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ON STRONG SEMISIMPLICITIES OF SEMIGROUPS WITH ZERO

by

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The fundamental notions of semigroups can be found in the books of A. H. CLIFFORD and G. B. PRESTON [4] and of E. S. LJAPIN [6]. In what follows, D. REES' [8] factor semigroups will play an important role. For semigroups various concrete radicals were discussed by J. BOSÁK [2], A. H. CLIFFORD [3], H. J. HOEHNKE [5], J. LUH [7], ST. SCHWARZ [9], H. SEIDEL [10], L. N. SHEVRIN [11] and the author [13]. The possibility to investigate general radicals of semigroups with zero has been shown e.g. by author [15], [16] and R. WIEGANDT [17].

Following author's [15], a class $\mathfrak{R}$ of semigroups S with zero is called a radical class if the following conditions are satisfied:

(i) Every homomorphic image of a semigroup from $\mathfrak{R}$ belongs to $\mathfrak{R}$.

(ii) Every semigroup S contains an ideal $R(S)$ belonging to $\mathfrak{R}$ such that $R(S)$ contains every other ideal belonging to $\mathfrak{R}$, of S.

(iii) We have $R(S/R(S)) = 0$ for the ideal $R(S)$ defined in condition (ii).

(Here and in what follows S/T denotes REES' factor semigroups.)

This $R(S)$ is said to be the $\mathfrak{R}$-radical of S. If $R(S) = S$ holds, then S is called an $\mathfrak{R}$-radical semigroup. If $R(S) = 0$ holds, then S is $\mathfrak{R}$-semisimple. An $\mathfrak{R}$-semisimple semigroup is said to be strongly $\mathfrak{R}$-semisimple, if every homomorphic image of S is $\mathfrak{R}$-semisimple. The groups with zero obviously are strongly $\mathfrak{R}$-semisimple for every general radical $\mathfrak{R}$. By author's paper [15] for every ideal I of S and for every radical $\mathfrak{R}$ the subsemigroup $R(I)$ is an ideal of S.

It is the purpose of this paper to prove that for every radical R, for which every $\mathfrak{R}$-semisimple semigroup also is strongly $\mathfrak{R}$-semisimple the mapping $\varphi : I + R(I)$, is a join-endomorphism of the lattice of all twosided ideals I of the semigroup. The similar ringtheoretical result was previously discussed by author [14]. The dualization of this semigroup-theoretical result, which also generalizes some results of Robert SHULKA [12], was investigated by author [16], and the similar ringtheoretical result by S. A. AMITSUR [1].

First we verify two preliminary propositions.

PROPOSITION 1. *The mapping* $\varphi : I \to R(I)$ *is monotone, i.e.* $I_1 \subseteq I_2$ *implies* $R(I_1) \subseteq R(I_2)$ *for the ideals* I_1 *and* I_2.

PROOF. Assume $I_1 \subseteq I_2$. Then trivially $R(I_1) \subseteq I_2$ holds. Let us consider the first isomorphism theorem (see D. REES [8]):

$$(R(I_1) \cup R(I_2))/R(I_2) \cong R(I_1)/(R(I_1) \cap R(I_2)). \tag{1}$$

On the left hand side of (1) we have a twosided ideal of the $\mathfrak{R}$-semisimple Rees factor semigroup $I_2/R(I_2)$ and therefore, by author's paper [15], this ideal is again $\mathfrak{R}$-semisimple. But on the right hand side of (1) one has a homomorphic image of the $\mathfrak{R}$-radical semigroup $R(I_1)$. Thus, by condition (i) on the right hand side of (1) stays an $\mathfrak{R}$-radical semigroup. These facts imply

$$R(I_1)/(R(I_1) \cap R(I_2)) = 0,$$

consequently $R(I_1) = R(I_1) \cap R(I_2) \subseteq R(I_2)$ which means the desired monotony of $\varphi : I \to R(I)$.

PROPOSITION 2. *If* I *and* S/I *are* $\mathfrak{R}$*-semisimple, then* S *itself is* $\mathfrak{R}$*-semisimple.*

The proof is, using the first isomorphism theorem and the definition of $\mathfrak{R}$-semisimplicity, almost trivial.

