Ruin Problem in Health Care Insurance with Random Interest Rate.

by

EHOUNOU SERGE ELOGE FLORENTIN ANGAMAN

MINOR DISSERTATION

Submitted in partial fulfilment of the requirements for the degree

MAGISTER COMMERCII

in

FINANCIAL ECONOMICS

in the

FACULTY OF ECONOMICS AND ECONOMETRICS

at the

UNIVERSITY OF JOHANNESBURG

SUPERVISOR: Dr FRANCK ADEKAMBI

2016

Dedication

I would firstly like to dedicate this thesis to the Lord Jesus Christ who gave me strength and kept my hope alive throughout my studies. Secondly, I also dedicate this to my 'little' mother, Angaman Epse Tanoh N'doua Odette, who believed in me and sent me to South Africa to pursue my studies. Thirdly, I also dedicate this dissertation to: Guillaume Tanoh; Christ-Daryl Angaman; Nshimwe Marie-Ciera Angaman; my siblings; George Niamkey; and, Andree Tanoh, my partner, for the motivation, encouragement and support throughout the whole project. Finally, this thesis is dedicated to my late mother, Sasso Marie Angaman.

Acknowledgements

I would like to thank the following people for supporting and guiding me throughout my studies:

I would like to express my profound gratitude to the National Research Foundation (NRF) for the funds allocated to me for this master’s programme; my supervisor, Doctor Franck Adekambi, whose strong knowledge in mathematical statistics and insurance guided me step by step right from the onset of this paper to the end; and Gustave Udahemuka, whose comprehensive background in programming gave me the practical views on grounds and facilitated the application of mathematics to these fields. The discussions with these two people have given me an overview of how the application of mathematical statistics, in finance and insurance fields, is important.

Special thanks to the Lord Jesus Christ, my parents, siblings and friends for their immense help throughout this project.

Declaration

I, Ehounou Serge Eloge Florentin Angaman, hereby declare that this minor dissertation is my own work and that all sources were accurately referenced. I have not submitted this minor dissertation to any other university for any other degree.

Table of Contents

List of Figures

List of Tables

Abstract

Ruin theory is a predominant topic in actuarial science and many researchers have brought their contributions in it to make the theory close to reality. Historically, the pioneers of ruin theory have assumed a risk model that is not affected by the force of interest. However, it is well-known that insurance companies operate in an economic environment where the force of interest plays an important role, hence the study of ruin probability without considering the force of interest is flawed in reality.

In this thesis, we consider Ramsay's model, modified by the inclusion of random interest rate on the surplus process, where we investigate the effect of random interest rate on the ruin probability. Autoregressive model is introduced to model the randomness of the interest rate. Based on the statistics' test, we found that The Three-Months South Africa Treasury Bill Rates (TBR) used in this study can be modelled by an autoregressive structure of order one. We then derive exponential type upper bounds for the ruin probability by using a renewal recursive technique. Finally, we provide numerical examples where the length of the sickness period and that of the healthy period follow an Erlang distribution to illustrate our results. We found that the probability of ruin is very small when we considered a random interest rate in the Ramsay's risk model unlike the results obtained by Adekambi (2013) when he considered a constant interest rate in the Ramsay model.

Chapter 1- Introduction

1.1 Background

Over the past two decades, due to financial difficulties, the economic environment has been marked by ruin cases in many financial institutions, and especially in insurance companies. For example, in May 2000, American Chambers Life Insurance Company was declared insolvent, and State Life and Health Insurance Guaranty Associations acquired it in the same year. Four years later, The Life and Health Insurance Company of America (LHICA) filed for bankruptcy as a result of financial difficulties. Following that announcement the Pennsylvania Department of Insurance took control of the company in July 2004. More recently, on February 28, 2015, precisely, CoOportunity Health was declared bankrupt due to financial difficulties and was liquidated to the Life and Health Insurance Guaranty Association. These are a few examples of insurance companies in the United States of America that went insolvent due to financial difficulties.

The main objective of an insurance company is to put together risks that policyholders may encounter by providing financial support or compensation that would help to reduce the financial consequences of the loss that may face them. Thus, state insurance regulators, policymakers and researchers have all become more concerned about the financial status of insurance companies and have tried to come up with an appropriate framework that would assist insurance companies to meet their commitments to policyholders. Hence, the notion of ruin theory appears as an important approach that insurance companies need to consider in order to measure the type of risk inherent in the insurance business, as well as assessing the financial situation of the company over a long period of time.

First introduced in the 1900s, by the Swedish actuary, Filip Lundberg, ruin theory can be defined theoretically as an approach that uses mathematical techniques to explain an insurer's exposure to insolvency or ruin (Lundberg, 1909). Moreover, in most actuarial literatures ruin theory is described as a method that investigates the probability that the insurer's surplus level –that is, the premium received over claims paid out- will eventually drop below zero making the insurance company insolvent (Tanasescu and Mircea, 2014). In other words, ruin theory deals with the approximation of the probability that the insurance company will go bankrupt. Therefore, by knowing the probability of ruin, the insurance companies can take measures to reduce the risk of bankruptcy. In line with this, Tanasescu and Mircea (2014) argued that one way an insurance company can measure the risk related to its business is to evaluate the ruin probability of the company. Furthermore, the authors stated that ruin probability is a useful means that can be used by the insurance company to control its disposable fund over a long period of time.

1.2 Research problem and research question

Following the work published by Lundberg (1903), ruin theory has become the main focus in actuarial science and researchers have extended it to reflect the economic environment in which the insurance companies operate more. According to Yao and Wang (2010), there are two kinds of the extension of ruin theory in actuarial science. The first kind of extension of the theory is the one that assumes that the increments in the surplus process are 'dependent', which advocated that dependence within claims and premiums can occur at any time in reality; the second type of extension is to consider a risk model that is influenced by the force of interest.

As far as the first extension is concerned, Kit Mo (2002) argued that, "Dependence within the risk model can happen many times in reality; hence the assumption of independent claims would be inadequate in modelling real world insurance processes." Therefore, in the aim of modelling real world insurance processes, many authors, such as: Kit Mo (2002); Frostig; Haberman and Levikson (2003); Chan and Yang (2006); Lefèvre and Loisel (2009); Biard *et al* (2011); Peng, Huang and Wang (2011); Cheung and Landriault (2012); and, more recently, Li, Wu and Zhuang (2015); Li and Yang (2015); Albrecher *et al*. (2015), as well as Raducan *et al*. (2015); investigated the effect of dependence within the risk model on ruin probability, where they assumed that either the premiums or the claims were dependent.

The second type of extension of ruin theory - which is of particular interest to us since most of insurance companies operate their business in a world that is deeply influenced by the force of interest - is based on developing a risk model that takes into account the forces of interest, reflecting the reality of insurance companies more. In line with that, authors, such as: Cai (2002a, 2002b); Yang and Zhang (2003); Yuen, Wang and Wu (2006); Zhang, Yuen, and Li (2007); Mitric and Sendova (2010); Yao and Wang (2010); Mitric, Badescu and Stanford (2012); Adekambi (2013); Gao and Yang (2014); Thampi (2015); as well as Li and Yang (2015); all investigated the effect of the force of interest on ruin probability in the aim of having a risk model close to reality .

In this study, we follow the model proposed by Ramsay (1984), where he studied ruin probability for a health care insurance risk model without considering the effect of interest rate on the risk model and the probability of ruin. In his study, the author assumed that the policyholder pays a premium to the insurance company at a constant rate, π , during the healthy period; on the other hand, during the period of sickness, if the length of sickness goes beyond a waiting period, w , the insurer pays out claims continuously at a constant rate, β , which is proportional to the remaining duration of sickness. He finally derived the ruin probability of the insurer by using an alternating renewal process.

The main motivation of our study is that, to the best of our knowledge, it is only Adekambi (2013) who has extended the model proposed by Ramsay (1984) by considering the effect of constant interest rate. However, Cai (2002b) argued that the assumption of constant rates of interest does not hold in reality 'since interest rate is usually statistically dependent over time'. Therefore, the main contribution of this paper is to extend the work done by Ramsay (1984) and Adekambi (2013) by investigating the effect of random interest rate on the probability of ruin in the Ramsay risk model. Given that most of the insurance companies operate in a world that is deeply influenced by the force of interest, and, due to the uncertainty associated with the insurance business; most insurers invest their surplus to generate some interest income. Therefore, the inclusion of random interest rate on the surplus process in our risk model depicts a real life scenario. Furthermore, Yao and Wang (2010), contended that it is unrealistic to assess the ruin probability of an insurance company without taking into account interest rate. The authors argue that a considerable amount of the surplus of the insurer is derived from interest income. In line with the work of Yao and Wang (2010), Cheng and Wang (2011) investigated the effect of dependent interest rate on the risk model to evaluate the probability of ruin for an insurance company. These authors believed that the hypothesis of a constant does not hold in real world.
This study will therefore seek to answer the following question: What is the effect of random interest rate on the probability of ruin in Ramsay's model?

1.3 Objective of the study and research design

The main purpose of the study, therefore, is to investigate the effect of random interest rate on the probability of ruin in the Ramsay model. In other words, we want to extend the results obtained by Ramsay (1984) and Adekambi (2013). Regarding the randomness of interest rate, the econometric estimation of the interest rate used in this study tells us that it can be well-modelled by an autoregressive structure of order one. One should recall that an autoregressive process is a process that assumes that the past values of a variable have an impact on its current values. This corroborates the study done by Cai (2002b), where he argued that interest rates are statistically proven to depend on each other over time.

Therefore, in order to investigate the effect of random interest rate on the probability of ruin in Ramsay's model, we will first use recursive technique to derive integral equations of the probability of ruin and then derive probability inequalities of the ruin probability. The inequalities are going to be used as upper bound to evaluate the ruin probability of our model.

1.4 Chapters overview

The remainder of the paper is structured as follows. Besides the introduction, chapter 2 discusses the literature review while chapter 3 presents Ramsay's model, then Chapter 4 presents the econometric estimation of interest rate. Thereafter, Chapter 5 describes the model and the methodology used in this study, while chapter 6 gives numerical examples to illustrate the application of the results under random interest rate, and, finally, chapter 7 presents the conclusion.

Chapter 2- Literature Review

2.1 Introduction

In this chapter, we look at some theoretical and empirical literatures regarding the assessment of ruin probability for an insurance company with the objective of identifying a gap in the literature that this study will attempt to fill. One of the limitations of the Lundberg risk model is based on the assumption of independence made on the claims and the premiums as well as the use of a risk model that does not consider the force of interest. Even though these two assumptions facilitate the determination of ruin probability, they are flawed and hence cannot be applied in real life. That is the reason why, in the effort to make the results more applicable in reality, many studies have been carried out to assess ruin probability using different approaches, such as the effect of dependent claims and premiums on ruin probability, in the case of ordinary renewal processes and the effect of the force of interest on this. Some researchers have evaluated ruin probability in the context of ordinary renewal processes where premiums and claims are dependent; while others have focused their studies on the asymptotic behaviour of ruin probability when the interest rate is considered.

This literature review consists of three sections. In the first section, we discuss studies that investigated the effect of the dependent claims and premiums on the probability of ruin. In the second section, we provide an overview of the economic environment of the insurance company. In the subsequent section, we discuss some theoretical and empirical literatures on ruin probability when the interest rate is considered.

2.2 Theoretical and empirical literature on the effect of dependent claims and premiums on ruin probability

Many studies have investigated the effect of dependent claims and premiums on the probability of ruin. Among others, Kit Mo (2002) argued- that the assumptions made on the Lundberg classical risk model that individual claims are independent of each other over time- does not always hold in reality, because claims generated by one policy in an insurance portfolio may induce claims from other policies in the same portfolio. Therefore, the author analysed the effects of dependence on both the probability of ruin and time to ruin. He used copulas to introduce the dependent structure on claim occurrence and then performed a simulation study to investigate the effects of different dependent structures on the aggregate claims number, the probability of ruin and the time to ruin. Finally Kit Mo (2002) found that the probability of ruin increases when dependence is considered in the risk model; however, this increase has a slight impact on the time to ruin. He concluded that the results obtained can be applied in real life.

In their study, Frostig *et al.* (2003) investigated the probability of ruin for a generalised life insurance models including whole life and long-term care contracts. The authors contended that, in such models, the claims' occurrences are dependent over time; hence the structure of their study is different from that of ruin probability in the Lundberg classical risk model where it is assumed that claims over time are independent. In order to obtain the results, they developed an algorithm to model the dependence structure on the claim, which, in turn, enables them to find recursively new upper and lower bounds for the ruin probability of their dependent risk model. Frostig *et al.* (2003) sated that the new upper bounds derived in their model perform much better than the Lundberg-type upper bounds; hence, their risk model is more adequate in real world insurance business.

