Bounds on the Game Transversal Number in Hypergraphs

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Abstract

Let $H = (V, E)$ be a hypergraph with vertex set V and edge set E of order $n_{_H} = |V|$ and size $m_{_H} = |E|$. A transversal in H is a subset of vertices in H that has a nonempty intersection with every edge of H. A vertex hits an edge if it belongs to that edge. The transversal game played on H involves of two players, *Edge-hitter* and *Staller*, who take turns choosing a vertex from H. Each vertex chosen must hit at least one edge not hit by the vertices previously chosen. The game ends when the set of vertices chosen becomes a transversal in H. Edge-hitter wishes to minimize the number of vertices chosen in the game, while Staller wishes to maximize it. The *game transversal number*, $\tau_g(H)$, of H is the number of vertices chosen when Edge-hitter starts the game and both players play optimally. We compare the game transversal number of a hypergraph with its transversal number, and also present an important fact concerning the monotonicity of τ_g, that we call the Transversal Continuation Principle. It is known that if H is a hypergraph with all edges of size at least 2, and H is not a 4-cycle, then $\tau_g(H) \le \frac{4}{11}(n_{_H} + m_{_H})$; and if H is a (loopless) graph, then $\tau_g(H) \le \frac{1}{3}(n_{_H} + m_{_H} + 1)$. We prove that if H is a 3-uniform hypergraph, then $\tau_g(H) \le \frac{5}{16}(n_{_H} + m_{_H})$, and if H is 4-uniform, then $\tau_g(H) \le \frac{71}{252}(n_{_H} + m_{_H})$.

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1 Introduction

In this paper, we continue the study of the transversal game in hypergraphs which was first investigated in [7]. The results obtained there implied the proof of the $\frac{3}{4}$-Game Total Domination Conjecture, which was posted by Henning, Klavžar and Rall [19], over the class of graphs with minimum degree at least 2.

Hypergraphs are systems of sets which are conceived as natural extensions of graphs. A *hypergraph* $H = (V(H), E(H))$ is a finite set $V(H)$ of elements, called *vertices*, together with a finite multiset $E(H)$ of nonempty subsets of $V(H)$, called *hyperedges* or simply *edges*. If the hypergraph H is clear from the context, we may write $V = V(H)$ and $E = E(H)$. We shall use the notation $n_{_H} = |V(H)|$ and $m_{_H} = |E(H)|$, and sometimes just n and m without subscript if the actual H need not be emphasized, to denote the order and the size of H, respectively. We say that two edges in H *overlap* if they intersect in at least two vertices. A hypergraph is *linear* if it has no overlapping edges.

A *k-edge* in H is an edge of cardinality k. The hypergraph H is said to be *k-uniform* if every edge of H is a k-edge. Every loopless graph is a 2-uniform hypergraph. Thus graphs are special hypergraphs. The *degree* of a vertex v in H, denoted by $d_H(v)$, is the number of edges of H which contain v. The maximum degree among the vertices of H is denoted by $\Delta(H)$.

Two vertices x and y of H are *adjacent* if there is an edge e of H such that $\{x, y\} \subseteq e$. The *neighborhood* of a vertex v in H, denoted $N_H(v)$ or simply $N(v)$ if H is clear from the context, is the set of all vertices different from v that are adjacent to v. A vertex in $N(v)$ is a *neighbor* of v.

A subset T of vertices in a hypergraph H is a *transversal* (also called *hitting set* or *vertex cover* or *blocking set* in many papers) if T has a nonempty intersection with every edge of H. A vertex *hits* or *covers* an edge if it belongs to that edge. The *transversal number* $\tau(H)$ of H is the minimum size of a transversal in H. In hypergraph theory the concept of transversal is fundamental and well studied. The major monograph [1] of hypergraph theory gives a detailed introduction to this topic. We refer to [6, 8, 14, 20, 21, 22, 23, 27] for recent results and further references.

The Game Transversal Number. The transversal game belongs to the growing family of *competitive optimization* graph and hypergraph games. Competitive optimization variants of coloring [2, 13, 15, 24, 25, 30], list-colouring [4, 28, 31], matching [12], domination [5, 26], total domination [18, 19], disjoint domination [9], Ramsey theory [10, 16, 17], and more [3] have been extensively investigated.

The transversal game played on a hypergraph H involves two players, *Edge-hitter* and *Staller*, who take turns choosing a vertex from H. Each vertex chosen must hit at least one edge not hit by the vertices previously chosen. We call such a chosen vertex a *legal move* in the transversal game. The game ends when the set of vertices chosen becomes a transversal in H. Edge-hitter wishes to end the game with the smallest possible number of vertices chosen, and Staller wishes to end the game with as many vertices chosen as possible. The *game transversal number* (resp. *Staller-start game transversal number*), $\tau_g(H)$ (resp. $\tau_g'(H)$), of H is the number of vertices chosen when Edge-hitter (resp. Staller) starts the game and both players play optimally.

A *partially covered hypergraph* is a hypergraph together with a declaration that some edges are already covered; that is, they need not be covered in the rest of the game. Once an edge has been covered, it plays no role in the remainder of the game and can be deleted from the partially covered hypergraph, as can all isolated vertices. Therefore, after those deletions we obtain a hypergraph being equivalent, from the transversal game viewpoint, to the partially covered hypergraph from which it has been derived; we call it a *residual hypergraph*. We will also say that the original hypergraph H, before any move has been made in the game, is a residual (and also partially covered) hypergraph.

Given a hypergraph H and a subset S of edges of H, we denote by $H|S$ the residual hypergraph[1] in which the edges contained in S do not appear anymore. We use $\tau_g(H|S)$ (resp. $\tau_g'(H|S)$) to denote the number of turns remaining in the transversal game on $H|S$ under optimal play when Edge-hitter (resp. Staller) has the next turn.

We will use the standard notation $[k] = \{1, \ldots, k\}$.

2 Known Results

Let H_1 be the hypergraph with vertex set $V(H_1) = \{x_1, x_2, x_3, y_1, y_2, y_4\}$ and edge set $E(H_1) = \{\{x_1, x_2, x_3\}, \{y_1, y_2, y_3\}, \{x_1, y_1\}, \{x_2, y_2\}, \{x_3, y_3\}\}$. For $k \geq 1$, let H_k consist of k vertex-disjoint copies of H_1, and let $\mathcal{H} = \{H_k \colon k \geq 1\}$. The hypergraph $H_3 \in \mathcal{H}$ is illustrated in Figure 1.

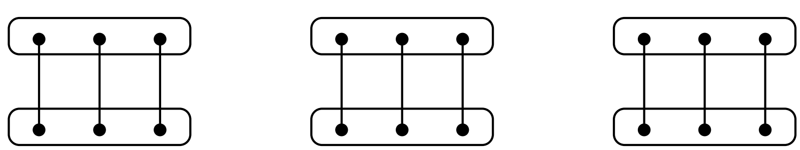

Figure 1: The hypergraph H_3 from the family $\mathcal{H}$.

The following upper bound on the game transversal number of a hypergraph is established in [7].

Theorem 1 ([7]) *If H is a hypergraph with all edges of size at least 2, and $H \ncong C_4$, then $\tau_g(H) \leq \frac{4}{11}(n_H + m_H)$, with equality if and only if $H \in \mathcal{H}$.*

As a special case of more general results due to Tuza [29] and Chvátal and McDiarmid [11],

[1]In the context of games we prefer to use the notation $H|S$, although its edge set coincides with that of the hypergraph denoted by $H - S$ in many hypergraph-theoretic papers.

if H is a simple graph, then $\tau(H) \le \frac{1}{3}(n_H + m_H)$. This bound is almost true for the game transversal number, as proved in [7].

