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SOME NOTES ON THE UPPER AND LOWER RADICALS

by

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§ 1. Introduction

In the following only associative rings are considered. A radical class or briefly a radical will mean a radical in the sense of Kuroš and Amitsur. For the basic concepts of the radical theory we refer to [2], [6] and [7].

For a given class M of rings, we denote the homomorphic closure of $\mathbf{M}$ by $H(\mathbf{M})$ and, the hereditary closure of $\mathbf{M}$ by $J(\mathbf{M})$, these are,

$$H(\mathbf{M}) = \{A \mid A \text{ is a homomorphic image of some } \mathbf{M}\text{-ring}\}$$

$$J(\mathbf{M}) = \{A \mid A \text{ is an accessible subring of some } \mathbf{M}\text{-ring}\}$$

$\mathcal{U}(\mathbf{M})$ denotes the upper radical class determined by $\mathbf{M}$ and $\mathcal{L}(\mathbf{L})$ denotes the lower radical class determined by $\mathbf{L}$.

The class $\mathbf{M}$ is said to be regular if it satisfies the following condition:

$$H(I) \cap \mathbf{M} \neq \varnothing, \text{ for every } 0 \neq I \triangleleft A \in \mathbf{M}$$

where $I \triangleleft A$ means I is an ideal of A. Note: we write I for the class $\{I\}$ containing I as its member.

A regular class may not contain the ring 0, for the sake of short statement we shall assume that regular classes contain the ring 0.

It is well-known that if the class $\mathbf{M}$ is regular then

$$\mathcal{U}(\mathbf{M}) = \{A \mid H(A) \cap \mathbf{M} = 0\}$$

In [5] W. G. Leavitt and Yu-Lee Lee have shown that if $\mathbf{L}$ is a homomorphically closed class of rings, then

$$\mathcal{L}(\mathbf{L}) = \{A \mid J(A/I) \cap \mathbf{L} \neq 0 \text{ for every } A/I \neq 0\}$$

In 2 we shall consider conditions for classes $\mathbf{L}_i$, $\mathbf{M}_I$, $i = 1, 2$, such that the upper and lower radical classes determine the same radical, that is,

$$\mathcal{L}(\mathbf{L}_i) = \mathcal{U}(\mathbf{M}_i)\ \ \mathcal{U}(\mathbf{M}_i) = \mathcal{U}(\mathbf{M}_j) \text{ and } \mathcal{L}(\mathbf{L}_i) = \mathcal{L}(\mathbf{L}_j).$$

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A class **M** of rings has been called special by V. A. ANDRUNAKIEVIČ [1] if it is a hereditary class of prime rings with the property:

If $I \lhd A$ with $I \in M$ then $A/I^* \in \mathbf{M}$, where I^* is the two-sided annihilator of I in A.

A radical R is called special if R is an upper radical determined by some special class.

A problem concerning the notion of the special radical can be naturally raised:

Find conditions for classes **M** and **L** auch that the upper radical determined by **M** and, the lower radical determined by **L** are special. This problem will be solved in § 3.

ANDRUNAKIEVIČ [1] has shown that every special radical is supernilpotent. The following theorem will be neccessary later on.

THEOREM 1 (cf. [1], Theorem 6, pp. 198). *Let R be a supernilpotent radical then the upper radical determined by the class of all prime R-semisimple rings is the smallest special radical containing R.*

§ 2. The coincidence of upper radical classes and lower radical classes

2.1 *Criterion for* $\mathfrak{L}(\mathbf{L}) = \mathfrak{U}(\mathbf{M})$

LEMMA 2. *Let* **L** *be a homomorphically closed class. Then a ring A is $\mathfrak{L}(\mathbf{L})$-semisimple if and only if $J(A) \cap \mathbf{L} = 0$ holds.*

PROOF. Assume a ring A be $\mathfrak{L}(\mathbf{L})$-semisimple since every semisimple class of a associative rings is hereditary, so every accessible non-zero subrings of A is $\mathfrak{L}(\mathbf{L})$-semisimple. This implies $J(A) \cap \mathbf{L} = 0$.