Moreover, Chang and Yang (2006) stated that the assumption made on the classical ruin theory that the risk model forms a random walk might not hold in reality due to the fact that annual incomes of an insurance company can be correlated. In order to illustrate their argument, the authors considered a risk model in which claims and premiums are dependent over time. They used the Granger's causality method to model the claim and premium and finally derived upper bounds for the ruin probabilities by using the martingale technique. They argued that their model can be applied in any domain other than in the domain of insurance by considering the premium as an input process of any system and the claim process as an output process of that system. According to Chang and Yang (2006:11) "The system can be monetary reserve or stock of a commodity, where we are interested in its reserve after some time and therefore one can apply the results obtained to find out the probability that the reserve will ever become negative".

Lefèvre *et al*. (2009), examined two generalised versions of the 'classical compound Poisson risk model'. In the first version of the model, they included a premium function that is not constant and claim processes that vary over time, and, in the latter, they assumed the possibility of interdependence between the claim occurrences. The authors argued that previous studies regarding the classical compound Poisson risk model assumed interdependence between the claims occurrences. However, according to Lefèvre *et al.* (2009), these assumptions may be found to be too restrictive and not applicable to real insurance portfolios. Therefore, in order to derive the probabilities of ruin of the two generalised versions of the classical compound Poisson risk model, the authors used a recursive method that relies on the use of remarkable families of polynomials which are of 'Appel' or 'generalised Appel' types. They contended that the results obtained can also be applied to an insurance portfolio with several interconnected risks, and to the case of continuous claim severities.

In addition, Biard *et al.* (2011) argued that many strong assumptions in the classical compound Poisson risk model may not accurately give the change of the reserve of an insurance business over time. This is the case in the insurance business, which experiences natural disaster risks like earthquake or flooding. Concerning these types of risks, claim amounts may be dependent on each other and they may also depend on previous natural disasters (Biard *et al.* 2011). Therefore, in order to take into account natural disaster risks, such as earthquakes or flooding, the authors considered a 'dependent Poisson risk model', where each claim amount depends on the previous claim inter-arrival time. Finally, they evaluated the finite time ruin probability of these type-dependent risk models by assuming that the claim sizes have a heavy tailed distribution. The technique used in order to derive the upper bound ruin probability is based on 'the analysis of spacings in the conditioned Poisson processes.

Moreover, Peng *et al*. (2011) argued that, because of the increasing complexity of insurance and reinsurance products, many researchers have been studying the modelling of dependent risks to reflect the real word insurance companies. In their study, they examined ruin probability in a dependent discrete risk model with a constant interest rate, in which the dependent claims are assumed to have a one-side linear structure. They found an explicit asymptotic formula for the ruin probability by martingale and inductive techniques. Peng *et al.* (2011) concluded that the results obtained are more applicable in reality.

Furthermore, Li and Yang (2015) contended that previous studies regarding 'multidimensional risk models' were based on the assumption that claims from different insurance portfolios are absolutely independent. Then, the authors argued that, even though this assumption makes the risk model easy to manipulate; it is nearly impossible to apply the results in real world insurance business. That is the reason why in their study, they considered a 'bidimensional renewal risk model' with dependent claims where they assumed that the insurance company proportionally allocates its aggregate initial capital to two types of insurance portfolios. The dependence among claim sizes comes from two perspectives: the first is based on the assumption that that the two types of claims have the same renewal claim number process; while the latter relies on the fact that each vector of claims follows a general dependence structure, which is given in terms of copulas having both asymptotic independence and asymptotic dependent situations. They finally found an exact asymptotic formula for the finite-time and infinite-time ruin probabilities.

2.3 Overview of the economic environment of the insurance company

The Lundberg classical risk model does not take into account the economic environment in which insurance companies operate, namely the inflation and interest rates. Both together have a significant impact on the economy of a country. The term 'inflation' is defined as a decline in purchasing power of a currency. It is often caused by the presence of increasing the prices of goods and services in an economy. In other words, inflation is a state where people realise that they need more money to buy goods and services than they needed last month or last year to buy the same goods and services. Therefore, the main consequence of inflation is that a currency of a country suffers the loss of its value. Hence, inflation is seen as a big problem in most of the economies. It is in this context that Castaner *et al.* (2013) argued that inflation and interest rates can, sometimes drastically, affect the evolution of the reserve of a company. "Claim amounts and premiums have often a tendency to increase for various socio-economic reasons (e.g. higher loss levels and larger compensations or coverages)"- (Castaner *et al.* 2013). Hence, it is imperative for insurance companies to know the impact of inflation on the total claim amounts when computing the premiums charged.

As far as interest rate is concerned, it is defined as the amount of interest due per period, as a proportion of the amount lent, deposited or borrowed. In other words, it is the rate a bank or other lender charges to borrow its money, or the rate a bank pays its savers for keeping money in an account. Given the importance of interest rates in an economy, many studies have been carried out in the objective of finding the main determinants of interest rates, and economists are divided between two of the most remarkable theories of interest rates, namely: the Irvin Fisher's classical approach and the liquidity preference theory first introduced by Keynes.

According to Fisher (1930), interest rate is the compensation or payment acquired by individual for loaning his savings to another person. In other words, it is the price of borrowing from a person and the recompense for lenders. Hence, according to Fisher's theory, the main determinant of interest rate dwells on how the supply and demand for savings interact; therefore, saving plays an important role in determining the interest rate. Fisher's theory of interest rates gained support from many economists. Fabozzi *et al.* (1998) support the view of Fisher in their study and describe the interest rate as the cost paid by debtors to a creditor for the utilisation of capital for the period of time.

However, many others criticised Fisher's theory and are in line with Keynes' liquidity preference theory. In his theory, Keynes (1936) defined interest rate as the interaction of money supply and money demand, rather than the interaction of supply and demand for savings. The reasoning behind his theory is as follows: Keynes believed that investors prefer to keep their money liquid for present transactions and they demand interest in returns for sacrificing their liquidity. In this context, he considers interest rates as an opportunity to hold cash (money), due to the fact that one may convert money into bonds and earn more. So, Keynes argued that the amount of money people hold varies inversely with interest rates - if rates decrease, they will hold more money until rates increase.

Therefore, both theories highlight the importance of the interest rate in modern economies, and the risk model should consider interest rates to model real life insurance companies. Yao and Wang (2010) argued that it is fundamental to consider the interest rate in the study of ruin probability since large amounts of the surplus of the insurance companies come from investment income. This is to say that the interest rate plays a vital role in the survival of the insurance company; so, studying the probability of ruin without taking into account rates of interest, could be too restrictive in real world.

2.4 Theoretical and empirical literature on the effect of interest rate on the probability of ruin

Given the importance of the interest rate in an economy, many researchers have investigated the effect of the force of interest on the ruin probability with the objective of modelling an insurance risk model that can be applied in real life. Authors, such as Shen and Lin (2010), studied the ruin probability in a continuous-time renewal model of upper-tailed independent and heavy-tailed random variables with the inclusion of interest rates in the model. The authors assumed that the claim sizes are identically distributed - but not necessarily independent and nonnegative variables - with the same distribution and their inter-arrival times; but there are another series of independent, identically distributed $(i.i.d)$ and nonnegative random variables. When assuming a model where the interest rate and premium are constant, they found a simple solution for ruin probability of the renewal process where the initial surplus is large.

Similarly, Gao and Yang (2014) investigated the effect of the force of interest on ruin probability by studying three kinds of finite-time ruin probabilities in a diffusion-perturbed bi-dimensional risk model with a constant force of interest, where the claims are pair-wise, strongly quasi-asymptotically independent and there are two general claim arrival processes. The authors derived a homogeneous asymptotic expression for finite-time ruin probability when they assumed the claims both to be long-tailed and dominatingly varying-tailed. They found that when considering a certain dependence structure among the inter-arrival times, the obtained formulas satisfy uniformly for all times when the claims are pair-wise, quasi-asymptotically independent and consistently varying-tailed.

More recently, Thampi (2015) studied the effect of the force of interest on ruin probability by investigating the asymptotic behaviour of the finite time ruin probability of a compound renewal risk process with a constant interest rate. The author assumed a risk model where the size of claims are 'weakly negatively dependent' (WND) and identically distributed random variables that belong to the group of variable tails. Based on the inclusion of a constant interest rate in the model and the assumption made on the claim sizes, he found an expression for the finite time ruin probability for this particular risk model and argued that the results obtained have extended and improved some corresponding results of related studies.

Furthermore, Li and Yang (2015) estimated the ruin probability for a bi-dimensional renewal risk model with a constant interest rate and dependent regularly varying claims. The authors assumed that the insurer invested its total surplus in two types of insurance businesses. Under the condition of having a constant rate of interest, the authors obtained an exact solution for the finite-time and infinite-time ruin probabilities.

However, Cai (2002b) argued that considering the force of interest in the classical risk model that is constant (or *i.i.d.*) does not reflect the true reality of the insurance company. For this reason, some researchers have shifted their study on the effect of a random interest rate on the surplus process to evaluate the ruin probability of the insurer.

Among others, Cai (2002b) evaluated ruin probabilities in two generalised risk model where he showed how the probability of ruin is affected by the timing of payments and interest rate. The author considered a dependent autoregressive process to model interest rate. Based on this assumption and using a renewal recursive method the author derived Lundberg inequalities for the ruin probabilities. He finally gave some numerical examples to illustrate the results in the case of the compound binomial risk process.

Similarly, Cheng and Wang (2011), assessed ruin probabilities for a discrete risk model in which they considered that the claims, the premiums and the interest rate had a dependent autoregressive process of order one. They then used the recursive and integral equations for the expected discounted penalty function to derive the generalised Lundberg inequality for the infinite time ruin probability as well as for the finite time ruin probability.

Furthermore, Yuen, Wang and Wu (2006) studied ruin probability related to the discounted penalty function in the renewal risk model. The authors derived an integro-differential equation for the Gerber-Shiu expected discounted penalty function when they asserted that the inter-occurrence times of claims followed an Erlang distribution. The main finding of their study is as follows: they derived lower and upper bounds for the ultimate probability of risk of the risk model, and, by using two special cases of stochastic interest processes - namely the Brownian motion case and the compound Poisson case - they found exact formula for the discounted density related to the expected discounted penalty function.

This chapter presents a review on ruin probability for an insurance company in broad, general sense. However, studies that evaluate ruin probability in health care insurance, in particular, are few. One such study conducted by Ramsay (1984) assessed ruin probability for a health care insurance concern using alternating renewal process without considering the effect of interest rate on the risk model of the insurer. Furthermore, Adekambi (2013) extended the model proposed by Ramsay (1984) by including a constant interest rate on the probability of ruin in the Ramsay model. He used martingale and recursive techniques to derive Lundberg inequalities for the ruin probability. However, the use of a constant interest rate does not necessarily hold in reality as it does in real life: insurance companies operate their businesses in an economic environment where interest rates are generally supposed to be dependent on one another over time (Cai, 2002b). Therefore, the major contribution of this study is to extend the work done by Ramsay (1984) and Adekambi (2013) by taking into consideration the effect of random interest rate, covered in the Ramsay model, to estimate ruin probability for a health care insurance company.

The next chapter presents the Ramsay model.

Chapter 3- Ramsay's Model

Ramsay (1984) proposed the alternating renewal process for the study of health insurance problems in his work entitled "The asymptotic ruin problem when the healthy and sick periods form an alternating renewal process". In what follows, the Ramsay model will be described, followed by the description of how the probability of health (and sickness) of the model was determined. After which, we finally present the main findings of the extension of the Ramsay model in the work done by Adekambi (2013). The model is described as: $T_1, Y_1, T_2, Y_2 ... T_{n,} Y_n ...$ is the sequence of independent random variables. According to Ramsay (1984), "The $T_i 's$ are assumed to have a common $c.d.f \ F(x)$ and the $Y_i 's$ have a common $c.d.f \ G(x)$. $c.d.f \ G(x)$. Both the $T_i 's$ and $Y_i 's$ are positive random variables."

