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ON THE DUALITY OF RADICAL AND SEMI-SIMPLE OBJECTS IN CATEGORIES

By

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§ 1.

A general theory of radicals and semi-simple objects in categories were studied in the papers of LIVŠIC [6], ŠULGEIFER [9], [10], RJABUHIN [7] and DICKSON [4], respectively.* In this note we lay stress on the duality between the concept of radical-ideals and that of semi-simple normal factorobjects. For this aim radical classes and semi-simple classes are defined axiomatically. In the categories of rings and groups, respectively, a radical class R defines a semi-simple class R^* which consists of all objects having zero R-radical. Moreover, a semi-simple class determines a radical class which consists of all objects having zero semi-simple images. Under certain (rather natural) conditions, we shall prove that a semi-simple class determines a radical class, however, we could not prove that to a radical class there belongs a semi-simple class (defined in the previous manner). The proof of the analogous statement for rings (cf. ANDERSON—DIVINSKY—SULIŃSKI [2]) makes strongly use of the operations defined on the ring. One could conjecture that generally a radical class does not determine a semi-simple class, further, that radical classes and semi-simple classes are dual, however, not equivalent classes (for the considered category is not selfdual).

Supposing a one-to-one correspondence between radical and semi-simple classes, we prove an intersection representation of radical ideals which were defined as a union of certain ideals. At last, applying Theorem 1 and 1^* of [11] we obtain structure theorems for objects belonging to a hereditary radical class and semi-simple class, respectively.

§ 2.

In this paper we adopt the notions and notations of the preceding paper [11], *and we assume that the considered categories satisfy all of the axioms* (C_1)—(C_{10}). In addition, we need also categories in which every epimorphism is a normal one. For such categories the so-called Isomorphism Theorems are valid. To formulate them we remark that in such a category for any map $\alpha: a \to b$ and for any ideal (m, μ) of b there exists a complete counterimage (d, δ) of (m, μ) by α; the complete

* *Added in proof (5 September 1968).* In August 1968 there appeared RJABUHIN's paper "Radicals in categories (Russian), *Mat. Issl. (Kishinev)*, **2** (1967), pp. 107—165" where, among others, similar investigations are made to those of § 3.

counterimage (d, δ) means such an ideal of a for which

$$\begin{array}{ccccc} k & \longrightarrow & d & \xrightarrow{\nu} & m \\ & \searrow^{\varkappa} & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\mu} \\ & & a & \xrightarrow{\alpha} & b \end{array}$$

is a commutative diagram, where ν is a (normal) epimorphism and $(k, \varkappa) = \operatorname{Ker} \alpha$ (cf. ŠULGEIFER [9] or SULIŃSKI [8]). A sequence $a \xrightarrow{\alpha} b \xrightarrow{\beta} c$ is called exact, if the normal image of α is just the kernel of β. By an exact diagram we understand a diagram consisting of exact rows and columns.

FIRST ISOMORPHISM THEOREM. *Let $(k, \varkappa)$ and (m, μ) be ideals of an object a and b, respectively, and let*

$$0 \longrightarrow k \xrightarrow{\varkappa} a \xrightarrow{\alpha} b \longrightarrow 0$$

be an exact sequence. Denote by (d, δ) the complete counterimage of (m, μ) by the epimorphism α. Then there are maps β and γ such that

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & k & \to & d & \to & m & \to & 0 \\ & & & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\mu} & & \\ 0 & \to & k & \to & a & \to & b & \to & 0 \\ & & & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} & & \\ & & 0 & \to & c & \to & c & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

is an exact commutative diagram.

SECOND ISOMORPHISM THEOREM. *Let $(k, \varkappa)$, (d_1, δ_1) and (d_2, δ_2) be ideals of an object $a \in C$ such that*

$$(k, \varkappa) = (d_1, \delta_1) \cap (d_2, \delta_2),$$

$$(a, \varepsilon_a) = (d_1, \delta_1) \cup (d_2, \delta_2)$$

hold. If

$$0 \to k \to d_1 \to b_1 \to 0$$

$$0 \to d_2 \to a \to b_2 \to 0$$

are exact sequences, then the diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & k & \to & d_1 & \to & b_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & d_2 & \to & a & \to & b_2 & \to & 0 \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

is exact and commutative (i.e. b_1 and b_2 are equivalent objects).

For these theorems we refer to [8], [12] or [13].

§ 3.