Conversely, suppose that a ring A satisfies the condition $J(A) \cap \mathbf{L} = 0$. Assume B be a $\mathfrak{L}(\mathbf{L})$-ideal of the ring A. cf $B \neq 0$ then every non-zero homomorphic image of B contains a non-zero accessible **L**-subring. In particular, B has a non-zero accessible **L**-subring. From this it follows $J(A) \cap \mathbf{L} \neq 0$, a contradiction. Thus $B = 0$ and the ring A is $\mathfrak{L}(\mathbf{L})$-semisimple.

THEOREM 3. *Suppose that the class M is regular and the class* **L** *is homomorphically closed. Then $\mathfrak{L}(\mathbf{L}) = \mathfrak{U}(\mathbf{M})$ if and only if the following conditions are satisfied*:

(1) $\mathbf{L} \cap \mathbf{M} = 0$,

(2) *For every non-zero ring A, if $J(A) \cap \mathbf{L} = 0$ then $H(A) \cap \mathbf{M} \neq 0$.*

PROOF. In view of Lemma 2, the necessity is straightforward.

Conversely, assume that the conditions of the theorem are satisfied. Since **L** is homomorphically closed, so from the first condition follows that no ring of **L** can be mapped homomorphically onto any non-zero **M**-ring. Hence the inclusion $\mathbf{L} \subseteq \mathcal{U}(\mathbf{M})$ holds. By the minimality of the lower radical we have $\mathfrak{L}(\mathbf{L}) \subseteq \mathcal{U}(\mathbf{M})$. Now, suppose that a ring A does not belong to the class $\mathfrak{L}(\mathbf{L})$. By Lemma 2 the non-zero $\mathfrak{L}(\mathbf{L})$-semisimple ring $A/\mathfrak{L}(\mathbf{L})(A)$ has no non-zero accessible **L**-subrings. By the second condition the ring $A/\mathfrak{L}(\mathbf{L})(A)$ can be mapped homomorphically onto some non-zero **M**-ring. This implies $H(A) \cap \cap \mathbf{M} \neq 0$ and so the ring A is not in $\mathcal{U}(\mathbf{M})$. Thus we have $\mathfrak{L}(\mathbf{L}) = \mathcal{U}(\mathbf{M})$.

2.2. *Criterion for* $\mathcal{U}(\mathbf{M}_1) = \mathcal{U}(\mathbf{M}_2)$

THEOREM 4. *Suppose* $\mathbf{M}_i$ $(i = 1, 2)$ *are regular classes of rings. Then* $\mathcal{U}(\mathbf{M}_1) = \mathcal{U}(\mathbf{M}_2)$ *if and only if*

$$H(A) \cap \mathbf{M}_j \neq 0$$

for every ring A *in* $\mathbf{M}_i$ $(i = 1, 2)$.

PROOF. The necessity is obvious.

Now assume that the conditions of theorem are valid. We have to show that $\mathcal{U}(\mathbf{M}_1) = \mathcal{U}(\mathbf{M}_2)$. Let A be an arbitrary ring in $\mathbf{M}_1$ and, B any non-zero ideal of A. Since the class $\mathbf{M}_1$ is regular so B can be mapped homomorphically onto some non-zero $\mathbf{M}_1$-ring C. By the hypothesis the ring C can be mapped homomorphically onto some non-zero $\mathbf{M}_2$-ring. This implies that every non-zero ideal of A can be mapped onto some non-zero $\mathbf{M}_2$-ring.

Thus the ring A is $\mathcal{U}(\mathbf{M}_2)$-semisimple, and so each ring A in $\mathbf{M}_1$ is $\mathcal{U}(\mathbf{M}_2)$-semisimle. Since $\mathcal{U}(\mathbf{M}_1)$ is the largest radical for which every ring in $\mathbf{M}_2$ is semisimple, we must have $\mathcal{U}(\mathbf{M}_2) \leq \mathcal{U}(\mathbf{M}_1)$. Similarly, also $\mathcal{U}(\mathbf{M}_1) \leq \mathcal{U}(\mathbf{M}_2)$ holds.