Ramsay considered the following insurance portfolio to build his model: the insurance company deals with insurance contracts. Each policyholder pays a premium to the insurance company at a constant rate of π during the healthy period. On the other hand, during the period of sickness, if the length of sickness goes beyond a waiting period of w, the insurer pays out claims continuously at a constant rate of β, which is proportional to the remaining duration of sickness. He assumed that the period of health and sickness alternate; hence, they form an alternating renewal process. If T_i denotes the length of the i^{th} period of health and Y_i, the length of the i^{th} period of sickness; then Ramsay (1984) assumed that, "The sequence $T_1, Y_1, T_2, Y_2 ...$ is an alternating renewal process in the sense of Cov (1962)." He further assumed that the insurance company sets up an initial reserve of u per contract. The reserve Q_n at the end of the n^{th} sickness period for an individual contract is

$Q_n = u + P_n - S_n$. $n = 1, 2, ...$ and $u \geq 0$. (3.1)

where

$S_n =$ is the discounted aggregate amount of benefit paid out up to the end of the n^{th} sickness period, $n =$ 1, 2, 3...

$$S_n = \sum_{k=1}^{n} \beta_k \left(Y_k^{'} - w\right)_+$$

$P_n =$ is the discounted aggregate amount of premium received up to the end of the n^{th} healthy period, $n = 1, 2, 3...$

$$P_n = \pi \sum_{k=1}^{n} Y_k$$

Note $P_0 \equiv S_0 \equiv 0$. Ramsay defined the random variable N, as a stopping time, as follows:

$N = Inf \left\{ n : Q_n < 0, n \geq 0 \right\}$.

He argued that ruin occurred at N iff $N < \infty$ and define $\psi(u) = \Pr\left\{ N < \infty \middle| Q_0 = u \right\}$

3.1 Probability of sickness and health

In this section, we provide some results obtained by Ramsay (1984) to illustrate how a health care insurance company operates. Ramsay assumed that the end of the preceding sickness period is denoted by time = 0.

Ramsay defined,

$$f^*(s) = \int_0^x e^{-st} dF(t) \quad (3.2)$$

$$g^*(s) = \int_0^x e^{-st} dG(t) \quad (3.3)$$

Considering $h_2(t)$ to be the renewal density for the process $f^*(s)$,

$$h_2(t) = \int_0^t dF(y) dG(t-y) + \int_0^t h_2(y) \int_0^{t-y} dF(z) dG(t-y-z) dy.$$

Using Laplace equation, he got

$$h_2^*(s) = f^*(s) g^*(s) + h_2^*(s) f^*(s) g^*(s)$$

or

$$h_2^*(s) = \frac{f^*(s) g^*(s)}{1 - f^*(s) g^*(s)} \quad (3.4)$$

Therefore, every time the term $\frac{f^*(s) g^*(s)}{1 - f^*(s) g^*(s)}$ is expressed, Ramsay (1984) assumed that $h_2^*(s) = \int_0^{+\infty} e^{-st} h_2(t) dt$ was a Lebesque integral.

The author made the same assumption for the process $g^*(s)$. If $h_1(t)$ is the renewal density for the process $g^*(s)$, then it follows that

$$h_1(t)=\int_0^t dF(y)+\int_0^t h_2(y)dF(t-y)dy.$$

Using Laplace equation, he got,

$$h_1^*(s)=f^*(s)+h_2^*(s)\,f^*(s)$$

or

$$h_1^*(s)=\frac{f^*(s)}{1-f^*(s)g^*(s)} \qquad (3.5)$$

Every time the term $\frac{f^*(s)}{1-f^*(s)g^*(s)}$ is expressed, Ramsay (1984) considered the following:

$p(t)=$ The probability that the policyholder is healthy. when $t>0$.

$q(t)=$ The probability that the policy holder gets sick. when $t>0$.

Clearly, $p(t)+q(t)=1$ and,

$$p(t)=\int_t^\infty dF(y)+\int_0^t h_2(y)\int_{t-y}^\infty dF(z)dy \qquad (3.6)$$

$$=1-F(t)+\int_0^t\left[1-F(t-y)\right]h_2(y)dy \qquad (3.7)$$

Ramsay (1984) argued that $p(t)$ exists if equation (3.7) is a Lebesque integral.

Now he considered,

$r(t)=$ is the probability that the policyholder gets sick and receives a benefit at time *t.*

Hence,

$$r(t)=\int_0^{t-w} h_2(y)\int_0^{t-w-y} dF(z)dG(t-w-y-z)dy \qquad (3.8)$$

Thus according to Ramsay (1984), these are the major probabilities required to define a health care insurance company.

However, the model proposed by Ramsay is not without limitations as he did not consider the effect of interest rate on the risk model, hence it is consistent with the classical risk model assumptions. In order to make the model proposed by Ramsay applicable in real world, Adekambi (2013) included the force of interest in the model, and he assumed that the force of interest is constant.

When considering constant force of interest Adekambi (2013) found the following,

$S_n=$ The discounted aggregate amount of benefit paid out up to the end of the n^{th} sickness period, $n=$1, 2, 3...

$$S_\delta(n)=\sum_{k=1}^{n}\beta e^{-\delta[Y_1+\ldots+Y_k+w]}\frac{\left[1-e^{-\delta[Z_k-w]_+}\right]}{\delta} \tag{3.9}$$

$P_n =$ The discounted aggregate amount of premium received up to the end of the n^{th} healthy period, $n = 1, 2, 3...$

$$P_\delta(n)=\pi\sum_{k=1}^{n}e^{-\delta[Y_1+Z_1+\ldots+Y_{k-1}+Z_{k-1}]}\frac{\left[1-e^{-\delta Y_k}\right]}{\delta} \tag{3.10}$$

Adekambi denote by ${}_n\psi_\delta(u)$ and $\psi_\delta(u)$ the finite and ultimate ruin probability when the force of interest is δ.

Then

$${}_n\psi_\delta(u)=\Pr\left\{\bigcup_{k=1}^{n}\left(V_\delta(T_k)<0\right)\right\} \text{ and } \psi_\delta(u)=\Pr\left\{\bigcup_{k=1}^{\infty}\left(V_\delta(T_k)<0\right)\right\} \tag{3.11}$$

Where

$$V_\delta(T_k)=u+P_\delta(n)-S_\delta(n). \tag{3.12}$$

Adekambi used two different techniques to find the upper bound for the ruin probability in the Ramsay model which was modified by the inclusion of constant interest rate. ${}_n\psi_\delta(u)\le e^{-R_1u}$ is the upper bound derived by the martingale techniques and $\psi_\delta(u)\le e^{-R_2u}$ is the one obtained by the recursive technique.

However, as mentioned in the previous chapter, the use of constant interest rate in the Ramsay's model does not necessarily portray the reality in which insurance companies operate. In order to build a risk model that is very close to reality, we decided to modify Ramsay's model by the inclusion of random interest rate. Another point that makes our risk model different from those proposed by Ramsay (1984) and Adekambi (2013) is that we did not use discounted aggregate amount of claim and discounted aggregate amount of premium. Therefore, we followed the same approach as Cai (2002b), where, in his study, he first derived an integral equation by using recursive technique, and then gave probability inequalities for ruin probability. Thus, our model becomes a special case of Cai's (2002b) risk model. The subsequent chapter presents the econometric estimation of the interest rate used in this study.

Chapter 4- Econometric Estimation of the Interest Rate

The purpose of this chapter is to illustrate the reason why a random interest rate was considered in this study. Several studies have estimated interest rate using different models to fit the data of specific countries. In this study, the econometric estimation of the interest rate is illustrated with reference to data relevant to the South Africa economy. This chapter is organised as follows: it commences with a discussion of the data used in the study; then how the model was estimated will be outlined; concluding with presenting the results.

4.1 Data

This section provides a description of the data used in the study. The study makes use of the annual data for the South Africa three-month Treasury Bill Rates from 1970 to 2015. Data was sourced from Quantec.

4.2 Model estimation

The following variables are used to evaluate the appropriateness of the randomness of the interest rate.

$$R_t = \delta R_{t-1} + K_t \tag{4.1}$$

Where R_t the current change in interest rate; R_{t-1} is the lagged change in interest rates; δ is called the coefficient of regression and K_t is the error term; which is an uncorrelated sequence of random variables with a probability distribution that has zero for mean and a finite variance, i.e. $K_t \sim (0, \sigma^2)$.

4.3 Empirical results

The first step in our empirical analysis is to investigate the appropriateness of the randomness of the interest rate. The result of our test is given by equation (4.2). It can be concluded that the lagged change in interest rate is a useful predictor of the current change in interest rate, as $t = 28.19 > 1.96$.Therefore, the randomness of interest rate used in this study can well be modelled by an autoregressive structure of order one.

$$\mathrm{R} = 0.97 * R\,(-1) \qquad R^2 = 0.66 \tag{4.2}$$

$$[0.03]$$

The second step in our empirical study is to perform some diagnostic tests for the residuals. These tests are very important since they are going to tell us whether or not the residuals are normally distributed, ensuring that they do not suffer from Heteroscedasticity and/or serial correlation. The first test performed is the Breusch-

Godfrey Serial Correlation LM Test and we found that residuals were not correlated to each other, as the null hypothesis at 5% significance level was rejected, since the probability of Chi-Square= $0.216 > 0.05$. The second test performed is Breusch-Pagan-Godfrey heteroscedasticity test, where we found that there was no presence of heteroscedasticity in the model as we rejected the null hypothesis at 5% significance level, since the probability of Chi-Square= $0.099 > 0.05$. Finally, we performed the normality test for the residuals and the finding was that the residuals are normally distributed. Thus the null hypothesis at 5% significance level was rejected again since the probability= $0.413 > 0.05$

We proved statistically that the randomness of the interest rate used in this study can be modelled by an autoregressive structure of order one. The next chapter describes the model and how we derived the integral equation and the probability inequalities for ruin probability in the Ramsay's model modified by the inclusion of random interest rate.

Chapter 5 - Research Methodology

This chapter presents the methodology used in our study. The aim of our study is to assess the probability of ruin in the Ramsay model as modified by the inclusion of random interest rate on the surplus process. In order to evaluate the ruin probability in our model, we first use the recursive technique to derive integral equations for the ruin probability and then give probability inequalities for the ruin probability. The upper bound is derived from the inequalities and serves in evaluation of the ruin probability. The first section of this chapter will describe our risk model. The next section will show the different steps to derive the integral equations for ruin probability, concluding this chapter by providing probability inequalities for ruin probability.

5.1 Description of the model

Let us assume a discrete time risk model in which the premiums and claims are noted only at times $t = 1, 2, \ldots$. We consider an insurance company that deals with sickness insurance contracts. Each policyholder pays a premium to the insurance company at a constant amount, λ, proportional to the healthy period. On the other hand, during the period of sickness, we assume that if the length of sickness goes beyond a waiting period, w, the insurer pays out claims to the policyholder at a constant amount, β, proportional to the remaining duration of sickness.

Let us consider $\{\lambda H_t, t=1,2,...,t\}$ and $\{\beta(S_t - w)_+, t=1,2,...,t\}$ as being two sequences of independent and identically distributed $(i.i.d.)$ positive random variables, where λH_t represents the aggregate premiums during t^{th}, that is, the healthy period, which is from time t−1 to time t. $\beta(S_t - w)_+$ denotes the aggregate claims during the t^{th} period of sickness. We set $\beta(S_t - w)_+ = \beta(S_t - w)$ if $S_t > w$ or $\beta(S_t - w)_+ = 0$ if $S_t \le w$. We assume that the amount of claims, $\beta(S_t - w)_+$, during the t^{th} period of sickness, is paid at the end of the t^{th} period, and the amount of premium, λH_t during the t^{th} period of health is received at the beginning of the t^{th} period, that is at time $t-1$.

Furthermore, we consider $H_1, S_1, H_2, S_2, ..., H_t, S_t$ to be a sequence of positive independent random variables. H_i, $i=1,...,t$ represents the length of the i^{th} healthy period. S_i, $i=1,...,t$ represents the length of the i^{th} period of sickness. One should notice that the healthy period and sick periods alternate; hence, we assume that they form an alternating renewal process. Thus, the process, $\{I_i, i \in \mathrm{N}\}$ with $I_i = H_i + S_i, i \ge 1$, $R_0 = 0$ forms an ordinary renewal process. We then set $N_q = \sum_{i=1}^{q} I_i$ and $N_0 = 0$ for $q \ge 1$.