Theorem 2 ([7]) *If H is a 2-uniform hypergraph, then $\tau_g(H) \le \frac{1}{3}(n_H + m_H + 1)$.*

3 Main Results

Since the transversal game played in a hypergraph H ends when the set of vertices chosen becomes a transversal in H, it is clear that $\tau(H) \le \tau_g(H)$ and $\tau(H) \le \tau_g'(H)$. If Edge-hitter fixes a minimum transversal set, T, in H and adopts the strategy in each of his turns to play a vertex from T if possible, then he guarantees that the game ends in no more than $2\tau(H) - 1$ moves in the Edge-hitter-start transversal game and in no more than $2\tau(H)$ moves in the Staller-start transversal game. We state this fact formally as follows.

Observation 1 *For every hypergraph H, the following holds.*
(a) $\tau(H) \le \tau_g(H) \le 2\tau(H) - 1$.
(b) $\tau(H) \le \tau_g'(H) \le 2\tau(H)$.

It is easy to see that the equalities $\tau(H) = \tau_g(H) = \tau_g'(H)$ hold if H is the disjoint union of complete k-uniform hypergraphs. Further, $\tau_g(H) = 2\tau(H) - 1$ and $\tau_g'(H) = 2\tau(H)$ are valid if $\tau(H) = 1$ and H contains at least two different edges. In Section 4 we present an infinite family of hypergraphs H with $\tau_g(H) = 2\tau(H) - 1$ and $\tau_g'(H) = 2\tau(H)$. These show that the lower and upper bounds given in Observation 1 cannot be improved even when $\tau(H)$ is large.

We next present a simple but fundamental and widely applicable lemma, named the *Transversal Continuation Principle*, that expresses the monotonicity of τ_g and τ_g' with respect to subhypergraphs. Its proof is given in Section 5.

Lemma 3 (Transversal Continuation Principle) *Let H be a hypergraph and let $A, B \subseteq E(H)$. If $B \subseteq A$, then $\tau_g(H|A) \le \tau_g(H|B)$ and $\tau_g'(H|A) \le \tau_g'(H|B)$.*

Let us mention without quoting the formal definitions that in any graph G the dominating sets are in one-to-one correspondence with the transversals of the hypergraph whose edges are the closed neighborhoods of the vertices in G; and similarly, the total dominating sets in a graph without isolated vertices are in one-to-one correspondence with the transversals of the open neighborhoods of the vertices. (These facts are immediate by definition.) In this way our Transversal Continuation Principle includes, as particular cases, the assertions called 'Continuation Principle' for the domination game in [26] and for the total domination game in [18], hence putting them on a higher level of generality.

As another consequence of the Transversal Continuation Principle, the number of moves in the Edge-hitter-start transversal game and the Staller-start transversal game when played optimally can differ by at most one. We state this formally as follows.

Theorem 4 *For every hypergraph H, we have $|\tau_g(H) - \tau_g'(H)| \le 1$.*

We remark that the hypergraphs that achieve equality in the bound of Theorem 1, namely the hypergraphs that belong to the family $\mathcal{H}$, contain both 2-edges and 3-edges. Our two main results in this paper show that the upper bound of Theorem 1 can be improved for 3-uniform and 4-uniform hypergraphs as follows.

Theorem 5 *If H is a 3-uniform hypergraph, then $\tau_g(H) \le \frac{5}{16}(n_H + m_H)$.*

Theorem 6 *If H is a 4-uniform hypergraph, then $\tau_g(H) \le \frac{71}{252}(n_H + m_H)$.*

Proofs of Theorem 4, Theorem 5 and Theorem 6 are given in Section 5, Section 6.1 and Section 6.2, respectively.

4 Family of Hypergraphs

By Observation 1, every hypergraph H satisfies $\tau_g(H) \le 2\tau(H) - 1$ and $\tau_g'(H) \le 2\tau(H)$. In this section, we present an infinite family of hypergraphs H with $\tau_g(H) = 2\tau(H) - 1$ and $\tau_g'(H) = 2\tau(H)$. For this purpose, we define a *k-corona* of a hypergraph H to be a hypergraph obtained by attaching k hyperedges (each of size at least 2) to each vertex of H, where the hyperedges attached to a vertex $v \in V(H)$ contain only degree-1 vertices apart from v.

Proposition 1 *For every positive integer k and for every hypergraph H of order at most $2^{k-1} - 1$, every k-corona H^k of H satisfies $\tau_g(H^k) = 2\tau(H^k) - 1$ and $\tau_g'(H^k) = 2\tau(H^k)$.*

Proof. Let us denote the vertices of H by $v_1, \ldots, v_n$ and the hyperedges attached to v_i by $e(1,i), \ldots, e(k,i)$. By our assumption, $n < 2^{k-1}$. Since $\tau(H^k) = n$, the inequalities $\tau_g(H^k) \le 2n - 1$ and $\tau_g'(H^k) \le 2n$ are valid by Observation 1. Therefore, it suffices to prove that Staller has a strategy to achieve at least $2n - 1$ turns if Edge-hitter starts the game, and at least $2n$ turns if Staller starts.

First, we associate a weight $\mathrm{w}(e)$ with each hyperedge e of H^k as follows. If e is a hyperedge of H, then we let $\mathrm{w}(e) = 0$. If $e = e(j,i)$ is a hyperedge attached to H for some $j \in [k]$ and $i \in [n]$, then we let $\mathrm{w}(e) = 2^{j-1}$. As the game is played, when a hyperedge is hit by a played vertex, the weight of such a hyperedge becomes zero. Hence, if $\mathrm{w}(H^k)$ denotes the sum of the weights of the edges in the residual hypergraph H^k, then the game starts with

$$\mathrm{w}(H^k) = \sum_{i=1}^{n} \sum_{j=1}^{k} 2^{j-1} = n(2^k - 1),$$

and is completed when $\mathrm{w}(H^k) = 0$; that is, the game is completed when the sum of the weights of the edges in the residual hypergraph H^k equals zero. We consider the following strategy of Staller.

Staller's Rule: She always plays a vertex of degree 1 such that the incident hyperedge has the smallest positive weight in the residual hypergraph.

We show that if Staller applies this rule, each of her moves together with the next move of Edge-hitter decreases the weight by at most 2^k. If Staller plays a vertex incident to an attached hyperedge of weight 2^s for some $s \in [k-1] \cup \{0\}$, then in the next turn Edge-hitter cannot choose a vertex which is incident to a hyperedge of smaller positive weight. Moreover, no single vertex of H^k is incident with two hyperedges of the same positive weight. Hence, Edge-hitter's move decreases the weight of the residual hypergraph by at most

$$\sum_{i=s}^{k-1} 2^i = 2^k - 2^s \,,$$

and, together with Staller's previous move which decreases the weight by 2^s, their two moves combined decrease the weight by at most 2^k.

If Edge-hitter begins the game, his first move decreases the weight of the residual hypergraph by at most

$$\sum_{i=1}^{k} 2^{i-1} = 2^k - 1,$$

while, if the weight of the residual hypergraph is not zero after Edge-hitter plays his $(n-1)$st move (that is, after the $(2n-3)$rd turn), then Staller's $(n-1)$st move in the $(2n-2)$nd turn decreases the weight by at most 2^{k-1}. Therefore, the weight of the residual hypergraph after the $(2n-2)$nd turn is at least

$$n(2^k-1) - (2^k-1) - (n-2)2^k - 2^{k-1} = 2^{k-1} - n + 1 \,,$$

which is at least 2, as we supposed $n \le 2^{k-1} - 1$. Since the obtained hypergraph has still positive weight, there exist some uncovered edges. Thus, Staller has a strategy which makes sure that the game is not complete after the $(2n-2)$nd turn, implying that $\tau_g(H^k) \ge 2n-1$. Consequently, $\tau_g(H^k) = 2n-1 = 2\tau(H^k) - 1$.

