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Abstract

The "mechanical systems" define a large and important class of highly nonlinear dynamical equations which, for example, encompasses all robots. In this report it is shown that a strict Lyapunov Function suggested by the simplest examplar of the class — a one degree of freedom linear time invariant dynamical system — may be generalized over the entire class. The report lists a number of standard but useful consequences of this discovery. The analysis suggests that the input-output properties of the entire class of nonlinear systems share many characteristics in common with those of a second order, phase canonical, linear time invariant differential equation.

1 Introduction

The "mechanical systems" define a large and important class of highly nonlinear dynamical equations. Considered as control systems specified in terms of an "internal" differential equation,

$$M[q]\ddot{q} + B[q, \dot{q}]\dot{q} = \tau, \tag{1}$$

the apparent complexity of all but the simplest examples of this class, would seem to preclude any possibility of input-output analysis reminiscent of the familiar linear time invariant case. In that context, of course, the availability of frequency domain techniques affords effective translation of such questions into a tractable algebraic form.

In recent years, the advent of mechanically complex and computationally powerful robotic systems has greatly spurred the search for *practicable* engineering insight into the techniques for their control. In such situations, where M may be interpreted as a generalized the "inertia matrix", $B\dot{q}$ represents the "coriolis and centripetal forces", and $q \in J$ is a vector of joint measurements, equation (1) generally contains thousands of transendental expressions. While this is an idealized model, it has been experimentally demonstrated that these nonlinearities are not analytical artifacts but correspond to physically pronounced variations in dynamical coupling coefficients which vary by orders of magnitude over the mechanical workspace [2].

At a recent conference, the author reported his construction of a strict Lyapunov function for the class of mechanical systems [12]. In this report it is shown that this function, suggested by the simplest examplar of the class — a one degree of freedom linear mechanical system — has stronger properties which will be termed "quadratic" in the sequel, and which may be generalized over the entire class as well. A standard application of Lyapunov theory leads to Theorem 2 in Section 1.3 which relates bounds derived from this Lyapunov function to bounds on the tracking error of mechanical systems forced by bounded reference signals. The class of mechanical systems is defined precisely in Section 2 and a number of useful structural properties of the class are derived there as well. This structure is used in Section 3 to "fix" Lord Kelvin's Lyapunov function for mechanical systems — total mechanical energy (Theorem 3) — by analogy to the linear case. Since the Lyapunov function bounds are, themselves, functions of the original nonlinear system parameters, algebraic relationships between feedback gains, plant parameters, and tracking performance are obtained and reported in Theorem 4 Theorem 5 , Theorem 6 , Theorem 7 of Section 4, under different assumptions about the information concerning reference signal and plant parameters available for use in the control scheme. The prospects for deriving a systematic engineering design methodology for mechanical systems analogous to the frequency domain techniques of linear time invariant systems theory are briefly explored in Section 5.

1.1 Second Order Linear Time Invariant Control Systems

In recent years, the bulk of theory concerning the control of nonlinear plants seems to have become fixed upon the idea of "global linearization". In the context of robotics, wherein one assumes the availability of n indpendent control inputs for each of n degrees of mechanical freedom, this idea translates into the exact cancellation of all the nonlinear dynamical terms, leaving behind a linear time invariant system whose Brunovsky Canonical Form takes the form of n independent double integrators. In the robotics literature, this methodology (commonly known as the "computed torque" technique [2,5,15,24]), is usually complemented by the addition

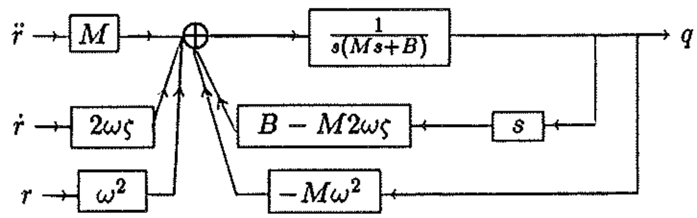

Figure 1: "Computed Torque" Technique Applied to LTI Plant.

of linear gains in the feedback path and preceded by a feedforward pre-compensator which inverts the resulting exponentially stable closed loop linear dynamics.

For example, consider the one degree of freedom mechanical system depicted in Figure 1, with mass, M, and viscous damping coefficient, B. Given some reference trajectory, r, the figure illustrates that this control strategy amounts to pole placement via state feedback preceded by inverse dynamics. The result is what might be termed "exponentially exact" tracking. Analytically, if we apply the control

$$\tau_{pda} \triangleq -M\omega^2 q + (B - M2\omega\varsigma)\dot{q} + M\left[\ddot{r} + 2\omega\varsigma\dot{r} + \omega^2 r\right],$$

and define the "error coordinates" in phase space,

$$e \triangleq \begin{bmatrix} q - r \\ \dot{q} - \dot{r} \end{bmatrix}, \tag{2}$$

then the resulting error dynamics take the form,

$$\dot{e} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\omega\varsigma \end{bmatrix} e,$$

of an exponentially stable system unforced by any disturbance. It follows that the error between the desired and true signal decays to zero at an exponential rate determined by the eigenvalues of the system matrix whose entries, $2\omega\varsigma, \omega^2$, have been chosen by the designer. Of course, this scheme makes a number of unusual assumptions about the availability of information concerning both the plant and the reference trajectory. For example, if the reference signal, r is not known ahead of time, it is generally impossible to obtain a reasonable version of $\dot{r}, \ddot{r}$, as required. Moreover, it is not generally possible to match plant parameters, M, B, exactly.

An alternative to the control scheme displayed in Figure 1 is presented in Figure 2 In this case the resulting error system may be seen to evince what might be called "exponentially small" tracking error. Analytically, the control takes the form

$$\tau_p \triangleq -\omega^2(q - r) - 2\omega\varsigma\dot{q} \tag{3}$$

If we define a "half translated" error system,

$$\bar{e} = \begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \end{bmatrix} \triangleq \begin{bmatrix} q - r \\ \dot{q} \end{bmatrix}, \tag{4}$$

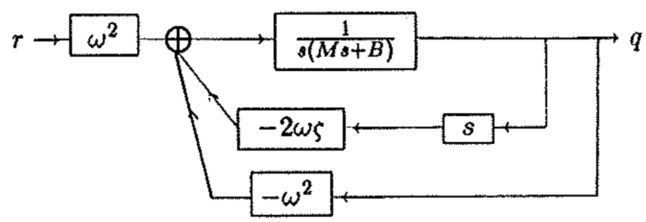

Figure 2: "Error-P Scheme": Traditional Approach Given Minimal Information.

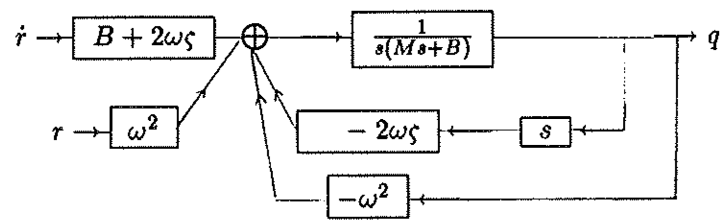

Figure 3: "Error-PD Scheme": Using Partial Information About the Plant and Reference Trajectory.

then the error dynamics may be written as,

$$\dot{\bar{e}} = \begin{bmatrix} 0 & 1 \\ -M^{-1}\omega^2 & -M^{-1}(B+2\omega\varsigma) \end{bmatrix} \bar{e} + \begin{bmatrix} \dot{r} \\ 0 \end{bmatrix}, \tag{5}$$

an exponentially stable system (assuming $\omega\varsigma > -\frac{1}{2}B$), forced by a disturbance due to the "unknown" reference velocity. Exponentially exact results obtain in the case that $\dot{r} \rightarrow 0$. Otherwise, the tracking error will decay at an exponential rate (which increases with the feedback gains) toward a steady state bound (proportional the magnitude $\dot{r}$ and inversely proportional to the magnitude of the feedback gains).

The underlying control concept is that of performance based upon high gain feedback rather than perfect information. Namely, with no à priori information concerning the plant or reference signal, "reasonable tracking" may be assured by sufficiently high gain feedback. Of course, as more information regarding the reference signal or plant parameters becomes available, it may be used to good effect in the sort of compromise control scheme depicted in Figure 3

In this case, the control input takes the form,

$$\tau_{pd} \triangleq -\omega^2(q-r) - 2\omega\varsigma(\dot{q}-\dot{r}) + B\dot{r} \tag{6}$$

and, in terms of the original error coordinates, (2),

$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \triangleq \begin{bmatrix} q-r \\ \dot{q}-\dot{r} \end{bmatrix},$$

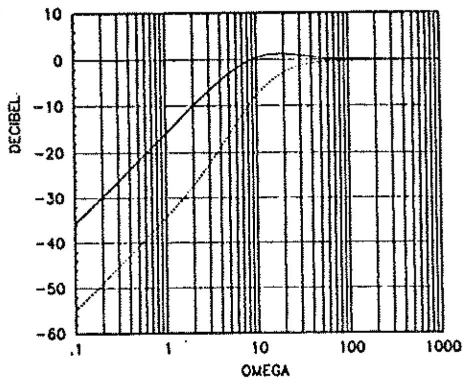

Figure 5: Comparison of Error-P (solid line) and Error-PD (dotted line) Transfer Functions

the error dynamics are given by

$$\dot{e} = \begin{bmatrix} 0 & 1 \\ -M^{-1}\omega^2 & -M^{-1}(B + 2\omega\varsigma) \end{bmatrix} e + \begin{bmatrix} 0 \\ \ddot{r} \end{bmatrix}, \tag{7}$$

which has the same structure as (5) with the exception that the disturbance is due solely to the reference acceleration. Clearly, if $\ddot{r} \to 0$, then exponentially exact results obtain, once again. Otherwise, a similar statement may be made concerning performance as in the previous case.

It is easy to see that the compromise "Error-pd" scheme, while still unable to deliver exponentially exact performance in the general case, is better than the minimal information scheme. For example, Figure 4 which compares the error transfer function,

$$E_p(s) \triangleq s \cdot \frac{Ms + (B + 2\omega\varsigma)}{Ms^2 + (B + 2\omega\varsigma)s + \omega^2},$$

from r to $\bar{e}_1 = e_1$ resulting from (5) (solid line in Figure 4) with

$$E_{pd}(s) \triangleq s^2 \cdot \frac{M}{Ms^2 + (B + 2\omega\varsigma)s + \omega^2},$$

resulting from (7) (dotted line in Figure 4) gives useful qualitative information concerning the value of the additional knowledge of B and $\dot{r}$ used in the latter.

In fact, similar information regarding the input-output behavior of linear time invariant systems may be obtained in a variety of ways. In this report, it is shown that recourse to the appropriate Lyapunov functions offers an approach toward this sort of information which, unlike the case of frequency domain methods, appears to generalize over the entire class of nonlinear mechanical systems. Perhaps more importantly, these arguments show that traditional linear proprotional and derivative feedback renders a mechanical system exponentially stable, so that adjusting feedforward pre-compensators to take advantage of increased information can never produce an unbounded response *no matter how inexact that information may be* so long as the pre-compensator output is bounded.

Specifically, consider the 2×2 array,

$$A \triangleq \begin{bmatrix} 0 & 1 \\ -M^{-1}\omega^2 & -2M^{-1}\omega\varsigma \end{bmatrix}.$$

Note that the eigenvalues are given by

$$-\frac{\omega}{M}\left(\varsigma \pm \sqrt{\varsigma^2 - M}\right).$$

When the system is underdamped, i.e., when $\varsigma < \sqrt{M}$, then

$$P \triangleq \begin{bmatrix} \omega^2 & \omega\varsigma \\ \omega\varsigma & M \end{bmatrix}$$

is a positive definite eigenvector of the "Lyapunov Operator" on $\mathbb{R}^{2\times 2}$,

$$PA + A^{\mathrm{T}}P = -\frac{\omega\varsigma}{M}P.$$

Thus, the fact that the tracking performance of a linear control system must improve as ω increases, is readily demonstrated with no appeal to a notion of the "system poles." It is this insight which we shall exploit in the rest of the report.

