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Technical Report on: Towards Reactive Control of Simplified Legged Robotics Maneuvers

Jeff Duperret
University of Pennsylvania, jdup@seas.upenn.edu

D. E. Koditschek
University of Pennsylvania

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Abstract
This technical report provides proofs and calculations for the paper "Towards Reactive Control of Simplified Legged Robotics Maneuvers," as well as implementation notes and a discussion on robustness.

Keywords
Robotics, legged locomotion, transitional maneuver

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Electrical and Computer Engineering | Electro-Mechanical Systems | Engineering | Systems Engineering

Technical Report on: Towards Reactive Control of Simplified Legged Robotics Maneuvers

Jeffrey Duperret and Daniel E. Koditschek

Department of Electrical and Systems Engineering, University of Pennsylvania, 200 South 33rd Street, Philadelphia, PA 19104
jdup@seas.upenn.edu

This technical report provides proofs and calculations for the paper [1], as well as implementation notes and a discussion on robustness.

1 Proposition and theorem proofs

Proposition 1 *A candidate K-step navigation plan $(\hat{\mathbf{q}},\hat{\mathbf{u}}) = (\{\mathbf{q}_0,...,\mathbf{q}_K\},\{\mathbf{u}_0,...,\mathbf{u}_{K-1}\})$ is admissible if and only if for every $i \in \{0,...,K-1\}$ it holds that $\mathbf{q}_i \in \mathcal{R}_{K-i}$ and $f(\mathbf{q}_i,\mathbf{u}_i) \in \mathcal{R}_{K-(i+1)}$.*

Proof. First assume $(\hat{\mathbf{q}},\hat{\mathbf{u}})$ is an admissible K-step navigation plan. We show that $\mathbf{q}_i \in \mathcal{R}_{K-i}$ and then use this to show $f(\mathbf{q}_i,\mathbf{u}_i) \in \mathcal{R}_{K-(i+1)}$.

Let $j = K-i$ and consider the change of variables $\bar{\mathbf{q}}_j = \mathbf{q}_{K-i}$, $\bar{\mathbf{u}}_j = \mathbf{u}_{K-i}$. The proposition $\mathbf{q}_i \in \mathcal{R}_{K-i}$ for $i \in \{0,...,K-1\}$ is then equivalent to $\bar{\mathbf{q}}_j \in \mathcal{R}_j$ for $j \in \{1,...,K\}$. Call $P(j)$ the proposition $\bar{\mathbf{q}}_j \in \mathcal{R}_j$ over the well-ordered index set $j \in \mathcal{J} = \{1,...,K\}$. We show $P(j), j \in \mathcal{J}$ is true using the principle of transfinite induction [2, p. 195]. Assume for the inductive hypothesis that $P(l)$ is true for all $l < j, l \in \mathcal{J}$. First consider $j = 1$. By admissibility of $(\hat{\mathbf{q}},\hat{\mathbf{u}})$ we have that $\mathbf{q}_K \in \mathcal{G} = \mathcal{R}_0$. Then $\mathbf{q}_{K-1} = f^{-1}(\mathbf{q}_K,\mathbf{u}_{K-1}) \in f^{-1}(\mathcal{R}_0,\mathbf{u}_{K-1})$ and because $\mathbf{q}_{K-1} \notin (\mathcal{G}\cup\mathcal{O})$ we have $\mathbf{q}_{K-1} \in \mathcal{R}_1$. So $\bar{\mathbf{q}}_1 \in \mathcal{R}_1$, or equivalently, $P(1)$ is true. Next consider $j > 1, j \in \mathcal{J}$. If $K < 2$ then $P(j), j > 1$ is vacuously true since $\mathcal{J} = \{1\}$. Assume $K \geq 2$. By hypothesis $P(j-1)$ is true for $j > 1$, or equivalently that $\bar{\mathbf{q}}_{j-1} \in \mathcal{R}_{j-1}$. We have $\bar{\mathbf{q}}_{j-1} = f(\bar{\mathbf{q}}_j,\bar{\mathbf{u}}_j)$ and so $\bar{\mathbf{q}}_j = f^{-1}(\bar{\mathbf{q}}_{j-1},\bar{\mathbf{u}}_j) \in f^{-1}(\mathcal{R}_{j-1},\bar{\mathbf{u}}_j)$. By assumption of admissibility $\bar{\mathbf{q}}_j \notin (\mathcal{G}\cup\mathcal{O})$, so we have $\bar{\mathbf{q}}_j \in \mathcal{R}_j$ and $P(j)$ is true for $j > 1, j \in \mathcal{J}$. This completes the successor step. We have shown that the assumption $P(l)$ is true for all $l < j, l \in \mathcal{J}$ implies $P(j), j \in \mathcal{J}$ is true, thus by the principle of transfinite induction $P(j)$ is true for all $j \in \mathcal{J}$, or equivalently, that $\mathbf{q}_i \in \mathcal{R}_{K-i}$ for $i \in \{0,...,K-1\}$. Since $\mathbf{q}_K \in \mathcal{G} = \mathcal{R}_0$ and $\mathbf{q}_i \in \mathcal{R}_{K-i}$ for all $i \in \{0,...,K-1\}$ we have $\mathbf{q}_{i+1} \in \mathcal{R}_{K-(i+1)}$ and so $f(\mathbf{q}_i,\mathbf{u}_i) \in \mathcal{R}_{K-(i+1)}$.

