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UTILIZING SEMIOTIC PERSPECTIVE TO INVESTIGATE ALGEBRA II STUDENTS' EXPOSURE TO AND USE OF MULTIPLE REPRESENTATIONS IN UNDERSTANDING ALGEBRAIC CONCEPTS

Isaac Gitonga

ACCEPTANCE

This dissertation, UTILIZING SEMIOTIC PERSPECTIVE TO INVESTIGATE ALGEBRA II STUDENTS' EXPOSURE TO AND USE OF MULTIPLE REPRESENTATIONS IN UNDERSTANDING ALGEBRAIC CONCEPTS, by ISAAC GITONGA, was prepared under the direction of the candidate's Dissertation Advisory Committee. It is accepted by the committee members in partial fulfillment of the requirements for the degree, Doctor of Philosophy, in the College of Education and Human Development, Georgia State University. The Dissertation Advisory Committee and the student's Department Chairperson, as representatives of the faculty, certify that this dissertation has met all standards of excellence and scholarship as determined by the faculty.

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UTILIZING SEMIOTIC PERSPECTIVE TO INVESTIGATE ALGEBRA II STUDENTS' EXPOSURE TO AND USE OF MULTIPLE REPRESENTATIONS IN UNDERSTANDING ALGEBRAIC CONCEPTS: AN EXPLORATORY CASE STUDY

by

ISAAC GITONGA

Under the Direction of CHRISTINE D. THOMAS Ph.D.

ABSTRACT

The study employed Ernest (2006) Theory of Semiotic Systems to investigate the use of and exposure to multiple representations in a 10th grade algebra II suburban high school class located in the southeastern region of the United States. The purpose of this exploratory case study (Yin, 2014) was to investigate the role of multiple representations in influencing and facilitating algebra II students' conceptual understanding of piecewise function, absolute-value functions, and quadratic functions. This study attempted to answer the following question: How does the use of and exposure to multiple representations influence algebra II students' understanding and transfer of algebraic concepts? Furthermore, the following sub-questions assisted in developing a deeper understanding of the question: a) how does exposure to and use of multiple representations influence students' identification of their pseudo-conceptual understanding of algebraic concepts?; b) how does exposure to and use of multiple representations influence students' transition from pseudo-conceptual to conceptual understanding?; c) how does exposure to and use of multiple representations influence students' transfer of their conceptual understanding to other related concepts? Understanding the notion of pseudo-conceptual understanding in algebra is significant in providing a tool for examining the veracity of algebra students' conceptual understanding, where teachers have to consistently examine if students accurately understand the meanings of the mathematical signs that they are constantly using. The following data collection techniques were utilized: a) classroom observation, b) task based interviews, and c) study of documents. The unit of analysis was students' verbal and written responses to task questions. Three themes emerged from the analysis of in this study: (a) re-imaging of conceptual understanding; (b) reflective approach to understanding and using mathematical signs; and (c) representational versatility in the use of mathematical signs. Findings from this study will contribute to the body of knowledge needed in research on understanding and assessing algebra students' conceptual understanding of mathematics. In particular the findings from the study will contribute to the literature on understanding; the process of algebraic concepts knowledge acquisition, and the challenges that algebra students have with comprehension of algebraic concepts (Knuth, 2000: Zaslavsky et al., 2002).

INDEX WORDS: Representations, Multiple representations, Pseudo-Conceptual understanding, Conceptual understanding, Semiotic systems.

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DEDICATION

But they that wait upon the Lord shall renew their strength; they shall mount up with wings as eagles; they shall run, and not be weary; and they shall walk, and not faint.

Isaiah 40:31 KJV

This journey would have not been possible without the prayers, help, support and encouragement from my family, friends, colleagues and mentor. This dissertation is especially dedicated to my family; my dear wife Dr. Lucy Gitonga and my three daughters Tammy, Tuana and Taylor. The four of you have encouraged me even when the journey looked tiring and impossible. To my dear wife thank you for encouraging me to: 'hang in there'; 'keep going'; 'stay strong' and reminding me that the Lord will renew my strength and give me wings to soar above the impossible. I feel honored to be married to such a wonderful and dear person as you and all the love you have for me and our three girls. To my dear children, Tammy, Tuana, and Taylor for your encouragement, patience, and understanding as daddy endeavored to accomplish this important goal. I am so proud to be your dad.

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Table of Contents

LIST OF TABLES

LIST OF FIGURES

1 INTRODUCTION

"…..what is mathematically simple and occurs at the initial stage of mathematical knowledge construction can be cognitively complex and requires a development of a specific awareness about the coordination of semiotic registers" (Duval 2006, pp. 126-127).

Background

In learning algebraic concepts which are normally generalized in various families of functions like the; linear, absolute-value, quadratic, cubic, exponential, and rational functions students often demonstrate a certain level of proficiency in manipulating algebraic symbols, solving equations and analyzing graphical representations of these functions. When encouraged some students can verbalize and explain the steps they perform in solving a given algebra problem involving these functions, therefore demonstrating an awareness of the necessary procedures required to solve an algebra problem. This awareness of algebraic procedural competence has been documented in several research studies (Davis, 2007; Knuth, 2000 and Carlson, 1998). However, several research study findings (Garofalo & Trinter, 2012; Brousseau, 1997; Carlson, 1998; Nabb, 2010; Kozulin, 2003; Davis, 2007; and Knuth, 2000) illuminate the following concerns: (a) students still memorize facts or procedures without understanding the underlying meaning structures in the mathematical concepts and in the procedures; and (b) students often demonstrate a fragile understanding and difficulties in explaining when or how they can use what they know. Consistent with the above concerns is Carlson (1998) argument that the primarily procedural orientation to using functions to solve specific problems has led to absence of meaning and coherence for students.

Learning mathematics and in particular the three functions mentioned above (i.e., piece-wise, absolute-value, and quadratic) is a communicative activity that involves the use of mathematical sign systems. This use of sign involves understanding how students receive the signs i.e., while listening and reading, and produce the signs while speaking, writing or inscribing or drawing. Ernest (2006) observes that creativity in students' sign reception and sign production at times may be conceptualized as the ultimate expression of conceptual understanding. One of my areas of interest as a teacher of algebra for the last twenty six years has been attempting to understand algebra students' mathematical signs receptions and productions. Specifically how algebra students' use and communicate their algebraic conceptual understanding. My interest also include understanding students' verbal and written responses to algebraic questions which on the surface demonstrate a full understanding of a particular mathematical notion yet their knowledge of the notion is riddled with contradiction and connections not based on logic (Berger, 2004b). Researchers (Sfard, 2000; Berger, 2005; Vygotsky, 1978) describe this type of pre-conceptual phase of understanding as pseudo-conceptual understanding which refers to students' use of words and mathematical symbols in communication without knowing exactly what they mean or represent (Sfard, 2000; Vinner, 1998; & Vygotsky, 1986).

Findings from several studies (Berger, 2004b; Ainsworth, 2006; & Knuth 2000) on the analysis of students' written and verbal responses to assigned algebraic tasks reveal evidence of pseudo-conceptual understanding of algebraic concepts. These findings have also been identified in the way students communicate their conceptual understanding with teachers and peers (Eraslan & Aspinwall, 2007; Garofalo & Trinter, 2012). Ernest (2006) considers this understanding as developmental and relates it to "using mathematical signs to refer to mathematical objects prior to fully understanding the sign, the transformational rules and the underlying meaning structure of those signs" (p. 3). In this phase of pseudo-conceptual thinking, students' primary focus of attention is the mathematical sign or a set mathematical signs procedures rather than the meaning or the ideas represented by the sign/s. Equally significant in this developmental phase of understanding is that students still perceive mathematical signs as an end in themselves as opposed to mathematical signs being representations of a given mathematical concepts (Sfard, 2000).

In this study I assumed Peirce (1998) triadic structure of a sign as the definition of a mathematical sign. Pierce's definition which is expounded in detail in the definition of terms observes that all signs have a triadic structure: (a) a representamen (inscriptions) which refers to the form a sign takes; (b) an object i.e., a physical thing or an abstract object; and (c) interpretant i.e., an idea or the meaning of an object. Examples of mathematical signs include; a mathematical symbol, a mathematical statement, mathematical expression, the name of a mathematical object, and so on (Berger, 2004b).

Concerns discussed above are consistent with algebra students' gravitating towards unreflective algebraic procedures and manipulations, which have in turn led to a call for instruction and assessment activities that promotes deeper conceptual understanding. It follows that deeper conceptual understanding of algebraic function has been a foremost concern in several research literature (CCSSI, 2010; Berthold et al., 2009; Carlson, 1998). In this study, I adopted Godino (1996) description of conceptual understanding as the implicit or explicit knowledge of the principles that govern a given mathematics domain, and the interaction between the various units of a given domain, as the definition of conceptual understanding. In an attempt to address this concern, exposure to and use of multiple representations has been proposed to potentially support and facilitate students' deeper conceptual understanding of mathematics in general (Ainsworth,

2006) and specifically in algebra (Monk, 2003; Goldin, 2003; and Smith, 2003). In addition research on learning with and exposure to multiple representations recognizes the potential benefits of facilitating students' deeper conceptual understanding (Berthold et al., 2009; Ainsworth, 2006; Ainsworth & van Labeke, 2004; Clement, 2004; and Tripathi, 2008). The findings from the several research studies (Amit & Fried, 2005; Sfard, 2000; White & Pea, 2011; Davis, 2007; and Knuth, 2000) that I looked into including Common Core State Standards Initiative (CCSSI, 2010), and the National Council of Teachers of Mathematics (NCTM, 2000) through the various publications have advocated for the development of curricula that are challenging and engaging for students. The findings from these studies also advocate for instruction that leads to deep conceptual understandings of mathematical concepts. CCSSI (2010) for example describes the hallmark of mathematical understanding as the "ability to justify, in a way appropriate to the student's mathematical maturity, why a particular mathematical statement is true or where a mathematical rule comes from" (p.2). Need for conceptual understanding is further supported by Berger (2004b) and Sfard (2000) observation that students who have attained a deeper conceptual understanding of a mathematical notion are capable of attending to a mathematical object (e.g., definition of functions or an absolute-value function) in its entirety and not just as a fragmented aspect of the object. However, several research study findings on exposure and use of multiple representations still reveal evidence of students' understanding of mathematics that do not extend beyond simple procedural competence (Knuth, 2000; Berger, 2005; Eraslan & Aspinwall, 2007; and Zaslavsky et al., 2002). This type of understanding is consistent with; Sfard (2000), Berger (2005) and Kozulin (2003) description of pseudo-conceptual understanding where students use words or symbols without knowing exactly what they mean or represents.

The following three families algebraic functions i.e., piecewise, absolute-value, and quadratic were the focus of the study. This study examined the role of multiple representations in facilitating 10^{th} grade algebra II students': (a) identification of pseudo-conceptual understandings in the three functions; transition from pseudo conceptual to conceptual understanding; and (c) transfer of their conceptual understanding to other related concepts. In this study the use of the term multiple representations will focus on the use of: verbal descriptions (oral or written words), tabular or pictorial representations (table of values), algebraic or symbolic representations (which include equations expressing the relationship between two or more quantities), and graphical representations (which include the Cartesian graphs).

Statement of the Problem

There is a need to examine how students' use of and exposure to different representations can potentially extend algebra students' conceptual understanding beyond procedural competence. Research on exposure to and use of multiple representations in mathematics education has demonstrated potential effectiveness in enhancing students' conceptual understanding (Ainsworth, 2006; Elia et al., 2007; Davis, 2007; Goldin, 2003; Monk, 2003; Smith, 2003; Goldin & Schteingold 2001; and Gagatsis & Elia, 2004). In addition, literature in mathematics teaching and learning is filled with examples of ways the use of multiple representations have had the potential to serve as resources for supporting students' conceptual understanding (Rider, 2007; Clement, 2004; Berthold & Renkl, 2009; Garofalo & Trinter, 2012; and Tripathi, 2008).

Though results from these research studies indicate potential positive outcomes in the use of multiple representations in promoting students' conceptual understanding in algebra, other studies (Davis, 2007; Knuth, 2000; Amit & Fried, 2005, Zaslavsky et al., 2002; Lobato &

Seirbert, 2002) have indicated the need for a closer examination of these conceptual understandings. In Garofalo and Trinter (2012) study for example, they describe how a significant number of algebra students capable of reciting the quadratic formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, often demonstrate difficulties in explaining the meaning of the various parts of the formulae (e.g., $\sqrt{b^2 - 4ac}$ or $x = \frac{-b}{2a}$) as well as describing what each variable in the formula represents.

The challenge of understanding students' conceptual understanding that does not extend beyond procedural competence is complicated by the theoretical framework used to study the problem. Studies investigating exposure to and use of multiple representations in enhancing students conceptual understanding (Amit & Fried, 2005; Ainsworth, 2006; Friedlander & Tabach, 2001; van der Meij & de Jong, 2006; diSessa, 2004; Zaslavsky, 2002; Knuth, 2000) are mainly rooted in theoretical frameworks in which conceptual understanding is regarded as deriving largely from interiorized actions (Berger, 2005, Sfard, 2000). In these theoretical frameworks the crucial role of mathematical signs and sign use in the teaching and learning of mathematical concepts is not integrated into the framework. This essential role of mathematical signs which according to Hoffman (2006) includes: (a) means by which students think about mathematical relations and objects (e.g., definitions of a function); and (b) product of students' mathematical thinking, is largely not recognized in the theoretical framework of the studies mentioned above. In an algebra class for example, students demonstrate an understanding of their algebraic concepts by means of visible signs and it is through the process of interpretation and transformation of these signs that students develop an understanding of mathematical concepts (Hoffman, 2006). Hence, it is appropriate that a sign-oriented perspective be integrated into a study that examines students' conceptual understanding. This study proposes the use of semiotics perspective from which I can investigate the exposure to and use of multiple representations. In particular, I will be utilizing Ernest (2006) Semiotic systems theory to examine how the notions of pseudo-conceptual and conceptual understanding of algebra concepts allied with the notion of mathematical sign provide a theoretical framework with which to examine algebra II students' construction of mathematical concepts.

Purpose of the Study

The purpose of this exploratory case study was to investigate the role of multiple representations in influencing 10^{th} grade algebra II students' conceptual understanding of: (a) absolute value functions; (b) piecewise functions; and (c) quadratics functions. Employing semiotic system theory, the study focused on understanding the role of exposure to and use of multiple representations in; identifying students' pseudo-conceptual understanding during the pre-conceptual phase of understanding these functions; influencing the transition from pseudo-conceptual to conceptual understanding of these algebraic concepts, and in facilitating students' transfer of their conceptual understanding to other related concepts. Ernest (2006) Theory of Semiotic Systems provided a theoretical framework that facilitated my understanding of how algebra II students' use and production of mathematical signs i.e., external representations of mathematical concepts contributed to their pseudo-conceptual, conceptual understanding and transfer of mathematical concepts to other concepts.

Research Question

Using a qualitative exploratory case study methodology, I examined the influence of the use of multiple representations as defined in the study i.e. use of; verbal descriptions (natural language), numerical (correspondence in a table of values), algebraic or symbolic representations (equations expressing the relationship between two or more quantities), and graphical representations (Cartesian graphs) as described in (Goldin, 2003), in exploring algebra II students' conceptual understanding of; absolute value functions, piecewise functions, and quadratics functions. This study attempted to answer the following question. How does the use of multiple representations influence algebra II students' understanding and transfer of their algebraic concepts? Specifically the following sub-questions were examined:

1. How does exposure to and use of multiple representations influence students' identification of pseudo-conceptual understanding of algebraic concepts?
2. How does exposure to and use of multiple representations influence students' transition from pseudo-conceptual to conceptual understanding?
3. How does exposure to and use of multiple representations influence students' transfer of their conceptual understanding to other related concepts?