1.2 Notation and Definitions

Definitions:

In the sequel, we will use the notation C^n to denote maps which are continuously differentiable n times. The set of analytic maps from $\mathcal{X}$ to $\mathcal{Y}$ is denoted $C^{\omega}[\mathcal{X}, \mathcal{Y}]$, and the subset of linear operators, $\mathcal{L}(\mathcal{X}, \mathcal{Y})$. If $f \in C^1[\mathbb{R}^n, \mathbb{R}^m]$ denote its $m \times n$ jacobian matrix as

$$Df \triangleq \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

If $m = 1$ then Df is a gradient; if $n = 1$ then Df is a tangent vector. It will often be necessary to obtain the jacobian of a matrix valued map — $m = p \times q$ — and the kronecker-stack notation presented in the Appendix is of great help in this regard. When we require only a subset of derivatives, e.g. when $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and we desire the jacobian of f with respect to the variables $x_1 \in \mathbb{R}^{n_1}$, as x_2 is held fixed, we may write

$$D_{x_1} f \triangleq Df \begin{bmatrix} I_{n_1 \times n_1} \\ 0 \end{bmatrix}.$$

The equations of motion (1) are defined on "phase space", $\mathcal{P} \triangleq T\mathcal{J}$, the tangent bundle over some "configuration space", $\mathcal{J}$. For ease of exposition, assume now, and in the sequel, that $\mathcal{J}$ is a simply connected subset of $\mathbb{R}^n$, so that we may introduce the euclidean metric, $\|\cdot\|$, and $\mathcal{P}$ may be identified with $\mathbb{R}^{2n}$. [1] Thus, in the sequel, the tangent space at q, $T_q\mathcal{J}$, will be identified with $\mathbb{R}^n$, and linear operators, $A \in \mathcal{L}(T_q\mathcal{J}, T_q\mathcal{J})$, will be identified with their matrix representations under the appropriate coordinate system.

[1] For example, in the case of a kinematic chain, we assume that there is some means of counting the number of absolute revolutions of non-prismatic joints.

Operator Bounds:

It will be important to obtain bounds on the operator norm of matrix valued maps. If $A : J \to \mathbb{R}^{n\times n}$ is a map taking matrix values then define

$$\mu_\eta(A) \triangleq \sup_{\|q\|\leq\eta} \sup_{\|x\|=1} |x^{\mathrm{T}} A(q) x| \qquad\qquad \nu_\eta(A) \triangleq \inf_{\|q\|\leq\eta} \inf_{\|x\|=1} |x^{\mathrm{T}} A(q) x|.$$

and

$$\mu(A) \triangleq \lim_{\eta\to\infty} \mu_\eta(A) \qquad\qquad \nu(A) \triangleq \lim_{\eta\to\infty} \nu_\eta(A).$$

It is always the case that

$$x^{\mathrm{T}} A(q) x \leq \mu_\eta(A)\|x\|^2,$$

and if A is continuous in q then $\mu_\eta(A)$ is always bounded for $\eta < \infty$. On the other hand,

$$\nu_\eta(A)\|x\|^2 \leq x^{\mathrm{T}} A(q) x$$

only if A takes symmetric positive definite values over the domain of radius η in J, and, in this case, $\nu_\eta(A) > 0$. For any constant symmetric matrix, $\mu(A)$ is the square root of the eigenvalue of greatest magnitude, while $\nu(A)$ is the square root of the eigenvalue of least magnitude of $A^{\mathrm{T}}A$, from which it follows that

$$\mu_\eta(A) = \sup_{\|q\|\leq\eta} \|A(q)\| \qquad\qquad 1/\nu_\eta(A) = \sup_{\|q\|\leq\eta} \|A^{-1}(q)\|,$$

where $\|\cdot\|$ denotes the operator norm induced by the euclidean norm of $\mathbb{R}^n$.

These bounds extend to bilinear operator valued maps in a straightforward fashion. If $B(q,\cdot) \in \mathcal{L}(T_qJ, \mathcal{L}(T_qJ, T_qJ))$ then given any basis of T_qJ, there is a "bilinear representation"

$$B(q,x)y = \begin{bmatrix} x^{\mathrm{T}} B_1(q) y \\ \vdots \\ x^{\mathrm{T}} B_n(q) y \end{bmatrix},$$

where $B_i : J \to \mathbb{R}^{n\times n}$ is a linear operator valued map for each i. It is natural to define

$$\mu_\eta(B) \triangleq \sup_{\|q\|\leq\eta} \sup_{\|x\|=1} \sup_{\|y\|=1} \|B(q,x)y\| \leq \left\| \begin{bmatrix} \mu_\eta(B_1) \\ \vdots \\ \mu_\eta(B_n) \end{bmatrix} \right\|,$$

and it follows that

$$\|B(q,x)y\| \leq \mu_\eta(B)\|x\| \cdot \|y\|.$$

for all q in a set of radius η within J.

Lyapunov Functions:

Given an open set $\mathcal{S} \subset \mathcal{P}$, a smooth (possibly time varying) scalar valued map, $v : \mathbb{R} \times \mathcal{S} \to \mathbb{R}$ is said to be *positive definite at a point* $p_d \in \mathcal{S}$ if, for all t, $v(t,p) = 0$, and $v > 0$ in some open neighborhood of p_d. Given a smooth (possibly time varying) vector field, f, on $T\mathcal{J}$, we shall say that, v, a positive definite map at $p_d \in \mathcal{S}$, constitutes a *Lyapunov function for f at p_d* if the time derivative along any motion of the vector field is non-positive,

$$\dot{v} = [D_p v]f + D_t v \leq 0,$$

in some neighborhood of p_d, for all t, and that it constitutes a *strict Lyapunov function for f* if the inequality is strict [10,14]. The *domain* of v with respect to p_d is the largest open subset of S containing p_d which satisfies the sign condition. A strict Lyapunov function will be called a *quadratic Lyapunov function for f on the domain, $\mathcal{S}$* if it is analytic and there exist three positive constants, $\alpha_1, \alpha_2, \alpha_3$, with the properties,

$$\alpha_1 \|p - p_d\|^2 \leq v(p,t) \leq \alpha_2 \|p - p_d\|^2 \text{ and } \dot{v}(p,t) \leq -\alpha_3 \|p - p_d\|^2 \tag{8}$$

for all $t \in \mathbb{R}^+$ and $p \in \mathcal{D}$.

1.3 Some Results from Lyapunov Theory

The existence of a strict Lyapunov function at a point is a sufficient condition for asymptotic stability of that equilibrium state. If a strict Lyapunov function has not been found, asymptotic stability may, nevertheless, be assured if a further condition on the possible limiting set holds. This is "LaSalle's Invariance Principle" [14]. It is possible, as well, to draw conclusions about the tracking capability of a forced dynamical system in consequence of of the stability properties of the unforced vector field at a particular equilibrium state. However, this seems to require the use of a strict Lyapunov function.

It has been known for over a century that the total energy of a mechanical system may be interpreted as a Lyapunov function [25]. Unfortunately, this choice of Lyapunov function is never strict, and an appeal to LaSalle's invariance principle is required. This report makes systematic use of a strict Lyapunov function in Section 4 which turns out to be quadratic (in the sense defined above). The tracking results follow as a standard consequence.

In order to introduce Lasalle's Invariance principle, one further definition is required. A *positive invariant set* relative to some vector field, f, is a set in state space with the property that any trajectory originating there stays there for all future time.

Theorem 1 (LaSalle's Invariance Principle [14]) *If v is a Lyapunov function for the time invariant vector field f on some pre-compact domain $\mathcal{D}$, then any trajectory originating in that domain approaches the largest positive invariant set contained within the subset of $\mathcal{D}$ with the property that $\dot{v} = 0$.*

In order to make use of quadratic Lyapunov functions, the following technical result is required.

Lemma 1 *If*

$$\dot{v} \leq \phi(v),$$

and $u(t)$ is a maximal solution to the differential equation, $\dot{u} = \phi(u)$, and $v(t_0) \leq u(t_0)$, then

$$v(t) \leq u(t),$$

for all $t > t_0$.

Proof: This is a standard application of a differential inequality. For example, see the refernce [9][Theorem III.4.1].

□

This useful fact leads to a variety of standard results (e.g. see [8]) involving transient and steady state behavior of disturbed dynamical systems. The following will prove particularly useful in the sequel.

Theorem 2 *Consider the disturbed dynamical system*

$$\dot{x} = A(t,x)x + d(t)$$

If v is a quadratic Lyapunov function at the point 0 for the undisturbed system (i.e. $d \equiv 0$) on some domain, $\mathcal{D}$, and there may be found a positive constant, α_4, such that

$$|D_x v\, d| \leq \alpha_4 \|x\|$$

then the response of the disturbed system from any initial condition, $x(t_0) \in \mathcal{D}$, is bounded by

$$\|x(t)\| \leq e^{-\frac{1}{2}\rho t}\chi + \beta$$

where

$$\rho \triangleq \frac{\alpha_3}{\alpha_2};\ \beta \triangleq \frac{\alpha_2\alpha_4}{\alpha_1\alpha_3};\ \chi \triangleq \frac{\alpha_2}{\sqrt{\alpha_1}}\|x(t_0)\|^2,$$

and the $\alpha_i, i = 1,3$, are defined by (8).

Proof: According to the hypothesis we have

$$\begin{aligned} \dot{v} &= D_x v\,[Ax + d] + D_t v \\ &\leq -\alpha_3\|x\|^2 + \alpha_4\|x\| \\ &\leq -\rho v + \tfrac{\alpha_4}{\sqrt{\alpha_1}}\sqrt{v}. \end{aligned}$$

The differential equation

$$\dot{u} = -\rho u + \frac{\alpha_4}{\sqrt{\alpha_1}}\sqrt{u},$$

is locally lipschitz around every positive initial condition. Thus,

$$u(t) \triangleq \left(e^{-\frac{1}{2}\rho t}u(0) + \frac{\alpha_4}{\sqrt{\alpha_1}}\right)^2$$

is its unique (and, hence, maximal) solution as long as $u(0) > 0$. It follows from Lemma 1 that

$$v(t) \leq \left(e^{-\frac{1}{2}\rho t}v(0) + \frac{\alpha_4}{\sqrt{\alpha_1}}\right)^2.$$

The result obtains, in turn, by noting that $\alpha_1\|x\|^2 \leq v$, according to (8).