Let us consider a class R of objects of a category C satisfying

(a) *If* $a \in R$ *and* $\alpha : a \to b$ *is a normal epimorphism, then* $b \in R$;

(b) *For each object* $a \in C$, *the union of all ideals* $(k, \varkappa)$ *with* $k \in R$, *belongs to* R; *this union will be called the R-radical of a and will be denoted by* R-rad a;

(c) *If* $\alpha : a \to b$ *is a normal epimorphism with* Ker $\alpha = R$-rad a, *then* R-rad $b = (0, \omega)$ *holds.*

Such a class R will be called a *radical-class, the objects belonging to R are called R-radical objects.*

An *R-ideal* of an object a shall mean an ideal $(k, \varkappa)$ with $k \in R$. According to (b) R-rad a is the union of all R-ideals of a. Since $\omega : a \to 0$ is a normal epimorphism, so (a) implies $0 \in R$.

The dual class of R leads to the notion of semi-simple class. Let S be a class of objects of satisfying

(a*) *If* $a \in S$ *and* $\alpha : b \to a$ *is a normal monomorphism, then* $b \in S$;

(b*) *For each object* $a \in C$ *the union of all normal factorobjects* (λ, l) *with* $l \in S$, *belongs to* S; *this union will be called the S-semi-simple image of a, and will be denoted by* S-ses a.

(c*) *If* $\alpha : b \to a$ *is a normal monomorphism with Coker* $\alpha = S$-ses a, *then* S-ses $b = (\omega, 0)$ *holds.*

We call such a class S a *semi-simple class*, and the objects belonging to S are the *S-semi-simple objects*. By an *S-normal factorobject* we understand a normal factorobject (λ, l) with $l \in S$.

Let R be a radical class, and consider the class R^* consisting of all objects $a \in C$ whose R-radical is a zero object. Similarly, for a semi-simple class S, let S^* denote the class of all objects $a \in C$, whose S-semi-simple image is a zero object. Obviously both of $R \cap R^*$ and $S \cap S^*$ consist only from the zero objects.

THEOREM 1. *Assume that in the category C the product of two normal epimorphism is a normal one.* If S is a semi-simple class of objects of C, then the class* $S^* = \{a \in C \mid S\text{-ses } a = (\omega, 0)\}$ *forms a radical class.*

PROOF. Let a be an arbitrary element of S^*, and $\alpha : a \to b$ a (normal) epimorphism. Suppose $b \notin S^*$, i.e. S-ses $b = (\lambda, l) \neq (\omega, 0)$. Now $(\alpha\lambda, l)$ is an S-normal factorobject of a, and therefore we obtain the contradiction S-ses $a \neq (\omega, 0)$. Hence the class S^* satisfies condition (a).

Let a be an arbitrary element of C and consider all ideals $(k_i, \varkappa_i)$, $i \in I$ of a with $k_i \in S^*$. Denote the union $\bigcup_{i \in I} (k_i, \varkappa_i)$ by $(k, \varkappa)$. We shall show $k \in S^*$. Assume $k \notin S^*$. This implies S-ses $k = (\lambda, l) \neq (\omega, 0)$, and so Ker $\lambda = (d, \delta)$ differs from (k, ε_k). Thus for $\delta_0 = \delta\varkappa$ we have $(d, \delta_0) < \bigcup_{i \in I} (k_i, \varkappa_i) = (k, \varkappa)$, therefore there exists an

* This condition is satisfied, for instance, if every map has a normal image.

index $j_0 \in I$ with $(k_{j_0}, \varkappa_{j_0}) \not\leqq (d, \delta)$. Making use of the Second Isomorphism Theorem for

$$(r, \varrho) = (k_j, \varkappa_j) \cap (d, \delta_0),$$
$$(s, \sigma) = (k_j, \varkappa_j) \cup (d, \delta_0)$$

we obtain an exact commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & r & \to & k_j & \to & b & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & d & \to & s & \to & b & \to & 0 \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

According to (a) (proved already for S^*), from $k_j \in S^*$ it follows $b \in S^*$, so the First Isomorphism Theorem yields the exact commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & d & \to & s & \to & b & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & d & \xrightarrow{\delta} & k & \xrightarrow{\lambda} & l & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & 0 & \to & c & \to & c & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Since $l \in S$ holds, so by condition (a^*) $b \in S$ follows. Thus $b \in S \cap S^*$ is valid and so $(d, \delta_0) = (s, \sigma)$, further $(k_j, \varkappa_j) \leqq (d, \delta_0)$ follows which is a contradiction. Hence the class S^* fulfills condition (b).