Now assume that for every $i \in \{0,...,K-1\}$ it holds that $\mathbf{q}_i \in \mathcal{R}_{K-i}$ and $\mathbf{q}_{i+1} = f(\mathbf{q}_i,\mathbf{u}_i) \in \mathcal{R}_{K-(i+1)}$. Since $\mathcal{R}_{K-i}\cap(\mathcal{G}\cup\mathcal{O}) = \emptyset$ it holds that $\mathbf{q}_i \notin \mathcal{O}\cup\mathcal{G}$. Since $\mathcal{R}_0 = \mathcal{G}$ it holds that $\mathbf{q}_K = f(\mathbf{q}_{K-1},\mathbf{u}_{K-1}) \in \mathcal{G}$. Then the K-step navigation plan $(\{\mathbf{q}_0,...,\mathbf{q}_K\},\{\mathbf{u}_0,...,\mathbf{u}_{K-1}\}) = (\hat{\mathbf{q}},\hat{\mathbf{u}})$ is admissible.

□

Corollary 1. *There exists an admissible K-step navigation plan from $\mathbf{q}$ if and only if $K \in \mathcal{I}_{\mathbf{q}}$. If a solution to the discrete navigation problem exists, the minimum number of steps that it can be completed in from $\mathbf{q}$ is* $\min(\mathcal{I}_{\mathbf{q}})$.

Proof. We first prove the equivalence between $K \in \mathcal{I}_{\mathbf{q}}$ and the existence of an admissible K-step navigation plan from starting state $\mathbf{q}$.

First assume there exists an admissible K-step navigation plan from $\mathbf{q}$ that will complete the task. Then, by Proposition 1, $\mathbf{q} \in \mathcal{R}_K$ and so by the definition of $\mathcal{I}_{\mathbf{q}}$ we have $K \in \mathcal{I}_{\mathbf{q}}$.

Now assume $K \in \mathcal{I}_{\mathbf{q}}$. Then $\mathbf{q} \in \mathcal{R}_K$ and so by Proposition 1 there exists an admissible K-step navigation plan from $\mathbf{q}$ that will complete the task.

Finally, we prove that if a solution to the task exists, the minimum number of steps the task can be completed in from $\mathbf{q}$ is $\min(\mathcal{I}_{\mathbf{q}})$. Assume a solution to the task exists starting from $\mathbf{q}$. By the previously proved equivalence $\mathcal{I}_{\mathbf{q}} \neq \emptyset$, where $\mathcal{I}_{\mathbf{q}}$ contains set of step numbers that any admissible navigation plan to the task can take. Then the minimum number of steps the task from $\mathbf{q}$ can be completed in is $\min(\mathcal{I}_{\mathbf{q}})$.

