THEN I DIVIDE 12^{x+1} BY 15^{x-1} TO GET $0,8^{x+1}$.

4. Learner S ... Can you explain your solution to question 6

I MULTIPLY THE BASES 10 AND 4 TO GET 40 AND THEN. ADDED THE EXPONENTS, 5 GOES INTO 40 AND 25 AS 8 AND 5. WHEN I ADD 2x - x IS x AND 3+1 IS 4, THEN x plus 4 IS 4x and 2 plus x is 2x.

5. Learner ..., Can you explain your solution question $\frac{8^{4x}}{5^{2x}}$

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APPENDIX F2: SCRIPT AND OUTCOME OF INTERVIEW

4   10A

APPENDIX A

.... DURATION: ONE HOUR

...NAIRE S .... ALGEBRAIC FRACTION

INSTRUCTIONS:

A. Simplify the following if possible and show all the workings.
B. Write neatly and legibly.
C. All answers will be treated with strictest confidence..

No	QUESTIONS AND LEARNERS' RESPONSES	YEAR
1.	$\left(\frac{x}{2}-\frac{y}{3}\right)\left(\frac{x}{2}+\frac{y}{3}\right)$ $= \frac{x^2}{4} + \frac{xy}{6} - \frac{xy}{6} - \frac{y^2}{9}$ $= \frac{x^2}{4} - \frac{y^2}{9}$ $= \left(\frac{x}{2}-\frac{y}{3}\right)\left(\frac{x}{2}+\frac{y}{3}\right)$	2009
2.	$36y^2 - \frac{x^2}{25}$ $= 1296 - \frac{x^2}{25}$ $= 1296 - 0$ 12961 - 0 $= 1296$ (CE) - 12961	2008
3.	$\frac{2x-5}{3} - \frac{3x-2}{4}$ $= \frac{2(x-5)}{3} - \frac{3(x-2)}{4}$ $= \frac{x-5}{3\times1} - \frac{x-2}{2\times2}$ (CE) $= \frac{x-5}{1} - \frac{x-2}{2}$	2010

6	$\frac{{}^{+3}.\ 4^{1-x}}{25^{2+x}}$ $\frac{10^{2x+3} . 4^{1-x}}{25^{2+x}}$ $= \frac{40^{2x-x+3+1}}{25^{2+x}}$ $= \frac{8^{x+4}}{5^{2+x}}$ (CE) $= \frac{8^{4x}}{8^{2x}}$ $=$	2015
7	$\frac{(2^2\ X\ 3)^{x+1}}{2^{2x} .3^x}$ $= \frac{(2^2 x 3)^{x+1}}{2^{2x} . 3^x}$ $= \frac{4x3^{x+1}}{4x.3^x}$ (CE) $= \frac{12^{x+1}}{4x.3^x}$	2006
8.	$\frac{5(x-1)}{6} - \frac{x-1}{2} + \frac{x-3}{3}$ $= \frac{5x-5}{6} - \frac{x}{2} + \frac{-3x}{3}$ $= \frac{x}{6} - \frac{x}{2} + \frac{-3x}{3}$ (CE) $= 6x - 2x - x$	2011

11	$\frac{3}{x-4} - \frac{2}{x+3} - \frac{21}{x^2-x-12}$ $= \frac{3}{-4x} - \frac{2}{3x} - \frac{21}{x^2-x-12}$ (LE) $= -0{,}75x - 0{,}6x - x^2 - x - 12$ (LE)	2015
12	$\frac{x^3+1}{x^2-x+1} - \frac{4x^2-3x-1}{4x+1}$ $= \frac{1x^3}{x^2+1x} - \frac{4x^2-3x-1}{4x+1}$ $= \frac{1x^3}{2x^2} - \frac{16x-3x-1}{4x+1}$ (LE) $= \frac{1x}{4x} - \frac{13x-1}{4x+1}$ $= 4x - 3{,}25x - 1$	2012

Accessed on 15/11/2015 from http://mathsandscience.com/tests-exams/gr-10-maths-exams/

Accessed on 15/11/2015 from http://www.ecexams.co.za/2015_November_Gr_10_Exams.htm