Significance of the Study

This study will contribute to the body of knowledge needed in research on understanding and assessing students' conceptual understanding of mathematics especially in domains like algebra. In particular the findings from the study will contribute to the literature on understanding; the process of algebraic concepts knowledge acquisition, and the challenges that algebra students have with comprehension of algebraic concepts. Equally important the findings from study will also contribute to the current literature on the role and potentially the effective use of multiple representations in promoting algebra students conceptual understanding. Specifically this study will contribute new insights into the research work on analyzing learning with multiple representations by highlighting areas of mathematical research study that are relatively under-investigated. These areas of research study include: (a) utilizing semiotic perspectives which involves the understanding of the role of mathematical sign and sign use in analyzing the use of multiple representations in promoting mathematics students' conceptual understanding of algebra concepts; and (b) analysis of instructional and assessment resources that provoke and promote the transition from pseudo-conceptual to conceptual understanding of algebra concepts and act as tools for examining the integrity of students' conceptual understanding.

Definition of Terms

Representation

The term representation used in the study is derived from Goldin (2003) description of a representation as a "configuration of signs, character, icons, or objects that can somehow stand for, or "represent" something else. According to the nature of the representing relationship the term represent can be interpreted in many ways, including the following (the list is not exhaustive): correspond to, denote, depict, embody, encode, evoke, label, mean, produce, refer to, suggest, or symbolize" (p. 276). This description is consistent with (Duval, 2006) description of representations as signs and their complex associations which are produced according to rules and which allow the description of a system, a process or a set of phenomena. Following is a definition of multiple representations that was adopted in this study.

Multiple Representation

The term multiple representations follows from the definition of representations and can be described as providing the same information in more than one form of external mathematical representations (Goldin & Shteingold, 2001). Hence a mathematical concept like quadratic function $y = -2(x+2)^2 - 5$ can be represented in a variety of modes i.e., as a verbal description, a correspondence in a table or a mapping, algebraic function, or as a graph. In this study the use of the term multiple representations will focus on the use of; verbal descriptions (natural language), numerical (correspondence in a table of values), algebraic or symbolic representations (equations expressing the relationship between two or more quantities), and graphical representations (Cartesian graphs) as described in (Goldin, 2003).

Mathematical Signs

In this study I will assume Peirce (1998) triadic structure of a sign as the definition of a mathematical sign. Pierce (1998) observes that all signs have a triadic structure: (a) a representamen (inscription) which refers to the form a sign takes; (b) an object i.e., a physical thing or an abstract object; and (c) interpretant i.e., an idea or the meaning of an object. In this regard Pierce (1998) defines a sign as'

> anything ... which mediates between an object and an interpretant; since it is both determined by the object relatively to the interpretant and determines the interpretant in reference to the object, in such wise as to cause the interpretant to be determined by the object through the mediation of this "sign" (p. 410).

In mathematics, examples of representamen include; graphs, verbal descriptions, and symbols. Examples of mathematical objects would consist of definition of; a function, a derivative, and a parallelogram. Examples of interpretant would include the ideas or interpretations generated in a student's mind by the representamen of for example the graph of an absolute–value function with a vertex (2,3) in the domain (-10,10). In this example, the graph is the representamen of the mathematical object absolute value function where different students may construct different interpretants like equation $y = -2|x - 2| + 3$, or the “V” shape of the of this graph.

Mathematical Text

Mathematical text will refer to a simple or a compound sign that can be represented as a selection or combination of spoken words, gestures, objects, inscriptions using paper, chalkboards or computer displays as well as recorded or moving images (Ernest, 2008).

Pseudo-Conceptual Understanding

The use of the term pseudo-conceptual understanding in this study follows from Vinner (1997) description of using words or mathematical symbols in communication without knowing exactly what they mean or represent. This explanation is consistent with Vygotsky (1994) description of learners understanding that is coherent and objective, but the bonds formed between the various units of a given domain are derived from experiences rather than systematic or logic based. In addition Berger (2005) and Sfard (2000) observe that students demonstrating pseudo-conceptual understanding in mathematical activity “when they are able to use and communicate about a mathematical notion as if they fully understand that notion even though their knowledge of that notion is riddled with contradictions and connections not based on logic” (Berger, 2005, p.14).

Conceptual Understanding

The implicit or explicit knowledge of the principles that govern a given mathematics domain, and the interaction between the various units of a given domain (Godino, 1996). Students demonstrating conceptual understanding are capable of attending to a mathematical object (e.g., definition of a function) in its entirety and not just as a fragmented aspect of the object (Berger 2004).

2 REVIEW OF THE LITERATURE

"Dismissing the importance of plurality of registers of representations comes down to acting as if all representations of the same mathematical objects had the same content or as if the content of one could be seen from another as if by transparency". Duval (2006, p. 14)

Brief Overview of The Literature

The literature review begins with a broad look at the notion of representation, in particular the role of semiotic representations in influencing students' conceptual understanding. An extensive discussion of multiple representations will follow including an analysis of; the potential benefits associated with the use of multiple representations, and research studies on the use of multiple representations. The notion of pseudo-conceptual and conceptual understanding will then follow including a semiotic perspective broad review on the role of multiple representations in influencing students' conceptual understanding.

Representation

Various definitions of the term *representations* in mathematics education have emerged Smith (2003). Goldin (2003) defines representation as a "configuration of signs, character, icons, or objects that can somehow stand for, or "represent" something else. According to the nature of the representing relationship the term *represent* can be interpreted in many ways, including the following (the list is not exhaustive): correspond to, denote, depict, embody, encode, evoke, label, mean, produce, refer to, suggest, or symbolize" (p. 276). Individual representations cannot be understood in isolation but, rather belong to a wider system known as representational system Goldin (2002). For example given the quadratic function $y = -2(x-4)^2 + 5$, the graph of this function may provide a visual representation of the function, and alternatively the signs and symbols in the given function (i.e. -2, 4, 5) may be used to describe the graph of the function (i.e., concave down, vertex (4,5), absolute maximum, and the x & the y-intercepts).

This study adopted Goldin (2003) notion of representational system as the definition of representation. Goldin (2003) notion of representational systems involves the presence of signs (or characters) with well-defined configurations or rules for combining them and a structure that basically provides meaning and fundamentally sets up the relationship between relevant signs and their rules. To further understand the notion of representational systems, Goldin (2003) distinguishes between two types of representations systems; (a) individual person's internal psychological, and (b) individual person's external representational systems.

Internal psychological systems include: "individual's natural language; their visual imagery, and spatial tactile, and kinesthetic representation; problem solving heuristics and strategies; their personal capabilities, including conceptions and misconceptions, in relation to conventional mathematical notations and configurations; their personal symbolization constructs and assignments to all these; and their effect in relation to mathematics." (p. 277). External representation systems include: normative natural language (e.g., standard English); conventional graphical, diagrammatical, and notational systems of mathematics, structured learning environments that may include concrete manipulative materials or computer-based micro-worlds…" (p277).

It follows that understanding the role of representations in student's conceptual understanding of mathematics is important. Mathematical objects (e.g., functions, groups, relations, sets, fields and geometric figures) can be described as abstract, unobservable, and non-physical objects (Duval, 2006; Godino, 1996). From this ontological position of mathematics, mathematical objects as opposed to other domains of scientific knowledge (physics, chemistry, biology, astronomy etc.) are not accessible by perception or by instruments such as microscope, telescope, and/or measurement apparatus (Duval, 2006). Further mathematical knowledge being classified as a priori knowledge (Ernest, 1991) means that it consists of propositions asserted on the basis of reason alone. Ernest (1991) extends this idea further and asserts that this reason,

> ..includes deductive logic and definitions which are used in conjunction with assumed set of mathematical axioms or postulates, as basis from which to infer mathematical knowledge. Thus the foundation of mathematical knowledge, that is the grounds for asserting the truth of mathematical propositions, consists of logical deductive proof. The proof of a mathematical proposition is a finite sequence of statements ending in the proposition which satisfies the following property. Each statement is an axiom drawn from a previously stipulated set of axioms, or is derived by a rule of inference from one or more statements occurring earlier in the sequence. The term ‘set of axiom’ is conceived broadly to include whatever statements are admitted into a proof without demonstration, including axioms, postulates and definitions (Ernest, 1991, p. 5).

It follows that the indispensable role of representations is to provide access to mathematical objects by supporting students to directly identify, communicate, manipulate and work with mathematical objects. Hence, Duval (2006) observes that the use of signs and semiotics representations provides access to the mathematical objects and shows us how to deal with these objects and argues that “ no kind of mathematical processing can be performed without using a semiotic systems of representation, because mathematical processing always involves substituting some semiotic representation for another” (p. 107).

Recent Common Core State Standard Curriculum Initiatives (CCSSI, 2010) and professional organization such as NCTM through their various publications has placed considerable emphasis on the use of representations and argues that “the term representation refers to both the process and to product-in other words, to the act of capturing a mathematical concept or relationship in some form and to the form itself… Moreover, the term applies to process and products that are observable externally as well as to those that occur “internally,” in the minds of people doing mathematics”. (NCTM, 2000, p. 67). Representations such as graphs, algebraic equations, diagrams, tables and charts, numerals are external manifestation of mathematical concepts. In the NCTM publication *Principles and Standards for School Mathematics* (NCTM, 2000), representation was introduced as a process standard. Although still integral to each of the content standards, the representation standard has been separated from the individual content standards presented in the initials edition (NCTM, 1989). This shift in prominence (Pape & Tchoshanov, 2001) demonstrates an increase interest in understanding the notion of representations among mathematics education researchers. Hence in the new process standard, NCTM (2000) advocates that instructional programs from prekindergarten through grade 12 should enable all students to: (a) create and use representations to organize, record, and communicate mathematical ideas; (b) select, apply, and translate among mathematical representations to solve problems; and (c) use representation(s) to model and interpret physical, social, and mathematical phenomena (p. 67).

Multiple Representations

Earlier research on learning with external representations, focused on the ways presenting pictures alongside text improved readers memory for text comprehension (Ainsworth, 2006). However the proliferation of multi-media learning environment in the last two decades has broadened the debate to include combinations of representations (multiple representations) such as diagrams, equations, text, tables, graphs, sounds, animations, video, and dynamic simulations. Any mathematical concept, for example; an absolute-value function or a quadratic function, can be represented in a variety of modes. Given the numerous modes of representations that are available, this study focused on the exposure to and the use of the following multiple representations; verbal descriptions (natural language), numerical, algebraic (formal notational systems) and conventional graphical representations (Goldin 2003).

Figure 1 is an illustration of how the same concept of a quadratic function $g(x) = 10x - x^2$ can be presented in a variety of modes. In the figure the use of multiple representations is demonstrated in the; verbal descriptions, numerical table of values, formal algebraic notational and conventional graphical representations. Several research studies findings on the use of and exposure to multiple representations conclude that, the ability to present the same concept in different ways provides students with the opportunity to; (a) build abstraction about mathematical concepts (Ainsworth, 2006), (b) highlight different features or characteristics, and (c) provide distinct conceptual resources and problem solving capability (Larkin & Simon, 1987; Parnafes & DiSessa, 2004; DiSessa, 2004). NCTM (2000) standards advocate a curriculum based on multiple representations, arguing that by encouraging students to incorporate many different types of representations into their sense-making approaches, they will improve in their ability to understand underlying mathematical concepts and problem solving capability.

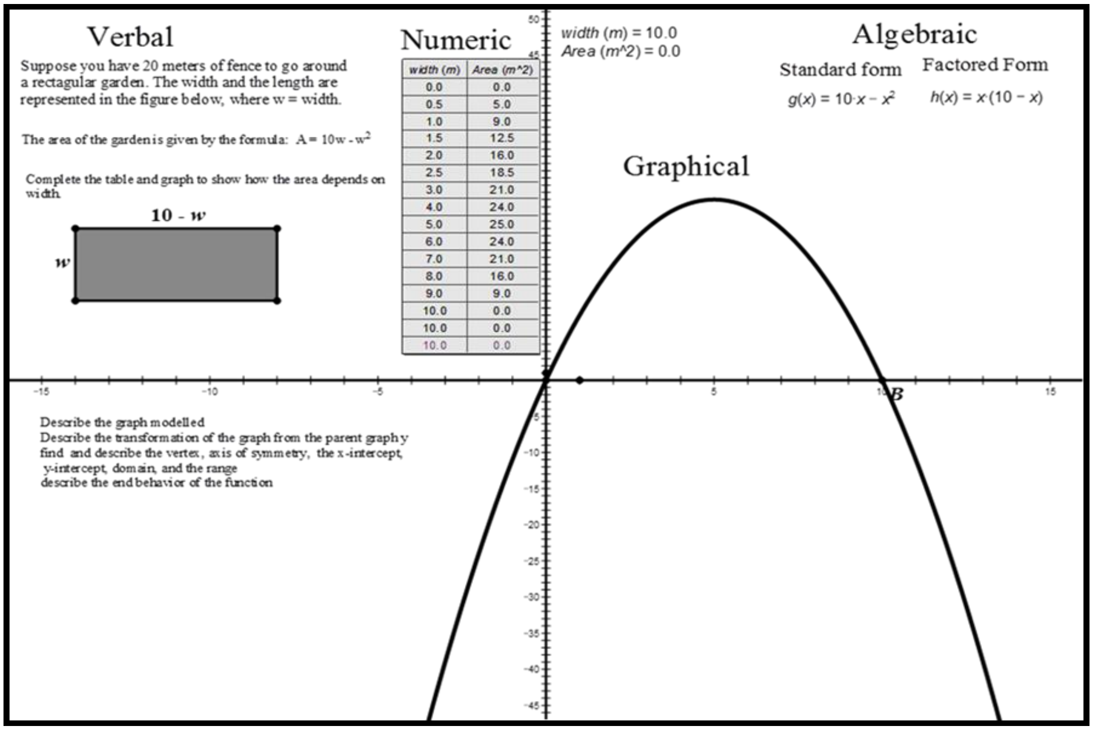


Figure 1. Sample Task Utilizing Multiple Representations of a Quadratic Function.

Potential Benefits of Multiple Representations

Potential benefits associated with exposure to and the use of multiple representations have been documented. These benefits include facilitating students' deeper conceptual understanding (Berthold et al., 2009; Ainsworth, 2006; Ainsworth & van Labeke, 2004; Clement, 2004; and Tripathi, 2008). Duval (2006) argues that a single representation cannot fully describe a mathematical concept. Kaput (1992) supports Duval's position by arguing that each representation has different advantages in facilitating conceptual understanding, and advocates the use of various representations for the same concept as an effective tool for instruction. In supporting this position Kaput (1989) also asserts that "the cognitive linking of representations creates a whole that is more than the sum of its parts" (p. 179), and argues that this cognitive linking enables students to see complex ideas in new ways and to apply them effectively. Meaningful and effective learning of a mathematics content domain like algebra has also been documented as a potential benefit associated with the use of multiple representations (Friedlander & Tabach, 2001). This potential benefit is demonstrated by how the use of multiple representations has played a significant role in the recent algebra instructional paradigm shift that emphasizes not only the symbolic manipulations skills but also a deeper understanding of families of functions (Gutierrez & Boero, 2006). Families of functions (e.g., linear, quadratic, absolute value, and exponential) share certain traits as demonstrated in the topic of transformation. In algebra instruction, transformation can be described as "action on the classes of functions that provide a measure of generalizability across the families" (Gutierrez & Boero, 2006, p.326). Use of multiple representations (graphs, algebraic equations, tables, and verbal descriptions) allow students to form deeper generalization of the families of functions when they explain their understanding and make predictions.

Role of Multiple Representations

The role of multiple representations can be summarized using Ainsworth (2006) functional taxonomy shown in figure 2. In a review of the potential benefits that multiple representations bring to a learning situation, Ainsworth (2006) utilizes these taxonomy of multiple representations to clearly identify three main pedagogical functions that multiple representations serve in the teaching and learning of mathematics. An analysis of these functions associated with the existing use of multiple representations in learning situations, suggests that there are three main roles that multiple representations play in supporting learning. These functions include; (a) complementing each other, (b) constraining interpretations, and (c) supporting construction of deeper conceptual understanding (Ainsworth, 2006).