□

We also note that – by Bellman's Optimality Principle – any path using the minimum number of possible steps to the goal has the property that all sub-paths to the goal also use the minimum number of possible steps. If this were not true and a quicker sub-path existed then we would get the contradiction that this quicker sub-path could be substituted into the minimum path to yield a path with fewer steps than the minimum path.

Theorem 1. *If a solution to the discrete navigation problem exists then the discrete navigation problem is solved in the minimum number of possible steps if and only if the following reactive control relation is observed at every step:*

$$\mathbf{u} \in \begin{cases} \mathcal{U}_{\mathbf{q},\min(\mathcal{I}_{\mathbf{q}})} & \mathcal{I}_{\mathbf{q}} \neq \emptyset, \\ \mathcal{U} & \text{else}, \end{cases} \tag{1}$$

where $\mathbf{q}$ is the state at any given iteration and $\mathbf{u}$ is the chosen control action at that iteration.

Proof. Assume a solution to the discrete navigation problem exists.

First assume the task is completed in the minimum number of possible steps. Without loss of generality assume this number of steps is $K \in \mathbb{N}^+$[1] so that the task is completed with some admissible K-step navigation plan $(\hat{\mathbf{q}}, \hat{\mathbf{u}}) = (\{\mathbf{u}_0, \ldots, \mathbf{u}_{K-1}\}, \{\mathbf{q}_0, \ldots, \mathbf{q}_K\})$. By Proposition 1 we have that $\mathbf{q}_i \in \mathcal{R}_{K-i}$ and $f(\mathbf{q}_i, \mathbf{u}_i) \in \mathcal{R}_{K-(i+1)}$ for every $i \in \{0, \ldots, K-1\}$. Since $\mathbf{q}_i \in \mathcal{R}_{K-i}$ we have that $\mathcal{I}_{\mathbf{q}_i} \neq \emptyset$ at every iteration before reaching the goal. Furthermore, because $\mathbf{q}_i \in \mathcal{R}_{K-i}$ and $f(\mathbf{q}_i, \mathbf{u}_i) \in \mathcal{R}_{K-(i+1)}$ we have that $\mathbf{u}_i \in \mathcal{U}_{\mathbf{q}_i, K-i}$. The task is completed in the minimum number of steps K, so – by Corollary 1 – at each iteration i leading up to the goal the minimum number of steps to complete the task is $K - i = \min(\mathcal{I}_{\mathbf{q}_i})$ and thus $\mathbf{u}_i \in \mathcal{U}_{\mathbf{q}_i, \min(\mathcal{I}_{\mathbf{q}_i})}$.

Now assume that the reactive control relation given in Equation 1 is observed such that $(\hat{\mathbf{q}}, \hat{\mathbf{u}}) = (\{\mathbf{q}_0, \ldots\}, \{\mathbf{u}_0, \ldots\})$ is the resulting state and control sequence, where $\mathbf{q}_0$ is the starting state. Since a solution to the task exists $\mathcal{I}_{\mathbf{q}_0} \neq \emptyset$ and $\mathbf{q}_0 \in \mathcal{R}_{\min(\mathcal{I}_{\mathbf{q}_0})}$ by definition of $\mathcal{I}_{\mathbf{q}_0}$. Call $\min(\mathcal{I}_{\mathbf{q}_0})$ the number $K \in \mathbb{N}^+$ such that $\mathbf{q}_0 \in \mathcal{R}_K$. The controller

[1] $\mathbb{N}^+$ denotes the positive integers.

then chooses an action u_0 such that $\mathbf{q}_1 \in \mathcal{R}_{K-1}$. Now assume that $\mathbf{q}_i \in \mathcal{R}_{K-i}$ for some $i \in \{0,...,K-1\}$. The controller chooses a control such that $f(\mathbf{q}_i,\mathbf{u}_i) \in \mathcal{R}_{K-(i+1)}$ and so by Proposition 1 the task is completed in K steps. The number of steps taken was $K = \min(\mathcal{I}_{\mathbf{q}_0})$ and so by Corollary 1 the task was completed in the minimum number of steps.