COROLLARY. *Let* **N** *be a subclass of a regular class* **M**. *Then* $\mathcal{U}(\mathbf{N}) = \mathcal{U}(\mathbf{M})$ *if the following condition is satisfied*:

For every non-zero ring $A \in \mathbf{M}$,

$$H(A) \cap \mathbf{N} \neq 0. \qquad (\alpha)$$

PROOF. It is easy to see that if the condition (α) is valid then the subclass N is regular. So the conditions of Theorem 3 are satisfied.

REMARK. In general, the converse is not true. For instance, let A be a non-zero simple ring. We take $\mathbf{M} = \{A, A + A\}$ and $\mathbf{N} = \{A + A\}$. Clearly, the class M is regular and $\mathcal{U}(\mathbf{N}) = \mathcal{U}(\mathbf{M})$ but the condition (α) is not valid.

2.3. *Criterion for* $\mathfrak{L}(\mathbf{L}_1) = \mathfrak{L}(\mathbf{L}_2)$

THEOREM 2. *Let* $\mathbf{L}_i$, $i = 1, 2$, *be homomorphically closed classes. Then* $\mathfrak{L}(\mathbf{L}_1) = \mathfrak{L}(\mathbf{L}_2)$ *if and only if the following condition is satisfied*:

(β) *For every non-zero ring* $A \in \mathbf{L}_i$, $J(A) \cap \mathbf{L}_j \neq 0$ $(i, j = 1, 2)$.

PROOF. Suppose $\mathfrak{L}(\mathbf{L}_1) = \mathfrak{L}(\mathbf{L}_2)$. Then every ring A in $\mathbf{L}_i$ is a $\mathfrak{L}(\mathbf{L}_j)$-radical and by Lemma 2 it follows $J(A) \cap \mathbf{L}_j \neq 0$.

Conversely, assume that the classes $\mathbf{L}_i$, $i = 1, 2$, satisfy the condition of theorem. Let A be an arbitrary ring in $\mathbf{L}_1$. Since the class $\mathbf{L}_1$ is homomorphically closed, so every homomorphic image of A is in $\mathbf{L}_1$. Therefore, by the condition (β) every homomorphic image of A has a non-zero accessible $\mathbf{L}_2$-subring. Hence the ring A is in $\mathfrak{L}(\mathbf{L}_2)$. From that follows $\mathfrak{L}(\mathbf{L}_1) \subseteq \mathfrak{L}(\mathbf{L}_2)$. Similarly, also $\mathfrak{L}(\mathbf{L}_2) \subseteq \mathfrak{L}(\mathbf{L}_1)$ holds.

COROLLARY. *Let* $\mathbf{L}_0$ *be a subclass of a homomorphically closed class* $\mathbf{L}$. *If* $J(A) \cap \mathbf{L}_0 \neq 0$ *holds for every non-zero ring* A *in* $\mathbf{L}$, *then* $\mathbf{L}(L_0) = \mathbf{L}(L)$, *provided that* $\mathbf{L}_0$ *is homomorphically closed.*

§ 3. Criterion for the upper and lower radical to be special

LEMMA 6. *Let* $\mathbf{L}$ *be a homomorphically closed class of rings such that the lower radical* $\mathfrak{L}(\mathbf{L})$ *determined by* $\mathbf{L}$ *is supernilpotent. Then the radical* $\mathfrak{L}(\mathbf{L})$ *is special if and only if the following condition is satisfied*:

(γ) *For a non-zero ring* A *if* $J(A) \cap \mathbf{L} = 0$ *then*

$$H(A) \cap P(\mathbf{L}) \neq 0$$

where

$$P(\mathbf{L}) = \{A \mid A \text{ is a prime ring and } J(A) \cap \mathbf{L} = 0.$$

PROOF. Let $\mathbf{L}$ be a homomorphically closed class of rings such that $\mathfrak{L}(\mathbf{L})$ is supernilpotent. By Lemma 2 every ring in $P(\mathbf{L})$ is prime $\mathfrak{L}(\mathbf{L})$-semisimple. By Theorem 1 the radical $\mathfrak{L}(\mathbf{L})$ is special if and only if $\mathfrak{L}(\mathbf{L}) = \mathfrak{U}(P(\mathbf{L}))$.