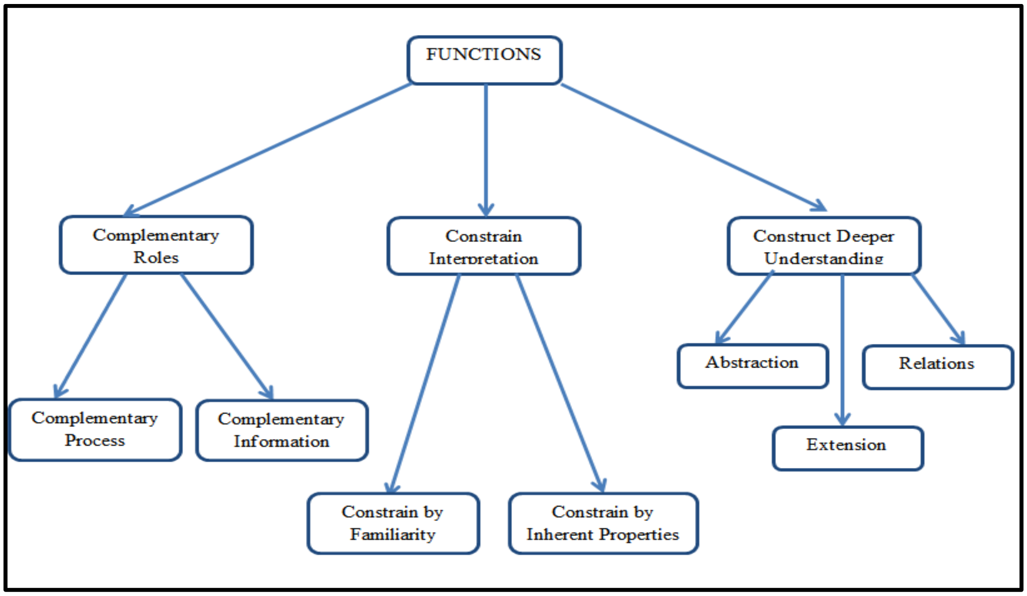


Figure 2. Functional Taxonomy of Multiple Representations Model. Adapted from “DeFT: A conceptual framework for considering learning with multiple representations,” by S. Ainsworth, 2006, *Learning and Instruction*, 16, p 187. Copyright 2006 by the Elsevier Ltd.

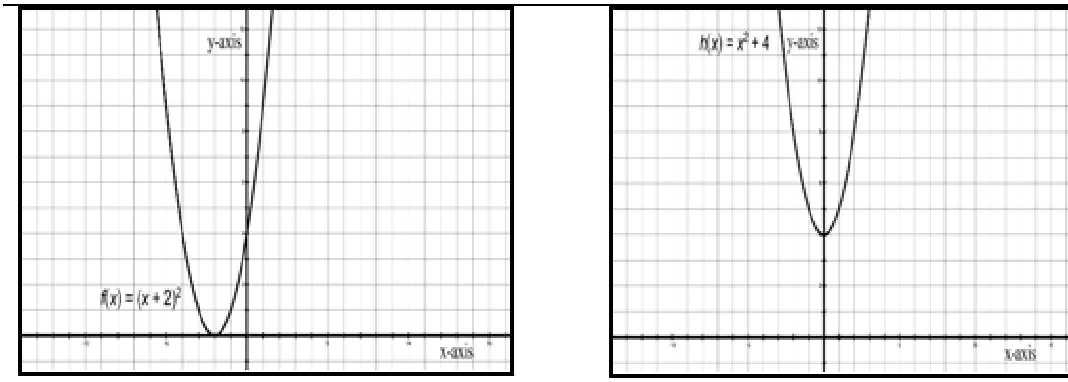
Exposure to and use of multiple representations provide complementary information when a single representation is insufficient in carrying all the information about a particular domain (Ainsworth, 2006). On the complementary role of multiple representations each representation complements each other because they differ in the information they each express or in the process that each representation supports. Use of multiple representations creates an environment where students can benefit from their combined advantages. Example: Given the quadratic function expressed in algebraic symbol vertex form as $g(x) = -3(x-8)^2 + 6$, this representation allows students to find the value of y for any given value of x, regardless of how large the value of x. However, this representation does not explicitly reveal other significant information that the equivalent graph provides.

Table 1. Sample of information provided by graphical representation of a quadratic function

	Question/information	Solution
1.	Find the end behavior of the graph of the function $g(x) = -3(x-8)^2 + 6$:	$as\ x \to \infty, y \to -\infty$ $as\ x \to -\infty, y \to -\infty$
2.	What is the concavity of the graph	Concave down: quadratic coefficient 'a' is a negative $a < 0$.
3.	What is the domain in interval notation	$[-\infty, +\infty]$
4.	What is the range in inequality notation	$y \leq 6$,
5.	Find the value of x for which g(x) > 0	since the graph is a concave down , $(8+\sqrt{2},\ 8-\sqrt{2})$
6.	Find the value of x for which $g(x) \leq 0$	Since the graph is a concave down $(-\infty, 8-\sqrt{2}] \cup [8+\sqrt{2}, \infty)$
7.	Find the value of x for which $g(x) \geq 0$	$[8+\sqrt{2},\ 8-\sqrt{2}]$
8.	Describe the transformation of $g(x)$ from the parent function $y = x^2$	Horizontal shift 8 units right, vertical shift 6 units up, concave down, vertical stretch factor 3

Table 1 provides information advantageously served by having a graphical representation. The two representations support different processes and carry a set of different yet complementary information (e.g. end behavior of the graph of a function) from which students can benefit by achieving a better understanding of the concept of quadratic functions.

Use of multiple representations can be potentially useful in assisting students' development of an improved understanding of a domain by using one representation to constrain their interpretation of a second representation. This can be done by; (a) utilizing a familiar representation to support the interpretation of a less familiar one, or (b) using the inherent properties of one representation to limit the inferences drawn from a second representation. Example: consider the following sample of a common algebra equivalent expression misconception $(x+2)^2 \neq x^2 + 4$, graphing representations of the quadratic functions $g(x) = (x+2)^2$ and $h(x) = x^2 + 4$ (see figure 3) can be used to show that the two expressions are not equivalent.



Graph $f(x) = (x+2)^2$ Graph $h(x) = x^2 + 4$

Figure 3. Sample of graphical representations comparing equivalency of two quadratic functions

In this example, familiar graphing representations of the quadratic functions $g(x) = (x+2)^2$ and $h(x) = x^2 + 4$ are used to support students' reasoning about a less familiar representation i.e. equivalence of the expressions $(x+2)^2$ and $x^2 + 4$. In this function it is the students' familiarity with the constraining representation (graphing) or its ease of interpretation that is essential (Ainsworth, 2006).

Ainsworth (2006) describes graphical constraining benefit of multiple representations as the range of inferences that can be made about the represented concept. Graphical constraining is evident in learning environment where one representation limits the interpretation of a graph. An animation, for example can constrain the interpretation of a graph. Kaput (1989) observes a strong tendency among students to view graphs as pictures rather than symbolic representations. Monk (2003) observed a shift in educators' perception of graphs from carrier of information to a lens through which information can be explored. Van der Meij and de Jong (2006) in a study on the effect of different types of support for learning using multiple representations in a simulation-based learning environment, describes the use of dynamic representations instead of static representation. They describe the animation of a car riding up a hill with constant power as a graphical constrain of the interpretation of the speed shown in a line graph. The animation can show a student that the line graph is not representing a valley but the speed of the car; learners can see that the car slows down going up the hill and that it accelerates going down the hill. Van der Meij and De Jong (2006) argue that the purpose of the constraining representation is not to provide new information but to support the learners' reasoning about the less familiar representation (Ainsworth, 2006).

Thirdly multiple representations can support the construction of deeper understanding when students integrate information from different representations to achieve insights that might not be achieved from a single representation (Ainsworth, 2006). Deeper understanding occurs through; (a) abstraction, (b) extension (generalization), and (c) relational understanding. In regard to abstraction, use of multiple representations influences the creation of mental entities that serve as the basis for new procedures and concepts at higher level of organization (Ainsworth, 2006). During the extension process, students generalize (extend) their knowledge from known to unknown to representation without fundamentally altering the nature of that knowledge (Ainsworth, 2006). In relational understanding the focus in on how two known representations are associated without reorganization of knowledge (Ainsworth, 2006).

In the construction of deeper conceptual understanding computational offloading can serve as an indicator of students understanding of a mathematical concept. Computational offloading is a term used to refer to the extent to which the use of multiple representations reduce the amount of cognitive effort required to solve an equivalent problem (Ainsworth, 2006). As an example consider the following absolute value function $y = -2|x + 5| - 6$ in which students are to determine; (a) the x-intercept of the graph of the function, and (b) the input for which $f(x) \geq 0$, Students attempting to solve this problem by engaging in symbolic (algebraic) manipulation only realize that the x-intercept/s does not exist after ending up with the equation $|x + 5| = -3$. However students exposed to the simultaneous use of the algebraic, numeric (table of values), and graphical representations of this function have the potential to expand their thinking about absolute value function and deduce that when the parameter ‘a’ and the constant ‘k’ have the same sign, it implies that the absolute value function has no x-intercepts. In this example, the use of multiple representations influences the creation of mental attributes (abstraction) about the properties of absolute value functions that serve as the basis for generalizing (extension) their knowledge about this function.

In Garofalo and Trinter (2012) study, they analyzed mathematical tasks that encouraged high school students and pre-service secondary school mathematics teachers to think flexibly about mathematical concepts and problems. They observed that with well-designed tasks; students can expand their thinking about mathematical ideas, vary their approaches to solving mathematical problems, and value the use of multiple representations in problem solving. In the study, algebra students were required to derive the quadratics formula from the standard quadratic model $ax^2 + bx + c$ = 0. Many did so by completing the square, whereas others needed a hint. After students used the usual “completing the square” strategy to derive the formula, they were required to leave the last line as shown below without combining the fractional expressions $x = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$. The purpose of doing this was to encourage students to think about what the expression $-\frac{b}{2a}$ represents in the equation and how it relates to the parabolic graph of a quadratic function. Computational offloading was evident in this activity as demonstrated by the extent to which external representation (graphing) and the symbolic expression above combined to reduce the amount of cognitive effort required in; identifying the x-coordinate of the vertex, and the equation of the line of symmetry.

The use of algebraic equation (symbolic) and the equivalent graph in this study exploited the perceptual processes by grouping together relevant information (relational understanding) so making search recognition easier. Garofalo and Trinter (2012), observed that after students had the opportunity to think about the solution, $-\frac{b}{2a}$ most of them recognized that $-\frac{b}{2a}$ represented the x-coordinate of the vertex and $x = -\frac{b}{2a}$ represented the axis of symmetry by extension. In addition students also realized that the roots of the equation which are also the zeroes of the graphs can be found at equal distance from the axis of symmetry on the x-axis. i.e. $\pm\frac{\sqrt{b^2-4ac}}{2a}$.

In summary, there is a convergence of research literature perspectives supporting the need for instructional and assessment activities that include exposure to and use of multiple representations. These perspectives are grounded in the understanding that multiple representations offer potential benefits that can contribute to students' deeper conceptual understanding of algebraic concepts. Following is a discussion on the research studies on the use of and exposure to multiple representations.

Research Studies on use of Multiple Representations

This section is a discussion on the several research studies on the use of multiple representations. As part of a larger body of research designed to explore how students understand the concept of quadratic functions, Eraslan and Aspinwall (2007) conducted a task-based interview qualitative study with tenth-grade honors students at a public high school in the south east of United States. These interviews took place at the end of a three-week instructional unit on quadratic functions. Assessment materials which included student's tests, quizzes, and interview questionnaires were collected and analyzed. The purpose of the study was to gain an insight into; (a) students' conceptual understanding, and (b) the cognitive obstacles related to five aspects of the quadratic functions being taught (i.e., translating, determining, interpreting, solving quadratic equations, and modelling quadratic functions. Participants in the study were given tasks that required them to translate problems from equations to graphs and from graphs to equations. In light of their written work, the researcher designed new tasks in order to reveal the participant's thought processes throughout the interviews. The participants' work on four translation tasks and their response to interview questions were analyzed. The purpose of the study was to understand possible reasons for the nature of students understanding of a relationship between the vertex and the coefficients *b* and *c* in the standard form, $y = a^2 + bx + c$ of a quadratic function. Participants were presented with a quadratic function $y = a(x - 2)^2$ + 5 and asked to produce the graph.

Next, the participants were presented with the graphic representation of a quadratic function showing the vertex and two ordered pairs on the graph and required to name a function for a given parabola. This task was approachable either by substituting the coordinates of the vertex into the vertex form $y = a(x - h)^2 + k$ of quadratic function the equation and then checking another point to find the leading coefficient. An alternative solution to the problem involved setting up a system of three equations in three unknowns and using algebraic means to find the coefficients of the equation. Analysis of students' verbal and written responses identified the understanding of the underlying structures of the mathematical sign as a cognitive obstacles that hindered the participants' translation between the two representations. This study highlights the importance of instruction that not only introduces students to different forms of representations but also intentionally emphasizes an understanding of situations in which one representation has advantage over another.

Result of the study also supports the importance of in-depth analysis of student thinking. Knowing the nature of students' cognitive obstacles and taking a carefully designed plan of action to minimize cognitive obstacles which is crucial in developing a rich conceptual understanding. Finding from the analysis of students responses suggested that students' obstacles to conceptual understanding might not be accessible through routine assessments like test, quizzes and homework assignments. In assessing students' understanding of a particular concept the study recommend that teachers create opportunities to discuss with students about their thought processes on tasks that reveal students' adaptable thinking. Equally important Eraslan and Aspinwall (2007) study highlighted the significance of; monitoring students' thinking, identifying their knowledge structures, and addressing obstacles that emerge when students are solving problems.

Knuth (2000) study examined students' understandings of the connections between algebraic and graphical representations of functions. The purpose of this qualitative study was to understand the connections students develop when they interact with multiple representations. The study utilized task-based interview data collection technique. Participants in the study were 178 students, enrolled in the following college preparation mathematics courses: 1^{st} year algebra, 2^{nd} year algebra, pre-calculus, and advanced level placement calculus classes. Participants were presented with a problem that had both the algebraic and a corresponding graphical representation and required the use of Cartesian Connection (Moschkovich, et al, 1993) where understanding that both the equation-to-graph and graph-to-equation connection are considered fundamental in developing the flexibility among the representations. While the graphical representations in the problem explicitly provided information required to efficiently solve the problem, the algebraic representations provided only limited and implicit information. Responses from the participants revealed an overwhelming reliance on algebraic representations, even on tasks for which a graphical representation seemed more appropriate and efficient.

The findings from this study indicate that for familiar routine problems many students master the connections between the algebraic and graphical representations, however, such mastery appeared to be superficial at best an indication of pseudo-conceptual understanding. Knuth (2000) concludes by questioning the assumptions regarding the ease with which students are thought to master the Cartesian Connection. Knuth (2000) recommends that an expert's knowledge of the mathematic domains should extend beyond simple procedural competence, where the goal of mathematics instruction should be to move students beyond procedural competence and towards a more robust and flexible understanding of concepts like functions.

Amit and Fried (2005) qualitative study examined potential benefits realized from the use of multiple representations where learners used standard representations in real classroom environment. The purpose of the study was to explore whether; (a) teachers and students share an understanding of the significance of multiple representations, and (b) students are truly realizing the potential benefits from lessons explicitly designed with multiple representations in mind. The research setting was the *Learners' Perspective Study* (LPS), an international effort involving nine countries (Clarke, 2001; Fried & Amit, 2005) which expands on the work done in TIMMS video study which; (a) exclusively examined 8th grade teachers and only one lesson per teacher (Stigler

& Hiebert, 1999), (b) focused on student actions within the context of the whole-class mathematics practice, and (c) adopted a methodology whereby student reconstructions and reflections were considered in several videotaped mathematics lessons (Fried & Amit, 2005). Data collection included videotaped classroom instructions 15 lessons on systems of linear equations, field notes, relevant classroom materials, and student focus group interviews. Results from the study indicated that, though none of the students in the study appreciated the graphic representations as complementary to the algebraic representation of linear relations, majority of the students understood why the teacher attached significant importance to the use of different representations. Several students in the study produced statements in line with the teacher's approach to different representations yet they had limited conceptual understanding of what the teacher was attempting to communicate to them. The results also indicated that division between the teacher's intention of what she was doing and the students' interpretation of what was expected of them represented one of the reasons why the students in this class did not seem to get the idea that representations are to be selected, applied, and translated. Conclusion from the study was a need to provide students with considerable experiences in the kind of thinking that potentially promotes linking of several representations.