□

2 Reactive Control Relation with Linear Dynamics

We give an algorithmic specification of the reactive control relation introduced in [1, Theorem 1] for a linear apex map with polyhedron control constraints and polyhedra forms of $\mathcal{O}$ and $\mathcal{G}$ as was proposed in [1, Section 4] . Note that we will write all set boundaries as closed to avoid the cumbersome notation of keeping track of which set boundaries are open and closed.

We assume the linear iterated dynamics

$$\mathbf{q}_{n+1} = f(\mathbf{q}_n,\mathbf{u}_n) = A\mathbf{q}_n + B\mathbf{u}_n,$$

where $\mathbf{q}_n \in \mathcal{D} = \mathbb{R}^m$, $\mathbf{u}_n \in \mathcal{U} \subset \mathbb{R}^p$, $A \in \mathbb{R}^{m\times m}$, $det(A) \neq 0$, $B \in \mathbb{R}^{m\times p}$, and that $\mathcal{U}$ is a polyhedron embedded in $\mathbb{R}^p$ described by:

$$\mathcal{U} = \{\mathbf{u} \in \mathbb{R}^p | \bar{A}_{\mathcal{U}}\mathbf{u} \geq \bar{b}_{\mathcal{U}}\}.$$

Note that the computations presented in this section directly extend to affine iterated dynamics. We also assume that the goal set $\mathcal{G}$ is a polyhedron[2] and that the obstacle set is a finite union of polyhedra $\mathcal{O} = \mathcal{O}_1 \cup ... \cup \mathcal{O}_p$, for $p \in \mathbb{N}$[3], where

$$\mathcal{G} = \{\mathbf{q} \in \mathcal{D} | \bar{A}_{\mathcal{G}}\mathbf{q} \geq \bar{b}_{\mathcal{G}}\}, \quad \mathcal{O}_j = \{\mathbf{q} \in \mathcal{D} | \bar{A}_{\mathcal{O}_j}\mathbf{q} \geq \bar{b}_{\mathcal{O}_j}\}.$$

Note that as a convention we use an over-line over matrices and vectors to denote polyhedra constraints.

We show the computation of [1, Theorem 1] in three steps. Recall that $\mathcal{R}_{k+1} = f^{-1}(\mathcal{R}_k,\mathcal{U}) \setminus (\mathcal{G} \cup \mathcal{O})$ for $k > 0$ and $\mathcal{R}_0 = \mathcal{G}$. We first show the computation of $f^{-1}(\mathcal{R}_k,\mathcal{U})$ and then the set difference computation $f^{-1}(\mathcal{R}_k,\mathcal{U}) \setminus (\mathcal{G} \cup \mathcal{O})$, allowing the recursive computation of the set $\mathcal{R} = \bigcup_k \mathcal{R}_k$. Finally, we use this show the control relation computation $\mathcal{U}_{\mathbf{q},\min(\mathcal{I}_{\mathbf{q}})}$.

2.1 Computation of $f^{-1}(\mathcal{R}_k,\mathcal{U})$

Assume $\mathcal{R}_k = \mathcal{R}_{k,1} \cup ... \cup \mathcal{R}_{k,r}$ consists of the finite union of r polyhedra with representation $\mathcal{R}_{k,i} = \{\mathbf{q} \in \mathcal{D} | \bar{A}_{\mathcal{R}_{k,i}}\mathbf{q} \geq \bar{b}_{\mathcal{R}_{k,i}}\}$. Then $f^{-1}(\mathcal{R}_k,\mathcal{U}) = f^{-1}(\mathcal{R}_{k,1} \cup ... \cup \mathcal{R}_{k,r},\mathcal{U}) =$

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[2] This formulation can be generalized to work with a goal set consisting of a finite union of polyhedra.