With the description of the model above, we have the following:
The capital of the insurance business at time t - represented by V_t with the initial capital v - is such that:

$$V_t = V_{t-1} + \lambda H_t - \beta(S_t - w)_+, t = 1, 2, ..., \qquad (5.1)$$

with $V_0 = v \ge 0.$

$V_t, t = 1, 2, ...,$ is the surplus process at time t of the risk model (5.1). In actuarial science, the preferred risk model (5.1) is 'the classical risk model' because it characterises an insurer's surplus, which experiences two opposing flow of money: namely, the premiums received from the policyholder and the claims paid out to the policyholder, with both of these increments considered to be independent. Furthermore, in the classical risk model, it is assumed that no investment is made by the insurance company; that is, the risk model (5.1) does not consider any interest rate. The ultimate probability of ruin with the initial capital, v in the risk process (5.1) is characterised by:

$$\omega(v) = \Pr\left\{\bigcup_{t=1}^{\infty}(V_t < 0)\right\} \tag{5.2}$$

As mentioned above, the risk model (5.1) is not influenced by any force of interest as we assumed that there is no investment made by the insurance company. However, in reality, most of the insurance companies invest their surplus in the stock market or in any investment portfolio that might generate some interest income on the surplus. Yao and Wang (2010) argue that a huge amount of the surplus of the insurer is derived from investment income. Thus, using a risk model that does not consider the force of interest does not portray the reality of insurance companies. Since interest rate (R_t, $t = 1, 2, \ldots$) in our study can be modelled by an autoregressive structure of order one, therefore R_t is such that,

$$R_t = \delta R_{t-1} + K_t, \; t = 1, 2, \ldots, \tag{5.3}$$

With $0 \le \delta \le 1$ and $R_0 = r_0 \ge 0$ as constants, and $\{K_t, t = 1, 2, \ldots\}$, a sequence of $(i.i.d.)$ nonnegative random variables. We can observe that if $\delta = 0$, model (5.3) is a random model of the $(i.i.d.)$ interest rate and turns out to be a model of constant rate if $\delta = 0$ and $K_t = c$, a constant, for all $K_t, t = 1, 2, \ldots,$. Therefore, according to Kellison (1991), model (3) has a dependent structure.

Moreover, $\{H_t, t = 1, 2, \ldots\}, \{\beta(S_t - w), t = 1, 2, \ldots\}$ and $\{K_t, t = 1, 2, \ldots\}$ are assumed to be independents and have common distribution functions:

$H(h) = \Pr\{H_1 \le h\}, F\{\beta(s-w)\} = \Pr\{\beta(S_1 - w) \le \beta(s-w)\}$ and $G(k) = \Pr\{K_1 \le k\}$, respectively, with $F(0) = 0$. Further, it is possible to show by the induction technique that (5.3) is equivalent to:

$$R_t = \delta^t R_0 + \delta^{t-1} K_1 + \ldots + \delta K_{t-1} + K_t. \tag{5.4}$$

Proof:

For $t = 1$, we have

$R_1 = \delta R_0 + K_1.$

Let's assume that the result is true for $t-1$, then:

$R_{t-1} = \delta^{t-1} R_0 + \delta^{t-2} K_1 + ... + \delta K_{t-2} + K_{t-1}.$

According to (5.3), the following is true:

$R_t = \delta R_{t-1} + K_t$

By substituting for R_{t-1}, we have

$R_t = \delta\left(\delta^{t-1} R_0 + \delta^{t-2} K_1 + ... + \delta K_{t-2} + K_{t-1}\right) + K_t$

$I_t = \delta\delta^{t-1} R_0 + \delta\delta^{t-2} K_1 + ... + \delta^2 K_{t-2} + \delta K_{t-1} + K_t$

Finally, the following is obtained:

$R_t = \delta^t R_0 + \delta^{t-1} K_1 + ... + \delta K_{t-1} + K_t.\ t = 1, 2, ...,$.

Equation (5.4) means that the force of interest is deeply influenced by the recent rates. When considering the model described on top and assuming that the insurance company could invest its surplus and receive some interests each year from the investment; we have the following surplus process at time t.

$$V_t = (V_{t-1} + \lambda H_t)(1 + R_t) - \beta\left(S_t - w\right)_+, t = 1, 2, ..., \tag{5.5}$$

Model (5.5) can be interpreted as a generalisation of the classical risk model (5.1) since it takes into account the interest rate described in model (5.3). In fact, model (5.1) becomes a special case of model (5.5) when it is assumed that the model is not influenced by any interest rate, that is, $R_t = 0$ for $t = 1, 2, ...,$. Moreover, it is possible to interpret model (5.5) in a risk theoretical framework. Let us consider that the insurance company invests its surplus and therefore receives some interest on it during each period. Let R_t represent the interest rate for the duration of the t^{th} period: that is, from time $t-1$ to time t. The following hypothesis can then be made about the payment of the claim amounts, $\beta(S_t - w)_+$, during the t^{th} period of sickness. We assumed that the insurance company pays out the claims at the end of the t^{th} period of sickness, that is, at time t. Regarding the amount of premiums, λH_t during the t^{th}

healthy period; it can be presupposed that the policyholder pays premiums at the beginning of the t^{th} period, that is, at time $t-1$. Therefore, the capital of the insurance company at time t, represented by V_t, with the initial capital, v; complies with model (5.5).

Hence, by using the induction technique, model (5.5) implies

$$V_t = v\prod_{q=1}^{t}(1+R_k) + \sum_{q=1}^{t}\left(\left(\lambda H_q(1+R_q) - \beta\left(S_q - w\right)_+\right)\prod_{n=q+1}^{t}(1+R_n)\right) n = 1,2,\ldots \qquad (5.6)$$

$\prod_{n=a}^{b}(1+R_n)=1$, if $a>b$

Proof:

By employing the induction technique:

For $t=1$, we have

$$V_1 = V_0(1+R_1) + \lambda H_1(1+R_1) - \beta(S_1 - w)_+.$$

Let us then assume that the result is true for $t-1$, then

$$V_{t-1} = v\prod_{q=1}^{t-1}(1+R_q) + \sum_{q=1}^{t-1}\left(\left(\lambda H_q(1+R_q) - \beta\left(S_q - w\right)_+\right)\prod_{n=q+1}^{t-1}(1+R_n)\right).$$

According to (5.5), it is found that:

$$V_t = (V_{t-1} + \lambda H_t)(1+R_t) - \beta\left(S_t - w\right)_+$$

$$V_t = V_{t-1}(1+R_t) + \lambda H_t(1+R_t) - \beta\left(S_t - w\right)_+$$

By substituting for V_{t-1}, one finds:

$$V_t = \left[v\prod_{q=1}^{t-1}(1+R_q) + \sum_{q=1}^{t-1}\left(\left(\lambda H_q(1+R_q) - \beta\left(S_q - w\right)_+\right)\prod_{n=q+1}^{t-1}(1+R_n)\right)\right](1+R_t) + \lambda H_t(1+R_t) - \beta\left(S_t - w\right)_+$$

$$V_t = v\prod_{q=1}^{t-1}(1+R_q)(1+R_t) + \sum_{q=1}^{t-1}\left(\left(\lambda H_q(1+R_q) - \beta\left(S_q - w\right)_+\right)\prod_{n=q+1}^{t-1}(1+R_n)(1+R_t)\right) + \lambda H_t(1+R_t) - \beta\left(S_t - w\right)_+$$

Since

$$\prod_{q=1}^{t-1}(1+R_q)(1+R_t) = (1+R_1)(1+R_2)\ldots(1+R_{n-1})(1+R_t) = \prod_{q=1}^{t}(1+R_q)$$

Then, the following equation is derived:

$$v\prod_{q=1}^{t-1}(1+R_q)(1+R_t)=v\prod_{q=1}^{t}(1+R_q),$$

and,
$$\begin{aligned}
&\sum_{q=1}^{t-1}\left(\left(\lambda H_q(1+R_q)-\beta\left(S_q-w\right)_+\right)\prod_{n=q+1}^{t-1}(1+R_n)(1+R_t)\right)+\lambda H_t(1+R_t)-\beta\left(S_t-w\right)_+\\
&=\sum_{q=1}^{t-1}\left(\left(\lambda H_q(1+R_q)-\beta\left(S_q-w\right)_+\right)\prod_{n=q+1}^{t}(1+R_n)\right)+\lambda H_t(1+R_t)-\beta\left(S_t-w\right)_+\\
&=\sum_{q=1}^{t}\left(\left(\lambda H_q(1+R_q)-\beta\left(S_q-w\right)_+\right)\prod_{n=q+1}^{t}(1+R_n)\right).
\end{aligned}$$

We finally obtain:

$$V_t=v\prod_{q=1}^{t}(1+R_q)+\sum_{q=1}^{t}\left(\left(\lambda H_q(1+R_q)-\beta\left(S_q-w\right)_+\right)\prod_{n=q+1}^{t}(1+R_n)\right),\ t=1,2,\ldots$$

The eventual probability of ruin in model (5.6) - associated with model (5.3), the initial capital and the initial rate, r_0 - is then characterised as:

$$\rho(v,r_0)=\Pr\left\{\bigcup_{t=1}^{\infty}(V_t<0)\right\}, \tag{5.7}$$

where V_t is specified in model (5.6). The process of $\rho\left(v,r_0\right)$ means that ruin happens no later than the t^{th} period of sickness. In fact, (5.7) implies the probability that the surplus process of the insurance company drops below zero for any period between one to infinity.

Since it is difficult to get an exact solution for the ruin probability, $\omega(v)$ it is even more challenging to obtain an exact solution for the ruin probability, $\rho\left(v,r_0\right)$. Thus, due to the difficulty of model (5.6), we will derive probability inequalities for $\omega\left(v\right)$ and $\rho\left(v,r_0\right)$. According to the Lundberg inequality, if $\lambda E(H_1)>\beta E(S_1-w)_+$ (net profit condition), and there exists a constant, $C>0$ such that:

$$\boldsymbol{E}\left[\boldsymbol{e}^{-C\left(\lambda H_1-\beta(S_1-w)_+\right)}\right]=\mathbf{1}. \tag{5.8}$$

then

$$\omega\left(v\right)\leq e^{-Cv}\ ,\ v\geq 0. \tag{5.9}$$

Given that the ruin probability, $\omega(v)$, in the classical risk model decreases when we consider the force of interest on the surplus as well as when we assume that the payment of the premiums is made at the beginning of each healthy period; we have the following relationship:

$$\rho(v,r_0) \le \omega(v), v>0 \tag{5.10}$$

In another words, (5.10) means that the probability of an insurance company getting ruined is less when the surplus process of the insurer is invested than when it is not because, by investing its surplus, the insurance company receives interest income during each period, which, in turn, allows it to pay out more claims, therefore decreasing the probability of getting ruined.

Then again, when considering (5.10), any correct upper bound for $\rho(v,r_0)$, for instance:

$\rho(v,r_0) \le Q(v,r_0)$ should be such that

$$Q(v,r_0) \le e^{-Cv}, v \ge 0. \tag{5.11}$$

Therefore, the aim of the next section is to derive integral equations for $\rho(v,r_0)$ by using a recursive technique and then giving probability inequalities for $\rho(v,r_0)$, which is a generalisation of the Lundberg inequality for ruin possibility, thereby satisfying (5.11).

5.2 Integral equations for ruin probabilities

In this section of the methodology, we will use the recursive technique to derive integral equations for ruin probability. These integral equations are very important in the sense that they will be used in the next section to provide probability inequalities for the probability of ruin.