In Zaslavsky et al. (2002) study, the purpose of the qualitative study was to examine some implicit assumptions regarding the connection between the symbolic representations (algebraic) and graphical (geometrical) representations of a mathematical concept (slope of a linear function). The 124 participants in the study were subdivided into five groups and included; eleventh-grade calculus students, prospective and in-service secondary mathematics teachers, and university mathematics educators and mathematicians. The participants were interviewed on two mathematical tasks that required them to analyze a slope appearing as an analytic attribute of a linear function, and as a geometric attribute of the line representing the graph of the function.

Task 1 required participants to identify the slope of a line graphed in a homogenous system of coordinates. In task 2 the participants were required to respond to a simple but non-standard task concerning the behavior of slope under a non-homogeneous change of scale. Analysis of the data (written responses, transcribed interviews, and field notes of group discussions), focused on the implicit and explicit concerns of the participants, the assumptions, and the nature of the experiences the participants went through in trying to cope with the cognitive conflicts in the two tasks assigned.

Results revealed that several participants had difficulties making connections between the algebraically (analytical) and the geometric (graphical) representations. The study draws attention to the several unquestioned assumptions concerning basic mathematical notions like slope, scales, and angles and graphical representations. The study calls for further re-examination and refinements of the underlying assumptions and conventions that teachers and students (individually or collectively) make when learning mathematics. These assumptions that the researchers in the study alluded to represents the pseudo-conceptual understandings that students (and in this study teachers) demonstrates and hence a need to address this concerns during mathematics instruction.

Pseudo-Conceptual Understanding

The notion of pseudo-conceptual understanding of mathematics referenced in this study is derived from Vygotsky's (1986) stages of pre-conceptual thinking and Berger's (2004) elaboration of stages of appropriation of mathematics object. When appropriating new mathematical objects for example an absolute value function like $y = a|x - H| + K$ either through direct instruction or in their interactions with mathematical text, students use different forms of pre-conceptual thinking (heaps, complexes and potential concepts) which roughly corresponds do a different stage of the development of generalization and abstraction in the student (Vygotsky, 1986 & Berger, 2004). Vygotsky (1994) describes pseudo-concepts as a special type of complex thinking that “from outside i.e. to an observer, it has the appearance of a true concept but on the inside (in terms of its genesis, the conditions under which it develops and the causal associations of these conditions) it is actually a complex thinking” (p.226). In mathematical activity, students in the pseudo-conceptual phase of understanding, “use and communicate mathematical notions as if they fully understand the notion, even though their knowledge of that notion is riddled with contradictions and connections not based on logic” (Berger, 2004, p.14).

Berger (2004) on elaborating on Vygotsky’s work on pre-conceptual thinking groups pre-conceptual thinking into three stages; a) heap thinking stage, b) complex thinking stage, and c) potential conceptual thinking stage. Each of these thinking corresponds to a different stage in the development of the generalization and abstraction of a concept. These stages represents Vygotsky’s interest in the genesis and development of concepts (Berger 2004). Since my study is focused on the complex thinking stage within which pseudo-conceptual thinking or understanding of a concept is situated, an understanding of the heap thinking stage is important in setting up a platform for supporting my understanding of the complex thinking stage. Heap thinking refers to learners linking of ideas or objects together as a result of an idiosyncratic association (Berger, 2004). In mathematics context this is the stage in which students associate one mathematical sign or text with another because of the physical context or circumstance rather than any mathematical property of the mathematical signs/text (Berger 2004).

An example of heap thinking in mathematics context involves students associating one sign, for example an absolute value function $y = a|x - H| + K$ with another sign for example a quadratic $y = a(x - H)^2 + K$ because of physical context or attributes of the signs rather than the mathematical properties of the signs. In another example of heap thinking, many students tend to believe that all functions should be definable by a single algebraic formula (Carlson, 1998) and tend to argue that a piecewise function defined like $f(x) = \begin{cases} 2x + 4, & x \leq 0 \\ \frac{1}{3}x - 2, & x > 0 \end{cases}$ represents two separate functions. Likewise a student using heap thinking would regard the functional notation f(x) as a product of two variables f and x. In these three examples students using heap thinking rely on the physical context such as the layout of the sign or other non-mathematical context to justify their mathematical understanding. Alternatively, in complex thinking stage students' ideas are based on experiences and associations with familiar concepts rather than on logic or any particular system. At this stage, students are capable of abstracting actual attributes of an idea or a concept (Berger, 2004) as their thinking is coherent and objective (Vygotsky, 1994). Vygotsky (1994) argues that this is a crucial stage in the formation of concepts as it allows students to think in coherent terms and communicate their ideas about a mental entity via words and symbols (Berger, 2004). It is during these students' communication with the teacher or with the more knowledgeable others that personal and meaningful concepts whose use is compatible with the wider mathematical community are developed.

Mathematics being quintessentially a study of abstract sign systems (Ernest, 1997), it is important that in this research I attend to how mathematical text and signs are exchanged in these communications including the construction of mathematical concepts (in this study algebraic concepts) that are based on the abstraction of attributes of signs (Berger, 2004). In this study I choose to be deliberate, explicit and focused on an important distinction between pseudo-conceptual and conceptual understanding. I believe this distinction is significant in providing a tool for examining the veracity of our students' conceptual understanding of mathematical concepts. In pseudo conceptual understanding the primary focus is on the mathematical representations as an end in itself. A notion that Sfard (2000) refers to as "signifier-as-an –object-in-itself" (p. 79). In this understanding, the mathematical sign or a set mathematical signs rather than the meaning of the sign/s is the primary focus of attention. The following is an example of the distinction between pseudo conceptual and conceptual understanding in the appropriation of a mathematical object. In this case the absolute value function $y = a|x - H| +$. Given an absolute value function $y = -2|x - 4| + 5$, a student using pseudo conceptual thinking in appropriating this function will focus only on the mathematical symbol (i.e. the variable, operations, the absolute value symbol, and the equality sign) as an end in itself. A student using conceptual thinking will focus on the ideas embedded in the mathematical symbol (Berger, 2004) that is the meaning of the variables in the absolute value function where in the above example, the meaning of the values of the variables -2, 4 and 5 represents an absolute value function graph that opens down, with a vertical stretch of scale factor 2, a horizontal shift of four units right, and a vertical shift of five units up from the graph of $y = |x|$. Equally import a student using conceptual thinking is able to link the absolute value function to other concepts like deducing various properties (e.g., the domain $(-\infty, \infty)$, the range $(-\infty, 5)$ and the y-intercept (0, -3). Having made a distinction between pseudo-conceptual and conceptual understanding of mathematical concepts, the following section is an attempt to demonstrate the significance of students' pseudo-conceptual understanding in mathematics and specifically in algebraic concepts. A discussion on how pseudo-concepts develop and their role in the development of mathematical concept will also be included.

As mentioned previously, instruction or student interaction with mathematical texts or signs that emphasize “signifier-as-an-object-in-itself” (Sfard, 2000, p.76) model or paradigm contribute to pseudo conceptual understanding of mathematical concepts. In this model of instruction the focus is on the mathematical inscriptions (words, text or symbols) only and less on the meaning of the concept or the underlying processes by which a concept is built. Vinner (1998) observed that pseudo-concepts are often caused by either the lack of or failure to activate the crucial elements of true conceptual thought process. Where the crucial elements represents students control mechanisms for examining whether their responses to a given question make sense or not in a given situation. In the following example, in an attempt to respond to a description of the transformation of the following absolute value function $y = -2|x + 5| - 6$ from the parent function $y = |x|$, students use the term ‘concave down’. Though quadratic functions and absolute value functions have graphs with some similar characteristic behavior, the response in this example questions whether students fully comprehend the concepts of absolute value function and if they used any control mechanism to examine their association (i.e., the difference between a quadratic function and an absolute value function), to determine if their response fits the question or not. Lack of control mechanism and hence pseudo conceptual understanding in this example is evident since absolute value functions are not ‘curves’ but two linear functions with a common vertex or end point.

Another area where pseudo-conceptual understanding of mathematical concepts is revealed, is when students are asked to identify a mathematical object. In the following example, students are expected to define the quadratic function written in standard form as

$y = ax^2 + bx + c,\ where\ a, b,\ \&\ c \in \mathbb{R}$ and $a \neq 0$. Failure to mention that $a \neq 0$ in the definition represents a pseudo-conceptual understanding because the leading term which is the dominant sign ax^2 defines the quadratic function. Challenges in students' distinction between an algebraically defined function and an equation (Carlson, 1998) also contributes to pseudo conceptual understanding of various algebraic functions. In an attempt to examine students' distinction between a quadratic function and a quadratic equation, identifying $y = x^2 + 15x + 56,$ as a quadratic equation represents a pseudo-conceptual understanding since a quadratic equation is an equation with one unknown variable only. In this example a control mechanism for examining whether a response to the identification of the above mathematical object as an equation makes sense or not is necessary. In this example, questions about the general form of a quadratic equation, and the number of different letters a specific quadratic equation should have, can activate a control mechanism that can lead to a student giving the correct response i.e. examining whether $y = x^2 + 17x + 72,$ is a quadratic function and not a quadratic equation. In this two examples terms, words and signs used by the teacher or the more knowledgeable others evoke in students' mind certain mathematical associations (Vinner, 1997). However in the absence of deeper conceptual understanding of the concept of quadratic functions, students do not attempt to examine their responses to know whether they constitute a correct answer or not. These examples and many other documented examples (Zaslavsky et al., 2002; Knuth; 2000; and Eraslan & Aspinwall, 2007) demonstrate a need for a closure examination of the role of pseudo-conceptual understanding in the development of students' conceptual understanding.

The role of pseudo concepts in the development of the meaning of a concept is crucial in; (a) formation of mathematical concepts, and (b) creating a platform for future generalization of a concept (Berger, 2004). In the formation of a concept the use of pseudo-concepts provides a student with initial access to new and unknown mathematical notions when she effectively communicates or productively engages in activities using mathematical signs even though that student might not fully comprehend the mathematical concept (Vygotsky, 1994). It is the use of pseudo-concepts that allow the student some form of communication either spoken or written about a mathematical object (e.g., an absolute value function, a quadratic function written in intercept form or a piece-wise function).

In creating a platform for future generalization, use of pseudo-concepts allows a student to abstracts or isolate different attributes of a mathematical object and organize ideas or objects with particular properties into groups (Vygotsky, 1994) in order to create platform for future generalizations. The following is an example of use of pseudo-conceptual understanding in this regard. In a lesson involving absolute–value functions, a teacher might require students to write an absolute-value model $y = a|x - H| + K$ and observe the relationship between the signs of both constant 'a' and 'k' and the x-intercept of the graph of the absolute-value function for different values of 'a' and 'k'. After several attempts including guess work and false starts (pseudo-concepts), a student who observes that when the value of 'a' and 'k' have the same sign then the graph of the absolute-value function has no x-intercepts can be said to have created a platform for future generalization. The significance of the duo roles of pseudo-conceptual understanding can be summarized as; (a) providing initial access to mathematical objects, and (b) acting as a bridge (Berger, 2004) between complex thinking (i.e., where mathematical ideas are based on experiences and associations with familiar concepts rather than on logic or any particular on logic) and conceptual thinking. The following is a description of conceptual understanding and its significance in the learning of algebraic concepts.

Conceptual Understanding

A student is in conceptual phase of understanding when she is capable of attending to a mathematical object (e.g., definition of a function) in its entirety and not just as a fragmented aspect of the object (Berger 2004). According to Sfard (2000) conceptual understanding is demonstrated when a student is capable of transitioning from "signifier-as-an-object-in-itself" phase of understanding to "signifier-as-a representation of-another object" (p.79) phase of understanding.

As elaborated in the previous discussion, pseudo-conceptual understanding phase of understanding is characterized by students' primary focus of attention on the mathematical signs where they perceive of mathematical sign as an end in itself. In the conceptual understanding phase, mathematical concepts whose existence is indicated by use of signs and symbols takes the role of representations of other mathematical objects, i.e., "signs are transparent so that the signified (mathematical ideas) shines through them" (Sfard, 2000, p.20). Consider the mathematical object, the quadratic function $y = -2(x-4)^2 + 3$ written in vertex form. A student using conceptual thinking to understand this mathematical object is expected to deduce various properties of quadratic function i.e., ideas embedded in the quadratics function in this mathematical object. For example the numbers -2, 4 and 3 represents a quadratic function with a vertex $(4,3)$ that is concave down, with a vertical stretch of scale factor 2, a horizontal shift of four units right, and a vertical shift of three units up from the graph of the parent function $y = x^2$. These are the internal links i.e., the links between the different properties and attributes of the function as deduced from the mathematical sign given. The student is also expected to link this function to other external concepts not explicit in the sign like the domain $(-\infty, \infty)$, the range $(-\infty, 3)$, the absolute maximum y = 3, input for which the range $f(x) \geq 0$, and the y-intercept (0, -29). To a student demonstrating conceptual understanding of this quadratic function, the mathematical sign $y = -2(x-4)^2 + 3$ is transparent enough so that the signified i.e., the mathematical idea of the object shines through the sign (Sfard 2000). To understand how conceptual understanding is internalized by students, I will refer to Vygotsky' (1978) learning theory.

Following is a discussion of Vygotsky (1978) Sociocultural learning theory that informs the theoretical framework (Theory of Semiotic Systems) adopted in this study. This discussion is essential in supporting my understanding of how conceptual understanding mentioned above is internalized through the use of socially elaborated symbol system e.g., mathematical signs.

Vygotsky Socio-cultural Learning Theory

Vygotsky's theory places significant emphasis on the dynamic interdependence of social and individual processes in the shared construction of knowledge and is based on the concept that learning as a human activity takes place in a cultural context, is mediated by language and other symbol systems, and can be best understood when investigated in their historical development (John-Steiner & Mahn, 1996). To understand the nature of the interdependence between individual and social processes in the shared construction of knowledge, Wertsch (1991) one of the first major scholars to interpret Vygotsky's work, identified three major themes of Vygotsky's theoretical approach; (a) individual development including higher mental functioning, has its origin in social processes, (b) human action, on both the social and individual planes, is mediated by semiotic means which are tools and signs, and (c) the need for reliance on genetic or developmental method in order to examine the origins and the history of mental phenomena. In this study, the utility of Vygotsky's theoretical framework provides a lens in my understanding of:

(a) the relevant features of the classroom social context and interpersonal relations including; teacher-student and student-student interaction that might occur in the research; (b) the role of exposure to multiple representations i.e., verbal descriptions, numerical, graphical and algebraic expressions, as semiotic mediators in influencing students' use of mathematical signs to gain access to mathematical objects.

The first theme concerning the social origins of higher mental functions emphasizes the significance of social interaction in human development. Vygotsky's general genetic law of cultural development emphasized his belief in the social formation of mind. In this theme, individual higher mental functioning originates on inter-mental plane i.e. social sphere or between people, and then on the individual or the intra-mental plane. Vygotsky believed that

> "Every function in the child's cultural development appears twice: first, on the social level, and later, on the individual level; first, between people (inter-psychological) and then inside the child (intra-psychological). This applies equally to voluntary attention, to logical memory, and to the formation of concepts. All the higher functions originate as actual relationships between individuals." (Vygotsky, 1978, p. 57).

Internalization of higher mental process then becomes a process of developmental transformation from the lower mental functions to the intra-mental i.e. within the individual. In this theme, Vygotsky is looking at learning as a process that is situated in, but not limited to social interaction. These theme supported my understanding of how students synthesize several influences into their novel modes of understanding and participation (John-Steiner & Mahn, 1996) by participating in a variety of classroom mathematical activities which in Vygotsky's work represent the social processes. Specifically, these theme supported my understanding of how students' interactions and exposure to multiple representations influenced: (a) their social norms for mathematics speaking and listening; (b) their judgment of adequacy of mathematical explanations and solutions; and (c) their habits of reasoning and sense making as elaborated by John-Steiner & Mahn (1996).

