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TECHNICAL NOTE

Stress–dilatancy relation for Mohr–Coulomb soils following a non-associated flow rule

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Rowe's stress–dilatancy relation for frictional (cohesionless) materials has been a cornerstone of soil mechanics. The original derivation of this relationship was based on incorrect energy minimisation considerations, but the relationship was proven later by De Josselin de Jong using friction laws, and has been confirmed by a large body of experimental results. In contrast, the validity of Rowe's stress–dilatancy relation for cohesive-frictional materials, which has also been used, although not as extensively, was never verified. This paper shows that Rowe's stress–dilatancy relation for Mohr–Coulomb soils (cohesive-frictional materials) is in fact incorrect. The paper also provides a correct stress–dilatancy relationship for non-associated Mohr–Coulomb soils that have both cohesive and frictional strength components. The derivation of the relationship for cohesive-frictional soils presented in this paper relies on use of the sawtooth model together with the application of the laws of friction.

KEYWORDS: constitutive relations; deformation; failure; friction; shear strength

La relation de contrainte – dilatance de Rowe pour matériaux de frottement (sans cohésion) a été une des chevilles ouvrières de la mécanique des sols. La dérivée originale de cette relation était basée sur des considérations erronées de minimisation de l'énergie, mais les rapports ont été démontrés plus tard par De Josselin de Jong, au moyen de lois sur le frottement, et a été confirmée par un grand nombre de résultats expérimentaux. Par contraste, la validité de la relation de contrainte – dilatance de Rowe pour matériaux cohésifs - de frottement, qui a été utilisée, elle aussi, mais dans une moindre mesure, n'a jamais été vérifiée. La présente communication démontre que la relation de contrainte – dilatance de Rowe pour sols Mohr-Coulomb non associés (matériaux cohésifs – de frottement) est en fait erronée. Cette communication présente également une relation de contrainte – dilatance correcte pour sols Mohr-Coulomb non associés, comprenant des composants à la fois cohésifs et à résistance de friction. La dérivée de la relation pour des sols cohésifs – de frottement présentée dans la présente communication est basée sur l'emploi du modèle en dents de scie ainsi que sur l'application des lois sur le frottement.

INTRODUCTION

Many natural soils are cohesive-frictional materials that have both cohesive and frictional strength components. Because the Mohr–Coulomb criterion is normally used for such soils, they are known as Mohr–Coulomb soils. In this paper it is shown that Rowe's stress–dilatancy relation for Mohr–Coulomb soils (Rowe, 1962) is incorrect, and a correct version of it based on the laws of friction is derived.

The stress–dilatancy relation proposed by Rowe (1962) has been widely used in simulating the stress–strain behaviour of uncemented sands and other granular materials (Hughes *et al.*, 1977; Molenkamp, 1981; Wan & Guo, 1998). By neglecting elastic strains and the strain due to particle crushing, and applying the principle of energy minimisation, Rowe (1962) arrived at

$$\frac{\sigma_1}{\sigma_3} = \tan^2\left(\frac{\pi}{4} + \frac{\phi_c}{2}\right)\left(1 - \frac{\dot{\varepsilon}_v}{\dot{\varepsilon}_1}\right) \tag{1}$$

where σ_1 and σ_3 are the major and minor principal effective stresses respectively (the customary primes are omitted, because this paper deals exclusively with effective stresses);

ϕ_c is the critical-state friction angle; and $\dot{\varepsilon}_1$ and $\dot{\varepsilon}_v$ are the major principal strain rate and volumetric strain rate respectively. Rowe's stress–dilatancy relation for frictional materials is supported by a large body of experimental results, but the derivation of the theory was questioned, because the principle of energy minimisation no longer applies when friction and the associated energy dissipation are involved. Despite questioning the applicability of the minimum energy principle to frictional materials, De Josselin de Jong (1976) did prove, with an alternative approach based on the laws of friction, that Rowe's final conclusions and his stress–dilatancy relationship were valid.