□

2 The Dynamical Structure of Mechanical Systems

Define a *mechanical system* to be any lagrangian dynamical system whose kinetic energy may be expressed locally as a quadratic form in generalized velocity and which is analytic in generalized position. For example, it will be shown in Lemma 2 that equation (1) arises from the application of the Euler-Lagrange derivative operator

$$\frac{d}{dt}D_{\dot{q}} - D_q \tag{9}$$

to the expression for total kinetic energy, κ, balanced by external torques or forces, τ, applied to every unconstrained degree of freedom. In more geometric terms, we assume that κ is an analytic Riemannian metric defined on some analytic manifold, J. Let $q : \mathbb{R} \rightarrow J$ be a smooth curve in J, whose tangent vector at a point, $q(t_0)$, along the curve will be denoted

$$\dot{q}(t_0) \triangleq \frac{d}{dt}q(t)\mid_{t=t_0} \in T_{q(t_0)}J.$$

Similarly, $\ddot{q}(u)$, denotes the "acceleration" — the tangent vector to the induced curve

$$(q,\dot{q}) : \mathbb{R} \rightarrow TJ.$$

The metric is locally described by the analytic matrix valued map, $M \in C^\omega[J, \mathcal{L}(TJ,TJ)]$, which takes values in the set of positive definite symmetric arrays, and it is now possible to write [2]

$$\kappa \triangleq \frac{1}{2}\dot{q}^{\mathrm{T}}M(q)\dot{q}, \tag{10}$$

Rather than introducing the notion of a Lie derivative, let the symbol $\dot{M}(q)$ be used to denote the derivative, of the map M along the curve q,

$$\dot{M}(q) \triangleq \frac{d}{dt}M \circ q.$$

2.1 Derivation of the Equations of Motion

It seems simplest to use the "stack-kronecker notation" (refer to the Appendix) in computing the equations of motion. Since it will be necessary to consider the derivative of M as a linear operator on the tangent space, the discussion profits from the introduction of a little more notation. Let $\acute{M}_q \in \mathcal{L}(T_qJ, \mathcal{L}(T_qJ, T_qJ))$ denote the linear map

$$\acute{M}_q : x \mapsto [x \otimes I]^{\mathrm{T}}\, D_q M^{\mathrm{S}}$$

taking tangent vectors at q, $x \in T_qJ$, to $\mathcal{L}(T_qJ, T_qJ)$ — the set of linear operators on tangent vectors at q. Thus, for each $q \in J$, $\acute{M}_q$ is a bilinear map on T_qJ Since M is analytic, $\acute{M}$, is also analytic with respect to q. Note that for any curve, $q(t)$, and tangent vector, $x \in T_qJ$,

$$\dot{M}(q)x = [x \otimes I]^{\mathrm{T}}\left(\frac{d}{dt}M^{\mathrm{S}}\right) = [x \otimes I]^{\mathrm{T}}\left(D_q M^{\mathrm{S}}\,\dot{q}\right) = \acute{M}_q(x)\dot{q} \tag{11}$$

[2] In the case at hand, it is an abuse of notation not to specify M in terms of the action of an image value on the point $p = (p_1, p_2) \in TJ$ — e.g. $M(p) \triangleq (p_1, \tilde{M}(p_1)p_2)$, where $\tilde{M}(p_1)$ has a matrix representation in $\mathbb{R}^{n\times n}$. For the present purposes, there seems to be considerable notational advantage and little conceptual danger in identifying M with the matrix representation of $\tilde{M}$.

while

$$D_q[M(q)x] = [x \otimes I]^{\mathrm{T}} \left(D_q M^{\mathbf{S}}\right) = \acute{M}_q(x). \tag{12}$$

It is worth pointing out that

$$\acute{M}_q(x) = [I \otimes x]^{\mathrm{T}} D_q M^{\mathbf{S}},$$

as may be seen by noting

$$\left[\dot{M}(q)x\right]^{\mathbf{S}} = \left[x^{\mathrm{T}}\dot{M}(q)\right]^{\mathbf{S}} = [I \otimes x]^{\mathrm{T}} D_q M^{\mathbf{S}}\, \dot{q},$$

since M is symmetric. We may now derive equation (1).

Lemma 2 *Given a lagrangian with kinetic energy, κ, as defined by (10), with no potential forces present, and with an external torque or force acting at every degree of freedom as specified by the vector, τ, the equations of motion may be written in the form (1),*

$$M(q)\ddot{q} + B(q,\dot{q})\dot{q} = \tau,$$

where

$$B(q,x) \triangleq \acute{M}_q(x) - \frac{1}{2}\acute{M}_q^{\mathrm{T}}(x). \tag{13}$$

Proof: If there is no potential energy then κ constitutes the entire Lagrangian, hence, application of the Euler-Lagrange operator, (9), to the kinetic energy,

$$\tau^{\mathrm{T}} = \left(\frac{d}{dt}D_{\dot{q}} - D_q\right)\kappa,$$

balanced by the externally applied torques and forces, τ, yields Newton's equations of motion [1,6]. This may be computed as follows:

$$\begin{aligned} \tfrac{d}{dt}D_{\dot{q}}\kappa &= \ddot{q}^{\mathrm{T}}M(q) + \dot{q}^{\mathrm{T}}\dot{M}(q) \\ &= \ddot{q}^{\mathrm{T}}M(q) + \dot{q}^{\mathrm{T}}\left[\acute{M}_q(\dot{q})\right]^{\mathrm{T}} \end{aligned}$$

from (11), while

$$\begin{aligned} D_q\kappa &= \tfrac{1}{2}\dot{q}^{\mathrm{T}}D_q[M(q)\dot{q}] \\ &= \tfrac{1}{2}\dot{q}^{\mathrm{T}}\acute{M}_q(\dot{q}) \end{aligned}$$

from (12). Collecting the terms which are quadratic in $\dot{q}$, we arrive at the conclusion,

$$\tau = M(q)\ddot{q} + \underbrace{\left[\acute{M}_q(\dot{q}) - \frac{1}{2}\left(\acute{M}_q(\dot{q})\right)^{\mathrm{T}}\right]}_{B(q,\dot{q})}\dot{q}. \tag{14}$$

□

The representation of coriolis and centripetal forces (i.e. those terms which are quadratic in the generalized velocity, $\dot{q}$)

$$\dot{M} - \frac{1}{2}\left(D_q M\dot{q}\right)^{\mathrm{T}}\dot{q},$$

in terms of the bilinear operator, B, is merely a convenience of exposition. Indeed, it is worth including the reminder at this juncture that two linear operators, A, B, which agree at one point will not, in general, agree at all points. In particular, even though

$$B(q,\dot{q})\dot{q} = \dot{M} - \frac{1}{2}\left(D_q M\dot{q}\right)^{\mathrm{T}}\dot{q}$$

for all $(q,\dot{q}) \in TJ$ in general, the operators, themselves, are not equal, i.e.,

$$B(q,\dot{q}) \neq \dot{M} - \frac{1}{2}\left(D_q M\dot{q}\right)^{\mathrm{T}}$$

In the sequel, it will be important to distinguish the "skew-symmetric" part of B and the part which is not. Accordingly, define the skew-symmetric valued operator,

$$J_q(x) \triangleq \acute{M}_q(x) - \acute{M}_q^{\mathrm{T}}(x). \tag{15}$$

Note that the action of B may be re-expressed as

$$B(q,x)y = \frac{1}{2}[\acute{M}_q(x) + J_q(x)]y.$$

At the expense of still more notation, it is now helpful to define another operator valued map on TJ which will express the "defect" from skew-symmetry of $\frac{1}{2}\dot{M} - B$. Corresponding to any tangent vector at q, $x \in T_qJ$, and any time, $t_0 \in \mathbb{R}$, there exists a smooth curve $q_x : \mathbb{R} \to J$ with the property that

$$q_x(t_0) = q \text{ and } \dot{q}_x(t_0) = x.$$

Thus, it is possible to define a linear map, $L_q \in \mathcal{L}(T_qJ, \mathcal{L}(T_qJ, T_qJ))$,

$$L_q(x) \triangleq \dot{M}(q_x) - \acute{M}_q(x),$$

whose action on any tangent vector, $y \in T_qJ$ is given by

$$L_q(x)y = \acute{M}_q(y)x - \acute{M}_q(x)y = \dot{M}(q_x)y - \dot{M}(q_y)x. \tag{16}$$

for every q, both J_q, L_q are bilinear maps on T_qJ, and both are are analytic in q. Note that [3]

$$J_q(x)y \in \text{ Ker } y^{\mathrm{T}}$$

for any tangent vectors, $x, y \in T_qJ$, while in general,

$$y \in \text{ Ker } L_q(x)$$

only if $y \in$ Im x.

[3] Another abuse of notation admits the representation of vectors in the cotangent bundle, T^*J using transposes.

Lemma 3 *Given any motion, $q(t) \in J$, any kinetic energy, κ, of the form (10), and the representation of "coriolis and centripetal" forces, B, as defined in (13), the action of $\frac{1}{2}\dot{M} - B$ evaluated along the motion, q, on any tangent vector $x \in T_qJ$ may be written as*

$$[\frac{1}{2}\dot{M}(q) - B(q,\dot{q})]x = \frac{1}{2}L_q(\dot{q})x - \frac{1}{2}J_q(\dot{q})x$$

Proof: Consider a tangent vector at time t, say $x \in T_{q(t)}J$. We have

$$(\frac{1}{2}\dot{M} - B)x = \frac{1}{2}\acute{M}_q(x)\dot{q} - \acute{M}_q(\dot{q})x + \frac{1}{2}\left(\acute{M}_q(\dot{q})\right)^{\mathrm{T}} x \tag{17}$$

according to (11) and (13). The result follows simply by applying the definitions (15) and (16).

□

Corollary 4 *For any motion $q : \mathbb{R} \rightarrow J$,*

$$\dot{q}^{\mathrm{T}}[\frac{1}{2}\dot{M} - B]\dot{q} \equiv 0.$$

Proof: From the previous lemma,

$$x^{\mathrm{T}}[\frac{1}{2}\dot{M}(q) - B(q,\dot{q})]x = \frac{1}{2}x^{\mathrm{T}}L_q(\dot{q})x - \frac{1}{2}x^{\mathrm{T}}J_q(\dot{q})x$$

and the second term is identically zero since $J_q(y)x \in \text{Ker } x^{\mathrm{T}}$ for any $x, y \in T_qJ$. When (and, in general, only when) x is a scalar multiple of $\dot{q}$, i.e. $x = \alpha\dot{q}$, then $x \in \text{Ker } L_q(\dot{q})$, and the first term is zero as well.

□

2.2 Total Energy as a Lyapunov Function

Corollary 4 leads to a quick proof of the following long understood property [25]. Later, we will express this as a solution of the "set point regulation" problem for mechanical nonlinear systems. Given the point,

$$p \triangleq \begin{bmatrix} r \\ 0 \end{bmatrix} \in \mathcal{P},$$

and the error coordinate system,

$$e \triangleq \begin{bmatrix} q - r \\ \dot{q} - \dot{r} \end{bmatrix} \equiv \begin{bmatrix} q - r \\ \dot{q} \end{bmatrix},$$

consider the nonlinear time invariant dynamical system, $\dot{e} = A(e)e$, on $\mathcal{P}$ where

$$A \triangleq \begin{bmatrix} 0 & I \\ -M^{-1}(e_1 + r)K_1 & -M^{-1}(e_1 + r)[B(e_1 + r) + K_2] \end{bmatrix}, \tag{18}$$

and K_1, K_2 are constant, positive definite, symmetric matrices.

Theorem 3 (**Lord Kelvin [25]**) *The origin of the system defined by (18) is asymptotically stable if K_1, K_2 are positive definite symmetric matrices.* [4]

Proof: The Lyapunov Function

$$v \triangleq \frac{1}{2}[e_1^T K_1 e_1 + e_2^T M(e_1 + r)e_2]$$

has time derivative

$$\dot{v} = e_1^T K_1 e_2 - e_2^T[(B + K_2)e_2 + K_1 e_1] + \frac{1}{2}e_2^T \dot{M} e_2$$

and since $e_2^T[\frac{1}{2}\dot{M} - B]e_2 \equiv 0$ as shown in Corollary 4 , this evaluates to

$$\dot{v} = -e_2^T K_2 e_2 \leq 0.$$

According to LaSalle's invariance principle, the attracting set is the largest invariant set contained in $\{(e_1, e_2) \in \mathcal{P} : \dot{v} \equiv 0\}$, which, evidently, is the origin since the vector field is oriented away from $\{e_2 \equiv 0\}$ everywhere else on that hyperplane.