We first express the finite time probability of ruin in model, (5.6) using model (5.3), the initial capital, v, and the initial rate, r_0, by:

$$\rho_t(v,r_0) = \Pr\left\{\bigcup_{q=1}^{t}(V_t<0)\right\}$$

$$= \Pr\left\{\bigcup_{q=1}^{t}\left(v\prod_{n=1}^{q}(1+R_n)+\sum_{m=1}^{q}\left(\lambda H_m(1+R_m)-\beta(S_m-w)_+\right)\prod_{n=m+1}^{q}(1+R_n)<0\right)\right\}.$$

Then

$$\lim_{t\to\infty}\rho_t(v,r_0) = \rho(v,r_0)$$

The subsequent integral equations for $\rho_t(v,r_0)$ and $\rho(v,r_0)$ are as follows:

Lemma 1: For $t = 1,2,\ldots$

$$\rho_{t+1}(v,r_0)=\int_0^\infty\int_0^\infty \overline{F}\left[(v+\lambda h)(1+\delta r_0+k)\right]dH(h)\,dG(k)$$
$$+\int_0^\infty\int_0^\infty\int_0^{(v+\lambda h)(1+\delta r_0+k)} \rho_t\left[(v+\lambda h)(1+\delta r_0+k)-\beta(s-w)_+,\delta r_0+k\right]dF\left\{\beta(s-w)\right\}dH(h)dG(k)$$

and

$$\rho(v,r_0)=\int_0^\infty\int_0^\infty \overline{F}\left[(v+\lambda h)(1+\delta r_0+k)\right]dH(h)\,dG(k)$$
$$+\int_0^\infty\int_0^\infty\int_0^{(v+\lambda h)(1+\delta r_0+k)} \rho\left[(v+\lambda h)(1+\delta r_0+k)-\beta(s-w)_+,\delta r_0+k\right]dF\left\{\beta(s-w)\right\}dH(h)dG(k)$$

Proof:

From (5.6), we have:

$V_1=(v+\lambda H_1)(1+R_1)-\beta(S_1-w)_+=(v+\lambda H_1)(1+\delta r_0+K_1)-\beta(S_1-w)_+$

Given that $S_1=s,H_1=h$, and $K_1=k$, let us consider $b=\delta r_0+k$ and assume that if:

$\beta(s-w)_+>(v+\lambda h)(1+\delta r_0+k)=(v+\lambda h)(1+b)$

Then

$\Pr\left\{V_1<0\,|\,S_1=s,H_1=h,K_1=k\right\}=1$, implying that ruin has occurred during the first period.

This means that:

$\Pr\left\{\bigcup_{q=1}^{t+1}(V_q<0)\,|\,S_1=s,H_1=h,K_1=k\right\}=1\cdot$

Let $\left\{H_t,t=1,2,...\right\},\left\{\beta(S_t-w)_+,t=1,2,...\right\}\,and\,\left\{K_t,t=1,2,...\right\}$ be independent duplicates of $\left\{H_t,t=1,2,...\right\},\left\{\beta(S_t-w)_+,t=1,2,...\right\}\,and\,\left\{K_t,t=1,2,...\right\}$, respectively.

Therefore, assuming that $K_1=k$, and by (5.4) we know that:

$R_t=\delta^t r_0+\delta^{t-1}K_1+\delta^{t-2}K_2...+\delta K_{t-1}+K_t$

$=\delta^{t-1}(\delta r_0+k)+\delta^{t-2}K_2...+\delta K_{t-1}+K_t$

$=\delta^{t-1}b+\delta^{t-2}K_2...+\delta K_{t-1}+K_t$, for $t=0,1,...$

R_t and $R_{t-1}=\delta^{t-1}b+\delta^{t-2}K_1+...+\delta K_{t-2}+K_{t-1}$ have a common distribution, where $t=1,2,...$ $\{R_t,t=1,2,...\}$ and $\{R_t,t=1,2,...\}$ have the same autoregressive process, that is to say:

$R_t=\delta R_{t-1}+K_t$, $t=1,2,..$ but with not the same initial interest rate $R_0=r_0=b=\delta r_0+k$

Hence, if $0<\beta(s-w)_+<(v+\lambda h)(1+\delta r_0+k)=(v+\lambda h)(1+b)$, Then $\Pr\left\{V_1 < 0 \mid S_1 = s, H_1 = h, K_1 = k\right\} = 0$

Which means by (5.5) that, for $0 < \beta\left(s-w\right)_+ < \left(v+\lambda h\right)\left(1+b\right)$,

$$\Pr\left\{\bigcup_{q=1}^{t+1}\left(V_q < 0\right) \mid S_1 = s, H_1 = h, K_1 = k\right\}$$

$$= \Pr\left\{\bigcup_{q=2}^{t+1}\left(V_k < 0\right) \mid S_1 = s, H_1 = h, K_1 = k\right\}$$

$$= \Pr\left\{\bigcup_{q=2}^{t+1}\left(\left((v+\lambda h)(1+b)-\beta(s-w)_+\right)\prod_{n=2}^{q}(1+R_n)+\sum_{m=2}^{q}\left(\lambda H_m(1+R_m)-\beta(S_m-w)_+\right)\prod_{n=m+1}^{q}(1+R_n)<0\right)\right\}$$

since R_t and R_{t-1} have the same distribution, it is found that:

$$\Pr\left\{\bigcup_{q=1}^{t+1}\left(V_q < 0\right) \mid S_1 = s, H_1 = h, K_1 = k\right\}$$

$$= \Pr\left\{\bigcup_{q=2}^{t+1}\left(\left((v+\lambda h)(1+b)-\beta(s-w)_+\right)\prod_{n=2}^{q}(1+R_{n-1})+\sum_{m=2}^{q}\left(\lambda H_m(1+R_{m-1})-\beta(S_m-w)_+\right)\prod_{n=m+1}^{q}\left(1+R_{n-1}\right)<0\right)\right\}$$

Let $p = q-1$, and then the following is obtained:

$$= \Pr\left\{\bigcup_{p=1}^{n}\left(\left((v+\lambda h)(1+b)-\beta(s-w)_+\right)\prod_{n=2}^{p+1}(1+R_{n-1})+\sum_{m=2}^{p+1}\left(\lambda H_{m-1}(1+R_{m-1})-\beta(S_{m-1}-w)_+\right)\prod_{n=m+1}^{p+1}\left(1+R_{n-1}\right)<0\right)\right\}$$

Finally, we arrive at this equation:

$$= \Pr\left\{\bigcup_{p=1}^{t}\left(\left((v+\lambda h)(1+b)-\beta(s-w)_+\right)\prod_{n=1}^{p}(1+R_t)+\sum_{m=1}^{p}\left(\lambda H_m(1+R_m)-\beta(S_m-w)_+\right)\prod_{n=m+1}^{p}\left(1+R_n\right)<0\right)\right\}.$$

For consistency, with $\Pr\left\{\bigcup_{q=1}^{t+1}\left(V_q < 0\right) \mid S_1 = s, H_1 = h, K_1 = k\right\}$, we assume that $q = p$ and

then have the following result:

$$\Pr\left\{\bigcup_{q=1}^{t+1}\left(V_q < 0\right) \mid S_1 = s, H_1 = h, K_1 = k\right\}$$

$$= \Pr\left\{\bigcup_{q=1}^{t}\left(\left((v+\lambda h)(1+b)-\beta(s-w)_+\right)\prod_{n=1}^{q}(1+R_n)+\sum_{m=1}^{q}\left(\lambda H_m(1+R_m)-\beta(S_m-w)_+\right)\prod_{n=m+1}^{q}\left(1+I_n\right)<0\right)\right\}$$

$$= \rho_t\left(\left(v+\lambda h\right)\left(1+b\right)-\beta\left(s-w\right)_+, r_0\right)$$

$$= \rho_t\left(\left(v+\lambda h\right)\left(1+b\right)-\beta\left(s-w\right)_+, b\right).$$

Therefore, when one places conditions on S_1, H_1 and K_1:

$$\rho_{t+1}(v, r_0) = \Pr\left\{\bigcup_{q=1}^{t+1}\left(V_q < 0\right)\right\}$$

$$= \int_0^\infty \int_0^\infty \int_w^\infty \Pr\left\{ \bigcup_{q=1}^{t+1} \left(V_q < 0 \mid S_1 = s, H_1 = h, K_1 = k \right) \right\} dF\left\{\beta\left(s-w\right)_+\right\} dH\left(h\right) dG\left(k\right)$$

$$= \int_0^\infty \int_0^\infty \int_{(v+\lambda h)(1+b)}^\infty dF\left(s\right) dH\left(h\right) dG\left(k\right)$$
$$+ \int_0^\infty \int_0^\infty \int_0^{(v+\lambda h)(1+b)} \rho_t\left(\left(v+\lambda h\right)\left(1+b\right) - \beta\left(s-w\right)_+, b\right) dF\left\{\beta\left(s-w\right)_+\right\} dH\left(h\right) dG\left(k\right)$$

$$= \int_0^\infty \int_0^\infty \overline{F}\left(\left(v+\lambda h\right)\left(1+b\right)\right) dH\left(h\right) dG\left(k\right)$$
$$+ \int_0^\infty \int_0^\infty \int_0^{(v+\lambda h)(1+b)} \rho_t\left(\left(v+\lambda h\right)\left(1+b\right) - \beta\left(s-w\right)_+, b\right) dF\left\{\beta\left(s-w\right)_+\right\} dH(h) dG(k) \quad (5.12)$$

Since $b = \delta r_0 + k$, we have:

$$\rho_{t+1}\left(v, r_0\right) = \int_0^\infty \int_0^\infty \overline{F}\left(\left(v+\lambda h\right)\left(1+b\right)\right) dH\left(h\right) dG\left(k\right)$$
$$+ \int_0^\infty \int_0^\infty \int_0^{(v+\lambda h)(1+b)} \rho_t\left(\left(v+\lambda h\right)\left(1+b\right) - \beta\left(s-w\right)_+, \delta r_0 + k\right) dF\left\{\beta\left(s-w\right)_+\right\} dH(h) dG(k)$$

Therefore, the integral equation for $\rho\left(v, r_0\right)$ in Lemma (1) follows from letting $t \to \infty$ in (5.12) and $\lim_{t\to\infty} \rho_t\left(v, r_0\right) = \rho\left(v, r_0\right)$.

5.3 Inequalities for ruin probabilities

The aim of section 5.2 was to derive an integral equation for $\rho\left(v, r_0\right)$ by using a renewal recursive technique. In this section, the integral equation obtained in the previous section will be utilised to derive probability inequality for $\rho\left(v, r_0\right)$. We will use the inductive technique in order to obtain the result.

The following theorem is to be considered:

Theorem 1: Assume that there exists a positive constant, $C_1 > 0$ such that:

$$E\left[e^{-C_1\left(\lambda H_1 (1+K_1) - \beta (S_1 - w)_+\right)} \right] = 1 \quad (5.13)$$

Then,

$$\rho\left(v, r_0\right) \le \gamma E\left[e^{C_1 \beta (S_1 - w)_+} \right] E\left[e^{-C_1\left(v + \lambda H_1\right)\left(1 + \delta r_0 + K_1\right)} \right], v \ge 0 \quad (5.14)$$

where

$$(\gamma)^{-1} = \inf_{t \ge 0} \frac{\int_t^\infty e^{C_1 \beta (s-w)_+} dF\left(s\right)}{e^{C_1 t} \overline{F}\left(t\right)} \quad (5.15)$$

Proof:

For any, $c \ge 0$, we have:

$$\overline{F}(c)=\frac{e^{-C_1c}\int_c^{\infty}e^{C_1\beta(s-w)_+}dF(s)}{e^{-C_1c}\int_c^{\infty}e^{C_1\beta(s-w)_+}dF(s)}\overline{F}(c)$$

$$=\left(\frac{e^{-C_1c}\int_c^{\infty}e^{C_1\beta(s-w)_+}dF(s)}{\overline{F}(c)}\right)^{-1}e^{-C_1c}\int_c^{\infty}e^{C_1\beta(s-w)_+}dF(s)$$

$$=\left(\frac{\int_c^{\infty}e^{C_1\beta(s-w)_+}dF(s)}{e^{C_1c}\overline{F}(c)}\right)^{-1}e^{-C_1c}\int_c^{\infty}e^{C_1\beta(s-w)_+}dF(s)$$

Knowing that

$$(\gamma)^{-1}=\inf_{t\geq 0}\frac{\int_t^{\infty}e^{C_1\beta(s-w)_+}dF(s)}{e^{C_1t}\overline{F}(t)}$$

implies that

$$\gamma=\sup_{t\geq 0}\left(\frac{\int_t^{\infty}e^{C_1\beta(s-w)_+}dF(s)}{e^{C_1t}\overline{F}(t)}\right)^{-1}$$

We then have

$$\overline{F}(c)=\left(\frac{\int_c^{\infty}e^{C_1\beta(s-w)_+}dF(s)}{e^{C_1c}\overline{F}(c)}\right)^{-1}e^{-C_1c}\int_c^{\infty}e^{C_1\beta(s-w)_+}dF(s)$$

$$\leq\gamma e^{-C_1c}\int_c^{\infty}e^{C_1\beta(s-w)_+}dF(s) \qquad (5.16)$$

since

$$\int_c^{\infty}e^{C_1\beta(s-w)_+}dF(s)\leq\int_{-\infty}^{c}e^{C_1\beta(s-w)_+}dF(s)+\int_c^{\infty}e^{C_1\beta(s-w)_+}dF(s).$$

and $\int_{-\infty}^{c}e^{C_1\beta(s-w)_+}dF(s)+\int_c^{\infty}e^{C_1\beta(s-w)_+}dF(s)=E\left[e^{C_1\beta(s-w)_+}\right]$

