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A reduction theorem for the Kripke-Joyal semantics: Forcing over an arbitrary category can always be replaced by forcing over a complete Heyting algebra

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A Reduction Theorem for the Kripke-Joyal Semantics: Forcing over an arbitrary category can always be replaced by forcing over a complete Heyting algebra

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Abstract. It is assumed that a Kripke-Joyal semantics $\mathcal{A} = \langle \mathbb{C}, \mathrm{Cov}, F, \Vdash \rangle$ has been defined for a first-order language $\mathcal{L}$. To transform $\mathbb{C}$ into a Heyting algebra $\overline{\mathbb{C}}$ on which the forcing relation is preserved, a standard construction is used to obtain a complete Heyting algebra made up of cribles of $\mathbb{C}$. A pretopology $\overline{\mathrm{Cov}}$ is defined on $\overline{\mathbb{C}}$ using the pretopology on $\mathbb{C}$. A sheaf $\overline{F}$ is made up of sections of F that obey functoriality. A forcing relation $\overline{\Vdash}$ is defined and it is shown that $\overline{\mathcal{A}} = \left\langle \overline{\mathbb{C}}, \overline{\mathrm{Cov}}, \overline{F}, \overline{\Vdash} \right\rangle$ is a Kripke-Joyal semantics that faithfully preserves the notion of forcing of $\mathcal{A}$. That is to say, an object a of $\mathbb{C}Ob$ forces a sentence with respect to $\mathcal{A}$ if and only if the maximal a-crible forces it with respect to $\overline{\mathcal{A}}$. This reduces a Kripke-Joyal semantics defined over an arbitrary site to a Kripke-Joyal semantics defined over a site which is based on a complete Heyting algebra.

We will begin by recapitulating the definition of the Kripke-Joyal semantics since the details of the definition will be needed to prove the reduction theorem in the second part of this paper.

1. The Kripke-Joyal Semantics

Robert Goldblatt's version of a site from [1] and A. Kock and G.E. Reyes notion of forcing over a site [2] are used to establish the definition of the Kripke-Joyal semantics used in this paper. Alternative expositions of forcing in categorical contexts can be found in [3], [4], [5], and [6].

In this paper, except for set membership and inclusion, all compositions of arrows are written in the order of composition. Also, a category is considered to be *small* if its collection of arrows is a set. This allows us to form the first definition.

Definition 1.1. A *stack* or *presheaf* of sets over a small category $\mathbb{C}$ is a contravariant functor $\mathbb{C} \xrightarrow{F} S$ where S is the category of sets. The functor category $S^{\mathbb{C}^{op}}$ is the category of all stacks over $\mathbb{C}$. For $a \in \mathbb{C}Ob$, $s \in aF$ is called a germ. □

If A is a set then $A\mathcal{P}$ is its power set. The capital letters, I, X, Y, and so on, represent index sets.

Definition 1.2. A *pretopology* on a small category $\mathbb{C}$ is an assignment $\mathbb{C}Ob \xrightarrow{\text{Cov}} ((\mathbb{C}Ar)\mathcal{P})\mathcal{P}$ which takes $a \in \mathbb{C}Ob$ to a collection of sets of arrows in $\mathbb{C}$ with codomain a satisfying the following conditions:

(i) the empty set $\phi \notin a\text{Cov}$.

(ii) the singleton $\left\{a \xrightarrow{1a}\right\} \in a\text{Cov}$.

(iii) if $\left\{a_x \xrightarrow{f_x} a | x \in X\right\} \in a\text{Cov}$ and for each $x \in X$ $\left\{a_y^x \xrightarrow{f_y^x} a_x | y \in Y_x\right\} \in a_x\text{Cov}$,then $\left\{a_y^x \xrightarrow{f_y^x f_x} a | y \in Y_x \text{ and } \text{x} \in \text{X}\right\} \in a\text{Cov}$.

(iv) if $\left\{a_x \xrightarrow{f_x} a | x \in X\right\} \in a\text{Cov}$ and $b \xrightarrow{g} a \in \mathbb{C}Ar$ then for each $x \in X$ the pullback $b \times_a a_x \xrightarrow{f'_x} b$ of f_x along g

$$\begin{array}{ccc} b \times_a a_x & \xrightarrow{g'} & a_x \\ {\scriptstyle f'_x}\downarrow & Pb & \downarrow{\scriptstyle f_x} \\ b & \xrightarrow[g]{} & a \end{array}$$

exists and $\left\{b \times_a a_x \xrightarrow{f'_x} b | x \in x\right\} \in b\text{Cov}$. □

Both Goldblatt and Kock and Reyes leave condition (i) out of their definitions of pretopologies. However, it is necessary for the development of a semantics for a formal language.

A collection $\left\{a_x \xrightarrow{f_x} a | x \in X\right\} \in a\text{Cov}$ is called a *cover* of a. If $\mathbb{C}$ is the category of open sets of a topological space (with a (unique) arrow from U to V if and only if $U \subseteq V$ for U and V open sets) then the assignment $U\text{Cov} = \{\text{open covers of U}\}$ satisfies the above conditions.

The function Cov can be extended to a contravariant functor by assigning to $b \xrightarrow{g} a$ the function that takes a cover of a to its corresponding cover on b by (iv) of the above definition. Thus $\text{Cov} \in (\text{S}^{\mathbb{C}^{\text{op}}})Ob$.