[3] $\mathbb{N}$ denotes the nonnegative integers.

$f^{-1}(\mathcal{R}_{k,1},\mathcal{U})\cup\ldots\cup f^{-1}(\mathcal{R}_{k,r},\mathcal{U})$. Each $f^{-1}(\mathcal{R}_{k,i},\mathcal{U})$ is computed as follows:

$$\begin{aligned}
f^{-1}(\mathcal{R}_{k,i},\mathcal{U}) &= \{\mathbf{q}\in\mathcal{D}|f(\mathbf{q},\mathbf{u})\in\mathcal{R}_{k,i} \;\;\wedge\;\; \mathbf{u}\in\mathcal{U}\}\\
&= \{\mathbf{q}\in\mathcal{D}|\bar{A}_{\mathcal{R}_{k,i}}f(\mathbf{q},\mathbf{u})\geq\bar{b}_{\mathcal{R}_{k,i}} \;\;\wedge\;\; \bar{A}_{\mathcal{U}}\mathbf{u}\geq\bar{b}_{\mathcal{U}}\}\\
&= \{\mathbf{q}\in\mathcal{D}|\bar{A}_{\mathcal{R}_{k,i}}(A\mathbf{q}+B\mathbf{u})\geq\bar{b}_{\mathcal{R}_{k,i}} \;\;\wedge\;\; \bar{A}_{\mathcal{U}}\mathbf{u}\geq\bar{b}_{\mathcal{U}}\}\\
&= \{\mathbf{q}\in\mathcal{D}|\bar{A}_{\mathcal{R}_{k,i}}A\mathbf{q}+\bar{A}_{\mathcal{R}_{k,i}}B\mathbf{u}\geq\bar{b}_{\mathcal{R}_{k,i}} \;\;\wedge\;\; \bar{A}_{\mathcal{U}}\mathbf{u}\geq\bar{b}_{\mathcal{U}}\}\\
&= \left\{\mathbf{q}\in\mathcal{D}|\begin{bmatrix}\bar{A}_{\mathcal{R}_{k,i}}A & \bar{A}_{\mathcal{R}_{k,i}}B\\ 0 & \bar{A}_{\mathcal{U}}\end{bmatrix}\begin{bmatrix}\mathbf{q}\\ \mathbf{u}\end{bmatrix}\geq\begin{bmatrix}\bar{b}_{\mathcal{R}_{k,i}}\\ \bar{b}_{\mathcal{U}}\end{bmatrix}\right\}.
\end{aligned}$$

This can be made more compact by explicitly projecting out the $\mathbf{u}$ coordinates of the polyhedron via Fourier-Motzkin elimination [3]. Redundant constraints introduced by Fourier-Motzkin elimination can then be eliminated using linear programming. Specifically, let $S(\mathcal{R}_{k,i})$ denote the polyhedron:

$$\left\{(\mathbf{q},\mathbf{u})\in\mathcal{D}\times\mathbb{R}^{p}|\begin{bmatrix}\bar{A}_{\mathcal{R}_{k,i}}A & \bar{A}_{\mathcal{R}_{k,i}}B\\ 0 & \bar{A}_{\mathcal{U}}\end{bmatrix}\begin{bmatrix}\mathbf{q}\\ \mathbf{u}\end{bmatrix}\geq\begin{bmatrix}\bar{b}_{\mathcal{R}_{k,i}}\\ \bar{b}_{\mathcal{U}}\end{bmatrix}\right\},$$

and $\Pi_{\mathbf{q}}(S)$ the polyhedron given by the projection of the polyhedron S onto its coordinates $\mathbf{q}$. Then:

$$f^{-1}(\mathcal{R}_{k,i},\mathcal{U})=\left\{\mathbf{q}\in\mathcal{D}|\mathbf{q}\in\Pi_{\mathbf{q}}(S(\mathcal{R}_{k,i}))\right\}. \tag{2}$$