Thus

$$\gamma e^{-C_1c}\int_c^{\infty}e^{C_1\beta(s-w)_+}dF(s)\leq\gamma e^{-C_1c}\,E\left[e^{C_1\beta(s-w)_+}\right]$$

Therefore

$$\overline{F}(c)\leq\gamma e^{-C_1c}E\left[e^{C_1\beta(s-w)_+}\right] \qquad (5.17)$$

So, for $t=1$, and for any $v\geq 0,$ and $r_0\geq 0$, it is discovered that:

$$\rho_1(v,r_0)=\Pr\{\beta(S_1-w)_+>(v+\lambda H_1)(1+R_1)\}=\Pr\{\beta(S_1-w)_+>(v+\lambda H_1)(1+\delta r_0+K_1)\}$$

$$\rho_1(v,r_0)=\int_0^{\infty}\int_0^{\infty}\int_{(v+\lambda h)(1+\delta r_0+k)}^{\infty}dF(s)dH(h)dG(k)$$

Knowing that

$$\int_{(v+\lambda h)(1+\delta r_0+k)}^{\infty} dF(s) = \left[F(s)\right]_{(v+\lambda h)(1+\delta r_0+k)}^{\infty}$$

$$\int_{(v+\lambda h)(1+\delta r_0+k)}^{\infty} dF(s) = F(\infty) - F\left[(v+\lambda h)(1+\delta r_0+k)\right]$$

$$\int_{(v+\lambda h)(1+\delta r_0+k)}^{\infty} dF(s) = 1 - F\left[(v+\lambda h)(1+\delta r_0+k)\right]$$

$$\int_{(v+\lambda h)(1+\delta r_0+k)}^{\infty} dF(s) = \overline{F}\left[(v+\lambda h)(1+\delta r_0+k)\right]$$

we find that:

$$\rho_1(v,r_0) = \int_0^{\infty}\int_0^{\infty} \overline{F}\left[(v+\lambda h)(1+\delta r_0+k)\right] dH(h)\,dG(k),$$

Therefore, according to (17),

$$\rho_1(v,r_0) \le \int_0^{\infty}\int_0^{\infty} \gamma E\left[e^{C_1\beta(S_1-w)_+}\right] e^{-C_1(v+\lambda h)(1+\delta r_0+k)} dH(h)\,dG(k)$$

$$\rho_1(v,r_0) \le \gamma E\left[e^{C_1\beta(S_1-w)_+}\right] \int_0^{\infty}\int_0^{\infty} e^{-C_1(v+\lambda h)(1+\delta i_0+k)} dH(h)\,dG(k)$$

and

$$\gamma E\left[e^{C_1\beta(S_1-w)_+}\right] \int_0^{\infty}\int_0^{\infty} e^{-C_1(v+\lambda h)(1+\delta r_0+k)} dH(h)\,dG(k) = \gamma E\left[e^{C_1\beta(S_1-w)_+}\right] E\left[e^{-C_1(v+\lambda H_1)(1+\delta r_0+K_1)}\right]$$

Thus

$$\rho_1(v,r_0) \le \gamma E\left[e^{C_1\beta(S_1-w)_+}\right] E\left[e^{-C_1(v+\lambda H_1)(1+\delta r_0+K_1)}\right]$$

Considering the inductive theory, we assume for any $v \ge 0$ and any $r_0 \ge 0$,

$$\rho_t(v,r_0) \le \gamma E\left[e^{C_1\beta(S_1-w)_+}\right] E\left[e^{-C_1(v+\lambda H_1)(1+\delta r_0+K_1)}\right] \qquad (5.18)$$

Equation (5.18) is derived from the fact that the probability of being ruined in the first period is higher than any other period in the future.

Thus $\rho_t(v,r_0) \le \rho_1(v,r_0) \le \gamma E\left[e^{C_1\beta(S_1-w)_+}\right] E\left[e^{-C_1(v+\lambda H_1)(1+\delta r_0+K_1)}\right]$ for any $t > 1$. Therefore, any correct upper bound for $\rho_1(v,r_0)$ should also be an upper bound for $\rho_t(v,r_0), t > 1$.

Hence, for any t, and $0 \le \beta(s-w)_+ \le (v+\lambda h)(1+\delta r_0+k)$; by (18), $\delta r_0 \ge 0$ and $K_1 \ge 0$, one arrives at the following:

$$\rho_t\left((v+\lambda h)(1+\delta r_0+k) - \beta(s-w)_+, \delta r_0+k\right)$$
$$\le \gamma E\left[e^{C_1\beta(S_1-w)_+}\right] E\left[e^{-C_1\left[\left((v+\lambda h)(1+\delta r_0+k)-\beta(s-w)_+ +\lambda H_1\right)\left(1+\delta(\delta r_0+k)+K_1\right)\right]}\right]$$

Based on the assumption made with respect to $\delta r_0 \geq 0$ and $K_1 \geq 0$, we have $\delta r_0 + k > 0$; and, if $p > 0$, then $e^{-pq} \leq e^{-q}$, and hence:

$$\gamma E\left[e^{C_1\beta(S_1-w)_+}\right]E\left[e^{-C_1\left[\left((v+\lambda h)(1+\delta r_0+k)-\beta(s-w)_+ + \lambda H_1\right)\left(1+\delta(\delta r_0+k)+K_1\right)\right]}\right]$$

$$\leq \gamma E\left[e^{C_1\beta(S_1-w)_+}\right]E\left[e^{-C_1\left[\left((v+\lambda h)(1+\delta r_0+k)-\beta(s-w)_+ + \lambda H_1\right)(1+K_1)\right]}\right]$$

$$\leq \gamma E\left[e^{C_1\beta(S_1-w)_+}\right]E\left[e^{-C_1\left((v+\lambda h)(1+\delta r_0+k)-\beta(s-w)_+\right)-C_1\lambda H_1(1+K_1)}\right]$$

It then follows that:

$$\gamma E\left[e^{C_1\beta(S_1-w)_+}\right]E\left[e^{-C_1\left((v+\lambda h)(1+\delta r_0+k)-\beta(s-w)_+\right)-C_1\lambda H_1(1+K_1)}\right] = \gamma E\left[e^{C_1\beta(S_1-w)_+}\right]E\left[e^{-C_1\lambda H_1(1+K_1)}e^{-C_1\left((v+\lambda h)(1+\delta r_0+k)-\beta(s-w)_+\right)}\right]$$

Because $\{H_t, t=1,2,...\}, \{S_t, t=1,2,...\}$ and $\{K_t, t=1,2,...\}$ are independents and have a common distribution function, we find:

$$\gamma E\left[e^{C_1\beta(S_1-w)_+}\right]E\left[e^{-C_1\left((v+\lambda h)(1+\delta r_0+k)-\beta(s-w)_+\right)-C_1\lambda H_1(1+K_1)}\right] = \gamma E\left[e^{C_1\beta(S_1-w)_+}\right]E\left[e^{-C_1\lambda H_1(1+K_1)}\right]E\left[e^{-C_1\left((v+\lambda h)(1+\delta r_0+k)-\beta(s-w)_+\right)}\right]$$

$$\gamma E\left[e^{C_1\beta(S_1-w)_+}\right]E\left[e^{-C_1\left((v+\lambda h)(1+\delta r_0+k)-\beta(s-w)_+\right)-C_1\lambda H_1(1+K_1)}\right] = \gamma E\left[e^{-C_1\lambda H_1(1+K_1)+R_1\beta(S_1-w)_+}\right]E\left[e^{-R_1\left((v+\lambda h)(1+\delta r_0+k)-\beta(s-w)_+\right)}\right]$$

$$\gamma E\left[e^{C_1\beta(S_1-w)_+}\right]E\left[e^{-C_1\left((v+\lambda h)(1+\delta r_0+k)-\beta(s-w)_+\right)-C_1\lambda H_1(1+K_1)}\right] = \gamma E\left[e^{-C_1\left[\lambda H_1(1+K_1)-\beta(S_1-w)_+\right]}\right]E\left[e^{-C_1\left((v+\lambda h)(1+\delta r_0+k)-\beta(s-w)_+\right)}\right]$$

Hence

$$\rho_t\left((v+\lambda h)(1+\delta r_0+k)-\beta(s-w)_+, \delta r_0+k\right) \leq \gamma E\left[e^{-C_1\left[\lambda H_1(1+K_1)-\beta(S_1-w)_+\right]}\right]E\left[e^{-C_1\left((v+\lambda h)(1+\delta r_0+k)-\beta(s-w)_+\right)}\right]$$

By (5.13), we have $E\left[e^{-C_1\left(\lambda H_1(1+K_1)-\beta(S_1-w)_+\right)}\right] = 1$

Therefore

$$\rho_t\left((v+\lambda h)(1+\delta r_0+k)-\beta(s-w)_+, \delta r_0+k\right) \leq \gamma e^{-C_1\left((v+\lambda h)(1+\delta r_0+k)-\beta(s-w_+)\right)} \qquad (5.19)$$

Hence, by Lemma 1, (5.16) and (5.19) we have:

$$\rho_{t+1}(v, r_0) \leq \gamma E\left[e^{C_1\beta(S_1-w)_+}\right]E\left[e^{-C_1(v+\lambda H_1)(1+\delta r_0+K_1)}\right] + \gamma e^{-C_1\left((v+\lambda h)(1+\delta r_0+k)-\beta(s-w_+)\right)}$$

$$\rho_{t+1}(v, r_0) \leq \gamma \int_0^\infty \int_0^\infty e^{-C_1(v+\lambda h)(1+\delta r_0+k)} \int_{(v+\lambda h)(1+\delta r_0+k)}^\infty e^{C_1\beta(s-w)_+} dF(s)\,dH(h)\,dG(k)$$

$$+\gamma \int_0^\infty \int_0^\infty e^{-C_1(v+\lambda h)(1+\delta r_0+k)} \int_0^{(v+\lambda h)(1+\delta r_0+k)} e^{C_1\beta(s-w)_+} dF(s)\,dH(h)\,dG(k)$$

And

$$\gamma\int_0^\infty\int_0^\infty e^{-C_1(v+\lambda h)(1+\delta r_0+k)}\int_{(v+\lambda h)(1+\delta r_0+k)}^{\infty} e^{C_1\beta(s-w)_+}dF(s)\,dH(h)\,dG(k)$$

$$+\gamma\int_0^\infty\int_0^\infty e^{-C_1(v+\lambda h)(1+\delta r_0+k)}\int_{-\infty}^{(v+\lambda h)(1+\delta r_0+k)} e^{C_1\beta(s-w)_+}dF(s)\,dH(h)\,dG(k)$$

$$=\gamma\int_0^\infty\int_0^\infty e^{-C_1(v+\lambda h)(1+\delta r_0+k)}\left(\int_{-\infty}^{(v+\lambda h)(1+\delta r_0+k)} e^{C_1\beta(s-w)_+}+\int_{(v+\lambda h)(1+\delta r_0+k)}^{\infty} e^{C_1\beta(s-w)_+}\right)dF(s)\,dH(h)\,dG(k)$$

$$=\gamma\int_0^\infty\int_0^\infty e^{-c_1(v+\lambda h)(1+\delta r_0+k)}\left(\int_{-\infty}^{\infty} e^{C_1\beta(s-w)_+}\right)dF(s)\,dH(h)\,dG(k)$$

$$=\gamma E\left[e^{C_1\beta(S_1-w)_+}\right]E\left[e^{-c_1(v+\lambda H_1)(1+\delta r_0+K_1)}\right]$$

With

$$\int_{-\infty}^{\infty} e^{C_1\beta(s-w)_+}dF(s)=E\left[e^{C_1\beta(S_1-w)_+}\right]$$

Therefore,

$$\rho_{t+1}(v,r_0)\le\gamma E\left[e^{C_1\beta(S_1-w)_+}\right]E\left[e^{-C_1(v+\lambda H_1)(1+\delta r_0+K_1)}\right]$$

Thus, for any $t=1,2,...,$ (5.18) holds.

Therefore, (5.14) follows from letting $t\to\infty$ in (5.18) and $\lim_{t\to\infty}\rho_t(v,r_0)=\rho(v,r_0)$.

Furthermore, it is possible to get a better upper bound in Theorem 1. This would be feasible when F is new worse than used in convex ordering (NWUC) - for the definition and properties of NWUC class, see Cao and Wang (1991).