Definition 1.3. A *site* $\langle \mathbb{C}, \text{Cov} \rangle$ is a small category $\mathbb{C}$ with a pretopology Cov. □

Let $\langle \mathbb{C}, \text{Cov} \rangle$ be a site and $\left\{a_x \xrightarrow{f_x} a : x \in X\right\} \in a\text{Cov}$. Then the pullback of f_x along f_y is called $a_x \times_a a_y \xrightarrow{f'_x} a_y$ and the pullback of f_y along f_x

is called $a_x \times_a a_y \xrightarrow{f'_y} a_x$ for each $x, y \in X$ as shown in the diagram:

$$\begin{array}{ccc} a_x \times_a a_y & \xrightarrow{f'_x} & a_y \\ {\scriptstyle f'_y}\downarrow & Pb & \downarrow{\scriptstyle f_y} \\ a_x & \xrightarrow[f_x]{} & a \end{array}$$

If F is a stack over $\mathbb{C}$ then the image of $a_x \times_a a_y \xrightarrow{f'_y} a_x$ is called $a_xF \xrightarrow{F^x_y} (a_x \times_a a_y)F$ and the image of $a_x \xrightarrow{x} a$ is called $aF \xrightarrow{F_x} a_xF$, for all $x, y \in X$.

Definition 1.4. A stack F is a *sheaf* over the site $\langle \mathbb{C}, \mathrm{Cov} \rangle$ if it satisfies the *compatibility condition*:

Given any cover $\left\{ a_x \xrightarrow{f_x} a | x \in X \right\} \in a\mathrm{Cov}$ for $a \in \mathbb{C}Ob$ and any collection of elements $\{s_x \in a_xF | x \in X\}$ that are pairwise compatible, i.e., $s_xF^x_y = s_yF^y_x$ for all $x, y \in X$, there is exactly one $s \in aF$ so that $sF_x = s_x$ for all $x \in X$. $\square$

The full subcategory of $S^{\mathbb{C}^{0p}}$ generated by those objects that are sheaves over the site $\langle \mathbb{C}, \mathrm{Cov} \rangle$ is called $\mathbb{C}Sh$. A *Grothendieck topos* is a category that is equivalent to one of the form $\mathbb{C}Sh$ for a site $\langle \mathbb{C}, \mathrm{Cov} \rangle$.

Let $\mathcal{L}$ be a first order formal language defined in the usual way [7]. We let v, v_1, v_2, and so on represent the variables of the language, w, w_1, w_2, and so on represent the constants of the language and u, u_1, u_2, and so on be metavariables for variables and constants. For an integer m, the set mWt is the collection of all constants and variables that are contained in the list $u_1, \ldots, u_m$. An integer m is *appropriate* to a well-formed formula Ξ if all of the variables and constants of Ξ appear in the list $u_1, \ldots, u_m$.

In the following definition, an $\mathcal{A}$-valuation at a for a well-formed formula of $\mathcal{L}$ with m appropriate and $a \in \mathbb{C}Ob$, is a function

$$mWt \xrightarrow{t} aF$$

that has action $v_i \longmapsto t_i \in aF$ for variables v_i with $i \leq m$ and

$$w \longmapsto w_a \in aF \text{ for constants.}$$

The notation $t(i/s)$ indicates the valuation obtained from t by replacing t_i by the element $s \in aF$. An element in the image of t is also written as "t".

The $\mathcal{A}$-valuations satisfy the following closure condition: Let t be an $\mathcal{A}$-valuation at $a \in \mathbb{C}Ob$ for a well-formed formula Φ with m appropriate to Φ and $(b \xrightarrow{f} a) \in \mathbb{C}Ar$ with codomain a. Then $w_a(fF)$ is the image of w for any constant w of $\mathcal{L}$ and any $\mathcal{A}$-valuation at b for any well-formed formula Ψ with $k \geq m$ appropriate to Ψ.

By an *a-evaluated well-formed formula* is meant a well-formed formula Ξ of $\mathcal{L}$, m appropriate to Ξ and an $\mathcal{A}$-valuation at a for Ξ. An a-evaluated well-formed formula Ξ with $\mathcal{A}$-valuation t is written as $\Xi[t]$. Now:

Definition 1.5. Let $\mathcal{L}$ be a first-order language as before and let $\mathcal{A} = \langle \mathbb{C}, \mathrm{Cov}, F, \Vdash \rangle$ be an ordered quadruple where $\langle \mathbb{C}, \mathrm{Cov} \rangle$ is a site, F is a sheaf and $\Vdash$ is a binary relation between objects a of $\mathbb{C}$ and a-evaluated well-formed formulae. Then $\mathcal{A}$ is a *Kripke-Joyal model* if any well-formed formula Ξ with m appropriate to Ξ, $a \in \mathbb{C}Ob$ and t an $\mathcal{A}$-valuation at a, satisfies the appropriate condition below:

for Π atomic, if $a \Vdash \Pi[t]$ and $b \xrightarrow{f} a$ is an arrow of $\mathbb{C}$, then $b \Vdash \Pi[t(fF)]$;

if $\left\{ a_x \xrightarrow{f_x} a | x \in X \right\} \in a\mathrm{Cov}$ so that $\forall x \in X$, $a_x \Vdash \Pi[t(f_xF)]$ then $a \Vdash \Pi[t]$;

$a \Vdash (\exists v_i)\Sigma[t]$ iff there is a cover $\left\{ a_x \xrightarrow{f_x} a | x \in X \right\} \in a\mathrm{Cov}$ and a collection $\{s_x \in a_xF | x \in X\}$ so that $a_x \Vdash \Pi[t(f_xF)(i/s_x)]$, $\forall x \in X$;

$a \Vdash (\Phi \vee \Psi)[t]$ iff there is a cover $\left\{ a_x \xrightarrow{f_x} a | x \in X \right\} \in a\mathrm{Cov}$ so that $a_x \Vdash \Phi[t(f_xF)]$ or $a_x \Vdash \Psi[t(f_xF)]$, $\forall x \in X$;