In the second theme, Vygotsky believed that the development of learner's higher mental functioning was mediated by social culturally-evolved tools and signs (Wertsch 1991). This list of tools and signs referred to as semiotic means include: "languages; various systems of counting; mnemonic techniques; algebraic symbols; works of art; writing; schemes, diagrams, maps and mechanical drawings; all sorts of conventional signs and so on" (Vygotsky,1981, p. 137). According to Vygotsky (1981), cognitive development and learning depends on; student's mastery of symbolic mediators, their appropriation, and internalization in the form of inner psychological tools (Kozulin, 2003). Action mediated by semiotic mediators also referred to as psychological tools or signs are the fundamental mechanism that link the social and individual processes. Hence in a mathematics classroom, semiotic mediators not only facilitate the mathematical activity but also define and shape students' inner processes (Berger, 2005). Wertsch and Stone (1985) argue that it is "by mastering semiotically mediated processes and categories in social interaction that human consciousness is formed in the individual" (p. 166). Vygotsky's second theme will support my understanding of the significance of semiotic mediation is influencing students' use of mathematical signs to gain access to mathematical objects.

The third theme is on reliance on genetic or developmental method in order to examine the origins and the history of mental phenomena. He argues that in order to understand mental phenomena we "need to concentrate not on the product of development but on the very process by which higher forms are established" (Vygotsky, 1978, p 64). According to this theme, since learning and development are situated in socially and culturally shaped contexts, as historical conditions change so does the context and the opportunities for learning. Meaning that for this study I needed to look beyond individual student's mental activity and include the situated practices that the participants found themselves. Specifically, looking at students' process of internalization of mathematical concepts in a learning environment that places significant emphasis in exposure to and use of multiple representations. This theme also made me cognizant of the constantly changing context and opportunity for learning presented in a multiple representations environment. Following is a discussion on Vygotsky's concept of internalization.

Vygotsky Concept of Internalization

One focus of Vygotsky's theory is the concept of internalization, which is conceived of as a representational activity that occurs simultaneously both in social practices and in the human mind (John-Steiner & Mahn, 1996). Internalization is a process by which shared construction of knowledge is appropriated, transmitted or transformed in formal and informal settings. Essential to the concept of internalization is the appropriation of socially elaborated symbol system. Semiotic representations that were the focus of this study are socially elaborated symbol systems. Vygotsky recognized the significance of the general transforming power of semiotic mediators and argued that by their very nature they have the "capacity to become cognitive tools" (Kozulin, 2003, p.25). However, in order to realize this capacity Vygotsky argues that, semiotic mediators should be appropriated under special conditions that emphasize their meaning as cognitive tools (Kozulin, 2003). This study examined how instructional and assessment activities that emphasize on students' conceptual understanding are appropriated.

Vygotsky theory recognizes that mediation of meaning is an essential component in the acquisition of psychological tools because semiotic signs derive their meaning from cultural conventions and argues that “symbolic tools e.g., mathematical signs have no meaning whatsoever outside the cultural conventions that infuses them with meaning and purpose” (Kozulin, 2003, p.26). In their study on the role of instructional conversations in classroom learning, Chang-Wells and Wells (1993) described the interdependence and transformative view of internalization: “It is at points of negotiation of meaning in conversation that learning and development occur, as each learner’s individual psychological processes mediate (and at same time are mediated by) the constitutive inter-mental processes of the group” (p. 86). In mathematics instruction, the meaning of a concept that is either represented verbally or using mathematical signs is not assimilated in a ready- made form (Berger, 2005) but undergoes significant development as the student uses signs in her/his communication with the more knowledgeable other. Students are expected to construct a concept whose use and meaning is compatible with its use in the mathematics community in so doing their conceptual construction is socially regulated (Berger, 2005). Vygotsky advocates for acquisition of semiotic tools within a learning paradigm that presupposes; (a) deliberate rather than spontaneous character of the learning process, (b) systematic acquisition of semiotic tools, and (c) emphasizes the generalized nature of semiotic tools and their application (Kozulin, 2003). In this study I focused on understanding how exposure to and use of multiple representations promote the acquisition of semiotic tools as described above. Following is a discussion on the use of and exposure to multiple representations from a semiotic perspective.

Semiotic Perspective and Multiple Representations

Several research studies (Goldin, 2003; White & Pea, 2006; Rider, 2007; Pape & Tchoshanov, 2001; and Monk, 2003) advocate for use of multiple representations and observes that mathematical concepts are learned powerfully when a variety of appropriate representations with appropriate relationships among them have been developed. Multiple representations such as use of; verbal descriptions, numerical algebraic symbolic expressions, and graphs are fundamental strategies of mathematical activity with a potential of providing both conceptual resources for problem solving process as well as social resources for communicating and coordinating interactions (White & Pea, 2011).

Central to school mathematics is the creation and use of standard system of representations e.g., use of signs and configuration of signs as seen in the school curriculum. However, learning mathematics without full understanding of the underlying meaning structures of these signs has long been a common outcome of school mathematics instruction. Findings from a considerable research base over the last three decades (Hiebert, 2003; Shaughnessy et, al, 2009; CCSSI, 2010) have shown a considerable gap between procedural and conceptual knowledge in mathematics students' performance (Kouba & Wearne, 2000), Students' knowledge and skills have been described as fragile and apparently learned without much depth and conceptual understanding (Oehrtman, Carlson & Thompson, 2008; Hiebert, 2003). Equally concerning is the mathematics curriculum in United States that has been typically characterized as focusing on learning skills and procedures and addressing many mathematics topics at a superficial level (Milgram & Wu, 2005). A long lists of state and local school district curricular expectations have led to teaching too much, too quickly, and with far too little depth (NCTM, 2000; CCSSI, 2010).

Studies on students' performance on related items that require reasoning, communication, conjecturing, and justifying answers have shown evidence of fragility and shallowness in conceptual understanding of mathematics (Kouba & Wearne, 2000).

Professional organizations like NCTM in their various publications recognizes that decisions made by teachers, school administrators, and other education professionals about the content and character of school mathematics have important consequences for students and recommend that such decisions should be based on sound professional guidance (NCTM, 2000). For three decades, the NCTM and recently in the Common Core State Standard Curriculum Initiatives (CCSSI, 2010) have been in the forefront advocating for the development of curricula that are challenging and engaging for students, including instruction that leads to deep understanding of the relationships among mathematical concepts. The National Council of Teachers of Mathematics (NCTM) in its *Principle and Standards for School Mathematics (2000)* has identified six key principles and ten standards of a high quality mathematics education for students.

The vision of school mathematics in the *Principles and Standards* is based on students' learning mathematics with understanding. The Learning Principle from the NCTM *Principles and Standards for School Mathematics (2000)* states that, "students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge" (NCTM, 2000, p.20). In addition to the Principles outlined in the NCTM literature, the ten Standards that span from prekindergarten to grade 12 are descriptions of what mathematics instruction should enable students to know and do. The Standards for mathematics practices advocated for in the CCSS initiative, describe a variety of skills that mathematics educators should seek to develop in their students. This skills include: NCTM process standards mentioned above; and the National Research Council (NRC, 2001) mathematical proficiency strands; "adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy)" (CCSSI p.1).

The focus of this study was on the conceptual understanding mathematical proficiency strand and the algebra content standard. Algebra as a mathematics domain has been a focal point of reform efforts in mathematics education with many mathematics educators and researchers (Pansuk, 2010; Oehrtman, Carlson & Thompson, 2008; NCTM, 2000; RAND Mathematics Study Panel, 2003; Shaughnessy et al. 2009; Kilpatrick & Izsák, 2008) advocating for instruction that emphasizes algebraic conceptual understanding. A significant overlap in the NRC (2001) and NCTM (2000) description of conceptual understanding, concludes that students demonstrate conceptual understanding,

> ..when they provide evidence that they can recognize, label, and generate examples of concepts; use and interrelate models, diagrams, manipulatives, and varied representations of concepts; identify and apply principles; know and apply facts and definitions; compare, contrast, and integrate related concepts and principles; recognize, interpret, and apply the signs, symbols, and terms used to represent concepts. Conceptual understanding reflects a student's ability to reason in settings involving the careful application of concept definitions, relations, or representations of either (NCTM, 2000, p.2).

It follows that the NCTM and the CCSSI initiative description of conceptual understanding focuses on characteristics and features of a student who has acquired conceptual understanding. The literature from these organizations does not explicitly explain how students acquire and develop mathematical conceptual understanding. What is needed is a framework that explains how students construct mathematical concepts that; (a) are personally meaningful to them, and (b) their usage is commensurate with that of the mathematical community. In an attempt to understand how schools create and use the standard system of representations as seen in the school curriculum, a sign oriented perspective from which to examine school mathematics would be appropriate (Ernest, 2006).

Ernest (2006), semiotic perspective is the theoretical lens I used using to examine how mathematical sign systems (representations) are developed, and elaborated in the educational process. Semiotic perspective as elaborated in the theoretical framework takes into account three necessary components of a semiotic system; (a) the set of mathematical signs, (b) set of rules of sign production, and (c) the set of relationship between the signs and their meanings embodied in an underlying meaning structure. Within the educational context students meet a whole new range of signs and symbolizing functions in mathematics. The historical and cultural development in mathematics have given rise to the semiotic systems that provide the underlying structure of the mathematics curriculum in our schools. Semiotic perspective outlines a number of different but interrelated semiotic systems that are important in the leaning of mathematics and closely align with the NCTM and Common Core Standards. These systems area; (a) numbers, counting and computation, (b) rational numbers (fractions) and their operations, (c) measures and their means of computation, (d) geometry, probability and statistics, (e) algebra including solving simple linear and quadratics equations, and (f) abstract systems such as calculus, analysis and abstract (axiomatic) group theory (Ernest, 2006). According to research (Ernest 2006; & Sfard, 2000), for individual to be successful in mathematical activity, mastery of semiotic systems is essential. This mastery involves; learning the implicit rules of correct use of mathematical sign, and application of these rules in the production and reading of mathematical text. This study will examine the role of multiple representations in promoting the mastery of semiotic systems. From a semiotic perspective, it is not just about the use of multiple representations but how utilizing multiple representations promotes the mastery of semiotic systems where students can potentially benefit in the: (a) identification of mathematical concepts; (b) acceptance and internalization of the goals of a given task; (c) selection of skills and procedures to perform a given mathematical task (Ernest, 2008).

Instructional practices that teach standard system (i.e., mathematical sign use) as an end in themselves may fail to develop students' mathematical power (Goldin, 2003). Research on students' conceptual understanding (Sfard, 2000) conclude that students demonstrate conceptual understanding when they are capable of transitioning from a mathematical-sign-as-an-object-in-itself phase of understanding to mathematical-sign-as-a-representation-of-another-object phase of understanding. In the conceptual thinking phase of understanding mathematical symbols and signs are perceived as: (a) representations of other mathematical objects and not an end by themselves; and (b) processes as well as objects. The transition from instructional model where mathematical objects (signs or symbols) are the primary focus of attention to a model where mathematical objects (signs or symbols) are transparent (Sfard, 2000) represents a paradigm shift in the way mathematical truths or concepts should be presented.

Summary of the Literature Review

The essential role of representations is to provide access to mathematical objects in students' sign receptions and sign productions activities. Duval (2006) observes that the use of signs and semiotics representations provides access to the mathematical objects and shows us how to deal with these objects. Further Duval argues that " no kind of mathematical processing can be performed without using a semiotic systems of representation, because mathematical processing always involves substituting some semiotic representation for another" (p. 107). Potential benefits associated with the use of multiple representations have been documented. These benefits include facilitating students' deeper conceptual understanding (Berthold et al., 2009; Ainsworth, 2006; Ainsworth & van Labeke, 2004; Clement, 2004; and Tripathi, 2008). Kaput (1992) in supporting Duval's positions argues that each representation has different advantages in facilitating conceptual understanding, and argues for the use of various representations for the same concept as an effective tool for instruction. The role of multiple representations can be summarized as providing three main functions; (a) complementing each other, (b) constraining interpretations, and (c) supporting construction of deeper conceptual understanding (Ainsworth, 2006)

Research studies on exposure to and use of multiple representations in algebra understanding (Ainsworth, 2006; Amit and Fried, 2005; Knuth, 2000; & Zaslavsky et al. 2002) support the need for algebra instructional and assessments activities that promote exposure to and use of multiple representations. The findings from this studies indicate that for familiar routine problems many students master the connections between the algebraic and graphical representations, however such mastery appeared to be superficial at best, which researchers Knuth(2000) and Berger (2004) describe as an indication of pseudo-conceptual understanding. Knuth (2000) recommendation that an expert's knowledge of the mathematic domains that extends beyond simple procedural competence sums up the findings of the research studies in this literature review. Further findings from these studies continue to emphasize the goal of mathematics instruction and assessments that should shift students' mathematical understanding beyond procedural competence and towards a more robust and flexible understanding of concepts like functions.

With the goal of shifting students understanding beyond simple procedural competence, literature reviewed further emphasized the need for instruction and assessment models or activities that aim at identifying students' pseudo-conceptual understanding in mathematics (Berger, 2004, 2005, Vygotsky, 1994, and Vinner, 1997). Pseudo-conceptual understanding refers to students' use of words and mathematical symbols in communication without knowing exactly what they mean or represent (Sfard, 2000; Vinner, 1998; & Vygotsky, 1986). The literature further indicated the significant purposes of pseudo-conceptual understanding include; (a) effective communication, and (b) promising engagement in activities that utilize mathematical signs even before students fully comprehend the relevant mathematical object they are studying. It can be argued that the use of pseudo-concept enables students to access new and unknown mathematical objects and hence allow students some form of communication about a given mathematical concept.

The literature further emphasized the need for transition from pseudo-conceptual understanding to conceptual understanding in mathematics in general and in algebra in particular. Students utilizing conceptual thinking have transitioned from 'signifier-as-an-object-in-itself' model of thinking to 'signifier-as-a-representation-of-another-object' model of thinking (Sfard, 2000, p.79). In the conceptual thinking stage mathematical symbols and signs are perceived as a representation of other mathematical objects and not an end by themselves. The transition from instructional model where mathematical objects (signs or symbols) are the primary focus of attention to a model where mathematical objects (signs or symbols) are transparent (Sfard, 2000) represents a paradigm shift in the way mathematical truths or concepts should be presented.

According to Sfard (2000) conceptual understanding is demonstrated when a student is capable of transitioning from "signifier-as-an-object-in-itself" phase of understanding to "signifier-as-a representation of-another object" (p.79) phase of understanding. Use of multiple representations (as defined in this study) either during instruction or assessment phase of learning can provide an avenue or a platform with which educators can assess students' conceptual understanding. A tool with which educators can assess whether students' understanding of mathematical concepts is in "signifier-as-a-representation-of-another object" phase of understanding.

Informed by this literature, this study sought to closely examine the conceptual understanding mathematics practice standard advocated for in for example CCSS Initiative. In particular the study focused on the role of exposure to and use of multiple representations in the transition from pseudo-conceptual understanding where mathematical objects are the focus of attention, to conceptual understanding where the meanings embedded in mathematical signs are the primary focus. In addition this study also focused on exploring the role of multiple representations in facilitating algebra students' conceptual understanding during mathematical sign receptions and sign productions. In particular the study focused on understanding how exposure to and use of multiple representations influenced students conceptual understanding where mathematical "signs are transparent enough so that the signified (mathematical ideas) shines through them" (Sfard, 2000, p.20).