Corollary 1:

Considering the assumptions of Theorem 1, if F is new worse than used in the convex ordering, it follows that,

$$\rho(v,r_0)\le\left[e^{-C_1(v+\lambda H_1)(1+\delta r_0+K_1)}\right],v\ge 0 \qquad (5.20)$$

Proof:

Willmot and Lin (2001)'s proposition of 6.11 shows that if F is NWUC, then

$$\gamma=\left(Ee^{C_1\beta(S_1-w)_+}\right)^{-1}.$$

Hence (5.20) is derived from (5.14).

It is possible to demonstrate that the upper bound in Theorem 1 is less than the Lundberg upper bound. In finding this to be so, we obtain the following result about C_1 and C.

Proposition 1:

If $\lambda E(H_1) > \beta E(S_1 - w)_+$ and $C_1 > 0$ in (5.13) and $C > 0$ in (5.8) exist, it follows that $C_1 \geq C$ - especially if both λH_1 and K_1 are not reduced to 0, then $C_1 > C$.

Proof:

Let's assume the following functions:

$$y(c) = E\left[e^{-c\left(\lambda H_1(1+K_1)-\beta(S_1-w)_+\right)}\right]-1$$

and

$$z(c) = E\left[e^{-c\left(\lambda H_1-\beta(S_1-w)_+\right)}\right]-1,$$

It follows that:

$$y'(c) = E\left[-\left(\lambda H_1(1+K_1)-\beta(S_1-w)_+\right)e^{-c\left(\lambda H_1(1+K_1)-\beta(S_1-w)_+\right)}\right],$$

and

$$y''(c) = E\left[\left(\lambda H_1(1+K_1)-\beta(S_1-w)_+\right)^2 e^{-c\left(\lambda H_1(1+K_1)-\beta(S_1-w)_+\right)}\right] \geq 0,$$

which means that the function $y(c)$ is convex. Furthermore, $y(0)=0$ and

$$\begin{aligned} y'(0) &= \beta E(S_1-w)_+ - \lambda E H_1(1+K_1) \\ &= \beta E(S_1-w)_+ - \lambda E(H_1) - \lambda E(H_1 K_1) \\ &\leq \beta E(S_1-w)_+ - \lambda E(H_1) < 0 \end{aligned}$$

Therefore, if $C_1 > 0$ and $C > 0$ exist, then they are unique positive roots of $y(c)$ and $z(c)$, respectively, on $(0,\infty)$. In addition, if $c > 0$, such that $z(c) \geq 0$, then $c \geq C$.

Conversely,

$$e^{-C_1\left(\lambda H_1(1+K_1)-\beta(S_1-w)_+\right)} \leq e^{-C_1\left(\lambda H_1-\beta(S_1-w)_+\right)}$$

Hence

$$1 = Ee^{-C_1\left(\lambda H_1(1+K_1)-\beta(S_1-w)_+\right)} \leq e^{-C_1\left(\lambda H_1-\beta(S_1-w)_+\right)}$$ (the Lundberg inequality theory)

This is the same as writing

$$z(C_1) = Ee^{-C_1\left(\lambda H_1-\beta(S_1-w)_+\right)} - 1 \geq 0,$$

which means that $C_1 \geq C$ - especially if both λH_1 and K_1 are not reduced to 0. Then:

$$1 = Ee^{-C_1\left(\lambda H_1(1+K_1)-\beta(S_1-w)_+\right)} < e^{-C_1\left(\lambda H_1-\beta(S_1-w)_+\right)},$$

Or

$$z(C_1) = Ee^{-C_1\left(\lambda H_1-\beta(S_1-w)_+\right)} - 1 > 0,$$

implying that $C_1 > C$.

Now the upper bound in Theorem 1 is to be represented by $Q(v, r_0)$, that is:

$$Q(v, r_0) = \gamma E\left[e^{C_1\beta(S_1-w)_+}\right]E\left[e^{-C_1(v+\lambda H_1)(1+\delta r_0+K_1)}\right], v \geq 0$$

Then the result is as follows.

Proposition 2:

For any $v \geq 0$, $Q(v, r_0) \leq e^{-Cv}$

Proof:

By $K_1 \geq 0, \delta i_0 \geq 0,$ Theorem 1 and Proposition 1; we get $v \geq 0$,

$$Q(v, r_0) = \gamma E\left[e^{C_1\beta(S_1-w)_+}\right]E\left[e^{-C_1 v(1+\delta r_0+K_1)-C_1\lambda H_1(1+\delta r_0+K_1)}\right]$$

$$\leq \gamma E\left[e^{C_1\beta(S_1-w)_+}\right]E\left[e^{-C_1 v(1+\delta r_0)-C_1\lambda H_1(1+K_1)}\right]$$

$$\gamma E\left[e^{C_1\beta(S_1-w)_+}\right]E\left[e^{-C_1 v(1+\delta r_0)-C_1\lambda H_1(1+K_1)}\right] = \gamma E\left[e^{C_1\beta(S_1-w)_+}\right]E\left[e^{-C_1 v(1+\delta r_0)}e^{-C_1\lambda H_1(1+K_1)}\right]$$

Since $\{H_t, t=1,2,...\}, \{S_t, t=1,2,...\}$ and $\{K_t, t=1,2,...\}$ are independents and they have a common distribution functions, one finds:

$$\gamma E\left[e^{C_1\beta(S_1-w)_+}\right]E\left[e^{-C_1 v(1+\delta r_0)-C_1\lambda H_1(1+K_1)}\right] = \gamma E\left[e^{C_1\beta(S_1-w)_+}\right]E\left[e^{-C_1\lambda H_1(1+K_1)}\right]e^{-C_1 v(1+\delta r_0)}$$

$$\gamma E\left[e^{C_1\beta(S_1-w)_+}\right]E\left[e^{-C_1 v(1+\delta r_0)-C_1\lambda H_1(1+K_1)}\right] = \gamma E\left[e^{-C_1\lambda H_1(1+K_1)}e^{C_1\beta(S_1-w)_+}\right]e^{-C_1 v(1+\delta r_0)}$$

$$\gamma E\left[e^{C_1\beta(S_1-w)_+}\right]E\left[e^{-C_1 v(1+\delta r_0)-C_1\lambda H_1(1+K_1)}\right] = \gamma E\left[e^{-C_1\left(\lambda H_1(1+K_1)-\beta(S_1-w)_+\right)}\right]e^{-C_1 v(1+\delta r_0)}$$

Thus,

$$Q(v, r_0) \leq \gamma E\left[e^{-C_1\left(\lambda H_1(1+K_1)-\beta(S_1-w)_+\right)}\right]e^{-C_1 v(1+\delta r_0)}$$

According to (5.13), we arrive at $E\left[e^{-C_1\left(\lambda H_1(1+K_1)-\beta(S_1-w)_+\right)}\right] = 1$.

Therefore,

$$Q(v, r_0) \leq \gamma e^{-C_1 v(1+\delta r_0)} \leq e^{-C_1 v(1+\delta r_0)}$$

And

$$e^{-C_1 v(1+\delta r_0)} \leq e^{-C_1 v}.$$

Finally,

$$Q(v, r_0) \leq e^{-C_1 v}$$

Proposition 2 implies that the upper bound in Theorem 1 is less than the Lundberg upper bound.

The following chapter provides a numerical example to illustrate the upper bound for ruin probability derived.

Chapter 6- Numerical Examples

6.1 Introduction

In this chapter, we use an example to illustrate the application of the upper bound derived in chapter 5. It is generally known that it is difficult to calculate the probability of ruin. However, the upper bound derived in this study is of great importance, as, in most of the practical cases, it can be used to estimate the probability of ruin. Therefore, the main purpose of this chapter is to calculate the adjustment coefficient as it is an important parameter to approximate the upper bound for the ruin probability.

The numerical results for the upper bound and Lundberg's upper bound for the ruin probability of this research are given in four tables, from Table 1 to Table 4.The following calculations are obtained by Matlab software.

6.2 Examples and results

Let us consider an insurance company that deals with disability contracts. We assume that the length of a sickness period is denoted by S and that of healthy period by H follow Erlang distribution whose density functions are given by,

$F(S < S_1) = 10e^{-10t}$ and $H(H < H_1) = \frac{e^{-\frac{t}{10}}}{10}$, respectively.

Equation (5.13) can be written as $mgf(r) = E\left[e^{-C_1(\lambda H_1(1+K_1)-\beta(S_1-w)_+)}\right] = 1$

Equation (5.13) can be rewritten as:

$mgf(r) = E\left[e^{-C_1\lambda H_1(1+K_1)}\right]E\left[e^{C_1\beta(S_1-w)_+}\right] = 1$, since the random variables H_1, K_1 and S_1 are $i.i.d.$

Consequently, by working out equation (5.13) in Matlab, with the assumption that $\lambda E(H_1) > \beta E(S_1 - w)_+$ holds, the convex function of $mgf(r)$ will have the following shape:

The convex function's shape $mgf(r)$ and the numerical value for the adjustment coefficient, with different values of the parameters, is presented as λ, β and w.

Case1: $\lambda = 2, \beta = 20$ and $w = 5$

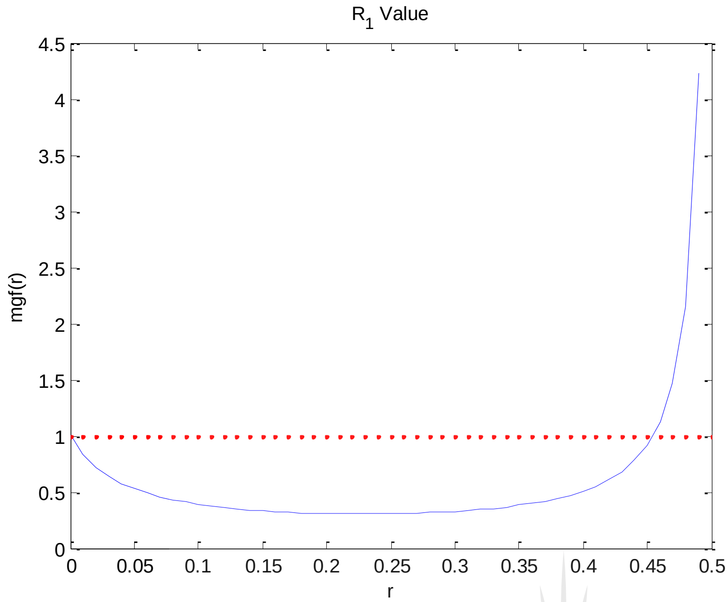

Figure 1: Illustration of the convex function of $mgf(r)$

Table 1: Adjustment coefficient C_1 when, $\lambda = 2$, $\beta = 20$

Parameters	$\lambda = 2$	$\beta = 20$	$w = 5$
Adjustment coefficient C_1	0.4500		

Table 2: Upper bound for ruin probability when $\lambda = 2$, $\beta = 20$, $w = 5$, and $C_1 = 0.4500$.

Initial surplus v	Upper bound for the ruin probability
$v = 0$	1
$v = 5$	0.105399
$v = 10$	0.011109
$v = 15$	0.001171
$v = 20$	0.000123
$v = 25$	0.000013

Case2: $\lambda = 50, \beta = 200$ and $w = 3$

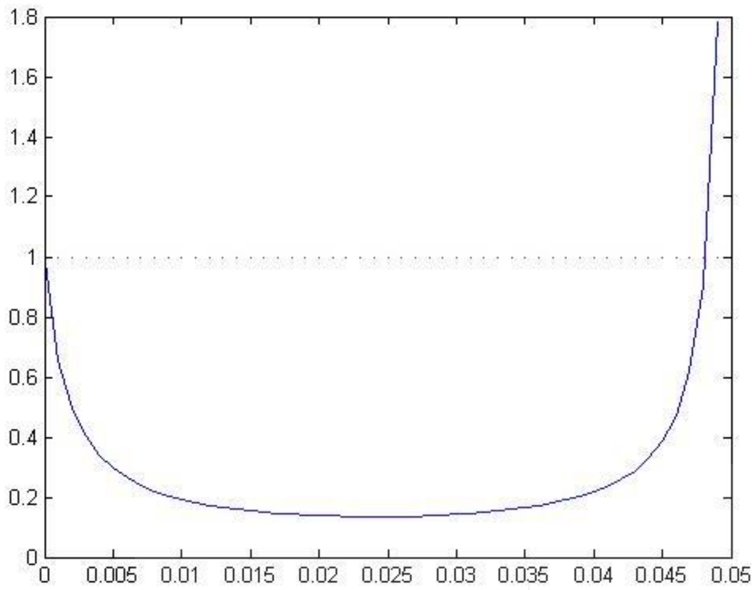

Figure 2: Illustration of the convex function of $mgf(r)$.