At last we are going to prove the validity of condition (c) for the class S^*. Again, let a denote an arbitrary element of C and consider the union $(k, \varkappa) = \bigcup_{i \in I} (k_i, \varkappa_i)$ of all ideals of a with $k_i \in S^*$. We have to prove that for Coker $\varkappa = (\lambda, l)$ the object l has no non-zero ideal (d, δ) with $d \in S^*$. In the contrary, assume that there exists an ideal $(d, \delta) \neq (0, \omega)$ of l with $d \in S^*$. Let (c, γ) denote the complete counterimage of (d, δ) by $\lambda : a \to l$. Obviously $(k, \varkappa) < (c, \gamma)$ holds, and so we get $c \notin S^*$, i. e. S-ses $c = (\sigma, s) \neq (\omega, 0)$. Consider Ker $\sigma = (r, \varrho)$ and the ideal $(k, \varkappa_1)$ of c $(\varkappa_1 \gamma = \varkappa)$, moreover, the ideals

$$(*) \qquad \begin{aligned} (k, \varkappa_1) \cap (r, \varrho) &= (q, \vartheta) \\ (k, \varkappa_1) \cup (r, \varrho) &= (t, \tau). \end{aligned}$$

Let (m, μ) denote the image of (t, τ) by σ. Now we have the exact commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & r & \to & t & \to & m & \to & 0 \\ & & & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\mu} & & \\ 0 & \to & r & \xrightarrow{\varrho} & c & \xrightarrow{\sigma} & s & \to & 0 \end{array}$$

and by (C_9) of [11] (m, μ) is an ideal of s. Hence from (a*) and $s \in S$ it follows $m \in S$. On the other hand the Second Isomorphism Theorem yields that

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & q & \to & k & \to & m & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & r & \to & t & \to & m & \to & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

is an exact commutative diagram. Thus making use of (a) for the (normal) epimorphism ν, the relation $k \in S^*$ implies $m \in S^*$. Hence we obtain $m \in S \cap S^* = 0$ and $(q, \vartheta) = (k, \varkappa_1)$. Hence $(\ast)$ yields $(k, \varkappa_1) \leqq (r, \varrho)$, so by the First Isomorphism Theorem we get the following exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & k & \to & r & \to & m & \to & 0 \\
 & & & & \downarrow{\scriptstyle \varrho} & & \downarrow & & \\
0 & \to & k & \overset{\varkappa_1}{\to} & c & \to & d & \to & 0 \\
 & & & & \downarrow{\scriptstyle \sigma} & & \downarrow & & \\
 & & 0 & \to & s & \to & s & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $d \in S^*$, so condition (a) implies also $s \in S^*$. Thus we have established $s \in S \cap S^* = 0$, in contradiction to the definition of s. Thus the theorem is proved.

REMARK. If C_R denotes the category of (associative) rings, then for any radical class R *the class* $R^* = \{a \in C_R \mid R\text{-rad } a = (0, \omega)\}$ *is a semi-simple class*. In ANDERSON—DIVINSKY—SULIŃSKI [2] it is proved that for any radical class R, the class R^* has property (a*) in the category of associative rings and alternative rings, as well.

In the proof of this statement in [2], the operations defined on the ring, play an important rôle, so it seems to be rather difficult to obtain a dual statement* of that of Theorem 1 (of course, with the assumption of Theorem 1, and without the assumption that the product of two normal monomorphisms is a normal one). In the first part of AMITSUR [1] it is shown that any general radical R-rad A of a ring A coincides with the intersection of all ideals I_i for which R-rad $(A/I_i) = 0$. This means that (b*) is satisfied. (c*) follows almost trivially from the definition of R and R^*.

Let us mention that the same holds also for the category C_G of groups (cf. KUROŠ [5]).

* Recently E. P. ARMENDARIZ and W. G. LEAVITT have shown that in the category of all rings not every class R^* satisfies property (a*) (*Proc. Amer. Math. Soc.*, **18** (1967), pp. 1114—1117).

§ 4.

In what follows we omit the assumption that the product of two normal epimorphism is a normal one, but we suppose that

(C_{11}) *R^* is a semi-simple class and S^* is a radical class for any radical class R and semi-simple class S.*

THEOREM 2. *If R is an arbitrary radical class then $R^{**}=R$.*

PROOF. First, suppose $a\in R$. According to (b^*), R^*-ses $a=(\lambda, l)$ is a normal factorobject with $l\in R^*$, i.e. R-rad $l=(0,\omega)$. On the other hand for the normal epimorphism λ condition (a) implies $l\in R$. Hence we have $(0,\omega)=R$-rad $l=(l_1\,\varepsilon_l)$. Thus R^*-ses $a=(\omega,0)$, i.e. $a\in R^{**}$.

Conversely, let $a\notin R$, then R-rad $a=(k,\varkappa)$ is a proper ideal of a. Put $(\lambda, l)=$ $=\operatorname{Coker}\varkappa$. $(k,\varkappa)<(a,\varepsilon_a)$ implies $(\lambda,l)>(\omega,0)$. By (c) we have R-rad $l=(\omega,0)$ i.e. $l\in R^*$. Thus (b^*) implies R^*-ses $a\geqq(\lambda,l)>(\omega,0)$. Hence $a\notin R^{**}$.