Clearly, the relation $\mathbf{L} \cap P(\mathbf{L}) = 0$ always holds. Thus, by Theorem 3 $\mathfrak{L}(\mathbf{L}) = \mathfrak{U}(P(\mathbf{L}))$ if and only if condition (γ) is valid.

THEOREM 7. *If* $\mathbf{L}$ *is a hereditary and homomorphically closed class containing all zero-rings then the lower radical* $\mathfrak{L}(\mathbf{L})$ *is special if and only if the property* (γ) *is valid.*

PROOF. In [4] HOFFMAN and LEAVITT have shown that if L is hereditary, then the lower radical $\mathfrak{L}(\mathbf{L})$ is hereditary. Hence, by the hypothesis, the radical $\mathfrak{L}(\mathbf{L})$ is supernilpotent. Thus the theorem is an immediate consequence of Lemma 6.

LEMMA 8. *Let* $\mathbf{M}$ *be a regular class of rings such that the upper radical* $\mathcal{U}(\mathbf{M})$ *is supernilpotent. Then the radical* $\mathcal{U}(\mathbf{M})$ *is special if the following condition is satisfied*:

(χ) *for every non-zero ring* $A \in \mathbf{M}$,

$$H(A) \cap \mathbf{M} \cap \mathbf{P} \neq 0,$$

where $\mathbf{P}$ *is the class of all prime rings.*

PROOF. Let $\mathbf{M}$ be a regular class satisfying the conditions of the lemma. Consider the class $\mathbf{N} = \mathbf{M} \cap \mathbf{P}$. By the Corollary of Theorem 4 we have $\mathcal{U}(\mathbf{M}) = \mathcal{U}(\mathbf{N})$ if condition (α) is satisfied. Next, we denote the class of prime $\mathcal{U}(\mathbf{M})$-semisimple ring by N_1 that is, $\mathbf{N}_1 = \overline{\mathbf{M}} \cap \mathbf{P}$, where

$$(*) \qquad \overline{\mathbf{M}} = \{A \mid H(I) \cap \mathbf{M} \neq 0, \text{ for every } 0 \neq I \triangleleft A\}.$$

Clearly $\mathbf{N} \subseteq \mathbf{N}_1$. Since class of prime rings and semisimple class are hereditary so the class $\mathbf{N}_1$ is hereditary.

Let a ring A be in $\mathbf{N}_1$. By ($*$) the ring A can be mapped homomorphically onto some non-zero ring A in $\mathbf{M}$. By condition (χ) the ring A has some non-zero homomorphic image A_2 in $\mathbf{N}$. From this it follows that, for every ring A in $\mathbf{N}_1$, $H(A) \cap \mathbf{N} \neq 0$ holds. By the corollary of Theorem 4 we have $\mathcal{U}(\mathbf{N}_1) = \mathcal{U}(\mathbf{N}) = \mathcal{U}(\mathbf{M})$. Thus, by Theorem 1 the radical $\mathcal{U}(\mathbf{M})$ is special.

THEOREM 9. *Let* $\mathbf{M}$ *be a regular class of rings. Then the upper radical* $\mathcal{U}(\mathbf{M})$ *is special if the following three conditions are satisfied*:

(i) $\mathbf{M}$ *does not contain non-zero zero-rings.*

(ii) *For each ring* A, *if* $0 \neq I \triangleleft A$ *and* $H(I) \cap \mathbf{M} \neq 0$, *then* $H(A) \cap \mathbf{M} \neq 0$.

(iii) *For every non-zero ring* $A \in \mathbf{M}$,

$$H(A) \cap \mathbf{M} \cap \mathbf{P} \neq 0.$$

PROOF. In [3] ENERSEN and LEAVITT have shown that if the class $\mathbf{N}$ satisfies the conditions (i) and (ii), then the upper radical $\mathcal{U}(\mathbf{M})$ is supernilpotent. Thus, the theorem is an immediate consequence of Lemma 8.

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