Rowe also provided a stress–dilatancy relation for cohesive-frictional materials based on the principle of energy minimisation (Rowe, 1962):

$$\frac{\sigma_1}{\sigma_3} = \tan^2\left(\frac{\pi}{4} + \frac{\phi_c}{2}\right)\left(1 - \frac{\dot{\varepsilon}_v}{\dot{\varepsilon}_1}\right) + \frac{2c}{\sigma_3}\tan\left(\frac{\pi}{4} + \frac{\phi_c}{2}\right)\left(1 - \frac{\dot{\varepsilon}_v}{\dot{\varepsilon}_1}\right) \tag{2}$$

where c is the interparticle cohesion. Equation (1) can be considered a special case of equation (2), resulting from making the cohesion term c equal to zero.

Typical cohesive-frictional materials include many natural soils, stabilised soils and rocks. One of the important properties of cohesive-frictional soils is that there exist cementation bonds between particles, and the contribution of these bonds to shear strength may be represented by interparticle cohesion. When external forces are applied to such a material, the input energy is used to change the volume of the material, overcome interparticle friction, and degrade the

cementation bonds between particles. Gerogiannopoulos & Brown (1978) applied critical state soil mechanics to study the cohesive component of shear strength in rocks. Carter *et al.* (1986) used a non-associated Mohr–Coulomb soil model to study cavity expansion in cohesive-frictional soils. Later, Harberfield (1997) suggested the use of the Rowe (1962) stress–dilatancy relation when studying the effects of cracking in soft rocks during the pressuremeter test. Cecconi & Viggiani (2001) analysed the experimental data on a pyroclastic weak rock by using equation (2).

As mentioned earlier, Rowe's derived stress–dilatancy relation was based on the incorrect assumption that energy minimisation would apply. Although equation (1) was proven to be correct by De Josselin de Jong (1976), the validity of equation (2) has never been verified. This must be done before it can be applied to cohesive-frictional materials. In this paper, it will be shown that equation (2) is not correct. Also, a stress–dilatancy relation for cohesive-frictional soils will be proposed, derived using the friction laws used by De Josselin de Jong (1976).

LAWS OF FRICTION AND STRESS TRANSFORMATION

The strength of a cohesive-frictional material has frictional, dilational and cohesive components, each of which has its own mobilisation rate with respect to strain. Dilatancy peaks at small strains, and both the cohesive and the dilative strength components degrade with increasing strain, until all that is left at large strains is frictional strength.

The shear strength of frictional materials can be analysed using the concept that sliding between two rigid blocks takes place only when the resultant force between them is sufficiently inclined to overcome the friction between them. Combining that with the notion of a toothed plane allows the incorporation of dilatancy into the shear strength expression. If the resultant force on the interface of the two blocks makes an angle λ with the normal to the contact surface, then the following laws of friction can be formulated.

$$\lambda \begin{cases} < \phi_c \Rightarrow \text{there is no sliding} \\ = \phi_c \Rightarrow \text{sliding is either imminent or under way} \\ > \phi_c : \text{not possible} \end{cases} \quad (3)$$

For purely frictional materials λ can be related to the normal stress σ and shear stress τ on the shear plane by $\lambda = \tan^{-1}(\tau/\sigma)$. For cohesive-frictional materials the resultant force needs to be inclined further, to overcome the additional shear strength from cohesion. This can be accounted for by using Caquot's principle, according to which normal stresses σ are transformed into modified normal stresses σ^* through

$$\sigma^* = \sigma + c \cot \phi_f \quad (4)$$

where ϕ_f is the friction angle at failure, which evolves with the soil state. Equation (4) allows use of the same formulation as used for purely frictional materials in the solution of problems involving cohesive-frictional soils.