$a \Vdash \neg\Phi[t]$ iff $\forall (b \xrightarrow{g} a)$, not $b \Vdash \Phi[t(gF)]$;

$a \Vdash (\Phi \supset \Psi)[t]$ iff $\forall (b \xrightarrow{g} a)$, if $b \Vdash \Phi[t(gF)]$ then $b \Vdash \Psi[t(gF)]$;

$a \Vdash (\Phi \wedge \Psi)[t]$ iff for every $b \xrightarrow{g} a$, $b \Vdash \Phi[t(gF)]$ and $b \Vdash \Psi[t(gF)]$;

$a \Vdash (\forall v_i)\Sigma[t]$ iff $\forall (b \xrightarrow{g} a)$ and any $s \in bF$, $b \Vdash \Sigma[t(gF)(i/s)]$. □

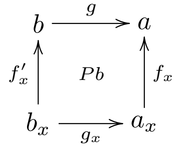
If Ξ has index n, let $a \Vdash \Xi[s_1, \ldots, s_n]$ if $a \Vdash \Xi[t]$ for some $\mathcal{A}$-valuation t for Ξ with m appropriate to Ξ, where $t_{i_1} = s_1, \ldots, t_{i_n} = s_n$. Then $a \Vdash \Xi$ if $a \Vdash \Xi[s_1, \ldots, s_n]$ for all $[s_1, \ldots, s_n] \in (aF)^n$. Finally, $\mathcal{A}$ is a model of Ξ, written $A \models \Xi$ if for all $a \in \mathbb{C}Ob$, $a \Vdash \Xi$.

We will need the property that truth persists in time. More specifically:

Lemma 1.6. *Let $\mathcal{A} = \langle \mathbb{C}, \mathrm{Cov}, F, \Vdash \rangle$ be a Kripke-Joyal model for $\mathcal{L}$ and Ξ a well-formed formula of $\mathcal{L}$. If $a \in \mathbb{C}Ob$, m is appropriate to Ξ and t is an $\mathcal{A}$-valuation of Ξ at a so that $a \Vdash \Xi[t]$, then $b \Vdash \Xi[t(gF)]$ for all $b \xrightarrow{g} a$.*

Proof. The proposition is true by definition if Ξ is an atomic formula. The remaining cases are proved by induction on the complexity or Ξ. The technique of the proof is illustrated here by establishing the cases where Ξ is $(\exists v_i)\Sigma(v_i)$ and where Ξ is $(\Phi \wedge \Psi)$.

Let $a \Vdash (\exists v_i)\Sigma[t]$. Then there is a cover $\left\{ a_x \xrightarrow{f_x} a | x \in X \right\} \in a\mathrm{Cov}$ and a collection $\{s_x \in a_xF | x \in X\}$ so that $a_x \Vdash \Sigma[t(f_xF)(i/s_n)]$. Let $b_x \xrightarrow{f'_x} b$ be the pullback of f_x along g for each $x \in X$, as shown in the diagram:



Note that $(f_xF)(g_xF) = (g_xf_x)F = (f'_xg)F = (gF)(f'_xF)$ for each $x \in X$ because F is a contravariant functor. By the induction hypothesis,

$b_x \Vdash \Sigma[t(gF)(f'_x F)(i/s_x(g_x F))]$ for each $x \in X$. Since $\left\{ b_x \xrightarrow{f'_x} b | x \in X \right\} \in b\mathrm{Cov}$ by condition (iv) of Definition 1.2, from Definition 1.5 it can be seen that $b \Vdash (\exists v_i)\Sigma[t(gF)]$.

Let $a \Vdash (\Phi \wedge \Psi)[t]$. Then for every arrow $c \xrightarrow{h} a$, $c \Vdash \Phi[t(hF)]$ and $c \Vdash \Psi[t(hF)]$. In particular, if $c \xrightarrow{h} a$ factors through $b \xrightarrow{g} a$, i.e. if h is the composition $c \xrightarrow{f} b \xrightarrow{g} a$ for some $c \xrightarrow{f} b$, then $c \Vdash \Phi[t((fg)F)]$ and $c \Vdash \Psi[t((fg)F)]$. And since F is a contravariant functor, $t((fg)F) = (t(gF))(fF)$. But then $c \Vdash \Phi[(t(gF))(fF)]$ and $c \Vdash \Psi[(t(gF))(fF)]$. Since $c \xrightarrow{f} b$ is any arrow with codomain b, from the definition of conjunction, $b \Vdash (\Phi \wedge \Psi)[t(gF)]$. □

2. A Reduction Theorem for the Kripke-Joyal Semantics

It is not clear at first what should be done to successfully turn the domain $\mathbb{C}$ of the model $\mathcal{A} = \langle \mathbb{C}, \mathrm{Cov}, F, \Vdash \rangle$ into a Heyting algebra. If the future state of an object a in $\mathbb{C}$ is defined to be the domain of an arrow with codomain a, then it may seem that ideals of objects closed with respect to the future should generate an appropriate Heyting algebra. That strategy works successfully where $\mathbb{C}$ is a preorder $I\!P$, so that for $p, q \in I\!POb$ there is at most a single arrow $p \leq q$. In the case of $a, b \in \mathbb{C}Ob$, however, there may be parallel arrows $a \overset{f}{\underset{g}{\rightrightarrows}} b$. Now, the topology generated by principal ideals made up of objects of $\mathbb{C}$ is too coarse. Given a well-formed formula Ξ and a valuation $t \in bF$ of its free variables for which $b \Vdash \Xi[t]$ there is no way of distinguishing between $a \Vdash \Xi[t(fF)]$ and $a \Vdash \Xi[t(gF)]$. This becomes a problem when trying to define forcing over the resulting Heyting algebra.