THEOREM 2^*. *If S is an arbitrary semi-simple class, then $S=S^{**}$.*

By definition, the R-radical $(k_0,\varkappa_0)$ of an object $a\in C$ is the union $\bigcup_{k\in R}(k,\varkappa)$ of all R-ideals of a. The following theorem gives an intersection representation of the R-radical. To formulate this, we shall call an ideal (d,δ) of an object $a\in C$ an *R^*-ideal*, if $\operatorname{Coker}\delta=(\lambda,l)$ is an R^*-normal factorobject (i.e. $l\in R^*$). Moreover, denote the R-radical and R^*-semi-simple image of a by $(k_0,\varkappa_0)$ and (λ_0,l_0), respectively. By Proposition 2 of [11] we obtain that $(d_0,\delta_0)=\operatorname{Ker}\lambda_0$ is the intersection of all R^*-ideals of a.

THEOREM 3. *The intersection of all R^*-ideals of $a\in C$ is equivalent to R-rad a, i.e. $(d_0,\delta_0)=(k_0,\varkappa_0)$.*

PROOF. Consider $\operatorname{Coker}\varkappa_0=(\beta,b)$. By condition (c) R-rad $b=(0,\omega)$, and so $b\in R^*$ holds. Therefore $(k_0,\varkappa_0)$ is an R^*-ideal and this implies

$$(d_0,\delta_0)\leqq(k_0,\varkappa_0).$$

According to (C_8) the map $\varkappa_0\lambda_0$ has an image, so we get the commutative diagram

$$\begin{array}{ccccc} & & k_0 & \xrightarrow{\mu} & m \\ & {\scriptstyle\varkappa_1}\nearrow & \downarrow{\scriptstyle\varkappa_0} & & \downarrow{\scriptstyle\nu} \\ d_0 & \xrightarrow{\delta_0} & a & \xrightarrow{\lambda_0} & l_0 \\ & & \downarrow{\scriptstyle\beta} & & \\ & & b & & \end{array}$$

where (m,ν) is the image of $\varkappa_0\lambda_0$ and by (C_9) ν is a normal monomorphism, and $(\beta,b)=\operatorname{Coker}\varkappa_0$. Since ν is a normal monomorphism, so (a^*) implies $m\in R^*$. On the other hand, (C_9) and $k_0\in R$ and (a) imply $(\mu,m)=(\omega,0)$. Since $(d_0,\delta_0)=$ $=\operatorname{Ker}\lambda_0$ and $\varkappa_0\lambda_0=\mu\nu=\omega$, so there exists a map $\varkappa_2: k_0\to d_0$ such that $\varkappa_2\delta_0=\varkappa_0$, and $\varkappa_2$ has to be a monomorphism. Therefore $(k_0,\varkappa_0)\leqq(d_0,\delta_0)$ is valid. Thus the theorem is proved.

Again, let $a \in C$ be an object with S-ses $a = (\lambda_0, l_0)$ and S^*-rad $a = (k_0, \varkappa_0)$. An S^*-normal factorobject (β', b') of $a \in C$ will mean a normal factorobject, whose kernel Ker $\beta' = (k, \varkappa)$ is an S^*-ideal i.e. $k \in S^*$. Denote by (β_0, b_0) the intersection of all S^*-normal factorobjects of a. By Proposition 2 of [11] $(\beta_0, b_0) = \text{Coker } \varkappa_0$ holds. Now the dual statement of that of Theorem 3 establishes the following

THEOREM 3*. *For the intersection (β_0, b_0) of all S^*-normal factorobjects $(\beta_0\ b_0) = (\lambda_0, l_0)$ is valid.*

Let us mention that in view of Proposition 2 of [11], Theorems 3 and 3* are equivalent statements.

We say that the semi-simple class S is *hereditary* if it satisfies

(d*) *For any object $a \in S$ and normal epimorphism $\alpha: a \to b$ it follows $b \in S$.*

Hereditary semi-simplicity is sometimes called strongly semi-simplicity (cf. ANDRUNAKIEVIČ [3]). For such a class S Theorem 1 of [11] imples immediately

THEOREM 4. *Let S be a hereditary semi-simple class. Any object $a \in S$ whose ideal-lattice L_a is compactly generated, can be subdirectly embedded in a direct product of S-semi-simple objects, moreover, any direct factor is subdirectly irreducible if and only if condition* (I) *of* [11] *is fulfilled.*

Dualizing, a radical class R is said to be *hereditary*, if

(d) *For any object $a \in R$ and normal monomorphism $\alpha: b \to a$ it follows $b \in R$.*

For hereditary radicals we obtain

THEOREM 4*. *Let R be a hereditary radical class. Any R-radical object a whose ideal-lattice L_a is co-compactly generated, is a transfree image of a free product of R-radical objects a_i, moreover, any free factor a_i is transfreely irreducible, if and only if condition* (I*) *of* [11] *is fulfilled.*

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