A state space approach to periodic convolutional codes*

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Abstract. In this paper we study periodically time-varying convolutional codes by means of input-state-output representations. Using these representations we investigate under which conditions a given time-invariant convolutional code can be transformed into an equivalent periodic time-varying one. The relation between these two classes of convolutional codes is studied for period 2. We illustrate the ideas presented in this paper by constructing a periodic time-varying convolutional code from a time-invariant one. The resulting periodic code has larger free distance than any time-invariant convolutional code with equivalent parameters.

Keywords: Convolutional codes, Periodically time-varying codes, Input-state-output representations

1 Introduction

Convolutional codes [10] are an important type of error correcting codes that can be represented as a time-invariant discrete linear system over a finite field [20]. They are used to achieve reliable data transfer, for instance, in mobile communications, digital video and satellite communications [10, 23].

Since the sixties it has been widely known that convolutional codes and linear systems defined over a finite field are essentially the same objects [20]. More recently, there has been a new and increased interest in this connection and many

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advances have been derived from using the system theoretical framework when dealing with convolutional codes, see [12, 17]. Most of the large body of literature on convolutional codes and on the relation of these codes with linear systems has been devoted to the "time-invariant" case.

In this work we aim at studying time-varying convolutional codes from a system theoretical point of view. These codes have attracted much attention after Costello conjectured in [5] that nonsystematic time-varying convolutional codes can attain can larger free distance than the nonsystematic time-invariant ones. Since then, several researchers have investigated such codes [3, 15, 16, 18]. Moreover, in combination with wavelets [6] time-varying convolutional codes yield unique trellis structures that resulted in fast and low computational complexity decoding algorithms. However, little is known on the relation of time-varying convolutional codes and time-varying linear systems and very few general constructions of time-varying convolutional codes with designed distances are known.

Here we deal with periodically time-varying convolutional codes (for short, periodic codes) using input-state-output representations, and investigate some of their special properties and structures. In particular, we associate periodic codes with suitably defined time-invariant convolutional codes. This allows us to derive constructions of (periodic) input-state-output representations for periodic codes from the better understood time-invariant class.

2 Preliminaries

In the sequel we shall follow the system theory notation and consider column vectors rather than row vectors.

2.1 Time-invariant convolutional codes

A time-invariant convolutional code is a set of finite support sequences, called *codewords*, obtained as the image of a polynomial shift operator (the *encoder*) acting on finite support sequences that correspond to the original *information*. More precisely this can be defined as follows.

Definition 1. *Let $\mathbb{F}$ be a finite field and n, k be positive integers with $k < n$. A* time-invariant convolutional code *$\mathcal{C}$ of rate k/n is a set of finite support sequences described as*

$$\mathcal{C} = \left\{ v : \, v(\ell) = \left(G\left(\sigma^{-1}\right) u \right)(\ell); \, \ell \in \mathbb{N}_0, u \in \left[\left(\mathbb{F}^k\right)^{\mathbb{N}_0} \right]_{\mathtt{FS}} \right\}$$

where $G(z) \in \mathbb{F}^{n \times k}[z]$ is a full column rank $n \times k$ polynomial matrix over $\mathbb{F}$, called the encoder*, u taking values in $\mathbb{F}^k$ is the* information *sequence and v is the* codeword*. Moreover, σ^{-1} denotes the shift $\left(\sigma^{-1} u\right)(\ell) = u(\ell - 1)$, and the subindex $\mathtt{FS}$ affecting a set of sequences indicates that only its finite support elements are considered.*

The encoders of a code $\mathcal{C}$ are not unique; however they only differ by right multiplication by unimodular matrices over $\mathbb{F}[z]$. An encoder matrix G is called *basic* if it has a polynomial right inverse; from now on we shall only consider basic encoders, and refer to them simply as encoders. The encoder G is called *minimal* if the sum of its column degrees attains the minimal possible value.

We define the *degree* δ of a convolutional code as the sum of the column degrees of one, and hence any, minimal encoder. Note that the list of column degrees (also known as Forney indices) of a minimal encoder is unique up to a permutation. The maximum of the Forney indices is called the *memory* of a code, and is denoted by m. A code $\mathcal{C}$ of rate k/n, degree δ and memory m is said to be an (n,k,δ) code or an (n,k,δ,m) code if the memory is to be specified.