Similarly, if Staller begins the game, then after her nth move played in the $(2n-1)$st turn, the weight of the residual hypergraph is at least

$$n(2^k-1) - (n-1)2^k - 2^{k-1} = 2^{k-1} - n \ge 1 \,.$$

Thus, Staller has a strategy which guarantees that the length of the game is at least $2n$, implying that $\tau_g'(H^k) \ge 2n$. Therefore, $\tau_g'(H^k) = 2n = 2\tau(H^k)$. $\square$

5 The Transversal Continuation Principle

In this section, we present a proof of the Transversal Continuation Principle. Recall its statement.

Lemma 3 (Transversal Continuation Principle). *Let H be a hypergraph and let $A, B \subseteq E(H)$. If $B \subseteq A$, then $\tau_g(H|A) \le \tau_g(H|B)$ and $\tau'_g(H|A) \le \tau'_g(H|B)$.*

Proof. Two games will be played, Game A on the hypergraph $H|A$ and Game B on the hypergraph $H|B$. The first of these will be the real game, while Game B will only be imagined by Edge-hitter. In Game A, Staller will play optimally while in Game B, Edge-hitter will play optimally.

We claim, by induction on the number of moves played, that in each stage of the games, the set of edges that are covered in Game B is a subset of the edges that are covered in Game A. Since $B \subseteq A$, this is true at the start of the games. Suppose now that Staller has (optimally) selected vertex u in Game A. This move of Staller hits at least one new edge, say e_u, in Game A. By the induction assumption, the edge e_u is not yet hit in Game B, and so the vertex u is a legal move in Game B. Edge-hitter now copies the move of Staller and plays vertex u in Game B, and then replies with an optimal move in Game B. If this move is legal in Game A, Edge-hitter plays it in Game A as well. Otherwise, if the game is not yet over, Edge-hitter plays any other legal move in Game A. In both cases the claim assumption is preserved, which by induction also proves the claim.

We have thus proved that Game A finishes no later than Game B. Suppose thus that k moves are played in Game B. Since Edge-hitter was playing optimally in Game B, $k \le \tau_g(H|B)$. On the other hand, because Staller was playing optimally in Game A and Edge-hitter has a strategy to finish the game in k moves, $\tau_g(H|A) \le k$. Therefore, $\tau_g(H|A) \le k \le \tau_g(H|B)$. Thus, if Edge-hitter is the first to play, the desired bound holds. In the above arguments we did not assume who starts first, hence in both cases Game A will finish no later than Game B, implying that $\tau'_g(H|A) \le \tau'_g(H|B)$. □

If two vertices are incident with precisely the same edges, then at most one of them can be played during the game. Now, assume that in the residual hypergraph H, vertex v hits all the edges that u hits, but $d_H(v) > d_H(u)$. As a consequence of the Transversal Continuation Principle, we may suppose that Edge-hitter never plays u and Staller never plays v.

Theorem 4 follows from the Transversal Continuation Principle. Recall its statement.

Theorem 4. *For every hypergraph H, we have $|\tau_g(H) - \tau'_g(H)| \le 1$.*

Proof. Consider the Edge-hitter-start transversal game and let v be the first move of Edge-hitter. Let A be the set of edges hit by v and let $B = \emptyset$, and consider the partially covered hypergraphs $H|A$ and $H|B$. We note that $H|B = H$ and $\tau_g(H) = 1 + \tau'_g(H|A)$. By the Transversal Continuation Principle, $\tau'_g(H|A) \le \tau'_g(H|B) = \tau'_g(H)$. Therefore, $\tau_g(H) = \tau'_g(H|A) + 1 \le \tau'_g(H) + 1$. Analogously, $\tau'_g(H) \le \tau_g(H) + 1$. □

6 Proof of Theorem 5 and Theorem 6

We remark that if H is a hypergraph, and H' is obtained from H by deleting all multiple edges in H (in the sense that if H has ℓ distinct edges $e_1, e_2, \ldots, e_\ell$ that are multiple edges, and so $e_1 = e_2 = \cdots = e_\ell$, then we delete $\ell - 1$ of these multiple edges), then $\tau_g(H') = \tau_g(H)$. Hence, it suffices to prove Theorem 5 and Theorem 6 in the case of hypergraphs with no multiple edges.

6.1 Proof of Theorem 5

In this section, we prove Theorem 5. For this purpose, we define a *colored hypergraph* with respect to the played vertices in the set D as a hypergraph in which every vertex is colored with one of four colors, namely white, green, blue, or red, according to the following rules.

- A vertex is colored *white* if it is incident with at least 3 edges uncovered by D.
- A vertex is colored *green* if it is incident with exactly 2 edges uncovered by D..
- A vertex is colored *blue* if it is incident with exactly 1 edge uncovered by D.
- A vertex is colored *red* if it is not incident with any edges uncovered by D.

Further, an edge is colored *white* if it is not covered by a vertex of D, and is colored *red* otherwise. Thus, an edge is colored red if it contains a red vertex.

By our definition given in the Introduction, the residual hypergraph does not contain red vertices and red edges. That is, the vertices of the residual hypergraph are colored with white, green and blue as defined above. Note that every edge of the residual hypergraph is white.

In a colored hypergraph, and also in a colored residual hypergraph, we associate a weight of 15 to each white edge and a weight of 0 to each red edge. Further, we associate a weight with every vertex as follows:

Color of vertex	*Degree in the residual hg.*	*Weight of vertex*
white	≥ 3	15
green	2	14
blue	1	11
red	—	0

Table 1. The weights of vertices according to their color.

Let W_H, G_H and B_H denote the set of white, green and blue vertices, respectively, in the residual hypergraph H. We define the *weight* of the residual hypergraph H as the sum of the weights of the vertices and edges in H and denote this weight by $\mathrm{w}(H)$. Thus,

$$\mathrm{w}(H) = 15|W_{_H}| + 14|G_{_H}| + 11|B_{_H}| + 15m_{_H}.$$

We note that as the game is played, if the color status of a vertex changes, then the color status of a green vertex can only change to blue or red, while the color status of a blue vertex can only change to red. We shall prove the following key theorem. From our earlier observations, it suffices for us to prove Theorem 7 in the case of hypergraphs with no multiple edges.

Theorem 7 *If H is a 3-uniform residual hypergraph, then $48\tau_g(H) \le \mathrm{w}(H)$.*

Proof. If $m_{_H} = 0$, then $\tau_g(H) = 0$ and the desired bound is immediate. Hence we may assume that $m_{_H} \ge 1$. We say that Edge-hitter can *achieve a 48-target* if he can play a sequence of moves guaranteeing that on average the weight decrease resulting from each played vertex in the game is at least 48. In order to achieve a 48-target, Edge-hitter must guarantee that a sequence of moves $m_1, \ldots, m_k$ are played, starting with his first move m_1, and with moves alternating between Edge-hitter and Staller such that if w_i denotes the decrease in weight after move m_i is played, then

$$\sum_{i=1}^{k} \mathrm{w}_i \ge 48 \cdot k\,, \tag{1}$$

where either k is odd and the game is completed after move m_k or k is any even number (in this latter case the game may or may not be completed after move m_k). Each played vertex must hit at least one edge not hit by the vertices previously chosen. Thus, every move decreases the weight by at least 26, since every move results in at least one vertex and at least one edge recolored red.