REMARK 3. Hitherto we need not have used our assumption that every $\mathfrak{R}$-semisimple semigroup is strongly $\mathfrak{R}$-semisimple.

In what follows we use the modularity of the lattice of all ideals of a semigroup. In fact, this lattice is distributive, since it is a sublattice of the Boolean algebra of all subsets of S. On the other side the proof of Theorem 4 is similar to author's [14] proof, taking set theoretical unions instead of sums.

THEOREM 4. *Let* R *be a radical such that every* $\mathfrak{R}$*-semisimple semigroup is strongly* $\mathfrak{R}$*-semisimple and* I *an arbitrary (twosided) ideal of the semigroup* S. *Then the mapping*

$$\varphi : I \to R(I)$$

is a join-endomorphism of the lattice of all (twosided) ideals of S, *i.e. we always have*

$$\varphi(I_1 \cup I_2) = R(I_1 \cup I_2) = R(I_1) \cup R(I_2) = \varphi(I_1) \cup \varphi(I_2). \tag{2}$$

PROOF. It is easier to prove, that the right side of (2) is contained on the left hand side of (2), since $I_j \subseteq I_1 \cup I_2$ for $j = 1$ and 2 by Proposition 1 implies $R(I_j) \subseteq R(I_1 \cup I_2)$ and therefore

$$R(I_1) \cup R(I_2) \subseteq R(I_1 \cup I_2),$$

indeed. The opposite inclusion will be verified in more steps, namely we shall show that both of $(I_1 \cup I_2)/(R(I_1) \cup I_2)$, and $(R(I_1) \cup I_2)/(R(I_1) \cup R(I_2))$ are $\mathfrak{R}$-semisimple Rees factor semigroups.

By $I_1 \supseteq R(I_1)$ and by the modularity of the lattice of all ideals of S one has

$$I_1 \cap (R(I_1) \cup I_2) = R(I_1) \cup (I_1 \cap I_2). \tag{3}$$

Therefore $I_1/(I_1 \cap (R(I_1) \cup I_2))$ is isomorphic to a homomorphic image of the strongly $\mathfrak{R}$-semisimple semigroup $I_1/R(I_1)$ which implies the $\mathfrak{R}$-semisimplicity of $I_1/(I_1 \cap (R(I_1) \cup I_2))$, too. Now by $R(I_1) \subseteq I_1$ and by (3) the first isomorphism theorem yields

$$(I_1 \cup I_2)/(R(I_1) \cup I_2)) \cong I_1/(I_1 \cap (R(I_1) \cup I_2)),$$

thus also $(I_1 \cup I_2)/(R(I_1) \cup I_2)$ is $\mathfrak{R}$-semisimple, as it has been pointed out previously.

Similarly $R(I_2) \subseteq I_2$ and the modularity of the lattice of all twosided ideals of S imply

$$I_2 \cap (R(I_1) \cup R(I_2)) = R(I_2) \cup (I_2 \cap R(I_1)). \tag{4}$$

Thus $I_2/(I_2 \cap (R(I_1) \cup R(I_2)))$ is $\mathfrak{R}$-semisimple, since by (4) it is a homomorphic image of the strongly $\mathfrak{R}$-semisimple Rees factor semigroup $I_2/R(I_2)$.

By the first isomorphism theorem and by $R(I_2) \subseteq I_2$ we have

$$(R(I_1) \cup I_2)/(R(I_1) \cup R(I_2)) \cong I_2/(I_2 \cap (R(I_1) \cup R(I_2))) \tag{5}$$

thus the left hand side of (5) is $\mathfrak{R}$-semisimple.

Now, by Proposition 2 and by the second isomorphism theorem (see D. REES [8]) it follows that $(I_1 \cup I_2)/(R(I_1) \cup R(I_2))$ is $\mathfrak{R}$-semisimple. But the $\mathfrak{R}$-semisimplicity of $(I_1 \cup I_2)/(R(I_1) \cup R(I_2))$ and the first isomorphism theorem imply also the (nontrivial) inclusion:

$$R(I_1 \cup I_2) \subseteq R(I_1) \cup R(I_2)$$

which yields at once also $R(I_1 \cup I_2) = R(I_1) \cup R(I_2)$, indeed.

This completes the proof of Theorem 4.

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