Note that the shear stresses are not affected by the transformation represented by equation (4) (i.e. $\tau^* = \tau$). As shown in Fig. 1, equation (4) maps the normal stress σ into a new, transformed normal stress σ^* by shifting the shear stress axis so that the Mohr–Coulomb yield envelope passes through the origin of the new system, thereby eliminating the cohesive intercept from the equation for the envelope in $\sigma^*–\tau^*$ space. For cohesive-frictional materials equation (3) can still be used to determine when sliding would start. However, it must be used with reference to a stress ratio in terms of the transformed stresses σ^* and τ^*, that is, the angle $\lambda = \tan^{-1}(\tau^*/\sigma^*)$.
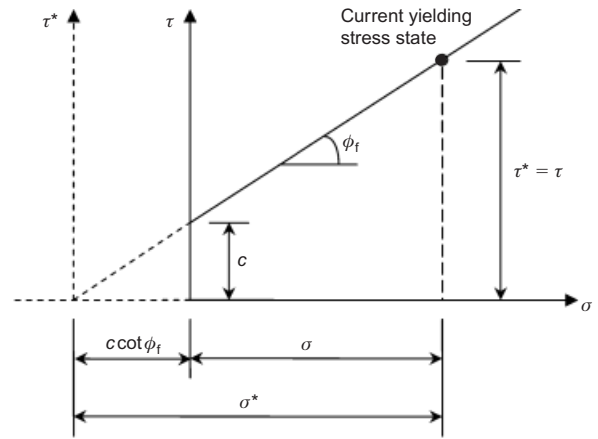

Fig. 1. Visualisation of stress transformation

SAWTOOTH MODEL

Consider a cylindrical sample of cemented sand with height h and cross-sectional area A in a triaxial compression test. As shown in Fig. 2(a), the stresses acting on the sample boundaries are the principal stresses $\sigma_1 > \sigma_2 = \sigma_3$. In Fig. 2(b) an equivalent representation of the sample and its loading is shown. In it, the transformed stresses $\sigma_1^* = \sigma_1 + c \cot \phi_f$ and $\sigma_3^* = \sigma_3 + c \cot \phi_f$ are shown applied to a sample identical to the one in Fig. 2(a) except for one detail: the sample in Fig. 2(b) is uncemented. The meaning of Fig. 2 is that the effects of the cohesion c may be modelled considering an equivalent soil with the same friction angle ϕ_f but with $c = 0$ to which an isotropic stress $c \cot \phi_f$ is applied, in addition to any other loadings.

Using Rowe's sawtooth model, the sliding occurs along separation planes between adjacent conglomerates of particles. As shown in Fig. 3, the separation plane has a stepped, sawtooth surface, and the direction of sliding is in the direction of the teeth. After sliding, a gap opens between the teeth, leading to an increase in the volume of the sample. The separation plane makes an angle α with the minor principal stress σ_3, and the teeth make an angle β with the major principal stress σ_1. The angle between the teeth and the separation plane is $\theta = \alpha + \beta - \pi/2$. Positive values of θ indicate volume increase.

The force transmitted through the teeth is denoted by F, and can be decomposed into vertical and horizontal components

$$F_v = (\sigma_1 + c \cot \phi_f) A$$
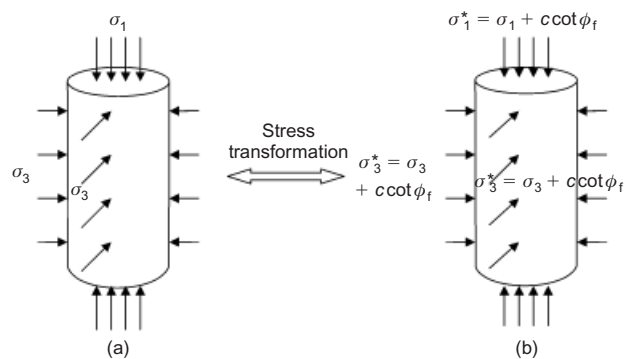$$F_h = (\sigma_3 + c \cot \phi_f) A \tan \alpha \quad (5)$$