In the discussion that follows, we analyse how Edge-hitter can achieve a 48-target. First of all we note that there is a trivial situation, namely when Edge-hitter can play a vertex that covers all remaining edges, and the current value of the residual hypergraph is at least 48. Then the 48-target is achieved with $k = 1$. This may happen in several cases below. We shall not mention it each time, we only discuss what happens otherwise.

We prove a series of claims that establish important properties that hold in the residual hypergraph H.

Claim 7.A *If $\Delta(H) \ge 4$, then Edge-hitter can achieve a 48-target.*

Proof. Suppose that $d_H(v) \ge 4$. If Edge-hitter plays the vertex v as his move m_1 in the residual hypergraph, this results in $\mathrm{w}_1 \ge 1 \cdot 15 + 4 \cdot 15 = 75 > 48 \cdot 1$ since after the move is played, at least one white vertex and at least four (white) edges are recolored red. Then Staller responds by playing her move m_2 which decreases the weight by $\mathrm{w}_2 \ge 11 + 15 = 26$

since her move results in at least one vertex and at least one edge recolored red. Thus, $\mathrm{w}_1 + \mathrm{w}_2 \ge 75 + 26 = 101 > 48 \cdot 2$, and so Inequality (1) is satisfied with $k = 2$. (□)

By Claim 7.A, we may assume that $\Delta(H) \le 3$, for otherwise Edge-hitter can achieve a 48-target.

Claim 7.B *If Edge-hitter can play a vertex that results in a weight decrease of at least* 68*, then he can achieve a* 48*-target.*

Proof. Suppose that Edge-hitter plays as his move m_1 a vertex in the residual hypergraph H which results in $\mathrm{w}_1 \ge 68 > 48 \cdot 1$. Then Staller responds by playing her move m_2 which decreases the weight by $\mathrm{w}_2 \ge 11 + 15 + 2 \cdot 1 = 28$ since her move results in the vertex she played and at least one edge recolored red, and at least two further vertices changing color. Therefore, $\mathrm{w}_1 + \mathrm{w}_2 \ge 68 + 28 = 96 = 48 \cdot 2$, and so Inequality (1) is satisfied with $k = 2$. (□)

Claim 7.C *If Edge-hitter can play a white vertex v that results in at least one of its neighbors recolored red, then Edge-hitter can achieve a* 48*-target.*

Proof. If Edge-hitter plays the vertex v as his move m_1 in H, this results in $\mathrm{w}_1 \ge 1 \cdot 15 + 3 \cdot 15 + 11 = 71$ since after the move is played, the vertex v and the three (white) edges incident with v are recolored red, while at least one neighbor of v is recolored red. Thus, by Claim 7.B, Edge-hitter can achieve a 48-target. (□)

By Claim 7.C, we may assume that there is no white vertex which, when played, results in at least one of its neighbors recolored red.

Claim 7.D *If there exist two overlapping edges that contain a common white vertex v, then Edge-hitter can achieve a* 48*-target.*

Proof. Since there are no multiple edges, the vertex v has three, four or five neighbors. Edge-hitter plays the vertex v as his move m_1 in the residual hypergraph H. Suppose firstly that $|N(v)| = 3$. By our earlier assumptions, no neighbor of v is recolored red, implying that all three neighbors of v are white vertices in H (of degree 3) and are recolored blue once v is played. Thus, in this case, $\mathrm{w}_1 \ge 1 \cdot 15 + 3 \cdot 15 + 3 \cdot 4 = 72$. Suppose secondly that $|N(v)| = 4$. At least two neighbors of v are recolored blue once v is played, implying that in this case, $\mathrm{w}_1 \ge 1 \cdot 15 + 3 \cdot 15 + 2 \cdot 4 + 2 \cdot 1 = 70$. Suppose thirdly that $|N(v)| = 5$. At least one neighbor of v is recolored blue once v is played, implying that in this case, $\mathrm{w}_1 \ge 1 \cdot 15 + 3 \cdot 15 + 1 \cdot 4 + 4 \cdot 1 = 68$. In all three cases, by Claim 7.B, Edge-hitter can achieve a 48-target. (□)

By Claim 7.D, we may assume that no white vertex belongs to the intersection of two overlapping edges. Recall that by our earlier assumptions, no white vertex which when played results in at least one of its neighbors recolored red. With these assumptions, the three edges that contain a white vertex are pairwise non-overlapping, implying that every white vertex has six neighbors. Further, these six neighbors are colored white or green.

Claim 7.E *If a white vertex v has a green neighbor u, then Edge-hitter can achieve a* 48*-target.*

Proof. By our earlier assumptions, $|N(v)| = 6$. If Edge-hitter plays the vertex v as his move m_1 in H, this results in the green neighbor u recolored blue and five further neighbors recolored. Therefore, $\mathrm{w}_1 \geq 1 \cdot 15 + 3 \cdot 15 + 1 \cdot 3 + 5 \cdot 1 = 68$. Thus, by Claim 7.B, Edge-hitter can achieve a 48-target. (□)

By Claim 7.E, we may assume that every neighbor of a white vertex is colored white. With this assumption, every component of H is one of the following three types:

- *Type-A:* A 3-regular, linear hypergraph.
- *Type-B:* A hypergraph with maximum degree 2.
- *Type-C:* A hypergraph consisting of a single edge.

We remark that a Type-A component of H consists entirely of white vertices, while a type-B component consists only of green and blue vertices, with at least one green vertex. A type-C component consists of three blue vertices. Since a type-B component contains only green and blue vertices, a move played in such a component decreases the weight by at least $11 + 15 + 2 \cdot 3 = 32$, since at least one vertex and one edge is recolored red, and at least two further vertices are recolored. A move played in a Type-C component decreases the weight by $3 \cdot 11 + 1 \cdot 15 = 48$, since three blue vertices and one edge are recolored red. We state this formally as follows.

Claim 7.F *A move played in a type-B component decreases the weight by at least* 32*, while a move played in a Type-C component decreases the weight by* 48*.*

Claim 7.G *If H contains a white vertex, then Edge-hitter can achieve a* 48*-target.*

Proof. Suppose that H contains a white vertex, v, that belongs to a component F. We note that F is a type-A component. Edge-hitter plays the vertex v as his move m_1 in the residual hypergraph H, which results in $\mathrm{w}_1 \geq 1 \cdot 15 + 3 \cdot 15 + 6 \cdot 1 = 66 > 48 \cdot 1$ since after the move is played, the vertex v and three edges are recolored red, while all six neighbors of v are recolored green. Then Staller responds by playing her move m_2. We note that $F - v$ is a linear (possibly disconnected) hypergraph that contains six green vertices with all other vertices colored white. If Staller plays her move m_2 in $F - v$ or in a Type-A component, then $\mathrm{w}_2 \geq 14 + 2 \cdot 15 + 4 \cdot 1 = 48$ since her played vertex (colored either white or green) and at least two edges are recolored red, while at least four further vertices are recolored. If Staller plays her move m_2 in a Type-B component, then, by Claim 7.F, $\mathrm{w}_2 \geq 32$. If Staller plays her move m_2 in a Type-C component, then, by Claim 7.F, $\mathrm{w}_2 \geq 48$. In all cases, $\mathrm{w}_2 \geq 32$. Therefore, $\mathrm{w}_1 + \mathrm{w}_2 \geq 66 + 32 = 98 > 48 \cdot 2$, and so Inequality (1) is satisfied with $k = 2$. (□)

By Claim 7.G, we may assume that every vertex is colored green or blue; that is, every component of H is of Type-B or Type-C. We have seen in Claim 7.F that in this situation Staller can never make a decrease smaller than 32. Thus, we obtain:

Claim 7.H *If Edge-hitter can play a vertex that results in a weight decrease of at least* 64*, then he can achieve a* 48*-target.*

Claim 7.I *If H contains two overlapping edges, then Edge-hitter can achieve a* 48*-target.*

Proof. Let e and f be two overlapping edges, with $e \cap f = \{v_1, v_2\}$. Edge-hitter plays the vertex v_1 as his move m_1 in the residual hypergraph H, which results in $\mathrm{w}_1 \ge 2 \cdot 14 + 2 \cdot 15 + 2 \cdot 3 = 64$ since after the move is played, both vertices v_1 and v_2 (currently colored green) and two edges are recolored red, while at least two further vertices are recolored (from green to blue, or from blue to red). Thus, by Claim 7.H, Edge-hitter can achieve a 48-target. (□)

By Claim 7.I, we may assume that every Type-B component is linear. Thus, H is a linear hypergraph.