To enable these distinctions to be made, it turns out to be more fruitful to take collections of arrows closed under preextension. That is to say, $\overline{\mathbb{C}}$ is defined to be the collection of all the *cribles* of $\mathbb{C}$.

Definition 2.1. A collection p of arrows of $\mathbb{C}$ is a *crible* if and only if $(\forall f \in p)(\forall g \in \mathbb{C}Ar)(g\partial = \mathit{б} f \supset gf \in p)$. A crible p is a *b-crible* if every arrow of p has codomain b. A *crible* p is an *f-crible* if every arrow of p factors through f. □

The maximal f-crible is denoted by $\overline{f}$ and is said to be the crible generated by f. These constitute the basis of a topology which is fine enough to keep individual cases of forcing distinct.

Definition 2.2. Let $\mathbb{C}$ be a category. Then $\overline{\mathbb{C}}$ is the topology generated by the choice of $\left\{ \overline{f} \,|\, f \in \mathbb{C}Ar \right\}$ for a basis. □

Notice in the above definition that only arbitrary unions are needed, and that the resulting topology is the collection of all the cribles of $\mathbb{C}$. We know that every topological space is a Heyting algebra closed under arbitrary joins. In this case $\overline{\mathbb{C}}$ is a complete Heyting algebra since it is closed under arbitrary meets as well.

The problem arises of defining a pretopology for $\overline{\mathbb{C}}$. The passage from $\mathbb{C}$ to $\overline{\mathbb{C}}$ determines to a large extent the definition of $\overline{\text{Cov}}$. In particular, for $p \in \overline{\mathbb{C}}Ob$, a cover of p will be a collection of inclusions $\{p_i \leq p | i \in I\}$ whose domains are themselves cribles. So that the resultant semantics matches the original, only those collections are accepted that have the property, that for any arrow $a \xrightarrow{f} b \in p$ there is a cover $\left\{a_x \xrightarrow{f_x} a | x \in X\right\} \in a\,\text{Cov}$ in the original pretopology so that for each $x \in X$, $f_x f \in p_i$ for some $i \in I$. More formally:

Definition 2.3. Given a site $\langle \mathbb{C}, \text{Cov}\rangle$, let $\overline{\text{Cov}}$ be the assignment

$$\overline{\mathbb{C}}Ob \longrightarrow (\overline{\mathbb{C}}Ar)\mathcal{P}^2$$

$$p \longmapsto \Big\{\{p_i \leq p|i \in I\}\,|(\forall (a \xrightarrow{f} b) \in p)$$

$$\Big(\exists \left\{a_x \xrightarrow{f_x} a|x \in X\right\} \in a\text{Cov}\Big)\,(\forall x \in X)(\exists i \in I)(f_x f \in p_i)\Big\}. \qquad \square$$

Lemma 2.4. $\overline{\text{Cov}}$ *is a pretopology for* $\overline{\mathbb{C}}$*.*

Proof. Definition 1.2 needs to be checked:

(i) $\phi \not\in p\overline{\text{Cov}}$ since $\phi \not\in a\text{Cov}$ for any $a \in \mathbb{C}Ob$.

(ii) Let $p \in \overline{\mathbb{C}}Ob$. Then $\{1_p\} \in p\overline{\text{Cov}}$ since, for $a \xrightarrow{f} b \in p$, the cover $\left\{a \xrightarrow{1_a} a\right\} \in a\text{Cov}$ gives $1_a f \in p$.

For (iii) assume that $D = \{p_i \leq p|i \in I\} \in p\,\overline{\text{Cov}}$ and that for each p_i, D_i is a cover $D_i = \left\{p^i_j \leq p_i|j \in J_i\right\} \in p_i\,\overline{\text{Cov}}$. It must be shown that $D' = \left\{p^i_j \leq p|j \in J_i \wedge i \in I\right\} \in p\,\overline{\text{Cov}}$. Let $a \xrightarrow{f} b \in p$. Then there is a cover $\left\{a_x \xrightarrow{f_x} a|x \in X\right\} \in a\,\text{Cov}$ so that $\forall x \in X$ there is an $i \in I$ so that $f_x f \in p_i$. Because D_i is a cover of p_i, for $f_x f \in p_i$ there is a cover $\left\{a^x_y \xrightarrow{f^x_y} a_x|y \in Y_x\right\} \in a_x\,\text{Cov}$ so that $\forall y \in Y_x$ there is $j \in J_i$ so that $f^x_y f_x f \in p^i_j$. Because Cov is a pretopology for $\mathcal{C}$ it satisfies condition (ii) and so $\left\{a^x_y \xrightarrow{f^x_y f_x} a|y \in Y_x\right\} \in a\,\text{Cov}$ which, by the preceding argument has the property that for each $y \in Y_x$ and $x \in X$ there is $i \in I$ and $j \in J_i$ so that $f^x_y f_x f \in p^i_j$. But then D' is a cover of p.