□

Unfortunately, the "skew-symmetry" properties explored in the previous section, and exploited in the proof of Lord Kelvin's result seem to have been misinterpreted in the robotics literature [19,13,21]. In particular, it is critical to note that the "defect from skew-symmetry", L_q must be taken into account when the motion with respect to which differentiation is occuring is expressed in terms of a coordinate translation as is always the case using error coordinates. An example of a misunderstanding of Lemma 3 is provided by the author's own earlier statement

Corollary 5 (**[13]**) *If r is a desired motion then the output of system (1) in response to the control strategy*

$$\tau = -K_1(q - r) - K_2(\dot{q} - \dot{r}) + M\ddot{r} + B\dot{r}$$

asymptotically approaches r if K_1, K_2 are positive definite symmetric matrices.

While this statement proves to be correct given additional assumptions on the reference signal and the feedback gain magnitudes, as shown below in Proposition 12 , the proof proposed in [13] relies upon an incorrect application of Lemma 3 , hence, is wrong.

2.3 Two Simple Examples

Example 1: Planar PR Arm

Consider the planar prismatic-revolute robot arm in the horizontal plane with a point mass at the first joint, m_1, and at the gripper, m_2,

$$q \triangleq \begin{bmatrix} \xi \\ \theta \end{bmatrix} \qquad p(q) = \begin{bmatrix} a\sin\theta \\ -\xi - a\cos\theta \end{bmatrix}$$

[4] This result has been subsequently and independently "rediscovered" many times [23,13,18,20]. The proof here is taken from [13].

with kinetic energy

$$\kappa(q,\dot q) = \frac{1}{2}\left[\dot\xi, \dot\theta\right]\begin{bmatrix} m_1+m_2 & -m_2 a\sin\theta \\ -m_2 a \sin\theta & m_2 a^2 \end{bmatrix}\begin{bmatrix}\dot\xi \\ \dot\theta\end{bmatrix},$$

and zero potential energy. We have

$$\dot M = -m_2 a\cos\theta \begin{bmatrix} 0 & \dot\theta \\ \dot\theta & 0 \end{bmatrix}, \qquad (D_q M\dot q\,)^{\mathrm T} = -m_2 a\cos\theta\begin{bmatrix} 0 & 0 \\ \dot\theta & \dot\xi\end{bmatrix}$$

hence,

$$\dot M - \frac{1}{2}\left(D_q M\dot q\,\right)^{\mathrm T} = m_2 a\cos\theta\begin{bmatrix} 0 & -\dot\theta \\ -\frac{1}{2}\dot\theta & \frac{1}{2}\dot\xi\end{bmatrix}$$

and

$$\frac{1}{2}\dot M - \left[\dot M - \frac{1}{2}\left(D_q M\dot q\,\right)^{\mathrm T}\right] = m_2 a\cos\theta\begin{bmatrix} 0 & \frac{1}{2}\dot\theta \\ 0 & -\frac{1}{2}\dot\xi\end{bmatrix}.$$

The latter is clearly not a skew-symmetric linear operator.

However, notice that the action on $\dot q$,

$$\left(\frac{1}{2}\dot M - \left[\dot M - \frac{1}{2}\left(D_q M\dot q\,\right)^{\mathrm T}\right]\right)\dot q = m_2 a\cos\theta\begin{bmatrix} 0 & \frac{1}{2}\dot\theta \\ -\frac{1}{2}\dot\theta & 0\end{bmatrix}\dot q,$$

may, indeed, be expressed in terms of the product of $\dot q$ with a skew-symmetric matrix.

To express these terms using the bilinear operator valued maps, $B(q,\cdot)$, J_q, L_q, defined above, first note that

$$D_q M^{\mathrm S} = \begin{bmatrix} 0 & 0 \\ 0 & -m_2 a\cos\theta \\ 0 & -m_2 a\cos\theta \\ 0 & 0\end{bmatrix},$$

hence

$$\acute M_q(x) = (x\otimes I)^{\mathrm T}\, D_q M^{\mathrm S} = -m_2 a\cos\theta\begin{bmatrix} 0 & x_2 \\ 0 & x_1\end{bmatrix},$$

and

$$B(q,x) = -m_2 a\cos\theta\begin{bmatrix} 0 & x_2 \\ -\frac{1}{2}x_2 & \frac{1}{2}x_1\end{bmatrix}.$$

Note that while

$$\dot M - \frac{1}{2}\left(D_q M\dot q\,\right)^{\mathrm T} = -m_2 a\cos\theta\begin{bmatrix} 0 & \dot\theta \\ \frac{1}{2}\dot\theta & -\frac{1}{2}\dot\xi\end{bmatrix} \neq -m_2 a\cos\theta\begin{bmatrix} 0 & \dot\theta \\ -\frac{1}{2}\dot\theta & \frac{1}{2}\dot\xi\end{bmatrix} = B(q,\dot q)$$

it is certainly the case that

$$\dot M - \frac{1}{2}\left(D_q M\dot q\,\right)^{\mathrm T}\dot q = -m_2 a\cos\theta\begin{bmatrix}\dot\theta^2 \\ 0\end{bmatrix} = B(q,\dot q)\dot q.$$

Now define

$$J_q(x) \triangleq \acute{M}_q(x) - \acute{M}_q(x)^{\mathrm{T}} = m_2 a \cos\theta \begin{bmatrix} 0 & -x_2 \\ x_2 & 0 \end{bmatrix}$$

$$L_q(x) \triangleq \acute{M}_q(x) - \dot{M}(q_x) = -m_2 a \cos\theta \begin{bmatrix} 0 & 0 \\ x_2 & -x_1 \end{bmatrix}.$$

and note that, indeed,

$$\left[\frac{1}{2}\dot{M}(q) - B(q,\dot{q})\right]\dot{q} = \frac{1}{2}\left[L_q(\dot{q}) - J_q(\dot{q})\right]\dot{q} = -\frac{1}{2}J_q(\dot{q})\dot{q}.$$

Example 2: Planar "Polar" Arm

Consider the planar revolute-prismatic robot arm in the horizontal plane with a single point mass at the gripper, m_2,

$$q \triangleq \begin{bmatrix} \xi \\ \theta \end{bmatrix} \qquad p(q) = \begin{bmatrix} \xi\sin\theta \\ -\xi\cos\theta \end{bmatrix}$$

with kinetic energy

$$\kappa(q,\dot{q}) = \frac{1}{2}m_2\left[\dot{\xi},\dot{\theta}\right]\begin{bmatrix} 1 & 0 \\ 0 & \xi^2 \end{bmatrix}\begin{bmatrix} \dot{\xi} \\ \dot{\theta} \end{bmatrix},$$

and, again, with zero potential energy. In this case we have

$$\dot{M} = 2m_2\xi\begin{bmatrix} 0 & 0 \\ 0 & \dot{\xi} \end{bmatrix}, \qquad (D_qM\dot{q})^{\mathrm{T}} = 2m_2\xi\begin{bmatrix} 0 & \dot{\theta} \\ 0 & 0 \end{bmatrix}$$

hence,

$$\dot{M} - \frac{1}{2}(D_qM\dot{q})^{\mathrm{T}} = 2m_2\xi\begin{bmatrix} 0 & -\frac{1}{2}\dot{\theta} \\ 0 & \dot{\xi} \end{bmatrix}$$

and

$$\frac{1}{2}\dot{M} - \left[\dot{M} - \frac{1}{2}(D_qM\dot{q})^{\mathrm{T}}\right] = 2m_2\xi\begin{bmatrix} 0 & \frac{1}{2}\dot{\theta} \\ 0 & -\frac{1}{2}\dot{\xi} \end{bmatrix}.$$

Again, while this is not a skew-symmetric linear operator, we have

$$\left(\frac{1}{2}\dot{M} - \left[\dot{M} - \frac{1}{2}(D_qM\dot{q})^{\mathrm{T}}\right]\right)\dot{q} = 2m_2\xi\begin{bmatrix} 0 & \frac{1}{2}\dot{\theta} \\ -\frac{1}{2}\dot{\theta} & 0 \end{bmatrix}\dot{q}.$$

In order to compute the bilinear operator valued maps, $B(q,\cdot), J_q, L_q$, note that

$$D_qM^{\mathrm{S}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 2m_2\xi & 0 \end{bmatrix},$$

hence

$$\acute{M}_q(x) = (x \otimes I)^{\mathrm{T}} D_qM^{\mathrm{S}} = 2m_2\xi\begin{bmatrix} 0 & 0 \\ x_2 & 0 \end{bmatrix},$$

$$B(q,x) \triangleq \acute{M}_q(x) - \frac{1}{2}\acute{M}_q(x)^{\mathrm{T}} = 2m_2\xi \begin{bmatrix} 0 & -\frac{1}{2}x_2 \\ x_2 & 0 \end{bmatrix}.$$

Again, we see that even though

$$\dot{M} - \frac{1}{2}\left(D_q M\dot{q}\,\right)^{\mathrm{T}} = 2m_2\xi \begin{bmatrix} 0 & -\frac{1}{2}\dot{\theta} \\ 0 & \xi \end{bmatrix} \neq 2m_2\xi \begin{bmatrix} 0 & -\frac{1}{2}\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix} = B(q,\dot{q}),$$

it is certainly the case that

$$\dot{M} - \frac{1}{2}\left(D_q M\dot{q}\,\right)^{\mathrm{T}} = 2m_2\xi \begin{bmatrix} -\frac{1}{2}\dot{\theta}^2 \\ \dot{\theta}\dot{\xi} \end{bmatrix} = B(q,\dot{q})\dot{q}.$$

Now define

$$J_q(x) \triangleq \acute{M}_q(x) - \acute{M}_q(x)^{\mathrm{T}} = 2m_2\xi \begin{bmatrix} 0 & -x_2 \\ x_2 & 0 \end{bmatrix},$$

$$L_q(x) \triangleq \acute{M}_q(x) - \dot{M}(q_x) = 2m_2\xi \begin{bmatrix} 0 & 0 \\ x_2 & -x_1 \end{bmatrix}.$$

Note, again, that

$$\left[\frac{1}{2}\dot{M}(q) - B(q,\dot{q})\right]\dot{q} = \frac{1}{2}\left[L_q(\dot{q}) - J_q(\dot{q})\right]\dot{q} = -\frac{1}{2}J_q(\dot{q})\dot{q}.$$

3 A Quadratic Lyapunov Function

Given a reference trajectory in $\mathcal{P}$,

$$p(t) \triangleq \begin{bmatrix} r(t) \\ \dot{r}(t) \end{bmatrix},$$

formed from the smooth curve $r \in C^2[\mathbb{R}, \mathcal{J}]$, we may define two distinct systems of error coordinates on $\mathcal{P}$,

$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \triangleq \begin{bmatrix} q - r \\ \dot{q} - \dot{r} \end{bmatrix}, \bar{e} = \begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \end{bmatrix} \triangleq \begin{bmatrix} q - r \\ \dot{q} \end{bmatrix}.$$

Now adopt the following set of assumptions for the remainder of this paper:

Bounded Inertia Matrix (bm): The inertia matrix, M, remains strictly positive definite over the entirety of $\mathcal{J}$, i.e. $0 < \nu(M)$.

Bounded Initial Error (be): a bound is available on the initial error, $\|e(0)\|^2 \leq \epsilon$.