Claim 7.J *If H contains a green vertex v with a blue neighbor u, then Edge-hitter can achieve a* 48*-target.*

Proof. Since H is linear, we note that $|N(v)| = 4$. Playing the vertex v results in $\mathrm{w}_1 \ge 1 \cdot 14 + 2 \cdot 15 + 11 + 3 \cdot 3 = 64$, since after the move is played, the vertex v and two edges are recolored red, and u is recolored red. Thus, by Claim 7.H, Edge-hitter can achieve a 48-target. (□)

By Claim 7.J, we may assume that each component of H is either a 2-regular linear hypergraph (consisting entirely of green vertices) or an isolated edge (consisting of three blue vertices). Playing a vertex from an isolated edge decreases the weight by 48, therefore we obtain:

Claim 7.K *If every component in the residual hypergraph is an isolated edge, then Edge-hitter can achieve a* 48*-target.*

By Claim 7.K, we may assume that at least one component, say F, of H is a 2-regular, linear hypergraph. Edge-hitter now plays in such a way as to restrict his moves to vertices in $V(F)$, independently of Staller's responses to his moves, as long as a green vertex in $V(F)$ exists. Further, among all green vertices in $V(F)$ at each stage of the game, Edge-hitter selects a green vertex with as many blue neighbors as possible.

Suppose that a total of ℓ green vertices in $V(F)$ are played by Edge-hitter. The first move, m_1, of Edge-hitter results in $\mathrm{w}_1 = 1 \cdot 14 + 2 \cdot 15 + 4 \cdot 3 = 56$. Thereafter, each subsequent move m_{2i+1} of Edge-hitter, where $i \in \{1, \ldots, \ell - 1\}$, results in $\mathrm{w}_{2i+1} \ge 1 \cdot 14 + 2 \cdot 15 + 1 \cdot 11 + 3 \cdot 3 = 64$, since the subsequent (green) vertices played by Edge-hitter in $V(F)$ can all be chosen to have at least one blue neighbor. By Claim 7.F, each of Staller's moves m_{2i}, where $i \in \{1, \ldots, \ell - 1\}$, result in $\mathrm{w}_{2i} \ge 32$. If the game is complete after Edge-hitter's ℓth move, then

$$\begin{aligned}\sum_{i=1}^{2\ell-1} \mathrm{w}_i &= \sum_{i=0}^{\ell-1} \mathrm{w}_{2i+1} + \sum_{i=1}^{\ell-1} \mathrm{w}_{2i} \\ &\geq 56 + 64(\ell-1) + 32(\ell-1) \\ &> 48 \cdot (2\ell-1).\end{aligned}$$

Thus, Inequality (1) is satisfied with $k = 2\ell - 1$ and the game is completed after move m_k. Hence, we may assume that the game is not complete after Edge-hitter's ℓth move. We show next that Inequality (1) is satisfied with $k = 2\ell$. We consider the sequence of ℓ vertices played by Staller in response to Edge-hitter's ℓ moves. As observed earlier, every move played by Staller decreases the weight by at least 32.

Claim 7.L *If one of Staller's moves in response to Edge-hitter's ℓ moves is not a blue vertex in $V(F)$ with two green neighbors, then Edge-hitter can achieve a 48-target.*

Proof. Suppose that Staller plays a move that is not a blue vertex in $V(F)$ with two green neighbors. We consider the four possible moves of Staller. If at least one of the ℓ vertices played by Staller does not belong to $V(F)$, then her first such played vertex either belongs to a component of H, different from F, that is a 2-regular, linear hypergraph or belongs to a component of H that is an isolated edge. In the former case, her move decreases the weight by 56, while in the latter case, her move decreases the weight by 48. If Staller plays a green vertex in $V(F)$, then her move decreases the weight by at least 56. If Staller plays a blue vertex in $V(F)$ that has at least one blue neighbor, then her move decreases the weight by at least $2 \cdot 11 + 1 \cdot 15 + 3 = 40$. In all four cases, Staller's move decreases the weight by at least 40, while the other $\ell - 1$ moves played by her each decrease the weight by at least 32, implying that

$$\begin{aligned}\sum_{i=1}^{2\ell} \mathrm{w}_i &= \sum_{i=0}^{\ell-1} \mathrm{w}_{2i+1} + \sum_{i=1}^{\ell} \mathrm{w}_{2i} \\ &\geq (56 + 64(\ell-1)) + (40 + 32(\ell-1)) \\ &= 48 \cdot (2\ell).\end{aligned}$$

Thus, Inequality (1) is satisfied with $k = 2\ell$. ($\Box$)

By Claim 7.L, we may assume that each move played by Staller in response to Edge-hitter's ℓ moves is a blue vertex in $V(F)$ with two green neighbors. Thus, each of the ℓ moves of Staller decreases the weight by exactly 32.

Claim 7.M *At least one move played by Edge-hitter is a green vertex in $V(F)$ with at least two blue neighbors.*

Proof. Suppose that none of the ℓ moves played by Edge-hitter is a green vertex in $V(F)$ with at least two blue neighbors. Then, every move of Edge-hitter, except for his first move, plays a green vertex with exactly one blue neighbor and three green neighbors. Thus, the first move of Edge-hitter recolors exactly five green vertices, while each of the subsequent $\ell - 1$ moves of Edge-hitter recolors exactly four green vertices. By our earlier assumption, each move played by Staller in response to Edge-hitter's ℓ moves is a blue vertex in $V(F)$ with two green neighbors. Thus, each move of Staller recolors exactly two green vertices. Therefore, $|V(F)| = 5 + 4(\ell - 1) + 2\ell = 6\ell + 1$. However, F is a 2-regular, 3-uniform hypergraph, and so $m(F) = \frac{2}{3}|V(F)|$, implying that $|V(F)|$ must be divisible by 3, a contradiction. $(\Box)$

By Claim 7.M, at least one of the ℓ moves played by Edge-hitter is a green vertex in $V(F)$ with at least two blue neighbors. Such a move decreases the weight by at least $1 \cdot 14 + 2 \cdot 11 + 2 \cdot 15 + 2 \cdot 3 = 72$, implying that

$$\begin{aligned}\sum_{i=1}^{2\ell} \mathrm{w}_i &= \sum_{i=0}^{\ell-1} \mathrm{w}_{2i+1} + \sum_{i=1}^{\ell} \mathrm{w}_{2i} \\ &\geq (56 + 72 + 64(\ell - 2)) + 32\ell \\ &= 48 \cdot (2\ell).\end{aligned}$$

Thus, Inequality (1) is satisfied with $k = 2\ell$. This completes the proof of Theorem 7. $\Box$

For a hypergraph H, we let $n_{\geq 3}(H)$ denote the number of vertices of degree at least 3 in H. Further, we let $n_2(H)$ and $n_1(H)$ denote the number of vertices of degree 2 and 1, respectively, in H. We observe that Theorem 7 can be restated as follows.