3 THEORETICAL FRAMEWORK

"….the role of semiotics as the study of signs encompasses all aspects of human sign making, reading and interpretation, across the multiple contexts of sign usage. Mathematics is an area of human endeavor and knowledge that is known above all else for its unique range of signs and sign-based activity….. So it seems appropriate to adopt a sign-orientated perspective from which to examine school mathematics". Ernest (2006, p. 1)

Brief Overview of the Conceptual Framework

The theoretical perspective that I adopted for the study was Ernest (2006) Semiotics Systems Theory. Ernest (2006) theory is inspired by Vygotsky's (1978) Sociocultural Learning Theory and provided a theoretical understanding and conceptualization of mathematics that is driven by a primary focus on mathematical sign and sign use. Ernest (2006) theory of semiotic systems as a theoretical lens, zooms in into Vygotsky's (1978) second theme of semiotics mediation and offers insights into students' sign reception and sign production. Wertsch (1991) one of the first major scholars to interpret Vygotsky's work, identified three major themes of Vygotsky's theoretical approach; (a) individual development including higher mental functioning, has its origin in social processes, (b) human action, on both the social and individual planes, is mediated by semiotic means which are tools and signs, and (c) the need for reliance on genetic or developmental method in order to examine the origins and the history of mental phenomena.

In a mathematics classroom, students learning any algebra function e.g., absolute-value function are inducted into a discursive practice that involves the use of signs and an understanding of the rules of an algebra semiotic system. The teacher or the knowledgeable other presents tasks (whether from textbook or other mathematics resources) in the form of signs as well as the rules for working or transforming the signs to accomplish a given task. Ernest (2008) observes that "the learning of mathematics in schools presupposes the induction of the students into a particular discursive practice, which involves the use of signs and sign rules in school mathematics" (p. 69). Informed by this understanding, Ernest (2008) theory was appropriate in the study in: (a) providing an in-depth understanding of how mathematical signs and their transformational rules are applied in an algebra classrooms; (b) supporting my understanding of how the shared construction of the three algebraic functions was internalized, appropriated, and transformed in a formal learning environment; and (c) providing a theoretical insight into how algebra II students' reception and production of mathematical signs played a role in their pseudo-conceptual and conceptual understanding of piecewise functions, absolute value function, and quadratic functions.

Theory of Semiotic Systems for Mathematics

Several semiotic theories have emerged in the last decade (Hoffmann, 2006; Cobb, 2007) centered on the usefulness of semiotics as a theoretical perspective and a practical position in the teaching and learning of mathematics. Semiotic theories have gained and continue to gain a lot of attention because of their contribution to new perspectives on knowing and knowledge, representing and representation (SÁEnz-Ludlow & Presmeg, 2006). Ernest (2006) semiotic perspective is one of the aforementioned theories with a principal focus on both the appropriation of signs and the understanding of the underlying meaning structures that embody the relationship between signs. Following is a description of the semiotic perspective which is based on the concept of semiotic systems (Ernest, 2008) and takes into account three necessary components; (a) set of signs, (b) set of rules of sign production, and (c) set of relationships between the signs and their meanings embodied in an underlying meaning structure.

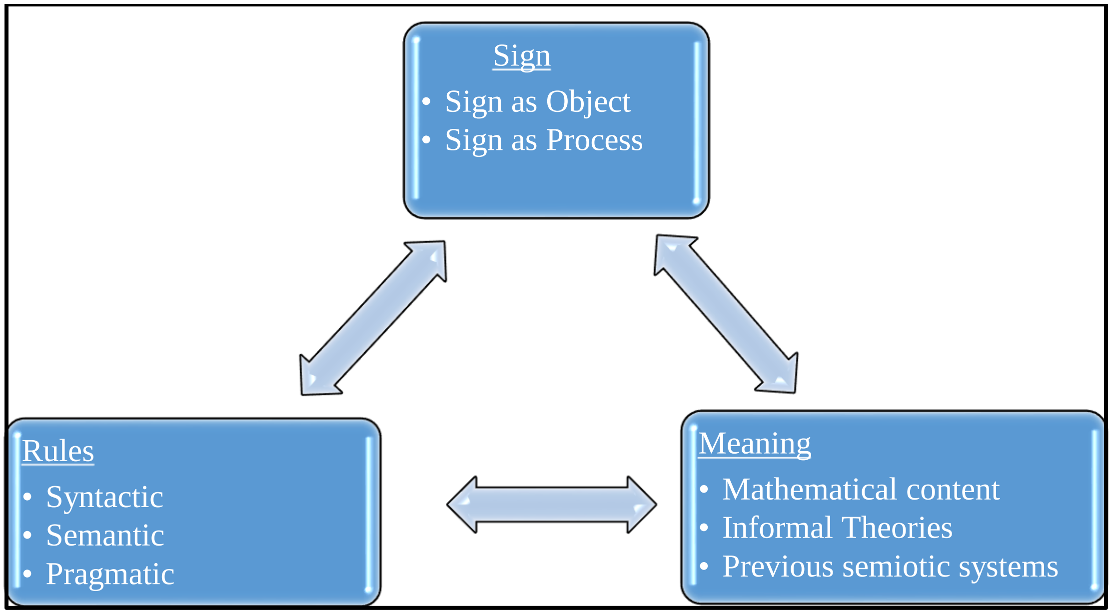


Figure 4. Semiotic Systems Theory Model (Ernest, 2008) based on the concept of three necessary components of a mathematical sign: (a) set of signs; set of transformation rules; and set of underlying meaning structures.

As illustrated previously, students learning any algebra function e.g., absolute value function are inducted into a discursive practice that involves the use of signs and an understanding of the rules of the absolute-value semiotic system. The teacher or the knowledgeable other presents tasks (whether from textbook or other mathematics resources) in the form of signs as well as the rules for working or transforming the signs to accomplish a given task. In most cases semiotic rules are exhibited implicitly through worked examples.

Set of Signs

The first component of a semiotic system involves a set of signs. Set of signs includes both elementary $(e.g.\ +, -, \times, \div, =, \leq, or \geq)$ and compound signs for example absolute value function $y = a|x - H| + K$ or quadratic function $f(x) = a(x - H)^2 + K$. These sets of sign were either : a) spoken or uttered via various media; written, drawn, represented by any material means, and encoded electronically; and b) multimodal involving a selection of sounds and spoken words; repetitive bodily movements; and use of artifacts like algebra tiles, manipulatives like pebbles and counters. In school algebra, mathematical signs (verbal description, symbolic or algebraic, graphical, and numerical data) are typically represented as textual inscriptions on the whiteboard, printed texts or worksheets or in students' written work (Ernest, 2006). Informed by Ernest semiotic theory I classified mathematical signs in the students' responses as: (a) sign as object; and (b) sign as process. For example the absolute value function $y = -2|x - 4| + 3$ viewed as an object represents a V shaped mathematical object with vertex at $(4, 3)$, y-intercept $(0, -5)$ and opens down. Similarly the same absolute value function sign can be viewed as a process when considering its parts and the transformations from the parent function (i.e. horizontal shift 4units right, vertical shift 3 units up, vertical stretch factor 2, and a reflection along the x-axis. Ernest (2008) also observes that in school algebra, the use of written language (verbal description) may be used to supplement the formal signs used at all level.

Set of Rules

The second component of semiotic systems is a set of rules of sign use and production. The set of rules for sign use, combination and production can be analyzed into 3 different types; syntactic, semantic, and pragmatic. For the study I focused on the syntactical and semantic rules. Syntactic rules are based on the signs for what they are i.e. rules for writing a well-formed formula. Ernest (2006) observes that these set of rules concerns both the understanding of the definition and determinants of a well-formed i.e. grammatically correct, sign as well as the sequencing of signs in conversations i.e. what sign utterances may legitimately follow on from prior signs in a given social contexts. For example writing a quadratic function as

$y = a|x - H|^2 + K$, does not constitute a well-formed formula because a quadratic function involves the square of the parenthesis and not the absolute value symbol. Semantic rules refers to the dimensions of sign interpretation and meanings (Ernest, 2006). For example in determining the x-intercept in the following function $y = -2|x - 4| + 3$, a student substitutes zero for y in the function and write the expression as $0 = -2(x - 4) + 3$, this could be interpreted as questionable understanding of the significance of the absolute values function sign | |. Arzarello (2006) extends this description beyond rules or algorithms to include any mode of sign production "a set of modes for producing signs and transforming them: such modes may possibly be rules or algorithms but may also be more flexible action or production modes used by subjects" (p. 279).

Observing the rules of a semiotic systems also involves following steps and procedures that legitimate certain text transformation as in the following example. In expanding the quadratic expression $(x - 4)^2$ students might be tempted to express it as $x^2 - 16$. The rule that legitimizes this transformation requires students to recognize that the binomial square expression $(x - 4)^2$ represent the product of the binomial $(x - 4)$ multiplied by itself, hence the correct expansion should be written as a trinomial $x^2 - 8x + 16$. In another example, simplifying the rational expressions $\frac{x+5}{2(x+5)}$, students might be tempted to simplify the expressions as $\frac{x+5}{2(x+5)} = \frac{0}{2}$ or $\frac{x+5}{2(x+5)} = \frac{0}{2}$. The rule that legitimizes this text transformation requires students to recognize that the numerator equals 1(x+5), so when (x + 5) is cancelled in the numerator and denominator, the resulting number in the numerator is 1 and not 0. Hence the simplified answer should be $\frac{1}{2}$. Ernest (2008) observes that the set of rules in a semiotic system whether implicit or explicit are the key operative mechanism and principles through which students; (a) form new signs, and (b) construct and elaborate on mathematical text. Pragmatic rules for example classroom stipulations on how to respond to or present to a mathematical tasks (e.g. underline final answers) are rhetorical and purely determined by social conventions.

Set of Underlying Meaning Structure

The third component of a semiotic system involves the underlying meaning structure of the relationship between the signs which can range from an unsubstantiated idea to a conjecture that a student might observe that is akin to an informal mathematical theory (Ernest, 2008). I used three different criteria to analyze the meaning structures of the semiotics systems which included identification of: a) a set of mathematical content; b) set of informal theories that students developed; and c) set of previously constructed semiotic systems that students referenced or made connections (Ernest, 2006). Set of mathematical contents includes loosely associated: "sign, concepts, objects, properties, functions, relationships, rules, procedures, methods, heuristics, classifications, problems, examples, ideas, images, metaphors, models, structures, representations, propositions, theorems, arguments, proofs, theories, etc." (Ernest, 2006, pg. 41). Informal mathematical theory refers to informal statements students used to evaluate a formal mathematical theory and served as the meaning structure of the several semiotic systems that students interacted with. For example, in the following absolute value function (semiotic system) $y = -2|x - 4| + 3$, a statement like "when $a = -2$ and $K = 3$ have different signs it indicates that there exist two x-intercepts" is an indication of informal mathematical theory. In another example involving quadratics functions, a teacher might require students to write a quadratic model in vertex form $y = a(x - H)^2 + K$ and observe how the values of parameter 'a' in the model controls the concavity of the parabola as well as the relationship between the signs of both constant ‘a’ and ‘k’ and the x-intercept of the graph of the quadratic function. In this example an informal ‘mathematical theory’ would be a statement that “when the value of ‘a’ and ‘k’ have the same sign then the graph of a quadratic function has no x-intercepts”. Finally previously constructed semiotic system served as a meaning structure for a new semiotic system. In the quadratic model example shown above, the meaning structure of the quadratic semiotic system can be described as an informal theory which intersects with the performance norms i.e. expected understanding of the relationship between the values of variables ‘a’ and ‘k’ and the x-intercept of the graph of a quadratics function. This example also illustrates that the act of meaning-making by students always draws upon their active mobilization of existing elements of meaning and understanding (Ernest, 2008). Arzarello (2006) describes this semiotic component as “a set of relationship among these signs and their meanings embodied in an underlying meaning structure” (p. 279). Sfard (1994) refers to meaning structure as a reservoir of meanings that can be drawn upon in formulating, developing and operating a semiotic system such as the metaphor. The underlying meaning structure of a semiotic system involves “repository of meanings and intuitions concerning the semiotic system that support its creation, development, and utilization” (Arcavi, 2005 pg. 12).

Significance of Semiotic Systems Theory

The significance of understanding and utilizing semiotic systems theory in this study, is the theoretical insight into the process of internalization of mathematical concepts that it provides. More important in the process of internalization, is the primary focus on mathematical sign and sign use. Ernest (2006) argues that,

> ….internalization necessitates the learner to be continually engaging in conversation, making public utterances and performances, deriving feedback from others, incorporating confirmations and corrections in his or her performance and functioning, which helps to shape the child's emerging powers…the learner also learns to read, understand and comprehend the social context. (Ernest, 2006, p. 91).

Ernest (2006) argument above is consistent with; Goldin (2003) notion of mathematical power, Sfard (2000) notion of conceptual understanding, and Vygotsky's (1978) description of the concept of internalization as the process by which shared construction of knowledge is appropriated, transmitted or transformed. Key to these descriptions is the appropriation of socially elaborated symbol systems (i.e., semiotic mediations process). Though Vygotsky (1978) offers a general description of concept of internalization, Ernest's (2006) semiotic theory focuses our attention to a more detailed description of the internalization process of mathematics concepts.

Internalization of mathematical concepts involves a gradual mastery of semiotic systems, which is described as the successful appropriation and deployment of; (a) mathematical signs, (b) mathematical sign rules, and (c) the underlying meaning structure of a semiotic system (Ernest, 2006). The focus of this study was on the algebraic semiotic systems and the potential benefits associated with the mastery of; piecewise, absolute value, and quadratic functions algebraic systems. Benefits associated with the mastery of these systems include the development of the following abilities: (a) identification of mathematical signs; (b) acceptance of tasks associated with these semiotic systems; and (c) selection of the central functions and structures of an algebra system that students maybe studying. A detailed explanation of these functions will follow.

In an educational context when semiotic systems are presented whether during instruction or in the resources like textbooks, only one component of the semiotic system is made explicit

(Ernest, 2006) i.e., the set of signs. The rules of sign production and the underlying meaning structures are in most cases implicit. Mastery of semiotic systems empowers students to critically read mathematical texts whether published, written by students themselves, their peers or by the teacher. Given any mathematical task that involves the use of algebraic conceptual understanding, the identification function of algebra semiotic systems enables students to read and interpret the context of the mathematical task assigned i.e., consciously or unconsciously students are aware when a task is signaled. Bennett et al (2012) observed that half of the time spent by students on tasks was spent on deciphering and understanding the nature of the tasks, rather than performing them. Therefore according to Ernest (2006), continuous and consistent engagement with semiotic systems leads to mastery of these systems which then enables students to unconsciously learn to recognize, understand, and engage with a task within familiar or unfamiliar context.

The second benefit of mastering the algebraic semiotic systems that I focused on in the study, involved acceptance and internalization of the goals of a given task (Ernest, 2006). Students who have mastered; (a) the implicit rules of sign usage in mathematical text whether written or spoken, and (b) the underlying meaning structures of the signs, are empowered to not only apply them in the production of texts but also in the reading and critiquing of the these text (Ernest, 1997). Hence, given a mathematical concepts like the; quadratic functions, absolute value function, or the piece-wise function, the ability to accept and internalize the goals of a given task allows students some form of communication whether spoken or written about these concepts. This acceptance function does at times occur without the students being aware of their tacit compliance of the task at hand. The acceptance function highlighted in this semiotic theory is consistence with Berger (2004) observation that the use of pseudo-concepts provides students with initial access to new and unknown mathematical notions even when they might not fully comprehend the mathematical concepts they are studying.

Selection function is the third benefit attributed to the mastery of semiotic systems. In this function students develop the ability to select from their own personal repertoire of knowledge, skills and procedures to perform appropriate functions in response to a given task (Ernest, 2006). These functions involve; texts transformation (showing their written algebraic steps), and production of oral responses (verbal descriptions of their understanding), and other means (graphing or tabular representations). In instruction where students are afforded the opportunity to utilize a variety of modes of representations, they develop the ability to not only read the mathematical tasks involving different functions (quadratic, linear and constant function) but also understand the rules governing the correct production of these functions with the given conditions/rules and apply them appropriately. Use of multiple semiotic representations; verbal descriptions, numeric, algebraic and graphical representations, then facilitates the successful appropriation and organization of semiotic systems, by providing students with a selection of their own personal stock of knowledge, and procedures to respond to a given task. The selection function highlighted in this section is consistent with Goldin (2003) notion of mathematical power and Berger (2005) notion of conceptual understanding where responding to a given task involves the ability to; recognize and visualize structured relationships, activate control mechanisms for examining their responses to a given task and employ of a variety of problem solving techniques.