2.2 Periodically time-varying convolutional codes

In this work we consider convolutional codes $\mathcal{C}$ with P-periodic encoders, i.e.:

$$\begin{aligned}\mathcal{C} = \Big\{v : \; v(P\ell+t) = \big(G^t\left(\sigma^{-1}\right)u\big)(P\ell+t);\; t=0,\ldots,P-1;\\ \ell\in\mathbb{N}_0, u\in\Big[\left(\mathbb{F}^k\right)^{\mathbb{N}_0}\Big]_{FS}\Big\},\end{aligned} \tag{1}$$

where each $G^t(z)$ is an $n\times k$ time-invariant (basic) encoder. Such codes will be called *P-periodic.*

Inspired by the ideas developed in [13] and [1] for the case of behaviors, considering the linear map

$$L_p : \left(\mathbb{F}^n\right)^{\mathbb{N}_0} \to \left(\mathbb{F}^{Pn}\right)^{\mathbb{N}_0}$$

defined by

$$(L_p v)(\ell) = \begin{bmatrix} v(P\ell) \\ v(P\ell+1) \\ \vdots \\ v(P\ell+P-1)\end{bmatrix},\; P\in\mathbb{N}$$

we associate with $\mathcal{C}$ a time-invariant convolutional code $\mathcal{C}^L$, the *lifted* version of $\mathcal{C}$, defined as

$$\mathcal{C}^L = \left\{\widetilde{v}\in\left(\mathbb{F}^{Pn}\right)^{\mathbb{N}_0} : \widetilde{v} = L_p v,\; v\in\mathcal{C}\right\}.$$

Note that, since

$$\big(G^t\left(\sigma^{-1}\right)u\big)(P\ell+t) = \big(\left(\sigma^t G^t\left(\sigma^{-1}\right)\right)u\big)(P\ell),$$

the equation in (1) can also be written as

$$\big(\varOmega_{P,n}\left(\sigma\right)v\big)(P\ell) = \big(G\left(\sigma,\sigma^{-1}\right)u\big)(P\ell),\; \ell\in\mathbb{N}_0,$$

where for $r \in \mathbb{N}$

$$\Omega_{P,r}(\sigma) = \begin{bmatrix} I_r \\ \sigma I_r \\ \vdots \\ \sigma^{P-1} I_r \end{bmatrix}$$

is a polynomial matrix operator in the shift σ and

$$G\left(\sigma, \sigma^{-1}\right) = \begin{bmatrix} G^0\left(\sigma^{-1}\right) \\ \sigma G^1\left(\sigma^{-1}\right) \\ \vdots \\ \sigma^{P-1} G^{P-1}\left(\sigma^{-1}\right) \end{bmatrix}$$

is a polynomial matrix operator in the shifts σ and σ^{-1}.

Moreover, it is possible to show that the matrix G can be decomposed as

$$G\left(\sigma, \sigma^{-1}\right) = G^L\left(\sigma^{-P}\right) \Omega_{P,k}(\sigma)$$

where

$$G^L\left(\sigma^{-1}\right) = \left[G^{L_0}\left(\sigma^{-1}\right) \mid G^{L_1}\left(\sigma^{-1}\right) \mid \cdots \mid G^{L_{P-1}}\left(\sigma^{-1}\right)\right]$$

and the blocks $G^{L_j}\left(\sigma^{-1}\right)$ have size $Pn \times k,\ j = 0, \ldots, P-1$.

Thus, the lifted code can be represented as

$$\mathcal{C}^L = \left\{\widetilde{v} : \widetilde{v}(\ell) = (G^L\left(\sigma^{-1}\right) \widetilde{u})(\ell),\ \ell \in \mathbb{N}_0,\ \widetilde{u} \in \left[\left(\mathbb{F}^{kP}\right)^{\mathbb{N}_0}\right]_{FS}\right\},$$

where $\widetilde{v} = L_P v$ and $\widetilde{u} = L_P u$.

2.3 Distance properties

In recent years great effort has been dedicated to developing constructions of non-binary convolutional codes having good distance [2, 9, 14]. However, in contrast to block codes, the theoretical tools for the construction of convolutional codes with good designed distance have not been fully exploited. In fact, most convolutional codes used in practice have been found by systematic computer search and their distance properties must be also computed by full search.

One of our objectives will be the construction of convolutional codes with a large free distance, which is defined as follows.