Fig. 2. Stress transformation in axial-symmetric condition: (a) before transformation; (b) after transformation
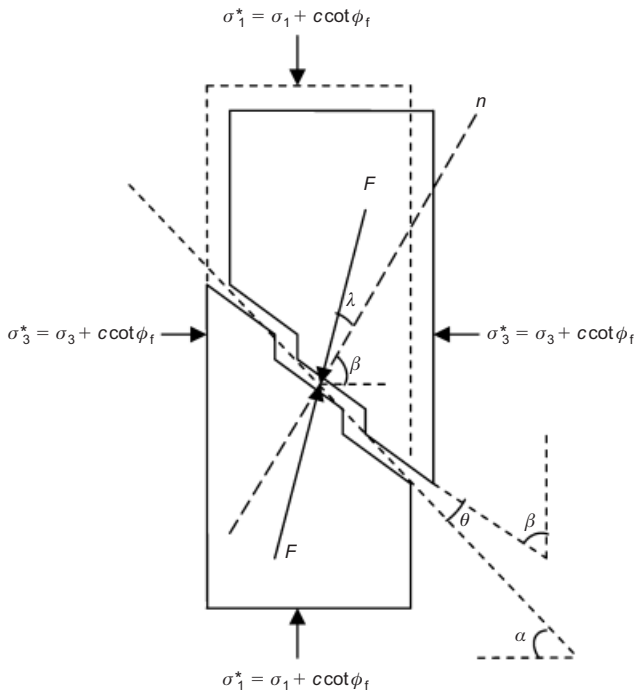
Fig. 3. Forces on teeth in a separation plane (modified from De Josselin de Jong, 1976)

According to Fig. 3, the normal line n makes an angle β with the horizontal, and the force F deviates from the n line by an angle λ. So F makes an angle $\beta + \lambda$ with the horizontal. From equation (5)

$$\tan(\beta + \lambda) = \frac{F_\mathrm{v}}{F_\mathrm{h}}$$

$$= \frac{\sigma_1 + c \cot \phi_\mathrm{f}}{(\sigma_3 + c \cot \phi_\mathrm{f}) \tan \alpha} \tag{6}$$

The Mohr–Coulomb criterion can be expressed as

$$R^* = \frac{\sigma_1 + c \cot \phi_\mathrm{f}}{\sigma_3 + c \cot \phi_\mathrm{f}}$$

$$= \tan^2 \left(\frac{\pi}{4} + \frac{\phi_\mathrm{f}}{2} \right) \tag{7}$$

where $R^* = \sigma_1^*/\sigma_3^*$ is the transformed stress ratio. Comparing equations (6) and (7)

$$R^* = \frac{\sigma_1 + c \cot \phi_\mathrm{f}}{\sigma_3 + c \cot \phi_\mathrm{f}}$$

$$= \tan \alpha \tan(\beta + \lambda) \tag{8}$$

The strain rate ratio $D = 1 - \dot{\varepsilon}_\mathrm{v}/\dot{\varepsilon}_1$ can also be expressed in terms of α and β. De Josselin de Jong (1976) showed that

$$D = \frac{\dot{V}_\mathrm{h}}{-\dot{V}_\mathrm{v}}$$

$$= \tan \alpha \tan \beta \tag{9}$$

where $\dot{V}_\mathrm{h}$ and $\dot{V}_\mathrm{v}$ are the volume change rates due to the horizontal and vertical displacement respectively. The negative sign in front of $\dot{V}_\mathrm{v}$ is in keeping with the geomechanics convention of having contraction and shortening be positive.

The assumption was made in the derivation of equation (9) that the total volume change rate $\dot{V}$ consists of the algebraic sum of $\dot{V}_\mathrm{v}$ and $\dot{V}_\mathrm{h}$. $\dot{V}$ and $\dot{V}_\mathrm{v}$ are relatively easily obtained from measurements made in common laboratory tests (notably, in triaxial tests). $\dot{V}_\mathrm{h}$ can be obtained by subtracting $\dot{V}_\mathrm{v}$ from the total volume change rate $\dot{V}$.