Concept of Mathematical Sign Appropriation and Use

Ernest's concepts of sign appropriation and use (Ernest, 2006b) is derived from his theory of Semiotic Systems (Ernest, 2006) and considers the development of mind, personal identity, language and knowledge. This concept as shown in figure 5, can be represented in a cycle of appropriation, transformation, publication, and conventionalization of mathematical signs (Ernest, 2006b). The model represents; (a) a micro view of learning and of knowledge production, and (b) illustrates how mathematical signs become appropriated by students through experiencing their public use.

This model was appropriate for this study because it provided a description of an overall process in which both; (a) students' individual and private meanings, and (b) collective and public expressions of their conceptual understanding are mutually shaped through conversations and interactions in a social settings like the classroom. Mathematical knowledge and the meaning of the full range of mathematical signs, texts and other forms of representations are distributed over all four quadrants of the model.

Mathematics and the teaching of mathematics are essentially symbolic practices in which signs are constantly being invented, used, or recreated to facilitate cognitive operations or purposes (Ernest, 2006; Hoffman, 2006). The term *Mathematization* for example refers to the process of representing problems or facts by means of symbols, indices and relational representation as provided by the history of mathematics (Hoffman, 2006).

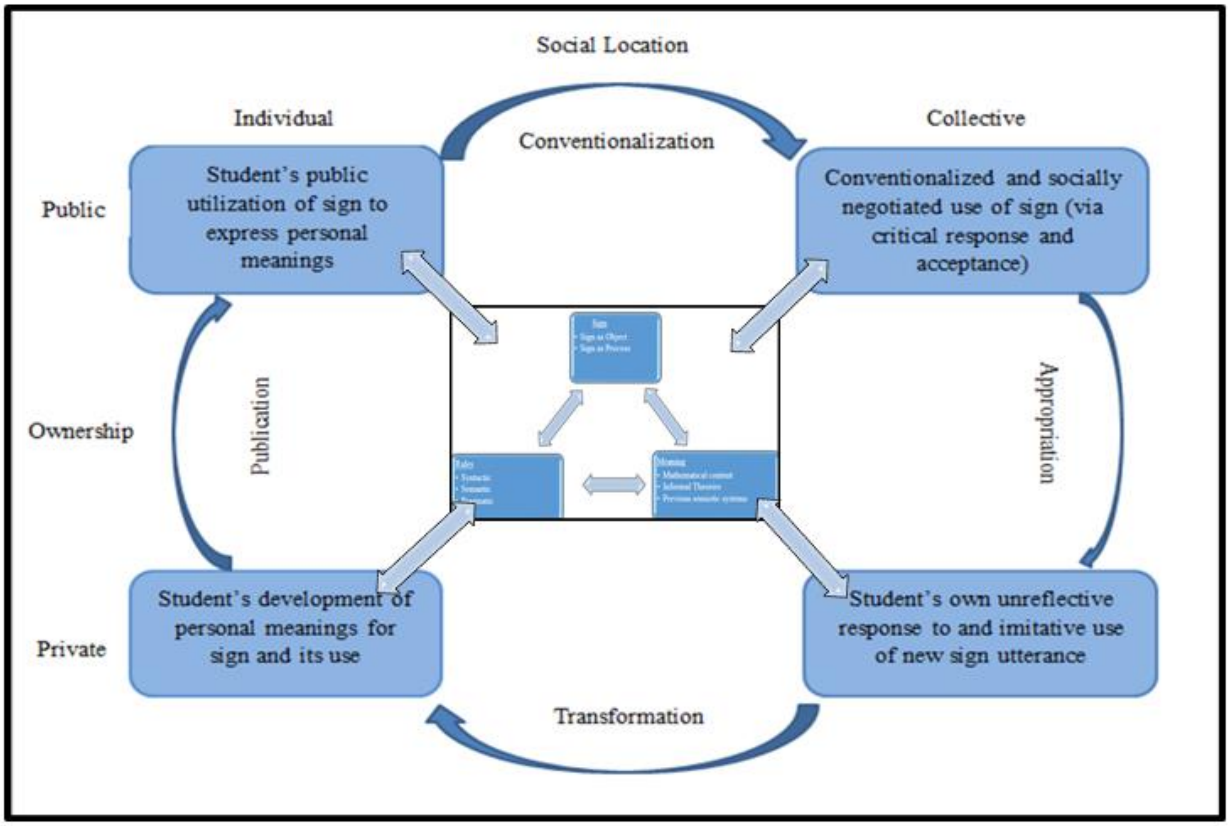


Figure 5. Schematic Model of Sign Appropriations and Use. Model represents; (a) a micro view of learning and of knowledge production, and (b) illustrates how mathematical signs become appropriated by students through experiencing their public use.

Other terms used like; *calculation* refers to the process of transforming representation according to rules of a certain system of representations, and *proving* refers to the process of representing a theorem as implied by other theorems within a consistent system of representations (Hoffman, 2006). It follows that signs and representations play an essential role in mathematics by providing access to mathematical objects (Berger, 2005). Hoffman (2006) argues that the essence of mathematics consists of working with representations. Starting at the top right and proceeding in a clockwise direction, a discussion of the theoretical micro view model of learning and knowledge production will follow.

Within the collective mathematics community (Hoffman, 2006; Ernest, 2008), conventionalized and socially negotiated mathematical signs are used to access mathematical objects as is evident in the mathematics textbooks and various curriculum resources available and used in schools. From a constructivist ontological perspective, mathematical objects are impossible to grasp and experience (Ernest, 1991) hence the essential need for signs and representations in mediating mathematical cognitions. Teachers perform mathematical activities like solving algebra problems by means of these visible signs which are external manifestation of mathematical concepts. These signs include; conventional graphical, diagrammatical, and formal notational systems of mathematics (Goldin, 2003). An example of a mathematics community is the NCTM which through its various publications recommends that high school students' algebra experience should enable them to create and use tabular, symbolic, graphical, and verbal representations to analyze and understand patterns, relations and functions (NCTM, 2000).

Brousseau (1997) observes that mathematicians do not communicate their results in the form in which they create them, instead they re-organize their results and give them a general form that is "de-contextualized, de-personalized, and de-temporalized" (p. 227). Examples of generalized form of mathematics are; absolute value function model $y = a|x - H| + K,$ quadratic function model $y = (x - H)^2 + K$, and the quadratic formulae $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The teachers' or the knowledgeable others' role is then to undertake the opposite action of facilitating students' ability to contextualize and personalize the knowledge created by mathematicians in order to give meaning to the knowledge to be taught (Hoffman, 2006). The teachers' role during this process compares to Hersh (1988) comparison of the teaching of mathematics to a restaurant or a theater that has a front end and a back end. With the activity displayed at the front for public viewing being tidied up according to strict norms of acceptability (Ernest 2008) while the back where preparatory work takes place often being chaotic and messy.

The collective and socially conventionalized mathematical sign use as observed in text-books and other mathematics resources, initially leads to students' own unreflective response to and imitative use of signs based on the perceived regularity (rule-based) and connectivity of use within classroom practices. This is the bottom right portion of the model. Several researchers (Berger, 2005; Vygotsky, 1978; Sfard, 2000; and Vinner, 1997) have used the term pseudo-conceptual understanding to describe this stage of the model. Pseudo-concepts occur whenever a student uses particular mathematical objects in a way that coincides with the use of a genuine concept even though the student has not fully constructed the concepts for themselves. Pseudo-concepts resemble true concepts in their use, but the thinking behind these pseudo-concepts is still non- logical or based on experiential association (Berger, 2005). At this stage mathematical signs rather than the meanings embedded in the signs are the primary focus of attention. Nabb (2010) observes students use of idiosyncratic devices (mnemonics) to assist them in memorizing formulas for later retrieval while admitting unfamiliarity of the formula conceptual foundation. Garofalo and Trinter (2012) describes how significant number of algebra students capable of reciting the quadratic formulae $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, demonstrate difficulties in explaining the meaning of the various parts of the formulae as well as describing what each variable in the formula represents.

This second stage of the cycle compares to Vygotsky's description of learning that leads to acquisition of spontaneous concepts which are the results of generalization of everyday personal experiences in the absence of systematic instruction (Kozulin, 2003). If instruction is not appropriately and effectively mediated, this stage maybe characterized by: (a) mastery of rote skills or understanding of mathematical concepts (Kozulin, 2003); (b) lack of higher-order mathematical understanding that allows students to apply their skills in different situations; (c) reliance on social cues (Brousseau, 1997) to provide the desired responses to a given mathematical task; and (d) potential of teachers accepting low level mathematical signs as evidence of general and higher level understanding (Brousseau, 1997). Action mediated by use of multiple representations like; verbal descriptions (natural language), numerical (correspondence in a table of values), algebraic or symbolic representations (equations expressing the relationship between two or more quantities), and graphical representations (Cartesian graphs) as described in Goldin (2003) are significant in linking students' social experiences in this case mathematics classroom interactions and their individual processing of mathematical information. Multiple representations act as semiotic mediators which facilitate students' mathematical activities and define and shape their inner processes.

After a series of interactions with mathematical signs which includes the imitative uses of mathematics signs, and through the process of mediation which includes; the appropriate use of mathematical signs, and social e.g., classroom mathematics activities interventions, students begin to develop an understanding of the implicit rules and associations in the various representations that they are studying. This is the third stage of the model (bottom left) where students develop personal meanings for mathematical signs and their uses, and transform them into something that is individually and privately owned (Ernest 2006b). At this stage mathematical ideas are transformed into mathematical conceptual understanding. This happens when students make consistent and logical linking of; (a) the various attributes and properties of a mathematical concept (e.g., the meaning of the various components of an absolute value function like

$y = -2|x + 5| - 6$), and (b) a mathematical concepts to other concepts or representations (e.g., understanding the domain, range, and end-behaviors of the absolute value function mentioned). At this stage conceptual understanding is characterized as the implicit or explicit knowledge of; (a) the principles that govern a given mathematics domain, and (b) the interaction between the various units of a given domain (Godino, 1996). Students are expected to construct concepts whose use and meaning are compatible with their use in the mathematics community (Berger, 2005).

In semiotic terms, acquiring conceptual understanding in mathematics can be seen as acquiring the ability to express oneself appropriately through the means of representations provided by the tradition of mathematics (Hoffmann, 2006). The term appropriate means not only mastering the conventions of common sign usage in mathematics, but also understanding the relation between mathematical knowledge and mathematical representations. In Ernest (2006) semiotic theory he describes this stage as a process of mastering the semiotic systems. Goldin (2003) describes this stage as enabling students to develop mathematical power. Students who have developed mathematical power: (a) are capable of understanding and manipulating standard representations; and (b) demonstrate ability to recognize and visualize structural relationships; (c) think spatially; and (d) formulate problem solving strategies. (p. 277).

At the students' public utilization of sign stage which is the top left portion of the model, students use mathematical signs acquired to express their personal meaning or understanding of mathematical concepts through; verbal expressions, spontaneous utterances and extended text (i.e., tests, examinations, homework, or projects). These publications can occur in autonomous conversation acts and are subject to the process of conventionalization. In the conventionalization process, students understanding expressed in various modes of communication (verbal or symbolic) can be subjected to attention and response (assessments and evaluations) which can be critiqued, negotiated, reformulated, and/or accepted (Ernest, 2006). However, for students who have developed mathematical power (Goldin, 2003), they have the potential to: (a) be continually engaged in mathematical conversations; (b) make public utterances of their mathematical understanding; (c) derive feedback from others; and (d) incorporate confirmations and corrections in their performance and functioning (Ernest, 2006, p. 91).

In the process of conventionalization, social norms for mathematics speaking and listening; judgment of adequacy of mathematical explanations and solutions; and habits of reasoning and sense making are created (John-Steiner & Mahn, 1996). The process of conventionalization is at the center of the Zone of Proximal Development ZPD (Vygotsky, 1978) which is described as “the distance between the actual developmental level as determined through independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peer” (Vygotsky, 1978. p 86). The process of appropriation and publication in the model are boundary operations between the public (also described as inter-psychological level of understanding) and private (also described as intra-psychological level of understanding) domain in which students participates in the communicative activity of sign reception and production (Ernest, 2006). In the private domain students transforms collectively appropriated mathematical signs into individual ones through the production of meaning, and is thus a pivotal location for learning.

4 METHODOLOGY

Brief Overview of the Research Design

This study employed an exploratory case study method Yin (2014). Yin (2014) describes a case study as a research method "that investigates a contemporary phenomenon the "case" in its real-world context especially when the phenomenon and the context may not be clearly evident" (p.16). The purpose of this study was to investigate the influence of the exposure to and use of multiple representations in algebra II students' conceptual understanding of; absolute value functions, piecewise functions, and quadratics functions. The choice of this method was guided by the following research question: How does the use of multiple representations influence algebra II students' understanding and transfer of their algebraic concepts? Specifically the following sub-questions were examined:

1. How does exposure to and use of multiple representations influence students' identification of pseudo-conceptual understanding of algebraic concepts?
2. How does exposure to and use of multiple representations influence students' transition from pseudo-conceptual to conceptual understanding?
3. How does exposure to and use of multiple representations influence students' transfer of their conceptual understanding to other related concepts?

Exploratory case study is located within the wider landscape of qualitative research methodology. Following Crotty (1998) four elements of a social research process, this chapter begins with a discussion of the philosophical underpinnings of a qualitative methodology within which case study is located. The chapter then follows with a detailed description of the exploratory case study including the rationale for employing this method. The chapter then discusses: research settings and participants, data collection procedures and instrumentation, data analysis methods, limitations of the study, and the trustworthiness of the research findings.

Qualitative Methodology Philosophical Underpinnings

To ensure a strong research design, researchers (Crotty, 1998; Guba & Lincoln, 1994; and Paul & Marfo, 2001) have advocated for a research paradigm that is congruent with the researchers' beliefs about the nature of reality. Crotty (1998) four elements of social research process (see figure 6) outlined the philosophical underpinnings that guided this research design. Crotty (1998), suggests that research process should include; (a) epistemology, (b) theoretical perspective, (c) methodology, and (d) methods.

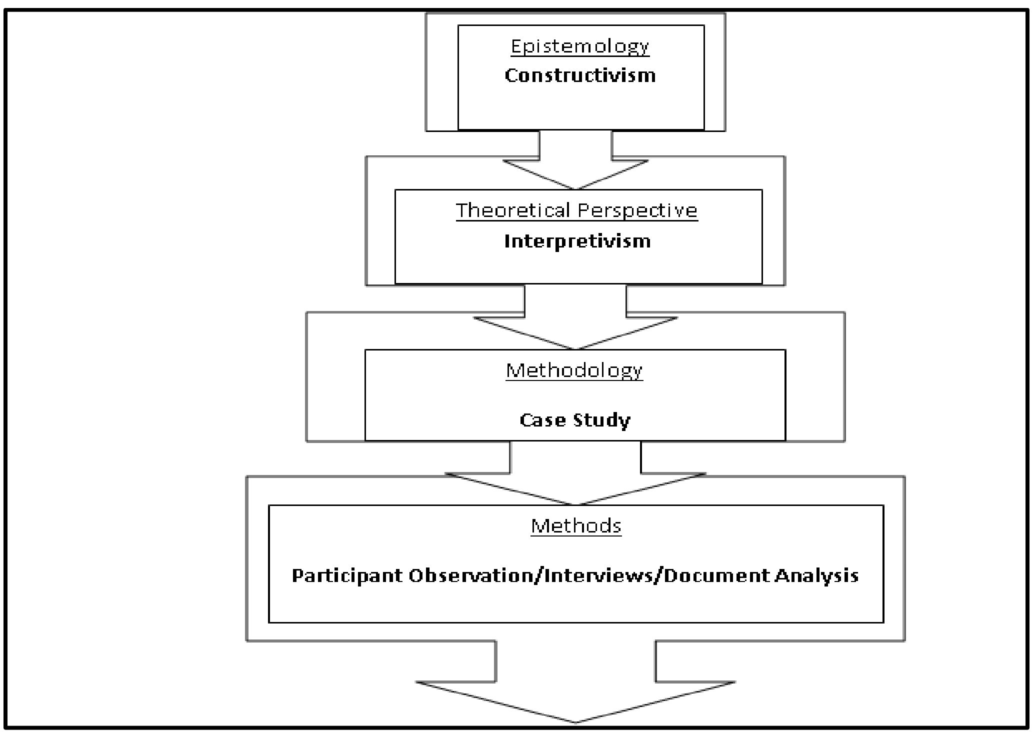


Figure 6. Four Elements of Social Research. A philosophical research design model used to guide my research design.