Table 3: Adjustment coefficient C_1 *when* $\lambda = 50$, $\beta = 200$, *and* $w = 3$.

Parameters	$\lambda = 50$	$\beta = 200$	$w = 3$
Adjustment coefficient C_1	0.0480		

Table 4: Upper bound for the ruin probability when $\lambda = 50$, $\beta = 200$ $w = 3$ *and* $C_1 = 0.0480$.

Initial surplus v	Upper bound for the ruin probability
$v = 0$	1
$v = 5$	0.7866
$v = 10$	0.6188
$v = 15$	0.4868
$v = 20$	0.3829
$v = 25$	0.3012

6.3 Interpretation of the results

Figure 1 and 2 show that the function $mgf(r)$ is a nonnegative continuous, convex function regardless of the value of the parameters λ, β and w, (such that $mgf(0)=1$ and $\lim_{r\to\infty} mgf(r)=\infty$**).** This is in line with the proof of proposition 1, where we showed that the function $y(c)=E\left[e^{-c\left(\lambda H_1(1+K_1)-\beta(S_1-w)_+\right)}\right]-1$ is convex and nonnegative. Therefore, if the adjustment coefficient exists, we continue to find the upper bound for the probability of ruin.

As far as Table1 and Table 3 are concerned, they give us the unique positive value of the adjustment coefficient C_1**.** It can be noticed that the value of the adjustment coefficient changes when the parameters λ, β and w change; supporting the uniqueness of the adjustment coefficient states in proposition 1.

It can be noted that the adjustment coefficient obtained in Table 3 has decreased compared to the one in Table 1. This is due to fact that we have increased the value of λ and β in case 2, implying that the adjustment coefficient is strongly affected by the value assigned to the parameter λ and β .

Finally, Table 2 and Table 4 provide us with the different values of the upper bound for the ruin probability. We notice that the upper bound computed in Table 2 is smaller than the one in Table 4. This is due to the fact that the adjustment coefficient obtained in case 1 is greater than the one in case 2. This implies that the bigger the value of the adjustment coefficient, the smaller the upper bound, which, in turn, means a smaller ruin probability.

Therefore, our results is in line with the results obtained by Adekambi (2013) when he used constant interest rate in the Ramsay's risk model. However the upper bound in our example is smaller than the one found by Adekambi (2013). The use of random interest rate in our risk model might be the reason why we obtained an upper bound for the ruin probability that is smaller than the one obtained by Adekambi (2013).

Chapter 7- Conclusion

In this study, we have evaluated the probability of ruin in the Ramsay model by considering a random interest rate. We have proved that the interest rate used in this study can be modelled by an autoregressive process of order one. For many researchers, such as Cai (2002b), this reflect the true economic environment in which insurance companies operate. Then, in order to assess the probability of ruin, we used recursive technique to derive integral equations and probability inequalities for the probability of ruin in our risk model. Our results are in line with intuitive thinking that the probability of ruin decreases when one incorporates random interest rate in the risk model. This means that the insurance company is allowed to invest its capital in order to receive some interest income, hence enabling her to face claims. Moreover, the adjustment coefficient obtained in the study heavily depends on the value assigned to the parameters λ and β . Our results, therefore, improved on the ones obtained by Adekambi (2013) when he used constant interest rate in the Ramsay model making our model is more practical in real world.

It must be noted, however, that our model is not without limitations.

First, we assumed a risk model in which claims and premiums are independent over time. This presumption might not hold in reality, as, in reality, claims and premiums may have a dependent autoregressive structure, that is, the value of the current claims and premiums may dependent on their previous ones.

Secondly, we assumed that claims are paid out at a constant amount β , which does not necessarily hold in real insurance world. Indeed, the claim amounts themselves may be dependent. It is not difficult to envisage a situation where these would hold. For instance, in reality, medical costs are more age-dependent, thus the amount, β should be paid accordingly.

Therefore, these limitations open doors for future research. For example, one could investigate the probability of ruin in the Ramsay risk model with random interest rate by considering claims and premiums to have a dependent autoregressive structure and by reckoning β to be paid out according to the age of the policyholder.

Reference

Adékambi, F. (2013). The Asymptotic Ruin Problem in Health Care Insurance with Interest. *Asia-Pacific Journal of Risk and Insurance*, 7(2): 143–162.

Albrecher, H., Constantinescu, C., Palmowski, Z., Regensburger, G., and Rosenkranz, M. (2015). Exact and Asymptotic Results for Insurance Risk Models with Surplus-dependent Premiums. *SIAM J. Appl. Math*, 73(1): 47–66.

Biard, R., Lefèvre, C., Loisel, S., and Nagaraja, H. N. (2011). Asymptotic finite-time ruin probabilities for a class of path-dependent heavy-tailed claim amounts using Poisson spacings. *Appl. Stochastic Models Bus. Ind,* 27: 503–518.

Cai, J. and Dickson, D., C., M. (2003). Upper Bounds for Ultimate Ruin Probabilities in the Sparre Andersen Model with Interest. *Insurance: Mathematics and Economics,* 32: 61–71.

Cai, J. (2002a). Discrete Time Risk Models under Rates of Interest. *Probability in the Engineering and Informational Sciences*, 16: 309–324.

Cai, J. (2002b). Ruin probabilities with Dependent Rates of Interest. *Journal of Applied Probability,* 39: 312–323.

Cao, J. and Wang, Y. (1991). The NBUC and NWUC Classes of Life Distributions. *J. Appl. Prob*, 28: 473-479.

Castener, A. Claramunt, M, M. Gathy, M. Lefèvre, C. and Marmol, M. (2013). Ruin problems for a discrete time risk model with non-homogeneous conditions. *Scandinavian Actuarial Journal*, 2: 83-102.

Chan, G, K, C and Yang, H. (2006). Upper Bounds for Ruin Probability under Time Series Models. *Probability in the Engineering and Informational Sciences*, 20(3): 529–542.

Cheng, J. and Wang, D. (2011). Ruin Problems for an Autoregressive Risk Model with Dependent Rates of Interest. *Applied Mathematics and Computation*, 218(7): 3822–3833.

Cheung, E, C, K and David, L. (2012). On a Risk Model with Surplus-dependent Premium and Tax Rates. *Methodol Comput Appl Probab,* 14: 233–251.

Cov. D. R. (1962). Renewal Theory. London: Methuen.

Diasparra, M. A. and Rosario, R. (2009). Bounds for the Ruin Probability of a Discrete-time Risk Process. *Appl. Prob*, 46: 99-112.

Fabozzi. Frank. Modigliani, F. Ferri, M. (1998). *Foundation of Financial Markets and Institutions*. NJ: Prentice Hall, Inc. 2nd edition.

Fisher, I. (1930). *The Theory of Interest*. New York: the Macmillan co. 1st edition.

Frostig, E. Haberman, S and Levikson, B. (2003). Generalized Life Insurance: Ruin Probability*. Scand Actuarial J*, 2: 136-152

Gao, Q. and Yang, X. (2014). Asymptotic Ruin Probabilities in a Generalized Bidimensional Risk Model Perturbed by Diffusion with Constant Force of Interest. *J. Math. Anal. Appl*. 419 (2): 1193–1213.

Kellison, S. (1991). *The Theory of Interest*. Boston: Irwin. 2nd Edition.

Keynes, J. M. (1936). *General Theory of Employment, Interest and Money*. London: MacMillan Company Ltd.

Kit Mo, K, C. (2002). Ruin Probabilities with Dependent Claims. Submitted in partial fulfilment of the requirements of the degree of bachelor of commerce with honours. Faculty of Commerce and Economics, University of New South Wales.

Lefèvre, C and Loisel, S. (2009), Finite-time ruin probabilities for discrete, possibly dependent, claim severities. *Methodol Comput Appl Probab*, 11: 425–441.

Li, J and Yang, H. (2015). Asymptotic ruin probabilities for a bidimensional renewal risk model with constant interest rate and dependent claims. *J. Math. Anal. Appl.* 426: 247–266.

Li, X. Wu, J and Zhuang, J. (2015). Asymptotic multivariate finite-time ruin probability with statistically dependent heavy-tailed claims. *Methodol Comput Appl Probab*, 17: 463–477.

Lundberg, F. (1903). I. Approximerad franstallning af sannolikhetsfunktionen. II. Aterforsakring af kollektivrisker. Almqvist and Wiksell, Uppsala, Sweden.

Lundberg, F. (1909). Uber die Theorie der Ruckversicherung. Berichte VI Int. Kong. Versich. Wiss. 1: 877-948.

Mitric, I. R., Badescu, A. L., and Stanford, D. A. (2012). On the absolute ruin problem in a Sparre Andersen risk model with constant interest. *Insurance: Mathematics and Economics,* 50: 167-178.

Mitric, I. R. and Sendova, K. P. (2010). On a multi-threshold compound Poisson surplus process with interest. *Scandinavian Actuarial Journal,* 1: 1–21.

Peng, J. Huang, J and Wang, D. (2011). The ruin probability of a discrete-time risk model with a one-sided linear claim process. *Communications in Statistics — Theory and Methods*, 40: 4387–4399.

Raducan, A. M., Vernic, R. and Zbaganu, G. (2015). Recursive calculation of ruin probabilities at or before claim instants for non-identically distributed claims. *Astin Bulletin*, 45: 421-443.

Ramsay, C. (1984). The asymptotic ruin problem when the healthy and sickness periods form an alternating renewal process. *Insurance: Mathematics and Economics,* 3: 139–143.

Shen, X. M. and Lin, Z. Y. (2010). The ruin probability of the renewal model with constant interest force and upper-tailed independent heavy-tailed claims. *Acta Mathematica Sinica, English Series,* 26(9): 1815–1826.

Tanasescu, P. and Mircea, I. (2014). Assessment of the ruin probability. *Economic Computation and Economic Cybernetics Studies and Research*, 48(3): 79-98.

Thampi, K. K. (2015). Finite Time Ruin Probability of the Compound Renewal Model with Constant Interest Rate and Weakly Negatively Dependent Claims with Heavy Tails. *International Game Theory Review*, 17(1): 1-15.

Yang, H, L. and Zhang, L, H. (2003). Martingale method for ruin probability in an autoregressive model with constant interest rate. *Probability in the Engineering and Informational Sciences,* 17: 183–198.

Yao, D and Wang, R. (2010). Upper bounds for ruin probabilities in two dependent risk models under rates of interest. *Appl. Stochastic Models Bus. Ind*, 26: 362–373.

Yuen, K. C. Wang, G. and Wu, R. (2006). On the renewal risk process with stochastic interest. *Stochastic Processes and their Applications,* 116: 1496–1510.

Zhang, Z., Yuen, K, C. and Li, W, K. (2007). A time-series risk model with constant interest for dependent classes of business. *Insurance: Mathematics and Economics,* 41: 32–40.

Appendix: Matlab code

```
%Moment generating function in calculating the value for the adjustment
%coefficient (R1)
Function
Mgf(r) = mg (R1, H1, K1, lambda, beta, w)
Mgf(r) = (-R1*lambda-1/10)*H1 - R1*lambda*H1*K1 - (K1^2 - 17.6304*K1 +
77.7078)/42.1748
End
%Main Matlab Codes for the implementation of the two Functions
clear all
clc
close all
%assigned values for the parameters (lambda, beta, w) in the Moment
%generating function for calculating the adjustment coefficient and
%Upper bounds of the ruin probability. These values can be varied.
Lambda = 50;
Beta = 200;
w = 3;
%r=0:1e-8:1e-3; % this value be any value from zero to infinity: Possible values of
R1
%r=0:1:10/beta;
r = linspace (0, 10/beta, 51);
%r = R1
%% Calculating for the approximate value of the adjustment coefficient (R1)
Mgf(r) = mg(r, lambda, beta, w); %moment generating function
Mgf(0)=1;% Value of the moment generating function when r=0
%ml=exp(r.*L);
figure (1)
plot(r, mgf(r), 'b-')
hold on
plot(r,m0, 'r-')
xlabel('r')
ylabel('mgf(r)')
title('R_1 Value')
```

```
hold off
% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%
[data index] = sort (abs (mgf(r)-1),'ascend');
R1 = r (index (2)) %Adjustment Coefficient
R1=0.0480
```