Theorem 8 *If H is a 3-uniform hypergraph, then*

$$48\tau_g(H) \leq 15n_{\geq 3}(H) + 14n_2(H) + 11n_1(H) + 15m_H.$$

Since the right side is at most $15n_H + 15m_H$, Theorem 5 is an immediate consequence of Theorem 7 and Theorem 8. Recall the statement of Theorem 5.

Theorem 5. *If H is a 3-uniform hypergraph, then $\tau_g(H) \leq \frac{5}{16}(n_H + m_H)$.*

As a further consequence of Theorem 8, we have the following upper bound on the game transversal number of a 3-uniform hypergraph with maximum degree at most 2.

Corollary 1 *If H is a 3-uniform hypergraph and $\Delta(H) \leq 2$, then the following holds.*

(a) $\tau_g(H) \leq \frac{3}{10}(n_H + m_H)$.
(b) $\tau_g(H) \leq \frac{1}{2}n_H$.
(c) $\tau_g(H) \leq \frac{3}{4}m_H$ *if H is 2-regular.*

Proof. If H is a 3-uniform hypergraph and $\Delta(H) \leq 2$, then $m_H \leq \frac{2}{3}n_H$, implying, by Theorem 8, that $\tau_g(H) \leq \frac{1}{48}(14n_H + 15m_H) \leq \frac{3}{10}(n_H + m_H) \leq \frac{1}{2}n_H$, which is equal to $\frac{3}{4}m_H$ whenever H is 3-uniform and 2-regular. $\Box$

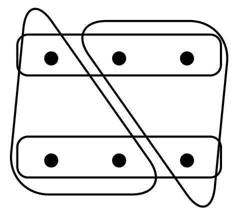

Figure 2: The 3-uniform hypergraph H with $n_H = 6$, $m_H = 4$ and $\tau_g(H) = 3$.

A small example attaining equation in all of Theorem 8 and Corollary 1 (a)–(c) is shown in Figure 2.

For the Staller-start game, we have the following consequence of Theorem 5.

Corollary 2 *If H is a 3-uniform hypergraph, then $\tau_g'(H) \le \frac{1}{16}(5n_H + 5m_H + 6)$.*

Proof. The first move of Staller decreases $n_H + m_H$ by at least 2, since at least one vertex and one edge are deleted by her move. Let H' denote the resulting residual hypergraph. Then $n_{H'} + m_{H'} \le n_H + m_H - 2$. By Theorem 5,

$$\begin{aligned} \tau_g'(H) &= 1 + \tau_g(H') \\ &\le 1 + \tfrac{5}{16}(n_{H'} + m_{H'}) \\ &\le 1 + \tfrac{5}{16}(n_H + m_H - 2) \\ &= \tfrac{1}{16}(5n_H + 5m_H + 6). \quad \Box \end{aligned}$$

6.2 Proof of Theorem 6

In this section, we prove Theorem 6. Again, we consider colored hypergraphs, where each edge and vertex is associated with a color. The colors of edges and vertices may change as the set D of chosen vertices is extended during the game.

An edge is colored *white* if it is not covered by a vertex of D, and is colored *red* otherwise. From the partially covered hypergraph red edges and isolated vertices are deleted. This way we obtain the residual hypergraph.

Each vertex of the hypergraph is associated with one from the following five colors: white, yellow, green, blue, and red. This coloring reflects to the degree of the vertex in the residual hypergraph; that is, the number of white edges incident to it.

- A vertex is colored *white* if it has degree at least 4.
- A vertex is colored *yellow* if it has degree 3.

- A vertex is colored *green* if it has degree 2.
- A vertex is colored *blue* if it has degree 1.
- A vertex is colored *red* if it is not incident with any white edges or equivalently, if it is deleted from the residual hypergraph.

We now define the parameter $\Delta^*(H)$ of the residual hypergraph H of the game as follows. If Edge-hitter is the next player to make a move on H, then $\Delta^*(H)$ is the maximum degree, $\Delta(H)$, of H. Otherwise, if Staller is the next player to make a move on H, then $\Delta^*(H)$ denotes the maximum degree of the residual hypergraph before Edge-hitter made his previous move. We associate a weight with every vertex in the residual hypergraph H that depends on $\Delta^*(H)$ and on the color of the vertex in H.

Color of vertex	*Degree in H*	*Weight of vertex*			
		$\Delta^*(H) \geq 5$	$\Delta^*(H) = 4$	$\Delta^*(H) = 3$	$\Delta^*(H) \leq 2$
white	≥ 4	852	852	–	–
yellow	3	852	845	845	–
green	2	852	838	750	750
blue	1	852	831	655	543
red	—	0	0	0	0

Table 2. The weights of vertices according to their color and $\Delta^*(H)$.

Further, the weight of an edge is 852 if it is white, and 0 if it is red. We shall prove the following key theorem. From our earlier observations, it suffices for us to prove Theorem 9 in the case of hypergraphs with no multiple edges.

Theorem 9 *If H is a 4-uniform residual hypergraph, then* $3024\tau_g(H) \leq \mathrm{w}(H)$.

Proof. If $m_H = 0$, then $\tau_g(H) = 0$ and the desired bound is immediate. Hence we may assume that $m_H \geq 1$. We say that Edge-hitter can *achieve a* 3024-*target* if he can play a sequence of moves guaranteeing that on average the weight decrease resulting from each played vertex in the game is at least 3024. In order to achieve a 3024-target, Edge-hitter must guarantee that a sequence of moves $m_1, \ldots, m_k$ are played, starting with his first move m_1, and with moves alternating between Edge-hitter and Staller such that if w_i denotes the decrease in weight after move m_i is played, then

$$\sum_{i=1}^{k} \mathrm{w}_i \geq 3024 \cdot k\,, \tag{2}$$

where either k is odd and the game is completed after move m_k or k is any even number (in this latter case the game may or may not be completed after move m_k).

We will analyze how Edge-hitter can achieve a 3024-target. Similarly to the proof of Theorem 7, there is a trivial situation, namely when Edge-hitter can play a vertex that covers all remaining edges, and the current value of the residual hypergraph is at least 3024. Then the 3024-target is achieved with $k = 1$. This may happen in several cases below. We shall not mention it each time, we only discuss what happens otherwise.

We prove a series of claims that establish important properties that hold in the residual hypergraph H.

Claim 9.A *If $\Delta(H) \geq 5$, then Edge-hitter can achieve a 3024-target.*

Proof. Let v be a (white) vertex of degree at least 5 in H. If Edge-hitter plays the vertex v, this results in at least five edges recolored red. Moreover, the white vertex v is recolored red. Hence, since $\Delta^*(H) = \Delta(H) \geq 5$ immediately before Edge-hitter plays v, $\mathrm{w}_1 \geq 5 \cdot 852 + 1 \cdot 852 = 5112 > 3024 \cdot 1$. Similarly, $\Delta^*(H) \geq 5$ before Staller makes her next move. Thus, Staller's move m_2 decreases the weight by $\mathrm{w}_2 \geq 852 + 852 = 1704$ and we have $\mathrm{w}_1 + \mathrm{w}_2 \geq 5112 + 1704 = 6816 > 3024 \cdot 2$. Therefore, Inequality (2) is satisfied with $k = 2$. (□)

By Claim 9.A, we may assume $\Delta(H) \leq 4$, for otherwise Edge-hitter can achieve a 3024-target.