To see that (iv) is satisfied, let $r \leq p$ and $D = \{p_i \in p|i \in I\} \in p\,\overline{\text{Cov}}$. The pullback of $r \leq p$ with $p_i \leq p$ is the lattice meet $r \wedge p_i, \forall i \in I$. It must be shown that $D_r = \{r \wedge p_i|i \in I\}$ is a cover of r. Let $a \xrightarrow{f} b \in r$. Then there is a cover $\left\{a_x \xrightarrow{f_x} a|x \in X\right\} \in a\,\text{Cov}$ so that, for each $x \in X$ there is an $i \in I$ with the property that $f_x f \in p_i$. But $f_x f \in r$ and so $f_x f \in r \wedge p_i$. This is true for every $x \in X$. Hence D' is a cover of r and property (iv) is satisfied. Hence $\overline{\text{Cov}}$ is a pretopology for $\overline{\mathbb{C}}$. $\square$

Next comes the question of what the right translation for a sheaf F is over $\langle\overline{\mathbb{C}}, \overline{\text{Cov}}\rangle$. Here one is guided by tradition. That is to say, one looks for a sheaf whose image of $p \in \overline{\mathbb{C}}Ob$ is a collection of sections of F and whose arrows are restrictions. By a section $\overline{s}$ of F is meant a function $p \xrightarrow{\overline{s}} S$ that chooses for each $(a \xrightarrow{f} b) \in p$ a germ from the stalk aF over a. That is to say, $f\overline{s} \in aF$. Now, these sections are used to evaluate variables in well-formed formulae and in order to obtain a semantics that is compatible with the original, the image $p\overline{F}$ is defined to be the collection of all the sections over p that satisfy functoriality. More formally:

Definition 2.5. Let F be a sheaf over the site $\langle \mathcal{C}, \mathrm{Cov} \rangle$, then $\overline{F}$ is the assignment

$$\begin{aligned} \overline{\mathbb{C}}Ob &\xrightarrow{\overline{F}} S \\ p &\longmapsto \Big\{ p \xrightarrow{\overline{s}} \cup_{f \in p} (\partial f)F | (\forall f \in p)(f\overline{s} \in (\partial f)F) \\ &\quad \wedge (\forall f \in p)(\forall (c \xrightarrow{g} \partial f)(gf)\overline{s} = (f\overline{s})(gF) \Big\} \\ q \leq p &\longrightarrow (p\overline{F} \longrightarrow q\overline{F}) \\ \overline{s} &\longrightarrow \overline{s}\lceil q \qquad \square \end{aligned}$$

Lemma 2.6. *$\overline{F}$ is a sheaf over $\langle \overline{\mathbb{C}}, \overline{\mathrm{Cov}} \rangle$.*

Proof. Clearly $\overline{F}$ is a contravariant functor. To check Definition 1.4 let $D = \{p_i \leq p | i \in I\} \in p\,\overline{\mathrm{Cov}}$ and $S = \{\overline{s}_i \in p_i\overline{F} | i \in I\}$ be a collection of sections that agree on the intersections of the p_i. Then it must be shown that there is a unique section over p whose restriction to each p_i is just $\overline{s}_i$. First, a candidate $\overline{s}$ is defined.

Let $a \xrightarrow{f} b \in p$. Then choose a cover $\left\{ a_x \xrightarrow{f_x} a | x \in X \right\} \in a\,\mathrm{Cov}$ and for each $x \in X$ choose $i \in I$ with the property that $f_x f \in p_i$. Let $s_x = (f_x f)\overline{s}_i$. Now $f_x f \in p_i$ and $f_y f \in p_j$. Let $s_x = (f_x f)\overline{s}_i$ and $s_y = (f_y f)\overline{s}_j$. By forming the pullback of f_x with f_y it can be seen that s_x and s_y are compatible. For suppose that f'_x is the pullback of f_x along f_y and f'_y is the pullback of f_y along f_x. Then, because the pullback square commutes, and because of the defining property for $\overline{s}_i$ and $\overline{s}_j$ and the fact that they agree on the intersection of p_i and p_j: $s_x(f'_y F) = (f'_y f_x f)\overline{s}_i = (f'_x f_y f)\overline{s}j = s_y(f'_x F)$. This is true for any $x \in X$, $y \in X$ and appropriate $i \in I$, $j \in I$, and so, because F is a sheaf over $\langle \mathbb{C}, \mathrm{Cov} \rangle$ there is a unique $s \in aF$ so that $s_x = s(f_x F)$, $\forall x \in X$.

Let $f\overline{s} = s$. Now observe that the definition of $f\overline{s}$ is independent of the choice of $\left\{ a_x \xrightarrow{f_x} a | x \in X \right\} \in a\,\mathrm{Cov}$ and p_i with $f_x f \in p_i$. To this end, let $\left\{ a_x \xrightarrow{f_x} a | x \in X \right\} \in a\,\mathrm{Cov}$ and i_x with $f_x f \in p_{i_x} \in p\,\overline{\mathrm{Cov}}$ and $\{s_x \in a_x F | x \in X\}$ give the value s at f and $B = \left\{ c_y \xrightarrow{g_y} a | y \in Y \right\} \in a\,\mathrm{Cov}$ and j_y with $g_y f \in p_{j_y} \in p\,\overline{\mathrm{Cov}}$ and $\{t_y \in a_y F | y \in Y\}$ give the value t at f, where s_x and t_y are defined above. Now pull back B along each f_x for $x \in X$. This gives a new cover $\left\{ a_x \wedge c_y \xrightarrow{g_y^x f_x} a | y \in Y \text{ and } x \in X \right\} \in a\mathrm{Cov}$

where $a_x \wedge c_y \xrightarrow{g_y^x} a_x$ is the pullback of g_y along f_x for each $y \in Y$ and $x \in X$. Then $s((g_y^x f_x)F) = (g_y^x f_x f)\overline{s}_{i_x}$ for each $y \in Y$ and $x \in X$. By a similar argument $t((f_x^y g_y)F) = (f_x^y g_y f)\overline{s}_{j_y}$, $\forall y \in Y$ and $x \in X$. But then

$$s((g_y^x f_x)F) = (g_y^x f_x f)\overline{s}_{i_x} = (f_x^y g_y f)\overline{s}_{j_y} = t((f_x^y g_y)F) = t((g_y^x f_x)F)$$

for all $x \in X$ and $y \in Y$. Because F is a sheaf, $s = t$.