Bounded Curve (bc): r is confined to a compact subset of $\mathcal{J}$ whose radius is less than β.

Bounded Velocity (bv): the magnitude of the reference signal velocity is uniformly bounded, i.e., $\|\dot{r}\| \leq \rho_d$

In this section, we will analyze the stability properties of a class of nonlinear time varying dynamical systems,

$$\dot{x} = A(t, x)x, \tag{19}$$

by recourse to Lyapunov theory. In the final portion of this paper, Section, 4, system (19) will be interpreted as the "undisturbed error dynamics", which result from a variety of tracking strategies to be introduced there. Depending upon the control strategy, we will require that and x be either e or $\bar{e}$. The structure of this class of error dynamics is specified by

$$A(q(t), \dot{q}(t)) \triangleq \begin{bmatrix} 0 & I \\ -\omega^2 M^{-1}[q(t)] & -M^{-1}[q(t)]\left(B[q(t), \dot{q}(t)] + 2\omega\varsigma I\right) \end{bmatrix}, \tag{20}$$

It should be noted that even under the assumptions above, the original the Lyapunov function used to prove Theorem 3 will not, in itself, avail. As mentioned in the introduction of the paper, since the dynamical system is non-autonomous (i.e., A varies with time independently of the state variable, x), LaSalle's invariance principle may not be applied in its traditional form. Instead of using arguments involving an avergaged version [14], it is possible to appeal to a *strict* Lyapunov function recently introduced in [12]. Here, the stronger, quadratic properties of this Lyapunov function will be stressed in order to obtain performance bounds as algebraic functions of the system parameters.

The following technical lemmas will be of use in the main result, below.

Lemma 6 *If $A, R \in \mathbb{R}^{n \times n}$ and A is a positive definite symmetric matrix then*

$$\nu(A)\nu(R)^2 \cdot \|x\|^2 \leq x^T R^T A R x \leq \mu(A)\mu(R)^2 \cdot \|x\|^2.$$

Proof: The right hand inequality follows since the norm of any inner product is less than the magnitude of the norms of its constituents. To obtain the left hand inequality, note that

$$\nu(A)\cdot\|Rx\|^2 \le x^{\mathrm{T}} R^{\mathrm{T}} A R x,$$

since A is positive definite.

□

Lemma 7 *Let A, B, C be continuous maps on J taking values in the set of positive definite symmetric matrices. If*

$$P(q) \triangleq \begin{bmatrix} A(q) & B(q) \\ B(q) & C(q) \end{bmatrix},$$

then

$$\nu(\check{P})\cdot\|x\|^2 \le x^{\mathrm{T}} P(q) x \le \mu(\hat{P})\cdot\|x\|^2$$

where

$$\check{P} \triangleq \frac{\nu_\eta(B)}{\mu_\eta(B)} \begin{bmatrix} \nu_\eta(A) & \mu_\eta(B) \\ \mu_\eta(B) & \nu_\eta(C) \end{bmatrix}, \qquad \hat{P} \triangleq \frac{\mu_\eta(B)}{\nu_\eta(B)} \begin{bmatrix} \mu_\eta(A) & \nu_\eta(B) \\ \nu_\eta(B) & \mu_\eta(C) \end{bmatrix},$$

as long as $\|q\| \le \eta$.

Proof: To obtain the left hand inequality, note that

$$\begin{aligned}
P &\ge \begin{bmatrix} \nu_\eta(A) I_n & B \\ B & \nu_\eta(C) I_n \end{bmatrix} \\
&= \begin{bmatrix} B^{\frac{1}{2}} & 0 \\ 0 & B^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \nu_\eta(A) B^{-1} & I_n \\ I_n & \nu_\eta(C) B^{-1} \end{bmatrix} \begin{bmatrix} B^{\frac{1}{2}} & 0 \\ 0 & B^{\frac{1}{2}} \end{bmatrix} \\
&\ge \begin{bmatrix} B^{\frac{1}{2}} & 0 \\ 0 & B^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \frac{\nu_\eta(A)}{\mu_\eta(B)} I_n & I_n \\ I_n & \frac{\nu_\eta(C)}{\mu_\eta(B)} I_n \end{bmatrix} \begin{bmatrix} B^{\frac{1}{2}} & 0 \\ 0 & B^{\frac{1}{2}} \end{bmatrix} \\
&= \tfrac{1}{\nu_\eta(B)} (I_2 \otimes B^{\frac{1}{2}})(\check{P} \otimes I_n)(I_2 \otimes B^{\frac{1}{2}}) \\
&= \tfrac{1}{\nu_\eta(B)} (\check{P} \otimes B) \\
&\ge \nu(\check{P}) I_{2n}
\end{aligned}$$

where the last two lines follow from Lemma 15 and Lemma 19 , respectively. The right hand inequality may be obtained similarly.

□

Corollary 8 *A sufficient condition for P to take only positive definite values on a set of radius η in J is that $\check{P}$ be a positive definite matrix.*

In situations where the off diagonal block is not symmetric, the following technical result proves useful.

Lemma 9 *Let A, B, C be continuous maps on J and let A, C take values in the set of positive definite symmetric matrices. Sufficient conditions for*

$$P \triangleq \begin{bmatrix} A & B \\ B^{\mathrm{T}} & C \end{bmatrix}$$

to take only positive definite values on a set of radius η in J are given by

$$\nu_\eta(C) > 1 \text{ and } \nu_\eta(A) > \mu_\eta(B)^2.$$

Proof: The result follows by re-writing

$$P = \begin{bmatrix} A - \frac{1}{4}B^{\mathrm{T}}B & 0 \\ 0 & C - I_n \end{bmatrix} + \begin{bmatrix} B^{\mathrm{T}} \\ I_n \end{bmatrix} [B \; I_n]$$

and noting that the sum of a positive definite matrix with a positive semi-definite matrix is always positive definite.

□

With these tools it is now possible to present a strict Lyapunov Function for the error system defined by (19) for the case that $x \triangleq \bar{e}$ which places no conditions on scalars that will be interpreted as feedback gain magnitudes in the next section.

Proposition 10 *Let $M(q)$ take positive definite values over the entirety of J. If r is a constant of magnitude $\|r\| \leq \beta$ then for any $\epsilon > 0$ there exists a scalar $\gamma > 0$ such that*

$$v(\bar{e}) \triangleq \frac{1}{2}\bar{e}^{\mathrm{T}}P(q)\bar{e} \; = \; \frac{1}{2}\bar{e}^{\mathrm{T}} \begin{bmatrix} \omega^2\gamma I & \omega\varsigma I \\ \omega\varsigma I & \gamma M(q) \end{bmatrix} \bar{e}$$

is a strict Lyapunov Function for the undisturbed error system defined by (20) on the "cylinder"

$$\mathcal{C} \triangleq \{\bar{e} \in \mathcal{P} : \|\bar{e}_1\| \leq \epsilon\}$$

Proof: The proof proceeds by choosing a value of γ which insures that v is positive definite while $\dot{v}$ is is strictly negative over some open interval of time after $t = 0$. It is then shown that this interval is unbounded (i.e., that the signs can never change thereafter).

According to Corollary 8 , P is positive definite as long as

$$\gamma > \frac{\varsigma}{\sqrt{\nu(M)}}.$$

Taking time derivatives along the solutions of system (19), we have

$$\dot{v} \; = \frac{1}{2}\bar{e}^{\mathrm{T}}[PA + A^{\mathrm{T}}P + \dot{P}]\bar{e},$$

which may be expanded as

$$\begin{aligned}\dot{v} &= -\omega\varsigma\bar{e}^{\mathrm{T}}\begin{bmatrix}\omega^2 M^{-1} & \omega\varsigma M^{-1}\\ \omega\varsigma M^{-1} & \gamma I\end{bmatrix}\bar{e}\\ &\quad -\omega\varsigma(\gamma-1)\bar{e}_2^{\mathrm{T}}\bar{e}_2 - \omega\varsigma\bar{e}_1^{\mathrm{T}}M^{-1}B\bar{e}_2\\ &\quad +\gamma\bar{e}_2^{\mathrm{T}}[\tfrac{1}{2}\dot{M} - B]\bar{e}_2.\end{aligned}$$

Since $\bar{e}_2 = \dot{q}$, the term in the last line vanishes according to Corollary 4 . Moreover, using Corollary 8 again, the block matrix in the first line is positive definite if

$$\gamma > \frac{\varsigma^2}{\nu(M)}.$$

Given some scalar, $\eta > 0$, to be chosen below, place the further condition on γ, [5]

$$\gamma > 1 + \epsilon\frac{\mu_\eta(B)}{\nu(M)}\sqrt{\frac{\mu_\eta(P)}{\nu_\eta(P)}}.$$

Since P is positive definite, under the assumption that $\eta > \|q(0)\|$, we have

$$\nu_\eta(P)\epsilon^2 \leq v(0) \leq \mu_\eta(P)\epsilon^2,$$

and, hence,

$$\begin{aligned}\gamma - 1 &> \sqrt{\tfrac{v(0)}{\nu_\eta(P)}}\tfrac{\mu_\eta(B)}{\nu(M)}\\ &> \|\bar{e}(0)\|\tfrac{\mu_\eta(B)}{\nu(M)}\\ &> \|\bar{e}_1(0)\|\tfrac{\mu_\eta(B)}{\nu(M)},\end{aligned}$$

which, in turn, implies

$$(\gamma - 1)\|\bar{e}_2(0)\|^2 > |\bar{e}_1(0)^{\mathrm{T}}M^{-1}B[q(0), \bar{e}_2(0)]\bar{e}_2(0)|. \tag{21}$$

This demonstrates that $\dot{v}(0) < 0$.

Since, by hypothesis, $\|q(0)\| < \beta + \epsilon$, if $\dot{v}(0) < 0$, then there is some open interval of time after $t = 0$ over which

$$v(t) < v(0) \leq \mu_{\beta+\epsilon}(P)\|e(0)\|^2 \leq \mu_{\beta+\epsilon}(P)\epsilon^2.$$

Over this interval,

$$\|\bar{e}(t)\|^2 \leq \frac{1}{\nu_{\beta+\epsilon}(P)}v(t) < \frac{\mu_{\beta+\epsilon}(P)}{\nu_{\beta+\epsilon}(P)}\epsilon^2.$$

It follows that

$$\|q(t)\| < \|\bar{e}_1\| + \|r\| < \sqrt{\frac{\mu_{\beta+\epsilon}(P)}{\nu_{\beta+\epsilon}(P)}}\epsilon + \beta \triangleq \eta$$

[5] While the ratio of eigenvalues of P in the following inequality is function of γ, it is easily verified that appears in the same power in both numerator and denominator: i.e. the ratio is globally bounded as a function of γ, hence, we may choose a large enough value of that constant to make the inequality correct.

over this interval.

Over any interval of time that $v(t) < v(0)$ and $\|q(t)\| < \eta$ we have

$$\gamma - 1 > \sqrt{\frac{v(0)}{\nu_\eta(P)} \frac{\mu_\eta(B)}{\nu(M)}} > \sqrt{\frac{v(t)}{\nu_\eta(P)} \frac{\mu_\eta(B)}{\nu(M)}}$$

and the inequality (21) continues to hold. This implies that $\dot{v}(t)$ is strictly negative over the interval, which must, in consequence, be unbounded to the right.