2.2 Computation of the set difference in $f^{-1}(\mathcal{R}_k,\mathcal{U})\setminus(\mathcal{G}\cup\mathcal{O})$

The form of the goal and obstacle sets given in [1, Section 4] has the property that the complement of the set $\mathcal{G}\cup\mathcal{O}$ forms (for a fixed y) a polyhedron. Specifically, the complement of $\mathcal{G}\cup\mathcal{O}$ is comprised of all apex states q whose (if $\sigma=+1$) epigraph of g_q^- and g_q^+ contain the obstacle endpoints or (if $\sigma=-1$) hypograph of g_q^- and g_q^+ contain the obstacle endpoints. This is given by:

$$(\mathcal{G}\cup\mathcal{O})^C=\left\{(x,\dot{x})\in\mathbb{R}^2|\forall i\in\{1,\ldots,p\}:\left(\begin{bmatrix}-\sigma & -\sigma\sqrt{\frac{2}{g}(y-y_{i,O1})}\\ -\sigma & \sigma\sqrt{\frac{2}{g}(y-y_{i,O1})}\\ -\sigma & -\sigma\sqrt{\frac{2}{g}(y-y_{i,O2})}\\ -\sigma & \sigma\sqrt{\frac{2}{g}(y-y_{i,O2})}\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\end{bmatrix}\geq\begin{bmatrix}-\sigma x_{i,O1}\\ -\sigma x_{i,O1}\\ -\sigma x_{i,O2}\\ -\sigma x_{i,O2}\end{bmatrix}\right)\right\},$$

so that the set-difference computation $f^{-1}(\mathcal{R}_k,\mathcal{U})\setminus(\mathcal{G}\cup\mathcal{O})=f^{-1}(\mathcal{R}_k,\mathcal{U})\cap(\mathcal{G}\cup\mathcal{O})^C$can be done simply by appending the $(\mathcal{G}\cup\mathcal{O})^C$ constraints to the polyhedra $f^{-1}(\mathcal{R}_k,\mathcal{U})$. Hence in practice each of the the $\mathcal{R}_k$ sets consist of a single polyhedron and the set difference computation is quite fast.

The more general problem of taking the set difference between a polyhedron and the union of polyhedra has been documented in the literature (for example in [4]). We have omitted the general computation since it wasn't used in the experiments.

2.3 Computation of $\mathcal{U}_{\mathbf{q},\min(\mathcal{I}_{\mathbf{q}})}$

Finally, we show the computation of the control relation of [1, Theorem 1] , in particular the set $\mathcal{U}_{\mathbf{q},\min(\mathcal{I}_{\mathbf{q}})}=\{\mathbf{u}\in\mathcal{U}|f(\mathbf{q},\mathbf{u})\in\mathcal{R}_{\min(\mathcal{I}_{\mathbf{q}})-1}\}$.

Let $\mathcal{R}_{\min(\mathcal{I}_{\mathbf{q}})-1}$ be given by the union of polyhedra $\cup_i \mathcal{R}_{\min(\mathcal{I}_{\mathbf{q}})-1,i}$, where each polyhedron is represented by the set of states $\{\mathbf{q} \in \mathcal{D} | \bar{A}_{\mathcal{R}_{\min(\mathcal{I}_{\mathbf{q}})-1,i}} \mathbf{q} \geq \bar{b}_{\mathcal{R}_{\min(\mathcal{I}_{\mathbf{q}})-1,i}}\}$. Let the control space be given by the polyhedron set $\mathcal{U} = \{\mathbf{u} \in \mathbb{R}^p | \bar{A}_{\mathcal{U}} \mathbf{u} \geq \bar{b}_{\mathcal{U}}\}$.