Now let $\overline{s}$ be the choice function over p that gives $f\overline{s} = s$ where s is defined above for $f \in p$. Does $\overline{s}$ satisfy the conditions of Definition 2.5? To see that it does, let $(a \xrightarrow{f} b) \in p$ and $c \xrightarrow{g} a$ be an arrow with codomain a. Then $gf \in p$. Let $s = f\overline{s}$ and $t = (gf)\overline{s}$ where s and t are defined by the construction given above. It must be shown that $t = s(gF)$.

To see that, let $A = \left\{a_x \xrightarrow{f_x} a | x \in X\right\} \in a\,\mathrm{Cov}$ and for $x \in X$ let $i_x \in I$ be such that $f_x f \in p_{i_x}$. Because covers are stable under pullbacks (that is to say, because of property (iv) of a pretopology), A can be pulled back along g to give $C = \left\{c_x \xrightarrow{f'_x} c | x \in X\right\} \in c\,\mathrm{Cov}$. Because of the way that t is defined, $(f'_x g f)\overline{s}_{i_x} = t(f'_x F)$ for all $x \in X$ since $f'_x g f = g_x f_x f$ and $g_x f_x f \in p_{i_x}$ for all $x \in X$. But then, because each $\overline{s}_i$ satisfies the defining property, $(g_x f_x f)\overline{s}_{i_x} = s(g_x f_x)F$, $\forall x \in X$, where g_x is the pullback of g along f_x, $\forall x \in X$. But $s((g_x f_x)F) = s((f'_x g)F) = s(gF)(f'_x F), \forall x \in X$ and so $s(gF)(f'_x F) = s((g_x f_x)F) = (g_x f_x f)\overline{s}_{i_x} = (f'_x g f)\overline{s}_{i_x} = t(f'_x F)$ for all $x \in X$. But C is a cover of c and the $t(f'_x F)$ are clearly compatible. Therefore there is only one $t \in cF$ so that $t(f'_x F) = (f'_x g f)\overline{s}_{i_x}$, for all $x \in X$. Hence $t = s_q F$.

Next, to see that $\overline{s}\lceil p_i = \overline{s}_i$, $\forall i \in I$, let $(a \xrightarrow{f} b) \in p_i$ and choose $a \xrightarrow{1_a} a \in a\,\mathrm{Cov}$. Then by definition $f\overline{s} = f\overline{s}_i$.

To see that $\overline{s}$ is unique, assume that there is a section $\overline{t}$ on p that obeys the functoriality condition with the property that $\overline{t}\lceil p_i = \overline{t}_i, \forall i \in I$. Let $(a \xrightarrow{f} b) \in p$. It must be shown that $f\overline{s} = f\overline{t}$. There is a cover $\left\{a_x \xrightarrow{f_x} a | x \in X\right\} \in a\,\mathrm{Cov}$ so that $\forall x \in X$ there is $i \in I$ with the property that $f_x f \in p_i$. But then $(f_x f)\overline{s} = (f_x f)\overline{s}_i = (f_x f)\overline{t}$, $\forall x \in X$ and appropriate $i \in I$. As shown before, the $(f_x f)\overline{s}_i$ are compatible. Hence, because F is a sheaf, there is a unique $s \in aF$ so that $s(f_x F) = s_x$, $\forall x \in X$. And so $f\overline{s} = s = f\overline{t}$. □

Finally, it remains to define a forcing relation over the site $\langle \overline{C}, \overline{\mathrm{Cov}}\rangle$, with respect to the sheaf $\overline{F}$, so that a Kripke-Joyal model can be obtained that is compatible with the original. There is a natural candidate:

Definition 2.7. Let $\mathcal{A} = \langle \mathbb{C}, \mathrm{Cov}, F, \Vdash\rangle$ be a Kripke-Joyal model for the first-order language $\mathcal{L}$. Let Ξ be a well-formed formula of $\mathcal{L}$. Then for $p \in \overline{\mathbb{C}}Ob$ and $\overline{t}$ a valuation for Ξ at p, consisting of sections of $\overline{F}$, define

$$p \;\overline{\Vdash}\; \Xi[\overline{t}] \qquad \text{iff} \qquad \forall (a \xrightarrow{f} b) \in p, \qquad a \Vdash \Xi[f\overline{t}].$$

Given Ξ with m appropriate and $p \in \overline{\mathbb{C}}Ob$, the valuation $\bar{t}$ is a choice function

$$mWt \xrightarrow{\bar{t}} p\overline{F}$$

that has action $\qquad v_j \longrightarrow \bar{t}_j \in p\overline{F}$ for variables $\mathrm{v_j}$ with $\mathrm{j \leq m}$

and $\qquad w \longrightarrow \overline{w} \in p\overline{F}$ for constants w

where $f\overline{w} = w_{6\,f} \in (6\,f)F$ for all $f \in p$, and where $w_{6\,f}$ is the $\mathcal{A}$-valuation of w at $6\,f$. To see that $\overline{w}$ obeys functoriality, recall that for $c \xrightarrow{g} 6\,f$, $g\overline{w} = w_{6\,f}(gF)$ interprets w at c. As a result, there is no problem with the treatment of constants of $\mathcal{L}$ in the transition from $\mathcal{A}$ to $\overline{\mathcal{A}}$. □