□

Corollary 11 *The Lyapunov Function, v, in the previous lemma is quadratic with*

$$\nu_\eta(\check{P})\|e\|^2 \leq v \leq \mu_\eta(\hat{P})\|\bar{e}\|^2 \text{ and } \dot{v} < -\nu_\eta(\check{Q})\|\bar{e}\|^2$$

where

$$\check{P} \triangleq \begin{bmatrix} \gamma\omega^2 & \omega\varsigma \\ \omega\varsigma & \gamma\nu(M) \end{bmatrix}, \ \hat{P} \triangleq \begin{bmatrix} \gamma\omega^2 & \omega\varsigma \\ \omega\varsigma & \gamma\mu_\eta(M) \end{bmatrix}, \ \check{Q} \triangleq \frac{\omega\varsigma}{\mu_\eta(M)} \begin{bmatrix} \frac{\nu(M)}{\mu_\eta(M)}\omega^2 & \omega\varsigma \\ \omega\varsigma & \gamma\nu(M) \end{bmatrix}.$$

It should be noted that this proof depends explicitly upon the skew-symmetry properties derived in Corollary 4 , and hence, requires an analysis based upon the $\bar{e}$ coordinate system, rather than the more desirable e system defined in the beginning of this section. Of course, when $\dot{r} = 0$ then both systems are identical. In general, however, in order for this Lyapunov function to work in the more desirable error coordinate system, then additional conditions must be placed upon the feedback constants, ω, ς, as shown in the following result.

Proposition 12 *Let $r : \mathbb{R} \to J$ be a smooth curve, with $\|r\| < \beta$ and $\|\dot{r}\| < \rho_d$. If γ is chosen as in Proposition 10 , and satisfies the additional inequality,*

$$\gamma > \frac{2\varsigma^2}{\nu(M)}$$

then v is a strict Lyapunov Function for system (19) as long as

$$\omega^2 > \mu_\eta(M)\frac{1}{2}\frac{\mu_\eta(M)\mu_\eta(B)^2\rho_d^2}{\nu(M)^2}, \tag{22}$$

and

$$\omega\varsigma > \frac{\rho_d\mu_\eta(L_q)}{1 - \frac{1}{\gamma} - \sqrt{\frac{\epsilon^2\mu_\eta(P)\mu_\eta(B)}{\gamma^2\nu_\eta(P)\nu(M)}}}. \tag{23}$$

Proof: Consider the Lyapunov candidate, from Proposition 10 ,

$$v \triangleq \frac{1}{2}e^{\mathrm{T}}Pe,$$

with derivative along the motion of the system,

$$\begin{aligned}\dot{v} &= -\omega\varsigma e^{\mathrm{T}}\begin{bmatrix}\omega^2 M^{-1} & \omega\varsigma M^{-1}\\ \omega\varsigma M^{-1} & \gamma I\end{bmatrix} e\\ &\quad -\omega\varsigma(\gamma-1)e_2^{\mathrm{T}}e_2 - \omega\varsigma e_1^{\mathrm{T}}M^{-1}Be_2\\ &\quad +\gamma e_2^{\mathrm{T}}[\tfrac{1}{2}\dot{M} - B]e_2.\end{aligned}$$

The last term may be expressed as

$$\begin{aligned}\gamma e_2^{\mathrm{T}}[\tfrac{1}{2}\dot{M} - B]e_2 &= \gamma e_2^{\mathrm{T}}[\tfrac{1}{2}\acute{M}_q(e_2)\dot{q} - \acute{M}_q(\dot{q})e_2 + \tfrac{1}{2}\left(\acute{M}_q(\dot{q})\right)^{\mathrm{T}} e_2]\\ &\quad (\text{from}(17))\\ &= \gamma e_2^{\mathrm{T}}[\tfrac{1}{2}\acute{M}_q(\dot{q})e_2 - \acute{M}_q(\dot{q})e_2 + \tfrac{1}{2}\left(\acute{M}_q(\dot{q})\right)^{\mathrm{T}} e_2]\\ &\quad +\tfrac{1}{2}e_2^{\mathrm{T}}[\acute{M}_q(e_2)\dot{r} - \acute{M}_q(\dot{r})e_2].\end{aligned}$$

Applying the definition in (16), we have

$$\begin{aligned}\gamma e_2^{\mathrm{T}}[\tfrac{1}{2}\dot{M} - B]e_2 &= \tfrac{1}{2}\gamma e_2^{\mathrm{T}}L_q(\dot{r})e_2\\ &\le \gamma\mu_\eta(L_q)\rho_d\|e_2\|^2.\end{aligned} \tag{24}$$

The middle term, after factoring out $e_2^{\mathrm{T}}e_2$, may be written

$$\begin{aligned}&-\omega\varsigma e_2^{\mathrm{T}}e_2\left((\gamma-1) + e_1^{\mathrm{T}}M^{-1}B(q,e_2)e_2/\|e_2\|^2 + e_1^{\mathrm{T}}M^{-1}B(q,\dot{r})e_2/\|e_2\|^2\right)\\ &\quad < -\omega\varsigma e_2^{\mathrm{T}}e_2\left(\gamma - 1 - \epsilon\sqrt{\tfrac{\mu_\eta(P)\mu_\eta(B)}{\nu_\eta(P)\nu(M)}}\right)\\ &\qquad -\omega\varsigma e_1^{\mathrm{T}}M^{-1}B(q,\dot{r})e_2.\end{aligned}$$

Substituting from these last two inequalities in the equation for $\dot{v}$ results in the relationship

$$\begin{aligned}\dot{v} &< -\omega\varsigma e^{\mathrm{T}}\left(\begin{bmatrix}\omega^2 M^{-1} & \omega\varsigma M^{-1}\\ \omega\varsigma M^{-1} & \gamma I\end{bmatrix} - \begin{bmatrix}0 & \tfrac{1}{2}M^{-1}B(q,\dot{r})\\ \tfrac{1}{2}B(q,\dot{r})^{\mathrm{T}}M^{-1} & 0\end{bmatrix}\right)e\\ &\quad -\omega\varsigma\gamma e_2^{\mathrm{T}}e_2\left(1 - \tfrac{1}{\gamma} - \sqrt{\tfrac{\epsilon^2\mu_\eta(P)\mu_\eta(B)}{\gamma^2\nu_\eta(P)\nu(M)}} - \tfrac{\mu_\eta(L_q)}{\omega\varsigma}\rho_d\right).\end{aligned}$$

Re-writing the matrix difference as

$$\begin{bmatrix}\tfrac{1}{2}(\omega^2 M^{-1} - \tfrac{1}{2}B^{\mathrm{T}}M^{-2}B) & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}\tfrac{1}{2}\omega^2 M^{-1} & \omega\varsigma M^{-1}\\ \omega\varsigma M^{-1} & (\gamma-1)I\end{bmatrix} + \begin{bmatrix}\tfrac{1}{2}B^{\mathrm{T}}M^{-1}\\ I\end{bmatrix}\left[\frac{1}{2}M^{-1}B, I\right], \tag{25}$$

we require condition (22) in order for the first matrix to be positive semi-definite, and

$$\gamma > \frac{2\varsigma^2}{\nu(M)}$$

in order for the second matrix to be positive definite. Since the sum of positive definite and semi-definite matrices is always positive definite, these inequalities guarantee that the first term in the expression for $\dot{v}$, above, is strictly negative. The second term is negative as long as condition (23) holds.

□

Corollary 13 *The Lyapunov Function, v, in the previous lemma is quadratic with*

$$\nu_\eta(\check{P})\|e\|^2 \leq v \leq \mu_\eta(\hat{P})\|e\|^2 \text{ and } \dot{v} < -\nu_\eta(\check{Q})\|e\|^2$$

where

$$\check{P} \triangleq \begin{bmatrix} \gamma\omega^2 & \omega\varsigma \\ \omega\varsigma & \gamma\nu(M) \end{bmatrix}, \hat{P} \triangleq \begin{bmatrix} \gamma\omega^2 & \omega\varsigma \\ \omega\varsigma & \gamma\mu_\eta(M) \end{bmatrix}, \check{Q} \triangleq \frac{\omega\varsigma}{\mu_\eta(M)} \begin{bmatrix} \frac{\nu(M)}{2\mu_\eta(M)}\omega^2 & \omega\varsigma \\ \omega\varsigma & (\gamma-1)\nu(M) \end{bmatrix}.$$

4 Consequences for Tracking Reference Signals

This section employs the results derived above to gain some insight into the tracking behavior of mechanical systems compensated by traditional "PD" linear feedback. To begin with, it is shown in Theorem 4 of Section 4.1, that the closed loop equations which result are exponentially stable Thus, as in the case of the "Error-P" scheme for linear plants (3), exponentially small tracking error obtains from forcing a PD compensated mechanical system with no more information than the reference signal itself, and performance is increased by higher feedback gains. In Theorem 5 of Section 4.2 it is shown that reference velocity and acceleration information may be used in the feedforward path with no resort to exact cancellation in the feedback path without compromising exponential stability as long as the feedback gains are sufficiently large. If information concerning plant parameters is available as well, then exact cancellation of L_q, (the "defect" from skew-symmetry introduced in Section 2) is shown to afford the same result in Theorem 6 with no requirements placed upon feedback gain magnitudes. Finally, assuming complete information about the reference trajectory and the plant parameters, Theorem 7 shows that exponentially exact results obtain from the equivalent of nonlinear inverse dynamics in the feedforward path. This result has important implications for the adaptive control of mechanical systems [11].

4.1 Error-P Controller: Minimal Feedforward Information

Consider the decoupled "PD" compensated system forced by a continuously differentiable reference signal, $r(t)$, introduced in Section 3, and let the assumptions introduced there continue to hold as well. In particular, assume that the reference trajectory is "unpredictable" — i.e. its first and second derivatives are unknown — but there is available an à priori bound on the maximum rate of change,

$$\|\dot{r}\| \leq \rho_d.$$

Assume, moreover, that no information regarding the plant dynamics, M, B, is available, so that further exact cancellation is impossible. Proceeding from analogy with the "Error-p" scheme of Section 1.1, the complete control law may be written exactly in the form of (3)

$$\tau_p = -\omega^2[r(t) - q] - 2\omega\varsigma\dot{q}. \tag{26}$$

The resulting closed loop error system takes the same form as (5) when written in the coordinate system, $\bar{e}$, defined by (2),

$$\dot{\bar{e}} = A[t, q]\bar{e} + d(t),$$

where, A is specified in (20),

$$A(q(t), \dot{q}(t)) \triangleq \begin{bmatrix} 0 & I \\ -\omega^2 M^{-1}[q(t)] & -M^{-1}[q(t)]\left(B[q(t), \dot{q}(t)] + 2\omega\varsigma I\right) \end{bmatrix},$$

and

$$d \triangleq \begin{bmatrix} \dot{r}(t) \\ 0 \end{bmatrix}$$

is the "disturbance" input due to the unknown but non-zero reference derivative.

Theorem 4 *The closed loop "disturbed" error system resulting from application of the control algorithm (26) has trajectories which are bounded in magnitude by*

$$\|e\| \leq e^{-\frac{1}{2}\rho t} \frac{\alpha_2}{\sqrt{\alpha_1}} \|e(0)\|^2 + \beta$$

where

$$\begin{aligned} \rho &\triangleq 2 \frac{\varsigma\omega}{\mu_\eta(M)} \cdot \frac{\nu(\check{Q})}{\mu(\check{P})} \\ &= 2 \frac{\varsigma\omega}{\mu_\eta(M)} \frac{\nu_\eta(M)}{\mu_\eta(M)} \cdot \frac{(\gamma\mu_\eta(M)+\omega) - \sqrt{(\gamma\mu_\eta(M)-\omega^2)^2 + 4\omega^2\varsigma^2 \frac{\mu_\eta(M)^2}{\nu_\eta(M)^2}}}{\gamma(\mu_\eta(M)+\omega^2) + \sqrt{\gamma^2(\mu_\eta(M)-\omega^2)^2 + 4\omega^2\varsigma^2}}, \end{aligned} \tag{27}$$

and

$$\begin{aligned} \beta &\triangleq \frac{\rho_d}{\rho} \cdot \frac{\omega\sqrt{\gamma^2\omega^2+\varsigma^2}}{\nu(\check{P})} \\ &= \frac{\rho_d}{\rho} \cdot \frac{\omega\sqrt{\gamma^2\omega^2+\varsigma^2}}{\gamma(\nu(M)+\omega^2) - \sqrt{\gamma^2(\nu(M)-\omega^2)^2+4\omega^2\varsigma^2}}, \end{aligned} \tag{28}$$

where $\eta > 0$ is defined in Proposition 10 .