Definition 2. *The* free distance *of a convolutional code $\mathcal{C}$ is given by*

$$d_{free}(\mathcal{C}) = \min\left\{\sum_{\ell=0}^{\infty} \mathrm{wt}\big(v(\ell)\big) : v \in \mathcal{C} \setminus \{0\}\right\},$$

where wt *denotes the* Hamming weight*, that is,* $\mathrm{wt}(v(\ell))$ *corresponds to the number of nonzero components of* $v(\ell)$*.*

Rosenthal and Smarandache [21] showed that the free distance of a time-invariant (n, k, δ) convolutional code is upper bounded by

$$d_{free}(\mathcal{C}) \leq (n-k)\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) + \delta + 1.$$

This bound is called the *generalized Singleton bound*. It is well-known [21] that over sufficiently large finite fields, there always exist convolutional codes that achieve this bound for any given set of parameters (n, k, δ).

However, in this paper we will consider instead the Griesmer bound defined in the next theorem. This bound is always less than or equal to the generalized Singleton bound, and can be considerably lower for codes over small fields, whereas it coincides with the Singleton for codes over sufficiently large finite fields.

Theorem 1. *[7, 10, 19] Let (n, k, δ, m) be a 4-tuple of nonnegative integers such that $k < n$, consider $q \in \mathbb{N}$ and $\hat{\mathbb{N}} = \begin{cases} \mathbb{N} & if \ \ km = \delta \\ \mathbb{N}_0 & if \ \ km > \delta \end{cases}$. Let further*

$$GB_q(n, k, \delta, m) = \max\left\{ d' : \sum_{j=0}^{k(m+i)-\delta-1} \left\lceil \frac{d'}{q^j} \right\rceil \leq n(m+i), \ \forall i \in \hat{\mathbb{N}} \right\}.$$

Then, every (n, k, δ, m)-convolutional code $\mathcal{C}$ over the field $\mathbb{F}_q$ is such that $d_{free}(\mathcal{C}) \leq GB_q(n, k, \delta, m)$.

$GB_q(n, k, \delta, m)$ is known as the *Griesmer bound*.

3 State space realizations

A state space system

$$\begin{cases} x(\ell+1) = Ax(\ell) + Bu(\ell) \\ \quad v(\ell) \quad = Cx(\ell) + Du(\ell) \end{cases}, \ l \in \mathbb{N}_0,$$

denoted by (A, B, C, D), where $A \in \mathbb{F}^{\delta\times\delta}$, $B \in \mathbb{F}^{\delta\times k}$, $C \in \mathbb{F}^{n\times\delta}$ and $D \in \mathbb{F}^{n\times k}$, is said to be a state space realization of the time-invariant (n, k, δ) convolutional code $\mathcal{C}$ if $\mathcal{C}$ is the set of finite support output sequences v corresponding to finite support input sequences u and zero inicial conditions, i.e., $x(0) = 0$.

Remark 1. This definition implicitly assumes that (A, B, C, D) is a minimal realization of $\mathcal{C}$, i.e., that A has the minimal possible dimension. This implies that A is nilpotent, (A, B) is controllable and (A, C) is observable, i.e., the polynomial matrices $\begin{bmatrix} z^{-1}I - A \mid B \end{bmatrix}$ and $\begin{bmatrix} z^{-1}I - A \\ C \end{bmatrix}$ have, respectively, right and left polynomial inverses (in z^{-1}).

State space realizations for convolutional codes can be obtained as minimal state space realizations of minimal encoders.

The next proposition, adapted from [8, Proposition 2.3], provides a state space realization for a given (not necessarily minimal) encoder.

Proposition 1. *Let $G \in \mathbb{F}^{n\times k}[z]$ be a polynomial matrix with rank k and column degrees $\nu_1, \ldots, \nu_k$. Consider $\bar{\delta} = \sum_{i=1}^{k} \nu_i$. Let G have columns $g_i = \sum_{\ell=0}^{\nu_i} g_{\ell,i} z^\ell$, $i = 1, \ldots, k$ where $g_{\ell,i} \in \mathbb{F}^n$. For $i = 1, \ldots, k$ define the matrices*

$$A_i = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ 1 & & & \vdots \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix} \in \mathbb{F}^{\nu_i \times \nu_i},\ B_i = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{F}^{\nu_i},\ C_i = \begin{bmatrix} g_{1,i} & \cdots & g_{\nu_i,i} \end{bmatrix} \in \mathbb{F}^{n\times \nu_i}.$$

Then a state space realization of G is given by the matrix quadruple $(A, B, C, D) \in \mathbb{F}^{\bar{\delta}\times\bar{\delta}} \times \mathbb{F}^{\bar{\delta}\times k} \times \mathbb{F}^{n\times\bar{\delta}} \times \mathbb{F}^{n\times k}$ where

$$A = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{bmatrix},\ B = \begin{bmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{bmatrix},\ C = \begin{bmatrix} C_1 & \cdots & C_k \end{bmatrix},\ D = \begin{bmatrix} g_{0,1} & \cdots & g_{0,k} \end{bmatrix} = G(0).$$

In the case where $\nu_i = 0$ the ith block is missing and in B a zero column occurs.