STRESS–DILATANCY RELATION

For the sawtooth model described in the previous section, the sample response can be completely described by the values of R^* (related to the yield or 'failure' surface for the material) and D (related to the dilatancy properties of the material) at any stage of a triaxial compression test. For these values of R^* and D, values of α, β and λ must be found that are consistent with the laws of friction. Taking R^* and D as known, equations (8) and (9) contain three unknowns: α, β and λ. To solve the unknowns, a third equation is required. The friction laws introduced as equation (3) provide this third equation: $\lambda_\mathrm{max} = \phi_\mathrm{c}$. Before this equation can be used, the angle α must be eliminated from equations (8) and (9) by introducing

$$\dot{E}^* = \frac{R^*}{D} = \frac{\tan(\beta + \lambda)}{\tan \beta} \tag{10}$$

The magnitude of $\dot{E}^*$ is fixed for known R^* and D. Solving for λ

$$\lambda = \tan^{-1} \left(\dot{E}^* \tan \beta \right) - \beta \tag{11}$$

Equation (11) expresses the relationship between λ and β for the given value of $\dot{E}^*$. In order to find the corresponding maximum value of λ, which is then made equal to ϕ_c to satisfy the friction laws, the value of β that maximises λ must first be found, which will be denoted as β_m. Setting λ_max equal to ϕ_c ensures that the friction laws are obeyed for all planes through the sample. Differentiating both sides of equation (11) with respect to β yields

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\beta} = \frac{1}{1 + \left(\dot{E}^* \tan \beta \right)^2} \cdot \frac{\dot{E}^*}{\cos^2 \beta} - 1 \tag{12}$$

Substituting equation (10) into equation (12) and setting $\mathrm{d}\lambda/\mathrm{d}\beta = 0$ then yields

$$\beta_\mathrm{m} = \frac{\pi}{4} - \frac{\lambda_\mathrm{max}}{2}$$

$$= \frac{\pi}{4} - \frac{\phi_\mathrm{c}}{2} \tag{13}$$

Using the values of λ_max and β_m in equation (10) yields

$$\dot{E}^* = \frac{R^*}{D} = \tan^2 \left(\frac{\pi}{4} + \frac{\phi_\mathrm{c}}{2} \right) \tag{14}$$

Combining equations (7) and (14)

$$\tan^2 \left(\frac{\pi}{4} + \frac{\phi_\mathrm{f}}{2} \right) = \tan^2 \left(\frac{\pi}{4} + \frac{\phi_\mathrm{c}}{2} \right) \left(1 - \frac{\dot{\varepsilon}_\mathrm{v}}{\dot{\varepsilon}_1} \right) \tag{15}$$

Rearranging equation (7)

$$\frac{\sigma_1}{\sigma_3} = \tan^2 \left(\frac{\pi}{4} + \frac{\phi_\mathrm{f}}{2} \right) + \frac{2c}{\sigma_3} \tan \left(\frac{\pi}{4} + \frac{\phi_\mathrm{f}}{2} \right) \tag{16}$$

Substituting equation (15) into equation (16) leads to

$$\frac{\sigma_1}{\sigma_3} = \tan^2 \left(\frac{\pi}{4} + \frac{\phi_\mathrm{c}}{2} \right) \left(1 - \frac{\dot{\varepsilon}_\mathrm{v}}{\dot{\varepsilon}_1} \right) + \frac{2c}{\sigma_3} \tan \left(\frac{\pi}{4} + \frac{\phi_\mathrm{c}}{2} \right) \sqrt{1 - \frac{\dot{\varepsilon}_\mathrm{v}}{\dot{\varepsilon}_1}} \tag{17}$$

By comparing equations (2) and (17) it can be seen that the

second term on the right-hand side of equation (2) is too large by a factor $\sqrt{1 - \dot{\varepsilon}_v/\dot{\varepsilon}_1}$. The correct form of the stress–dilatancy relation to use in the Rowe framework is thus equation (17).