Then $\mathcal{U}_{\mathbf{q},\min(\mathcal{I}_{\mathbf{q}})}$ is given by the set of controls from state q that can reach $\mathcal{R}_{\min(\mathcal{I}_{\mathbf{q}})-1}$ on the next iteration, or equivalently, the set of controls $\mathbf{u} \in \mathcal{U}$ for which $Aq + Bu \in \mathcal{R}_{\min(\mathcal{I}_{\mathbf{q}})-1}$. This is equivalent to:

$$\mathcal{U}_{\mathbf{q},\min(\mathcal{I}_{\mathbf{q}})} = \bigcup_i \left\{ \mathbf{u} \in \mathbb{R}^p | \begin{bmatrix} \bar{A}_{\mathcal{R}_{\min(\mathcal{I}_{\mathbf{q}})-1,i}} B \\ \bar{A}_{\mathcal{U}} \end{bmatrix} \mathbf{u} \geq \begin{bmatrix} \bar{b}_{\mathcal{R}_{\min(\mathcal{I}_{\mathbf{q}})-1,i}} - \bar{A}_{\mathcal{R}_{k-1,i}} A\mathbf{q} \\ \bar{b}_{\mathcal{U}} \end{bmatrix} \right\}. \tag{3}$$

2.4 Implementation Notes

In the experimental implementation we artificially bound all unbounded polyhedra “far” away from the local region of interest around the robot and obstacle, allowing us to swap between representing the sets $\mathcal{R}_k$ and as both polyhedra and the convex hull of vertices. This has the practical benefit of allowing us to compute the projection in Equation 2 simply by projecting the vertices of $S(\mathcal{R}_{k,i})$ onto the first m coordinates.

The control function used in the experiments was derived from the control relation of Theorem 1 by the following process. From an apex state q in any $\mathcal{R}_k$, $k \in \mathbb{N}^+$, the reachable set on the next iteration forms some line segment, part of which is contained in $\mathcal{R}_{k-1}$. The line resulting from extending this line segment will generically intersect the edges of the polyhedra $\mathcal{R}_{k-1}$ at two points, the average of which must be a point somewhere in the interior of $\mathcal{R}_{k-1}$. This interior point serves as a target point (being, in some sense, an intuitively “robust” target to aim for since it is the furthest from the two edge points of the polyhedra), and the control selected from $\mathcal{U}_{\mathbf{q},\min(\mathcal{I}_{\mathbf{q}})}$ is equal to the control that achieves the closest next state to this target point. So if the target point is within the reachable line segment then the control is selected which causes the next state to be the target point, else if the target point is outside the reachable line segment then the control is selected which causes the next state to be the line segment endpoint nearest to the target point. We consider this strategy to be an implementation detail. In principle any control input satisfying Theorem 1 will work, however some points can be considered to be more robust to errors in anchoring than others but work remains to rigorously characterize the nature of these errors in an experimental setting so as to make informed strategies to maximize robustness.

3 Discussion on robustness

We should note that implementing a minimum-step strategy can lead to otherwise avoidable robustness issues and is not always the correct strategy for implementation. For example, the case when the state is contained in a set $\mathcal{R}_k$ but very close to a corner can lead to cases where only a small range of control inputs satisfy the control relation of Theorem 1. We observed that in some of these cases, normal experimental error can all-but-guarantee that the next state will “miss” the set $\mathcal{R}_{k-1}$. In this case it may be wiser to spend a step leaping into a more interior point of $\mathcal{R}_k$ before initializing the algorithm so as to trade robustness for the property of taking the minimum steps to complete the task, a topic which will be the focus of future work.

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References

1. J. Duperret and D. E. Koditschek, “Towards reactive control of simplified legged robotics maneuvers,” in *International Symposium on Robotics Research*, December 2017, accepted.
2. P. Suppes, *Axiomatic Set Theory*. Dover Publications, 1972.
3. D. Bertsimas and J. N. Tsitsiklis, *Introduction to linear optimization*. Athena Scientific Belmont, MA, 1997, vol. 6.
4. S. V. Rakovi, E. C. Kerrigan, D. Q. Mayne, and J. Lygeros, “Reachability analysis of discrete-time systems with disturbances,” *IEEE Transactions on Automatic Control*, vol. 51, no. 4, pp. 546–561, 2006.