In this realization (A, B) is controllable, and if G is a minimal encoder, (A, C) is observable.

4 Constructing periodically time-varying convolutional codes

In comparison to the literature on time-invariant convolutional codes, there exist few algebraic constructions of time-varying convolutional codes with good properties [11, 22]. Here we present a new technique to build time-varying convolutional codes from time-invariant ones. In particular, in this section we focus on constructing 2-periodic codes with optimal free distance. We illustrate our approach by means of an example of a 2-periodic $(3, 2, 2, 1)$ code having larger distance than any $(3, 2, 2, 1)$ time-invariant convolutional code.

We first investigate the problem of finding periodic state space representations of periodic convolutional codes. As shown in Section 2.2, using a lifting technique one can transform a time-varying periodic linear system into an equivalent time-invariant one. Following [1], we study the relationship between the periodic state space representations of a given code and the time-invariant state space representations of its lifted version. For the sake of simplicity we assume that the period is $P = 2$. However, whereas in [1] only single-input/single-output systems were considered, here we deal with codes of general rate k/n that are closely related to multi-input/multi-output (MIMO) systems.

Assume that $\Sigma(\cdot) = (A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ is a δ-dimensional state space representation of a code $\mathcal{C}$, as present below:

$$\begin{cases} x(\ell+1) = A(\ell)x(\ell) + B(\ell)u(\ell) \\ \quad v(\ell) \quad = C(\ell)x(\ell) + D(\ell)u(\ell) \end{cases}, \ l \in \mathbb{N}_0 \tag{2}$$

where $(A(\cdot), B(\cdot), C(\cdot), D(\cdot)) \in \mathbb{F}^{\delta\times\delta} \times \mathbb{F}^{\delta\times k} \times \mathbb{F}^{n\times\delta} \times \mathbb{F}^{n\times k}$ are periodic functions with period 2. Letting

$$\begin{aligned} w(\ell) &= x(2\ell) \\ u^L(\ell) &= \begin{bmatrix} u(2\ell) \\ u(2\ell+1) \end{bmatrix} \\ v^L(\ell) &= \begin{bmatrix} v(2\ell) \\ v(2\ell+1) \end{bmatrix} \end{aligned}$$

we obtain the following time-invariant δ-dimensional state space representation $\Sigma^L = (E, F, H, J)$ for the lifted code $\mathcal{C}^L$:

$$\begin{cases} w(\ell+1) = Ew(\ell) + Fu^L(\ell) \\ \quad v^L(\ell) \quad = Hw(\ell) + Ju^L(\ell) \end{cases}, \tag{3}$$

with

$$\begin{aligned} E &= A(1)A(0) & F &= \begin{bmatrix} A(1)B(0) & B(1) \end{bmatrix} \\ H &= \begin{bmatrix} C(0) \\ C(1)A(0) \end{bmatrix} & J &= \begin{bmatrix} D(0) & 0 \\ C(1)B(0) & D(1) \end{bmatrix}. \end{aligned}$$

The representation $\Sigma^L = (E, F, H, J)$ of $\mathcal{C}^L$ is said to be *induced by* the representation $\Sigma(\cdot) = (A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ of $\mathcal{C}$, or equivalently, $\Sigma(\cdot)$ is said to induce Σ^L. Moreover, a time-invariant representation $\Sigma^L = (E, F, H, J)$ of $\mathcal{C}^L$ is called *induced* whenever it is induced by some periodic representation $\Sigma(\cdot)$ of $\mathcal{C}$.

The following proposition is a generalization of [1, Proposition 3.1] (with identical proof) and characterizes induced representations.

Proposition 2. *Let $\mathcal{C}$ be a 2-periodic code and $\mathcal{C}^L$ the lifted code associated to $\mathcal{C}$. Then a δ-dimensional state space representation $\Sigma^L = (E, F, H, J)$ of $\mathcal{C}^L$, with*

$$\begin{aligned} E &\in \mathbb{F}^{\delta\times\delta} & F &= \begin{bmatrix} F_1 & F_2 \end{bmatrix} \in \mathbb{F}^{\delta\times 2k} \\ H &= \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \in \mathbb{F}^{2n\times\delta} & J &= \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \in \mathbb{F}^{2n\times 2k}. \end{aligned}$$

is induced if and only if

$$\operatorname{rank} \mathcal{M} = \begin{bmatrix} E & F_1 \\ H_2 & J_{21} \end{bmatrix} \leq \delta.$$