Stresses are often expressed in terms of the first and second invariants of the stress tensor. In axisymmetric conditions, the mean stress $p = (\sigma_1 + 2\sigma_3)/3$ and the deviator stress $q = \sigma_1 - \sigma_3$ are typically used. Equation (17) can be rewritten in terms of these stress variables as

$$d = \frac{9(M - \eta) - 3m_c}{9 + M(3 - 2\eta) + m_c} \tag{18}$$

where $d = \dot{\varepsilon}_v^p/\dot{\varepsilon}_s^p$ is the dilatancy rate, with $\dot{\varepsilon}_v^p$ and $\dot{\varepsilon}_s^p$ being the plastic volumetric and deviatoric strain rates respectively; $\eta = q/p$ is the stress ratio; M ($= q/p$ at critical state) is related to the critical state friction angle ϕ_c; and m_c (which is related to the cohesive intercept c) is given by

$$m_c = \frac{6(3 - M)(c/p)^2}{3 - \eta} - \frac{2c(3 - M)}{p} \sqrt{\left(\frac{3c/p}{3 - \eta}\right)^2 + \frac{3 + 2\eta}{3 - \eta}} \tag{19}$$

CONCLUSIONS

Based on the laws of friction, a stress–dilatancy relationship has been derived for non-associated Mohr–Coulomb soils that have both cohesive and frictional strength components. At the same time it has been shown that the original Rowe stress–dilatancy relationship for cohesive-frictional soils is not correct. This was done in the transformed stress space, in which the material has the same friction angle ϕ_f but no cohesion. Application of a uniform hydrostatic stress field $c \cot \phi_f$ to the material compensates for making $c = 0$. The derivation further relies on use of the sawtooth model, together with application of the laws of friction.

Because of the incorrect hypothesis originally made at the time of its publication, equation (2) provided by Rowe (1962) is incorrect. The correct stress–dilatancy relation to use in modelling non-associated Mohr–Coulomb soils is equation (17). The proposed stress–dilatancy relationship can be used for both frictional and cohesive-frictional materials (i.e. for clean sands and other granular materials, natural soils with cementation bonds, stabilised soils, and even soft rocks).

NOTATION

A cross-sectional area of cylindrical sample of cemented sand
c interparticle cohesion
D strain rate ratio ($= 1 - \dot{\varepsilon}_v/\dot{\varepsilon}_1$)
d dilatancy rate ($= \dot{\varepsilon}_v^p/\dot{\varepsilon}_s^p$)

$\dot{E}^*$ ratio of R^* to D
F force transmitted through teeth
F_v, F_h vertical and horizontal components of F
h height of cylindrical sample of cemented sand
M $= q/p$ at critical state
m_c cohesion-related term in equation (18)
p mean stress ($= (\sigma_1 + 2\sigma_3)/3$)
q deviator stress ($= \sigma_1 - \sigma_3$)
R^* transformed stress ratio ($= \sigma_1^*/\sigma_3^*$)
V total volume change rate
$\dot{V}_h, \dot{V}_v$ volume change rates due to horizontal and vertical displacement
α angle between separation plane and minor principal stress σ_3
β angle between teeth and major principal stress σ_1
β angle between normal line n and the horizontal
β_m value of β that maximises λ
$\dot{\varepsilon}_1, \dot{\varepsilon}_v$ major principal strain rate and volumetric strain rate
$\dot{\varepsilon}_v^p, \dot{\varepsilon}_s^p$ plastic volumetric and deviatoric strain rates
η stress ratio ($= q/p$)
θ angle between teeth and separation plane ($= \alpha + \beta - \pi/2$)
λ angle between force F and normal line n
λ_{max} maximum value of λ
σ normal stress
σ^* modified normal stress
σ_1, σ_3 major and minor principal effective stresses
τ shear stress
τ^* modified shear stress
ϕ_c critical-state friction angle
ϕ_f friction angle at failure

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