Proof: According Proposition 10 , there exists a strict Lyapunov function for the undisturbed system, and this is shown to be quadratic with coefficients, $\alpha_i, i = 1, 3$, given in Corollary 11 . To obtain α_4, note that

$$D_{\bar{e}} v\, d(t) = \bar{e}^{\mathrm{T}} P \begin{bmatrix} \dot{r} \\ 0 \end{bmatrix} \leq \|\bar{e}\| \cdot \| \begin{bmatrix} \gamma\omega^2 \\ \omega\varsigma \end{bmatrix} \| \rho_d.$$

Thus, the conditions of Theorem 2 are satisfied, and the result follows from algebraic substitution of the eigenvalues of $\check{P}, \hat{P}, \check{Q}$ from Corollary 11 for the α_i as stipulated by that theorem.

□

Of course, in order for this analysis to be truly practicable, it would be necessary to address the idealities of the model. Equation (1), as derived in Section 2.1, presumes the absence of any uncontrolled external forces or torques. In reality, a mechanical system may be subject to a variety of uncontrolled external forces. For example, most commercially available robots deviate from the model derived above quite dramatically. Flexibility in the links, slippage in transmissions, and backlash in gear trains introduce stiction, hysteresis,and other nonlinear effects. Harmonic drives and compliance at the joints, introduce extra dynamics [22]. Demagnetization and potential damage to the windings place limits upon the maximum permissible armature current and, therefore, output torque, of a dc servo. Moreover, while it is traditional in the control community to model electric motors as if they were first order lags [4], it is not impossible to find commercial robot arms employing dc servos whose mechanical and electrical time constants are of similar magnitude, and which, in consequence, have second order dynamics, not uncommonly *oscillatory* [7]. Thus, not only may the model introduced in (1) have missing functional terms in practice, but its dimensionality, $2n$, may too low by at least again as much as the number of actuators, n. Here, only a token effort will be made to redress these omissions, merely in the way of providing an example of how the analysis might be altered.

The most practically significant omission in (1) is arguably effect of the potential forces of the earth's gravitational field. Assume, now that there are external and uncontrolled potential forces operating upon the mechanical system,

$$k(q) \triangleq D\varphi ,$$

where φ is a potential function whose gradient, k is bounded over the workspace, J. The complete set of dynamical equations must now be written

$$M(q)\ddot{q} + B(q,\dot{q})\dot{q} + k(q) = \tau.$$

Suppose, then, that potential forces are present, and there is no information available concerning the dynamical parameters of the plant sufficient to compute $k(q)$. The controller (26), gives rise to the same error system, (5), except that the disturbance input, d, now takes the form

$$d(t) = \begin{bmatrix} \dot{r} \\ k(q) \end{bmatrix}.$$

Since we assume k is analytic, there exists a bound,

$$\kappa_0 \triangleq \sup_{\|q\| \leq \eta} \|k(q)\|,$$

where η is the radius of J determined in the proof of Proposition 10 . It follows that the tracking response has the same form as in Theorem 4 above, with the substitution

$$\beta \triangleq \frac{\sqrt{\rho_d^2 + \kappa_0^2}}{\rho} \cdot \frac{\gamma(\mu_\eta(M) + \omega^2) + \sqrt{\gamma^2(\mu_\eta(M) - \omega^2)^2 + 4\omega^2\varsigma^2},}{\gamma(\nu(M) + \omega^2) - \sqrt{\gamma^2(\nu(M) - \omega^2)^2 + 4\omega^2\varsigma^2}.} \tag{29}$$

In the sequel, assume that external potential forces are either not present or have been cancelled exacatly by feedback.

4.2 Using More Information About the Plant and Reference Trajectory

We may now explore the situation where more information is available concerning the plant and the reference trajectory. Specifically, suppose that $r, \dot{r}$, are both available in real time, but that nothing is known regarding the desired acceleration apart from a bound on its magnitude, $\|\ddot{r}\| \leq \alpha_d$. Again, for ease of exposition, we may pre-suppose the absence of the uncontrolled potential forces, or their exact cancellation. Thus, consider the same PD feedback scheme as in Section 4.1, above, with reference velocity information added in the feedforward path:

$$\tau_{pd} = k(q) - \omega^2[r(t) \dot{-} q] - 2\omega\varsigma[\dot{q} - \dot{r}]. \tag{30}$$

Now, shift to the fully translated error coordinate system, e, defined by (2). It is easy to see that the closed loop error dynamics once again take the form (26), with A as specified in (20), but with a disturbance term of the form

$$\begin{aligned} d(t) &= -\begin{bmatrix} 0 \\ \ddot{r} + M^{-1}B(q,\dot{q})\dot{r} \end{bmatrix} \\ &= -\begin{bmatrix} 0 \\ \ddot{r} + M^{-1}B(q,\dot{r})\dot{r} \end{bmatrix} - \begin{bmatrix} 0 \\ M^{-1}B(q,e_2)\dot{r} \end{bmatrix} \\ &\triangleq d_1(t) + d_2(t) \end{aligned}$$

There are now two problems. On the one hand, since we are using the coordinate system e rather than the $\bar{e}$, we must appeal to Proposition 12 rather than Proposition 10 in our analysis of the undisturbed system. The former is much weaker in that à priori estimates of the initial error, ϵ, and estimates of the plant coefficient bounds, $\nu(M), \mu(M)$, are required not simply for analytical purposes (as in the latter proof) but impose potentially unrealistic demands for sufficiently large feedback gain magnitudes specified by the inequalities (22) (23). On the other hand, since d_2 is linear in e_2, the disturbance $d(t)$ is not uniformly bounded any longer, and no α_4 may be found to satisfy the requirements of Theorem 2 given the problem as stated in the present form.

In order to apply that theorem, it is necessary to reformulate the error dynamics by pulling $d_2(t)$ into $A(t,q)$ as follows. Since $B(q,\cdot)$ is bilinear, there exists a "transposed" bilinear operator valued map on J, $\bar{B}(q,\cdot)$ such that

$$B(q,x)y = \bar{B}(q,y)x.$$

Now, the closed loop dynamical system may be written as

$$\dot{e} = A_1(t)e + d_1(t),$$

where

$$A_1(t) = \begin{bmatrix} 0 & I \\ -\omega^2 M^{-1}[q(t)] & -M^{-1}[q(t)]\left(B[q(t),\dot{q}(t)] + \bar{B}[q(t),\dot{r}(t)] + 2\omega\varsigma I\right) \end{bmatrix}, \tag{31}$$

and d_1 is as defined above.

Theorem 5 *Let the feedback gains in (30) satisfy the inequalities*

$$\omega^2 > \mu_\eta(M)\frac{1}{2}\frac{\mu_\eta(M)\mu_\eta(B+\bar{B})^2\rho_d^2}{\nu(M)^2}, \tag{32}$$

and

$$\omega\varsigma > \frac{\rho_d\mu_\eta(L_q+\bar{B})}{1-\frac{1}{\gamma}-\sqrt{\frac{\epsilon^2\mu_\eta(P)\mu_\eta(B)}{\gamma^2\nu_\eta(P)\nu(M)}}}, \tag{33}$$

where γ is chosen as in Proposition 12 . The closed loop "disturbed" error system resulting from the control strategy (26) has trajectories which are bounded in magnitude as specified in Theorem 4 , where

$$\beta \triangleq \frac{\alpha_d + \frac{\mu_\eta(B)}{\nu_\eta(M)}\rho_d}{\rho}\cdot\frac{\omega\sqrt{\omega^2\varsigma^2+\gamma^2\mu_\eta(M)^2}}{\nu(\check{P})}. \tag{34}$$

Proof: Notice that Proposition 12 applies to A_1 in (31) as well as A in (20) as long as B is replaced by $B+\bar{B}$ in the proof, with the attendant changes in the gain inequalities, (22) and (23) as indicated above. The proof proceeds exactly as in Theorem 4 , using the quadratic coefficients, α_i, as displayed in Corollary 13 .

□

Suppose, in addition to $\dot{r}$, an exact computation of B is available in real time. Now modify the "Error-pd" control scheme, (6), to include this information in the form

$$\tau_{pd} = k(q) - \omega^2[r(t) - q] - [2\omega\varsigma I - L_q(\dot{r})] \cdot [\dot{q} - \dot{r}] + B(q, \dot{q})\dot{r}. \tag{35}$$

The closed loop dynamical system may be written as

$$\dot{e} = A_2(t)e + d_2(t), \tag{36}$$

where

$$A_2(t) = \begin{bmatrix} 0 & I \\ -\omega^2 M^{-1}[q(t)] & -M^{-1}[q(t)]\left(B[q(t), \dot{q}(t)] - L_q[\dot{r}(t)] + 2\omega\varsigma I\right) \end{bmatrix}, \tag{37}$$

and

$$d_2 = \begin{bmatrix} 0 \\ \ddot{r} \end{bmatrix}.$$

Theorem 6 *The closed loop "disturbed" error system, (36) has trajectories which are bounded in magnitude as in Theorem 4 where*

$$\beta \triangleq \frac{\alpha_d}{\rho} \cdot \frac{\omega\sqrt{\omega^2\varsigma^2 + \gamma^2\mu_\eta(M)^2}}{\nu(\check{P})}. \tag{38}$$

Proof: The proof proceeds as in the previous two theorems. Note, in this case, that the "defect" from skew-symmetry remaining in equation (24) within the proof of Proposition 12 is exactly cancelled, hence, the proof proceeds as in Proposition 10 with the substitution of $\mu_\eta(B) + \mu_\eta(L_q)$ for $\mu_\eta(B)$.

□

Finally, suppose, we are given full information concerning both the reference trajectory as well as the plant parameters. Modify the "Error-pd" control scheme, (6), by placing a full inverse model of of the nonlinear compensated plant as follows,

$$\tau_{pd} = k(q) - \omega^2[r(t) - q] - [2\omega\varsigma I - L_q(\dot{r})] \cdot [\dot{q} - \dot{r}] + B(q, \dot{q})\dot{r} + M(q)\ddot{r}. \tag{39}$$

The closed loop dynamical system may be written as (36) where A_2 is given by (37), but with $d_2 \equiv 0$. By arguing as in the previous theorem we obtain this last result.

Theorem 7 *The closed loop "undisturbed" error system resulting from the application of the control (39) is globally asymptotically stable.*

5 Concluding Remarks

Decoupled proportional and derivative feedback has been shown to render all mechanical systems globally exponentially stable. In consequence, bounded inputs are guaranteed to produce bounded inputs. Moreover, it is clear simply from inspecting the relative values of ρ and β in the preceding theorems that a crude analogy may be drawn between the tracking performance of the second order linear time invariant system of Section 1.1 and the general class of mechanical systems subject to this simple and traditional linear feedback compensation scheme. Namely,

- it appears that for fixed reference trajectory bandwidth (i.e. velocity and acceleration bounds, ρ_d, α_d) increasing the magnitude of the feedback gain, ω results in faster exponential error decay rate, ρ, and smaller steady state tracking bound, β;
- it appears that the addition of increasing information concerning the derivatives of the reference trajectory and plant dynamics produces improvements in ρ and β as well.