Moreover, in this case, decomposing the matrix $\mathcal{M}$ *as*

$$\mathcal{M} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix},$$

the 2-periodic δ*-dimensional state space representation of* $\mathcal{C}$ *that induces* Σ^L *is* $\Sigma(\cdot) = (A(\cdot), B(\cdot), C(\cdot), D(\cdot))$*, where*

$$\begin{array}{llll} A(0) = Q_1 & A(1) = N_1 & B(0) = Q_2 & B(1) = F_2 \\ C(0) = H_1 & C(1) = N_2 & D(0) = J_{11} & D(1) = J_{22}. \end{array}$$

Hence, Proposition 2 characterizes the state space realizations of time-invariant convolutional codes from which (2-periodic) time-varying codes can be constructed. In this way one can use the large body of literature and constructions for the time-invariant case in order to build time-varying convolutional codes with good properties. In the next section we illustrate this with an example.

4.1 2-periodic $(3, 2, 2, 1)$ convolutional code with free distance 4

It is know from the literature that $(3, 2, 2, 1)$ time-invariant convolutional codes have at most free distance 3, whereas the Griesmer bound for this kind of codes is 4. In this section we construct a 2-periodic $(3, 2, 2, 1)$ convolutional code with free distance 4 based on the construction of a time-invariant code whose state space realization is induced by a 2-periodic realization. This shows that time-varying convolutional codes can attain larger free distance than time-invariant ones.

Example 1. Consider the $(6, 4, 2, 1)$ time-invariant convolutional code, $\mathcal{C}^L$, over $\mathbb{F}_2$ with generator matrix $G = G_1 z + G_0$, where

$$G_0 = \begin{bmatrix} 1\,1\,0\,0 \\ 0\,1\,0\,0 \\ 1\,0\,0\,0 \\ 1\,0\,1\,0 \\ 1\,1\,1\,1 \\ 0\,1\,1\,0 \end{bmatrix} \quad \text{and} \quad G_1 = \begin{bmatrix} 0\,0\,1\,1 \\ 0\,0\,0\,1 \\ 0\,0\,0\,1 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \end{bmatrix}.$$

As $\begin{bmatrix} G_0 \\ G_1 \end{bmatrix}$ is an encoder of a $(12, 3)$ block code of free distance 4, we conclude that the free distance of $\mathcal{C}^L$ is at most 4. Moreover, it can be computed via a program that the free distance of $\mathcal{C}^L$ is indeed 4. Since the column degrees of G

are $\nu_1 = 0, \nu_2 = 0, \nu_3 = 1, \nu_4 = 1$, by Proposition 1, a state space realization of G is given by $(E, F, H, J) \in \mathbb{F}^{2\times 2} \times \mathbb{F}^{2\times 4} \times \mathbb{F}^{6\times 2} \times \mathbb{F}^{6\times 4}$ where

$$E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; F = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; H = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \; J = G_0. \tag{4}$$

Now, the matrix $\mathcal{M}$ defined in Proposition 2 is:

$$\mathcal{M} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since $\operatorname{rank} \mathcal{M} = 2 \leq \delta$, by this proposition we conclude that realization (4) is induced by a 2-periodic $(3, 2, 2, 1)$ convolutional code, with realization $\Sigma(\cdot) = (A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ where

$$A(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad A(1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad B(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C(0) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad C(1) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad D(0) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad D(1) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

This 2-periodic code can also be described as in (1) with $P = 2$, and $G^t(z) = C(t) \left(z^{-1}I - A(t)\right)^{-1} B(t) + D(t)$, for $t = 0, 1$, which yields

$$G^0(z) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} z + \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad G^1(z) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} z + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

This example is equivalent to the one presented by Palazzo in [18].

5 Conclusions

In this paper we have studied the relation between time-invariant and time-varying convolutional codes by means of input-state-output representations. Using a well known lifting technique we have shown how it is possible to transform a given periodically time-varying convolutional code into a time-invariant one. Moreover, we have provided conditions, in terms of input-state-output representations, to transform a time-invariant convolutional code into a time-varying one. Using these ideas, we have illustrated how to construct a 2-periodic $(3, 2, 2, 1)$

convolutional code with optimal free distance from a $(3, 2, 2, 1)$ time-invariant one. This showed that time-varying convolutional codes can attain larger free distance than time-invariant ones. Constructions of periodic convolutional codes of higher periods and with other parameters are currently under investigation.

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