Claim 9.B *If $\Delta(H) = 4$, then Edge-hitter can achieve a 3024-target.*

Proof. Suppose that Edge-hitter plays a (white) vertex v of degree 4 in H. We note that immediately before Edge-hitter plays v, $\Delta^*(H) = \Delta(H) = 4$. If the degree of a vertex x is decreased by ℓ after the vertex v is played, then the weight $\mathrm{w}(x)$ of x decreases by at least 7ℓ. When Edge-hitter plays v, four white edges and the white vertex v are recolored red. Further, $\sum_{w \in N(v)} d_H(w)$ decreases by exactly 12. Therefore, this move results in $\mathrm{w}_1 \geq 4 \cdot 852 + 1 \cdot 852 + 12 \cdot 7 = 4344 > 3024 \cdot 1$. In the next turn Staller plays a vertex, u say. We note that immediately before Staller makes her move, $\Delta^*(H) = 4$. Staller's move results in the vertex u recolored red, and so the weight $\mathrm{w}(u)$ of u decreases by at least 831. Her move also results in at least one edge recolored red. Further, $\sum_{w \in N(u)} d_H(w)$ decreases by at least 3. Thus, Staller's move m_2 decreases the weight by at least $831 + 852 + 3 \cdot 7 = 1704$. Therefore, $\mathrm{w}_1 + \mathrm{w}_2 \geq 4344 + 1704 = 6048 = 3024 \cdot 2$, and so Inequality (2) is satisfied with $k = 2$. (□)

By Claim 9.B, we may assume $\Delta(H) \leq 3$, for otherwise Edge-hitter can achieve a 3024-target.

Claim 9.C *If $\Delta(H) = 3$, then Edge-hitter can achieve a 3024-target.*

Proof. Suppose that $\Delta(H) = 3$ and Edge-hitter plays a (yellow) vertex v of degree 3 in H. Then, before Edge-hitter and Staller make their next moves, $\Delta^*(H) = 3$. If the degree of a vertex x is decreased by ℓ, the weight of x decreases by at least 95ℓ. When Edge-hitter plays v, three white edges and the yellow vertex v are recolored red. Further, $\sum_{w\in N(v)} d_H(w)$ decreases by exactly 9. Therefore, playing the vertex v results in $\mathrm{w}_1 \geq 3\cdot 852+1\cdot 845+95\cdot 9 = 4256 > 3024\cdot 1$. In the next turn Staller plays a vertex, u say. Staller's move results in the vertex u recolored red, and so the weight of u decreases by at least 655. Her move also results in at least one edge recolored red. Further, $\sum_{w\in N(u)} d_H(w)$ decreases by at least 3. Thus, Staller's move m_2 decreases the weight by at least $655 + 852 + 3 \cdot 95 = 1792$. Therefore, $\mathrm{w}_1 + \mathrm{w}_2 \geq 4256 + 1792 = 6048 = 3024 \cdot 2$, and so Inequality (2) is satisfied with $k = 2$. (□)

By Claim 9.C, we may assume $\Delta(H) \leq 2$, for otherwise Edge-hitter can achieve a 3024-target.

Claim 9.D *If $\Delta(H) = 2$, and H contains two overlapping edges or H contains a green vertex with a blue neighbor, then Edge-hitter can achieve a* 3024*-target.*

Proof. Suppose that $\Delta(H) = 2$. Then, before Edge-hitter and Staller make their next moves, $\Delta^*(H) = 2$. If the degree of a vertex x is decreased by ℓ after the vertex v is played, then the weight of x decreases by at least 207ℓ.

Suppose firstly that H contains two overlapping edges; say u and v are two vertices common to them. Edge-hitter now plays the vertex v. The two edges incident with v are recolored red, as are both green vertices u and v. Further, $\sum_{w\in N(v)\setminus\{u\}} d_H(w)$ decreases by 4. Therefore, playing the vertex v results in $\mathrm{w}_1 \geq 2\cdot 852+2\cdot 750+4\cdot 207 = 4032 > 3024\cdot 1$. In the next turn Staller plays a vertex, w say. Staller's move results in the vertex w and at least one edge recolored red. The degree sum of the neighbors of w decreases by at least 3. Thus, Staller's move m_2 decreases the weight by $\mathrm{w}_2 \geq 543+852+3\cdot 207 = 2016$. Therefore, $\mathrm{w}_1 + \mathrm{w}_2 \geq 4032 + 2016 = 6048 = 3024 \cdot 2$, and so Edge-hitter achieves a 3024-target.

Suppose secondly that H contains a green vertex, v, having a blue neighbor, w. Edge-hitter now plays the vertex v which results in $\mathrm{w}_1 \geq 750+543+2\cdot 852+5\cdot 207 = 4032 > 3024\cdot 1$, since after the move is played, the green vertex v and its blue neighbor w are recolored red, and the two white edges incident with v are recolored red. Further, the degree sum of the neighbors of v different from w decreases by 5. Therefore, analogously as before, Inequality (2) is satisfied with $k = 1$ or $k = 2$. (□)

By Claim 9.D, we may assume that every component of the residual hypergraph H is either a 2-regular, linear hypergraph or an isolated edge (consisting of four blue vertices), for otherwise Edge-hitter can achieve a 3024-target.

Claim 9.E *If there is an isolated edge in H, then Edge-hitter can achieve a* 3024*-target.*

Proof. If the assumption holds, Edge-hitter can play a vertex from an isolated edge, what results in $\mathrm{w}_1 = 4 \cdot 543 + 852 = 3024 = 3024 \cdot 1$. In the next turn Staller plays either a vertex from an isolated edge and $\mathrm{w}_2 = 3024$, or a vertex from a 2-regular, linear component. In the latter case, two white edges and the played green vertex are recolored red, and further six green vertices are recolored blue, implying that $\mathrm{w}_2 = 750 + 2 \cdot 852 + 6 \cdot 207 = 3696$. In both cases, $\mathrm{w}_1 + \mathrm{w}_2 \ge 6048 = 3024 \cdot 2$, and so Inequality (2) is satisfied with $k = 2$. (□)

By Claim 9.E, we may assume that every component of the residual hypergraph H is a 2-regular, linear hypergraph, for otherwise Edge-hitter can achieve a 3024-target. Edge-hitter now selects a component C of H, and will play inside C as long as a green vertex in $V(C)$ exists, independently of Staller's responses to his moves. More explicitly, among all green vertices in $V(C)$ at each stage of the game, Edge-hitter plays a green vertex with as many blue neighbors as possible. We note that subsequent to his first move, as long as a green vertex in $V(C)$ exists, Edge-hitter can play a green vertex having at least one blue neighbor.

Suppose that a total of s green vertices in $V(C)$ are played by Edge-hitter. The first move, m_1, of Edge-hitter results in $\mathrm{w}_1 = 750 + 2 \cdot 852 + 6 \cdot 207 = 3696$. Thereafter, each subsequent move m_{2j+1} of Edge-hitter, where $j \in \{1, \dots, s-1\}$, results in $\mathrm{w}_{2j+1} \ge 750 + 2 \cdot 852 + 543 + 5 \cdot 207 = 4032$. Every move played by Staller decreases the weight by at least $543 + 852 + 3 \cdot 207 = 2016$, since with each of her moves at least one edge and a vertex are recolored red, and the degree sum of the remaining vertices is decreased by at least 3. In particular, each of Staller's moves m_{2j}, where $j \in \{1, \dots, s-1\}$, results in $\mathrm{w}_{2j} \ge 2016$. If the game is complete after Edge-hitter's sth move, then

$$\begin{aligned}
\sum_{j=1}^{2s-1} \mathrm{w}_j &= \sum_{j=0}^{s-1} \mathrm{w}_{2j+1} + \sum_{j=1}^{s-1} \mathrm{w}_{2j} \\
&\ge 3696 + 4032(s-1) + 2016(s-1) \\
&= 3696 + 3024 \cdot 2(s-1) \\
&> 3024 \cdot (2s-1).
\end{aligned}$$

Thus, Inequality (2) is satisfied with $k = 2s - 1$ and the game is completed after move m_k. Hence, we may assume that the game is not complete after Edge-hitter's sth move.