Theorem 2.8. *Given $\mathcal{A} = \langle C, \mathrm{Cov}, \mathrm{F}, \Vdash\rangle$, the derived quadruple $\overline{\mathcal{A}} = \langle \overline{C}, \overline{\mathrm{Cov}}, \overline{F}, \overline{\Vdash}\rangle$ is a Kripke-Joyal semantics.*

Proof. It remains only to show that $\overline{\Vdash}$ satisfies the properties of the Kripke-Joyal semantics in Definition 1.5. The proof proceeds by induction on the complexity of a well-formed formula Ξ. Let $p \in \overline{\mathbb{C}}Ob$ and $\bar{t} \in (p\overline{F})^m$ where m is appropriate to Ξ. The atomic case is an easy consequence of functoriality. Let Ξ be the well-formed formula $(\exists v_i)\Sigma(v_i)$ where Σ contains at least the free variable v_i, and assume that the hypothesis holds for Σ. It must be shown that $p \overline{\Vdash} (\exists v_i)\Sigma[\,\bar{t}\,]$ if and only if there is a cover $\{p_i \leq p | i \in I\} \in p\overline{Cov}$ and a collection $\left\{\bar{s}_i \in p_i\overline{F} | i \in I\right\}$ so that $p_i \overline{\Vdash} [\Sigma(\bar{t}\lceil p_i)(i/\bar{s}_i)]$, $\forall i \in I$.

To show this last equivalence, assume first that $p \overline{\Vdash} (\exists v_i)\Sigma[\bar{t}\,]$. It is necessary to construct an appropriate cover of p. To do this, first observe that $\forall (a \xrightarrow{f} b) \in p$, $a \Vdash (\exists v_i)\Sigma[f\bar{t}\,]$. There is a cover $\left\{a_x \xrightarrow{f_x} a | x \in X_f\right\} \in a\,\mathrm{Cov}$ and a collection $\{s_x \in a_x F | x \in X_f\}$ so that $a_x \Vdash \Sigma((f\bar{t}\,)(f_x F))(i/s_x)F$, $\forall x \in X_f$. But then $\overline{f_x f} \leq p$, $\forall x \in X_f$. Let the cover of p be the collection $\left\{\overline{f_x f} | x \in X_f \wedge f \in p\right\}$. Clearly this belongs to $p\,\overline{\mathrm{Cov}}$. For each $\overline{f_x f}$ let the section $\bar{s}_x$ associated with it be the section generated by s_x. By that is meant that if $c \xrightarrow{g} a_x$ is any arrow then $(gf_x f)\bar{s}_x = s_x(gF)$. Then $\bar{s}_x$ satisfies the defining property of a germ in the stalk $(\overline{f_x f})\overline{F}$. But now, since $a_x \Vdash \Sigma((f\bar{t}\,)(f_x F))(i/s_x)$, by Lemma 1.6 if $c \xrightarrow{g} a_x$ is an arrow of C then $c \Vdash \Sigma[((f\bar{t})(f_x F)(gF))(i/s_x(gF))]$. Since $(f\bar{t})(f_x F)(gF) = (gf_x f)\bar{t}$ for all $x \in X_f$ then $c \Vdash \Sigma[(gf_x f)\bar{t}(i/(gf_x f)\bar{s}_x)]$. But then $\overline{f_x f} \overline{\Vdash} \Sigma[\bar{t}\lceil \overline{f_x f}(i/\bar{s}_x)]$. This is true for every $x \in X_f$ and $f \in p$. Hence there is a cover $\left\{\overline{f_x f} | x \in X_f \wedge f \in p\right\} \in p\,\overline{\mathrm{Cov}}$ and a collection $\{\bar{s}_x | x \in X_f \wedge f \in p\}$ so that $\overline{f_x f} \overline{\Vdash} \Sigma[\,\bar{t}\lceil\, \overline{f_x f}(i/\bar{s}_x)]$ for all $x \in X_f$ and $f \in p$.

To show the converse, assume that there is a cover $\{p_i \leq p | i \in I\} \in p\,\overline{\mathrm{Cov}}$ and a collection $\left\{\bar{s}_i \in p_i\overline{F} | i \in I\right\}$ so that $p_i \overline{\Vdash} \Sigma[\ \bar{t}\lceil p_i(i/\bar{s}_i)]$, $\forall i \in I$. Let $(a \xrightarrow{f} b) \in p$. Then there is a cover $\left\{a_x \xrightarrow{f_x} a | x \in X\right\} \in a\,\mathrm{Cov}$ $\forall x \in X$ there is an $i \in I$ with the property that $f_x f \in p_i$. But $a_x \Vdash \Sigma[(f_x f)\bar{t}(i/(f_x f)\bar{s}_i)]$ for every $x \in X$ and appropriate $i \in I$. Since $(f_x f)\bar{t} = (f\bar{t})(f_x F)$, $\forall x \in X$ then $a \Vdash (\exists v_i)\Sigma[f\bar{t}\,]$. But then $p \overline{\Vdash} (\exists v_i)\Sigma[\,\bar{t}\,]$.