Evidently, a more systematic examination of these results is required before such "appearances" may be stated precisely. For example, one might proceed by translating information obtained from a bode plot as in Figure 4, into statements about the relative magnitudes of the "quadratic" coefficients, $\alpha_i, i = 1,4$ obtained by applying Theorem 4 to the linear system (5) and applying Theorem 6 to the linear system (7). Since it is clear that these coefficients are almost identical in the general case probably the same insights would obtain. Thus, for example, in order to recover the complete understanding of tradeoffs between information and performance embodied in the traditional bode plot, e.g. Figure 4, there remains the (rather dreary) task of making precise algebraic comparisons between the constants ρ, β for the various cases discussed in the previous section. Presumably, given better models of the non-idealities in the plant dynamics, these comparisons would be well worth making.

Having made bold general claims regarding the apparent simplicity of a class of highly nonlinear dynamical systems in addition to strong statements regarding the efficacy of this mode of analysis in particular, it seems appropriate to end on a humbler note. As a cautionary example of how these tools might fail to alert the designer to dangerous problems, return to the case of the planar prismatic-revolute arm introduced in Section 2.3. Given the mechanical system,

$$\begin{bmatrix} m_1 + m_2 & -m_2 a \sin\theta \\ -m_2 a \sin\theta & m_2 a^2 \end{bmatrix} \begin{bmatrix} \ddot{\xi} \\ \ddot{\theta} \end{bmatrix} - m_2 a \cos\theta \begin{bmatrix} \dot{\theta}^2 \\ 0 \end{bmatrix} = \begin{bmatrix} \tau_\xi \\ \tau_\theta \end{bmatrix}$$

it is desired to track the reference trajectory,

$$r(t) \triangleq \begin{bmatrix} \beta \cos t \\ 0 \end{bmatrix}.$$

For ease of exposition, suppose $m_1 = 0, m_2 = 1$. According to the results of Theorem 4 , the control strategy

$$\tau_p = -\omega^2 q - 2\omega\varsigma\dot{q} + \omega^2 \begin{bmatrix} \beta \cos t \\ 0 \end{bmatrix}$$

results in "exponentially small" tracking error, whose magnitude may be reduced by increasing ω and ς. In the case that $\|q\|^2 + \|\dot{q}\|^2 \ll 1$, we have

$$M(q) = \begin{bmatrix} 1 & -a\sin\theta \\ -a\sin\theta & a^2 \end{bmatrix} \approx \begin{bmatrix} 1 & -a\theta \\ -a\theta & a^2 \end{bmatrix}$$

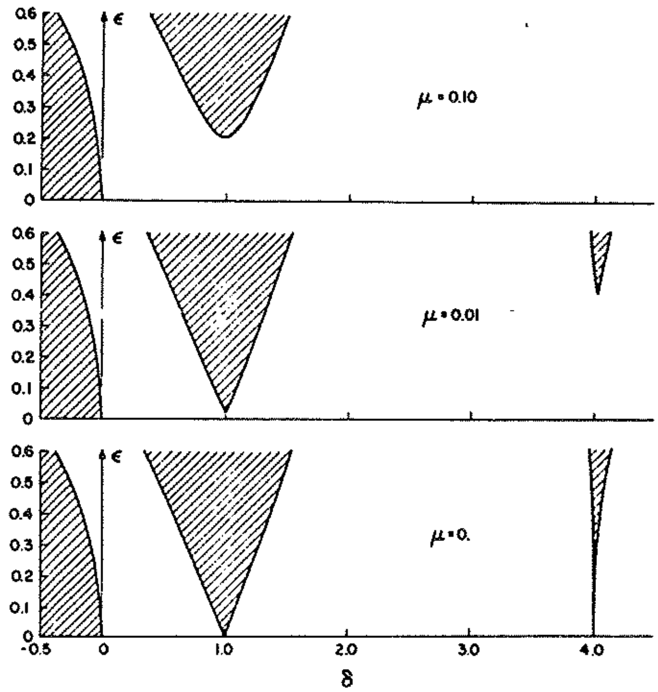

Figure 6: Regions of Instability of the Damped Mathieu Equation (copied from [17, p.301]).

and

$$B(q,\dot{q})\dot{q} \approx 0,$$

and the linearized dynamics closed loop dynamics is given by

$$\begin{aligned} \omega^2\beta\cos t &= \ddot{\xi}(1 - a\theta\ddot{\theta}) + 2\omega\varsigma\dot{\xi} - \omega^2\xi \\ 0 &= a^2\ddot{\theta} + 2\omega\varsigma\dot{\theta} + (\omega^2 - a\ddot{\xi})\theta. \end{aligned}$$

If we suppose that the tracking error between ξ and $\beta\cos t$ is small, then the second equation may be written

$$e_{\xi}(t) = a^2\ddot{\theta} + 2\omega\varsigma\dot{\theta} + (\omega^2 + a\beta\cos t)\theta.$$

This is a damped Mathieu Equation. It is well known [17], that the system,

$$\ddot{\theta} + \mu\dot{\theta} + (\delta - \epsilon\cos t)\theta,$$

may be made unstable by the appropriate choice of ϵ and δ regardless of the magnitude μ. Some sample coefficient choices are displayed in the accompanying figure (copied from [17, p.301]). Thus, as the tracking gets better along the first degree of freedom, ξ, the local stability properties along the second degree of freedom are compromised!

Of course this does not contradict our global stability result. Even if the tracking error in the first degree of freedom, ξ, is identically zero, these instability results are local in nature, and the arguments used in their proof become invalid as $\theta, \dot{\theta}$ become large. The example should serve as a warning, however, concerning our present inability to predict the nature of the response of a mechanical system subject to simple PD feedback once the errors have decayed to within the steady state "tunnel" whose existence is guaranteed and whose magnitude is even estimated by Theorem 4 and its successors.

A The Stack Representation and Kronecker Products

While most of the material in this section has been available for nearly a century, it seems never to have attained wide-spread familiarity within the engineering and applied mathematics community. Readers may consult Bellman [3] for a nicely motivated presentation which leaves many of the proofs to the reader. On the other hand, MacDuffee [16] offers a more thoroughgoing presentation which makes actual reference to the original work of the late nineteenth century mathematicians who developed these results.

If $A \in \mathbb{R}^{n\times m}$, the "stack" representation of $A \in \mathbb{R}^{nm}$ formed by stacking each column below the previous will be denoted A^{S} [3].

If $B \in \mathbb{R}^{p\times q}$, and A is as above then the ***kronecker product*** of A and B is

$$A \otimes B \triangleq \begin{bmatrix} a_{11}B & \dots & a_{1m}B \\ a_{21}B & \dots & a_{2m}B \\ & \vdots & \\ a_{n1}B & \dots & a_{nm}B \end{bmatrix} \in \mathbb{R}^{np\times mq}.$$

The kronecker product is not, in general, commutative. Note that while the transpose "distributes" over kronecker products [16],

$$(A \otimes B)^{\mathrm{T}} = (A^{\mathrm{T}} \otimes B^{\mathrm{T}}),$$

the stack operator, in general, does not.

Lemma 14 *If* $A \in \mathbb{R}^{n\times m}$ *then there exists a nonsingular linear transformation of* $\mathbb{R}^{nm}$, T, *such that*

$$\left(A^{\mathrm{T}}\right)^{S} = TA^{S}$$

Proof: For $p = nm$, let $\mathcal{B} \triangleq \{b_1, \dots, b_p\}$ denote the canonical basis of $\mathbb{R}^p$ — i.e., b_i is a column of p entries with a single entry, 1, in position i, and the other $p-1$ entries set equal to zero. The transpose operator is a reordering of the canonical basis elements, hence may be represented by the elementary matrix,

$$T \triangleq \left[b_1, b_{n+1}, b_{2n+1}, \dots, b_{(m-1)n+1}, b_2, b_{n+2}, b_{2n+2}, \dots, b_{(m-1)n+2}, \dots b_n, b_{2n}, b_{3n}, \dots, b_{mn}\right].$$

□

For $n = m$, if we define $P_+ \triangleq I + T$, $P_- \triangleq I - T$ then both operators are projections onto the set of "skew-symmetric" , "symmetric" operators of $\mathbb{R}^n$, respectively, since $P_\pm^2 = P_\pm$. Note that $\mathrm{Ker}\, P_\pm = \mathrm{Im}\, P_\mp$.

The kronecker product does "distribute" over ordinary matrix multiplication in the appropriate fashion.

Lemma 15 ([16]) *If* $A \in \mathbb{R}^{n\times m}, B \in \mathbb{R}^{p\times q}, C \in \mathbb{R}^{m\times k}, D \in \mathbb{R}^{q\times l}$ *then*

$$(A \otimes B)(C \otimes D) = (AC \otimes BD).$$

Lemma 16 ([3]) *If* $B \in \mathbb{R}^{m\times p}$, $A \in \mathbb{R}^{n\times m}$, *and* $C \in \mathbb{R}^{p\times q}$ *then*

$$[ABC]^{\mathrm{S}} = (C^{\mathrm{T}} \otimes A)B^{\mathrm{S}}.$$

Noting that for any column, $c \in \mathbb{R}^{p\times 1}$, we have

$$c^{\mathrm{S}} = \left[(c)^{\mathrm{T}}\right]^{\mathrm{S}} = c,$$

there follows the corollary

Corollary 17 *If* $B \in \mathbb{R}^{m\times p}$, $c \in \mathbb{R}^{p}$ *then*

$$\begin{aligned} Bc &= Bc^{\mathrm{S}} = (c^{\mathrm{T}} \otimes I)B^{\mathrm{S}} \\ &= \left([Bc]^{\mathrm{T}}\right)^{\mathrm{S}} = \left(c^{\mathrm{T}}B^{\mathrm{T}}\right)^{\mathrm{S}} = (I \otimes c^{\mathrm{T}})\left(B^{\mathrm{T}}\right)^{\mathrm{S}}. \end{aligned}$$

Noting, moreover, that

$$tr\ \{A\} = \left(I^{\mathrm{S}}\right)^{\mathrm{T}} A^{\mathrm{S}},$$

there follows the additional result

Corollary 18 *If* $A \in \mathbb{R}^{n\times m}, B \in \mathbb{R}^{p\times m}$ *then*

$$tr\ \left\{AB^{\mathrm{T}}\right\} = \left(A^{\mathrm{S}}\right)^{\mathrm{T}} \mathrm{B}^{\mathrm{S}}.$$

Proof:

$$\begin{aligned} tr\ \left\{AB^{\mathrm{T}}\right\} &= \left(I^{\mathrm{S}}\right)^{\mathrm{T}} \left(AB^{\mathrm{T}}\right)^{\mathrm{S}} \\ &= \left(I^{\mathrm{S}}\right)^{\mathrm{T}} (B \otimes I)A^{\mathrm{S}} \\ &= \left(A^{\mathrm{S}}\right)^{\mathrm{T}} (B^{\mathrm{T}} \otimes I)I^{\mathrm{S}} \\ &= \left(A^{\mathrm{S}}\right)^{\mathrm{T}} B^{\mathrm{S}}. \end{aligned}$$

□

Lemma 19 ([16]) *For any two square arrays,* $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{m\times m}$,

$$spectrum(A \otimes B) = spectrum(A) \cdot spectrum(B),$$

i.e., every eigenvalue of $(A \otimes B)$ *is the product of an eigenvalue of* A *with an eigenvalue of* B.

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