Claim 9.F *If there are no green vertices in $V(C)$ after Edge-hitter's sth move, then Edge-hitter can achieve a 3024-target.*

Proof. Suppose that after Edge-hitter's sth move, which is the $(2s-1)$st turn in the game, all vertices in $V(C)$ in the resulting residual hypergraph are colored blue. Let v be the vertex played by Edge-hitter in his sth move, and let e_1 and e_2 be the two edges incident with v. We show that v had at least two blue neighbors. Suppose, to the contrary, that $e_1 \cup e_2$ contains only one blue vertex, say $u \in e_1$, before Edge-hitter plays the vertex v. We now consider a vertex, u', from e_1 that is different from u and v. We note that u' is a green vertex. Let e' be the edge incident with u' that is different from e. After Edge-hitter's sth move, all remaining vertices are colored blue. In particular, the three vertices in $e' \setminus \{u'\}$ are all colored blue. Moreover, by the linearity of C, at most one of them belongs to $e_1 \cup e_2$. Hence, before Edge-hitter's sth move, the green vertex u' had at least two blue neighbors (in fact at least three together with u, but we don't need this now), which contradicts the rule that Edge-hitter plays a green vertex with the largest number of blue neighbors. Therefore, the vertex v had at least two blue neighbors. Thus, $\mathrm{w}_{2s-1} \geq 750+2\cdot 852+2\cdot 543+4\cdot 207 = 4368$. Staller's sth move results in $\mathrm{w}_{2s} \geq 2016$. Hence,

$$\begin{aligned}\sum_{j=1}^{2s} \mathrm{w}_j &= \sum_{j=0}^{s-1} \mathrm{w}_{2j+1} + \sum_{j=1}^{s} \mathrm{w}_{2j} \\ &\geq (3696 + 4032(s-2) + 4368) + 2016s \\ &= 3024 \cdot (2s).\end{aligned}$$

Thus, Inequality (2) is satisfied with $k = 2s$. ($\square$)

By Claim 9.F, we may assume that there is at least one green vertex in $V(C)$ after Edge-hitter's sth move, but after Staller's sth move there are no green vertices in $V(C)$, for otherwise Edge-hitter can achieve a 3024-target. Necessarily, Staller's sth move plays a vertex from $V(C)$.

Claim 9.G *If Staller's sth move does not play a blue vertex with three green neighbors, then Edge-hitter can achieve a* 3024*-target.*

Proof. Under the given assumptions, Staller's sth move either plays a green vertex, in which case $\mathrm{w}_{2s} \geq 750 + 2 \cdot 852 + 6 \cdot 207 = 3696$, or plays a blue vertex having a blue neighbor, in which case $\mathrm{w}_{2s} \geq 2 \cdot 543 + 852 + 2 \cdot 207 = 2352$. In both cases, $\mathrm{w}_{2s} \geq 2352$, implying that

$$\begin{aligned}\sum_{j=1}^{2s} \mathrm{w}_j &= \sum_{j=0}^{s-1} \mathrm{w}_{2j+1} + \sum_{j=1}^{s} \mathrm{w}_{2j} \\ &\geq (3696 + 3024 \cdot 2(s-1)) + 2352 \\ &= 3024 \cdot (2s).\end{aligned}$$

Thus, Inequality (2) is satisfied with $k = 2s$. ($\square$)

By Claim 9.G, we may assume that Staller's sth move plays a blue vertex, v, with three green neighbors, say u_1, u_2, and u_3. Let e be the edge incident with v, and let e_i be the edge incident with u_i that is different from e for $i = 1, 2, 3$. Since no green vertices remain after Staller plays the vertex v, we note that the three edges e_1, e_2 and e_3 are vertex-disjoint. Further, immediately before Staller plays her sth move, the vertex set $S = e_1 \cup e_2 \cup e_3 \cup \{v\}$ contains exactly ten blue vertices and three green vertices. In the $(2s-1)$st turn, Edge-hitter played as his sth move a green vertex of degree 2 which is not contained in S, and therefore his move recolored at most six vertices in S from green to blue. Thus, before the $(2s-1)$st turn, the set S contained at least four blue vertices and, by the Pigeonhole Principle, at least one of the vertices u_1, u_2, u_3 had at least two blue neighbors. According to Edge-hitter's rule, on the $(2s-1)$st turn when he played his sth move, he therefore selected a green vertex with at least two blue neighbors, implying that $\mathrm{w}_{2s-1} \geq 750+2\cdot 852+2\cdot 543+4\cdot 207 = 4368$.

Staller's sth move results in $\mathrm{w}_{2s} \geq 2016$. Hence,

$$\begin{aligned}\sum_{j=1}^{2s} \mathrm{w}_j &= \sum_{j=0}^{s-1} \mathrm{w}_{2j+1} + \sum_{j=1}^{s} \mathrm{w}_{2j} \\ &\geq (3696 + 4032(s-2) + 4368) + 2016s \\ &= 3024 \cdot (2s).\end{aligned}$$

Thus, Inequality (1) is satisfied with $k = 2s$. This completes the proof of Theorem 9. □

As an immediate consequence of Theorem 9, we have that if H is a 4-uniform hypergraph, then $3024\tau_g(H) \leq 852n_H + 852m_H$, and so Theorem 6 is an immediate consequence of Theorem 9. Recall the statement of Theorem 6.

Theorem 6. *If H is a 4-uniform hypergraph, then $\tau_g(H) \leq \frac{71}{252}(n_H + m_H)$.*

From the proof of Theorem 9 we also derive:

Corollary 3 *If H is a 4-uniform hypergraph and $\Delta(H) \leq 2$, then $\tau_g(H) \leq \frac{7}{18}n_H$, moreover $\tau_g(H) \leq \frac{7}{9}m_H$ if H is 2-regular.*

Proof. If H is 4-uniform and has $\Delta(H) \leq 2$, then $m_H \leq \frac{1}{2}n_H$. Recall that, under the assumption $\Delta^*(H) \leq 2$, the weight of a green vertex is 750, and that of a white edge is 852. Thus, by Theorem 9, we obtain:

$$3024\tau_g(H) \leq \mathrm{w}(H) \leq 750n_H + 852m_H \leq 750n_H + 426n_H = 1176n_H.$$

This means $\tau_g(H) \leq \frac{7}{18}n_H$, which is precisely $\frac{7}{9}m_H$ if H is 2-regular and 4-uniform. □

For the Staller-start game, Theorem 6 has the following further consequence.

Corollary 4 *If H is a 4-uniform hypergraph, then $\tau_g'(H) \leq \frac{1}{252}(71n_H + 71m_H + 110)$.*

Proof. The first move of Staller decreases $n_H + m_H$ by at least 2, since at least one vertex and one edge are deleted by her move. Let H' denote the resulting residual hypergraph. Then $n_{H'} + m_{H'} \leq n_H + m_H - 2$. By Theorem 6,

$$\begin{aligned}\tau_g'(H) &= 1 + \tau_g(H') \\ &\leq 1 + \tfrac{71}{252}(n_{H'} + m_{H'}) \\ &\leq 1 + \tfrac{71}{252}(n_H + m_H - 2) \\ &= \tfrac{1}{252}(71n_H + 71m_H + 110). \quad \square\end{aligned}$$

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