The case for disjunction is similar to that for the existential quantifier. Negation, implication and conjunction are similar to the case for the universal quantifier which is demonstrated below:

It must be shown that $p\overline{\Vdash}(\forall v_i)\Sigma[\,\overline{t}\,]$ iff for every $q \leq p$ and for every $\overline{s}_i \in q\overline{F}$, $q\overline{\Vdash}\Sigma[\,\overline{t}\lceil q(i/\overline{s}_i)]$. Assume that $p\overline{\Vdash}(\forall v_i)\Sigma[\,\overline{t}\,]$, $q \leq p$, $\overline{s}_i \in q\overline{F}$ and let $(a \xrightarrow{f} b) \in q$. Then $f \in p$ and $a \Vdash (\forall v_i)\Sigma[f\overline{t}]$. But then $a \Vdash \Sigma[(f\overline{t})(i/f\overline{s}_i)]$. Hence $q\overline{\Vdash}\Sigma[\,\overline{t}\lceil q(i/\overline{s}_i)]$. Now, to show the converse, assume for every $q \leq p$ and $\overline{s}_i \in q\overline{F}$ that $q\overline{\Vdash}\Sigma[\,\overline{t}\lceil q(i/\overline{s}_i)]$. Let $(a \xrightarrow{f} b) \in p$, let $c \xrightarrow{g} a$ be an arrow with codomain a and let $s \in cF$. Let $\overline{s}_i \in (\overline{gf})\overline{F}$ be the section generated by s. Since $\overline{gf} \leq p$ then $\overline{gf}\overline{\Vdash}\Sigma[\,\overline{t}\lceil q(i/\overline{s}_i)]$. But then $c \Vdash \Sigma[(gf)\overline{t}(i/s)]$ and so $a \Vdash (\forall v_i)\Sigma[f\overline{t}]$. □

Now, with any $a \in \mathbb{C}Ob$ is associated the maximal crible $\overline{1}_a$, and with any $t \in aF$ can be associated the section $\overline{t}$ generated by the germ t. This leads immediately to:

Corollary 2.9. *Given a Kripke-Joyal model $\mathcal{A} = \langle \mathbb{C}, \mathrm{Cov}, \mathrm{F}, \Vdash \rangle$ and derived Kripke-Joyal model $\overline{A} = \left\langle \overline{\mathbb{C}}, \overline{\mathrm{Cov}}, \overline{F}, \overline{\Vdash} \right\rangle$ then, for well-formed formulae Ξ of the first-order language $\mathcal{L}$, for any $a \in \mathbb{C}Ob$ and any valuation t for Ξ at a,*

$$a \Vdash \Xi[t] \qquad \textit{if and only if} \qquad \overline{1}_a \overline{\Vdash} \Xi[\,\overline{t}\,].$$

3. Conclusion

This paper began by considering a Kripke-Joyal model $\mathcal{A} = \langle \mathbb{C}, \mathrm{Cov}, \mathrm{F}, \Vdash \rangle$ defined over an arbitrary category $\mathbb{C}$. By taking the collection of cribles on $\mathbb{C}$, a complete Heyting algebra $\overline{\mathbb{C}}$ was obtained. A pretopology $\overline{\mathrm{Cov}}$ was defined for $\overline{C}$ that preserved the essential features of Cov. Furthermore, in the transition from $\langle \mathbb{C}, \mathrm{Cov}, \rangle$ to $\left\langle \overline{\mathbb{C}}, \overline{\mathrm{Cov}} \right\rangle$, the pretopology $\overline{\mathrm{Cov}}$ has acquired the property that any collection $\{p_i \leq p | i \in I\}$ with $\bigvee_{i \in I} p_i = p$ is a cover of p. By taking sections of F that obey functoriality, a sheaf $\overline{F}$ was obtained whose germs are used to interpret the constants and free variables of well-formed formulae of $\mathcal{L}$ in a manner that is compatible with the original $\mathcal{A}$-valuation. Finally, a forcing relation $\overline{\Vdash}$ was defined over $\left\langle \overline{\mathbb{C}}, \overline{\mathrm{Cov}} \right\rangle$ in terms of the original relation $\Vdash$. It was shown that $\overline{A} = \left\langle \overline{\mathbb{C}}, \overline{\mathrm{Cov}}, \overline{F}, \overline{\Vdash} \right\rangle$ is a Kripke-Joyal model and that it forces precisely the same well-formed formulae as the original model A. As a result, without loss of generality, forcing over an arbitrary category can always be replaced by forcing over a complete Heyting algebra.

On the face of it, this result seems impossible. How is it that the generality of a category can be replaced by the specificity of a complete Heyting algebra as a domain of forcing? The answer lies in the construction of the derived category. What we see is that, in the derived category, each object of the original category indexes all future states of that object, so that any pathway from the past to the future in the original category can be traced along any of a number of pathways in the derived category. This also reveals that the original category is not really “embedded” in the derived category or, more accurately perhaps, that it is "embedded" in multiple distorted variations. In that sense this is an "embedding theorem" whereby a general construction is multiply enfolded in a construction with more structure.

What is the significance of this result? This work arose in the course of attempting to internalize Cohen's forcing in a topos. Given that Cohen's forcing is an instance of a Kripke-Joyal semantics, it remained only to internalize a Kripke-Joyal semantics. The idea was to try to force along the lattice of subobjects of 1 of a topos and then to go from there. That could only be done if the domain of forcing of a Kripke-Joyal semantics had more structure to it than it appeared to have, namely, that the domain of forcing could always be replaced by a complete Heyting algebra. That is how we set about looking for a proof, which we found, as detailed in this paper. The remainder of our project was less successful in that we could only internalize a generalized version of forcing for very restricted languages [8].

There is also a philosophical point that could be of significance. Given that a complete Heyting algebra is isomorphic to the lattice of subobjects of 1 of the category of sheaves over it, the domain of forcing can always be thought of as a collection of possible truth values. And not just arbitrary truth values, but truth values that are situated between true and false. Thus, this theorem not only provides a reduction of the generality of the domain of forcing of a Kripke-Joyal semantics, but also reduces the generality of truth values so that they can always be said to lie between true and false for any semantics that are instances of a Kripke-Joyal semantics.

Authors' Note

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