Epistemology.

This exploratory case study methodology was based on a constructivism epistemological paradigm (Guba & Lincoln, 1994). In constructivism paradigm, realities are apprehended in the form of multiple, intangible mental constructions that are socially and experientially based and dependent for their form and content on the individual persons or groups holding the construction (Guba & Lincoln, 1994). In this paradigm my role as a researcher and the object of my investigation are assumed to be interactively linked so that the findings in the study are literary created as investigations proceeds (Denzin & Lincoln, 2005). In light of this understanding, I approached the study with a belief that my personal philosophy of the nature of knowledge was essential in understanding the world around us (Crotty, 1998). This understanding extended to the kind of knowledge that I believe was attained in my research as well as the characteristics of the knowledge that the study contributed.

Theoretical Perspective.

The theoretical perspective for conducting the study was interpretivism. This perspective was guided by constructivism epistemology mentioned above and represented statements of the assumptions that I brought to the study. This assumptions were also reflected in the case study methods. The study sought to give an insight into; the question of *how*, and the qualities of experiences gained by the students within the classroom social interaction. In assuming this perspective I was attempting to understand and explain my participants' social (classroom) realities. Any understanding of causation (Crotty, 1998) came through an interpretive understanding of social actions (e.g. classroom interaction) that the participants were engaged in. Social constructions vary and are personal (Guba & Lincoln, 1994), implying that the participants construction of social realities was elicited and refined through my interaction with them in the study. The aim of inquiry in this perspective was to understand and reconstruct the constructions that we (including myself) initially held about the use of multiple representations in an algebra II classroom. In the study the construct that I sought to understand was how the use of multiple representations as defined in the study influenced students' conceptual understandings of algebraic concepts. Since the study utilized the symbolic aspects of mediation, my objective was to gain and understanding of the potential changes in students' conceptual understanding that were attributed to the introduction and use of multiple representations. Equally important, since this study focused on semiotic aspects of mediation in sociocultural activities (e.g. mathematics classroom interaction), using interpretivism theoretical perspective, supported my aim of formulating a more informed and sophisticated (Guba & Lincoln, 1994) understanding of instructional and assessment practices that use multiple representations.

Case Study Methodology.

The underlying philosophy of an exploratory case study is to understand participants' construction of their own realities (Stake, 1995; and Phillips & Burbules, 2000). My goal therefore was to understand how the participants i.e. twenty one algebra II mixed gender class made sense of their realities i.e. the instructional and assessment practices that utilized multiple representations. Rationales for using case study (Yin, 2014; Cohen, 2000) included; strength in reality, attention to subtlety and complexity, support to alternative interpretations, represents a step to action, and forms an archive of descriptive material. Case study methodology was appropriate for this study because: (a) represented a bounded system (Merriam, 2009; & Yin, 2009); and (b) flexible and adaptable. First, bounded system included the participants i.e., the algebra II students who represented unique example of real people in real situations (Cohen, 2000), hence portraying the reality of being in a particular situation. Bounded system also referred to the temporal i.e., twelve weeks period, geographical institution located in the south east of the country that serve middle and high school student population and confined to one particular teacher and his algebra II students. Second, case study design methodology was also flexible and adaptable in terms of data collecting methods and participants' response to themes that emerged during the study.

Use of multiple representations in mathematics learning is a complex and multi-layered research discipline (van der Meij & de Jong, 2006) and requires a full understanding and the grappling of the many interweaved and overlapping issues and themes that emerge in such a study. Since case study is concerned with the rich and clear description of relevant events, the use of this methodology was well –suited for this study in order to gain a deeper insight into the complexities associated with the students' use of multiple representations. Additionally since the focus of case study was to understand participants' perceptions of their realities and experiences, by carefully attending to the classroom social interactions (subtle and complex features), this methodology did potentially shade some light into the various sometimes conflicting perceptions about the significance of multiple representations utility. By blending a description of events with their analysis, I was able to potentially portray what is like to be in a mathematics classroom where instruction utilizes multiple representations.

Equally important Cohen (2000) describes case study as a step to action where the researcher's insights may be directly interpreted and put to use. This use includes, educational policy making as well as providing institutional feedback on effective instructional practices. Observing and following the surface features of an instructional practice as well as superficial treatment of good ideas that emerge from these observations does not produce meaningful and effective results (Lewis, Perry, & Murata, 2006). Having an insightful understanding of instructional practices should provide consumers of this knowledge (curriculum designers, teacher educators, and teachers), important and meaningful information that they can potentially adopt in their practices. Purpose of this study is to develop insightful understanding of effective instructional practices that utilize multiple representations. This is in the hope that this study will provide an archive of descriptive material (Cohen, 2000) that could serve as data source for other researchers and users interested in this area of research. The rationale for choosing a single case study is therefore based on the understanding that the study will be attempting to understand and collect insightful data of a typical everyday phenomenon (regular mathematics class where multiple representations are utilized). The lessons learned and the insightful information gathered from this case will inform the readers about the experiences in a typical average high school algebra II mathematics classroom.

Methods.

The following data collection techniques were utilized; classroom observations, task-based interviews, and collection of artifacts. The data collection procedures and instrumentations are discussed in details in the research design section of this chapter.

Research Settings and Participants

The research site was an algebra II class in a high school located in the south east of the country. The high school is a college preparation institution that emphasizes academic excellence and integrity in the study of mathematics. The school follows a rotating schedule with classes meeting 4 days a week for 55 minutes. Effective communication of mathematics using correct mathematical notation in an organized and logical manner is strongly emphasized. Use of multiple representations in the instructional and assessment practices is encouraged in the school.

The mathematics department strongly advocates instructional practices that emphasizes conceptual understanding and students' ability to justify their answers. The Mathematics courses taught in both the regular and advanced classes include; geometry, algebra II, Pre-calculus, Calculus, Advanced Placement AB & BC Calculus, and Advanced Placement Statistics. The participants in the study were a 10th grade advanced algebra II mixed gender class of 21 students. The students had a prior algebra experience in 8th grade when they took advanced algebra I class which was a prerequisite for taking the algebra II class. The mathematics teacher in the class that I observed had 36 years of classroom teaching experience and is a strong advocate of instructional practices that emphasis the use of variety of approaches in solving mathematical problems. His responsibilities included; lesson planning, instructing, and assessing all the students in the class. He is also the Advanced Placement Calculus BC instructor at the school and annually holds calculus Advanced Placement AB and BC summer workshops for teachers from all over the country.

The teacher Mr. Steve (pseudonym) started each lesson with clear instructional objectives and clearly articulated the mathematical ideas or procedures students were expected to learn. The teacher's instructional strategies and the lesson design provided the students with the opportunity for discourse around important algebra II concepts. In addition the instructional strategies promoted students': justification of mathematical ideas or procedures; reflection on the correctness or sensibility of ideas and procedures; and the embrace of wrong answers as worthwhile learning opportunities. Mathematical tasks assigned or used during instruction promoted students': use of variety of methods (algebraic, verbal, numerical, visual, and graphical) to represent and communicate mathematical ideas and procedures; use of multiple solution strategies and representations to support their ideas or procedures used; thinking beyond immediate solution and making connections to other related mathematical concepts; and focus on understanding important mathematical concepts and processes.

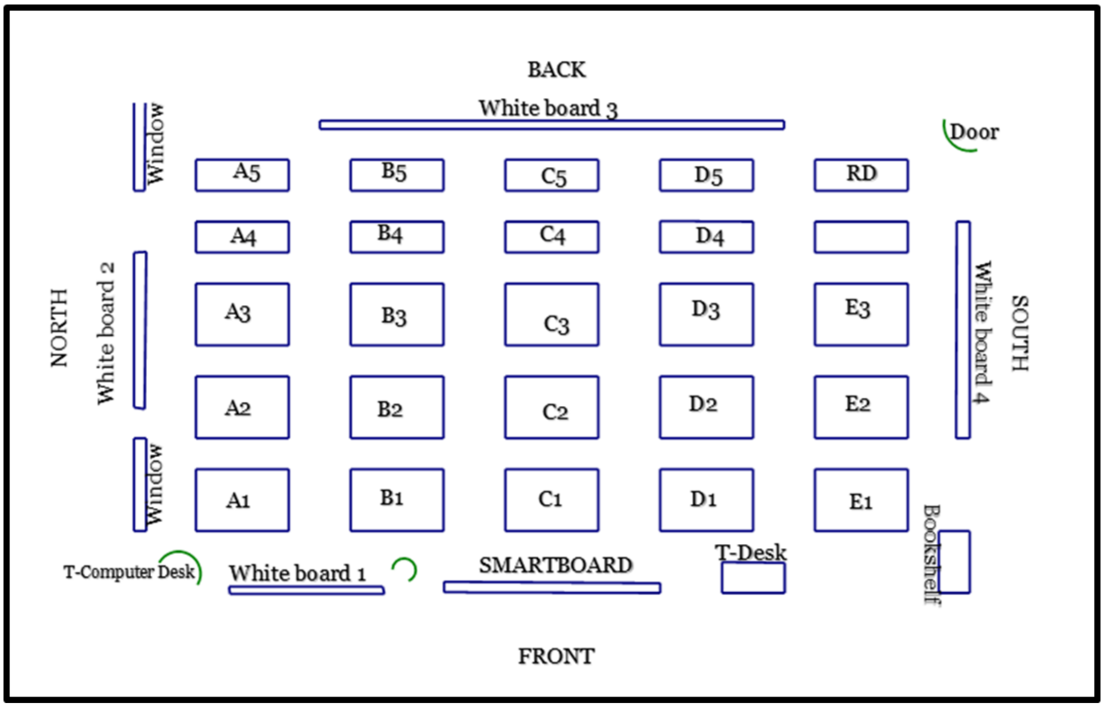


Figure 7. A model of the Research Site Advanced Algebra II classroom. A1 through E3 represents the assigned seats. RD represents the researcher's desk.

Participants.

The participants were all the twenty one students that agreed to participate in the study. Their written and verbal responses to the questions asked during instruction, group work activity and tests contributed to the massive data collected in the study. The study was built around the four students Ron, Tyron, Stacey, and Yolanda (pseudonyms) that volunteered to do the five task-based individual interviews at the end of each lesson unit as part of the review for the unit test. The four students' responses and experiences during the task-based interviews were then analyzed and cross- referenced with the other 17 students' classroom verbal and written responses during classroom observation and in their assessments materials.

Data Collection Procedures and Instrumentations

The first step in gaining access into a school is to become aware of the hierarchy and the rules of the institution (Bogdan & Biklen, 1998). I went through the process of negotiating entrée into the research site. This involved seeking permission from: (a) the Institute Review Board (IRB) of the university, a process that involved submitting to the board a proposal that detailed the procedures involved in the project; and (b) the principal of the school to conduct the research study. The school calendar at the Mountain Side High School (pseudonym) is divided into three terms. Term I from August to November, term II from December to February, and term III from March to May. At the beginning of the school year, and prior to any research activity I visited the advanced algebra II class that was the research site. The mathematics classroom teacher Mr. Steve (pseudonym) gladly welcomed and introduced me to the class. I informed the students the purpose of the research study verbally and passed out the informed assent form and the parental permission form (see appendix A). The following information was verbally communicated to the students and also included in the students assent and parental permission form: (a) a description of the central purpose of the study and data collection procedures; participants rights to voluntarily withdraw from the study at any time; confidentiality protection of the participants; statement about any risk associated with participation in the study. To participate in the study each student was required to assent to participate and the parental permission form had to be signed. Included also in these two documents was a copy of the research protocol from the Institute Review Board of the university. Use of pseudonyms ensured that the students' and the teachers' confidentiality was protected. All twenty one students signed and turned in the student assent form. All the parental permission form forms were also signed and turned after two days.

Data Collection Techniques

Yin (2014) observes that the data collection process within the case studies is complex and recommends the use of certain formal procedures to ensure quality control during the process. Following Yin (2014) recommendations on 'formal procedures to ensure quality control during the data collection process' (p.118), I used the following three principles of data collection to ensure quality control: (a) multiple source of evidence; (b) created a case study database; and (c) maintained a chain of evidence. Multiple sources of evidence ensured triangulation of data sources i.e. creation of converging lines of inquiry (Yin, 2014). Creation of case study database assisted in my organizing and documenting the data collected. I did this by creating two separate sets of collection of: (a) data or evidentiary base i.e., field notes, interview transcripts, observation transcripts and case study protocols; and (b) my own set of reports, memo, reflection, and articles read (Yin, 2014). Maintenance of a chain of evidence was achieved by maintaining a clear focus and set of information from initial research question to ultimate case study conclusions.

Since the emphasis in my study was to understand and interpret learning in a complex social setting e.g. mathematic classroom, multiple data collection techniques was used. With the use of multiple techniques of investigating I was ensured of the improvement of the likelihood of accuracy and objectivity (Dewalt, K & Dewalt, B., 2010). The techniques consistent with qualitative case study methodology that I used were; (a) classroom observation, (b) task-based interviews, and (c) collection of artifacts.

Classroom Observations.

Observation is a research method in which the researcher takes part in the daily activities, interactions, and events of a group of people as one of the means of learning the explicit and tacit aspects of their life routine (Dewalt, K. & Dewalt, B., 2010). The rationale for using observation as a method in this study was to develop a holistic understanding (Dewalt, K. & Dewalt, B., 2010) of the students' interactions with the various multiple representations i.e. verbal, graphical, numerical and algebraic semiotic resources as objectively and as accurate as possible. Holistic understanding include: (a) a unique and contextualized insight (Dewalt, K. & Dewalt, B., 2010) into the classroom events and activities; and (b) capturing the dynamics processes of the classroom interactions as they unfold over a period of time. My main focus was to examine the emerging dynamics that were evoked by the use of multiple representations including capturing nuances of students' thinking processes while engaged in problem solving using these representations (Chahine. I, 2013).

My role in the study was that of a non-participant observer. I observed and recorded the daily classroom activities for a period of one term i.e., twelve weeks. I took field notes of each lessons observed to record teacher actions, student actions, and any board work put up by the teacher or the students. I used an observational protocol as a method for recording the field notes. See figure 8 for a sample of day 6 observation log. Included in the observational protocol was: (a) header to record essential information i.e. date, time, unit/topic, resources, purpose of the study, representations observed, and research questions; (b) space to record descriptive notes on particular activities like smartboard notes, teacher verbal and written discussion on white board, and students' verbal and written discussions on the various white boards around the class and on their note books; and (c) space to record reflective notes that included my comments, reactions and experience in the field.

Appendix B is a summary of the advanced algebra II class instructional sequences. The instructional sequences covered the following four units or focus topics: matrices and their applications; piecewise functions and absolute value functions; quadratic functions and their applications; and introduction to polynomial functions. At the completion of each unit/focus topic, the students took a test. There were a total of four tests and one final exam at the end of the term in November. The purpose of the tests was to provide some information on; their conceptual understanding of the units covered, growth in knowledge, ability to represent their mathematical ideas, and reasoning ability. The final exam was intended to provide information on their conceptual understanding of all the four focus topics covered throughout the term. In the interest of time and the massive amount of data collected, the data in this study only focused on the following units: piecewise functions, absolute-value functions, and quadratics functions and their applications.

Observation Log

Day #: 6	M T W TH F	Date:
Location: S100	Time: 9:30-10:30	Resources: Textbook/ Smart note /worksheets
Unit/Topic: PIECE-WISE FUNCTION Review & Intro to Absolute	Type of MR: Verbal, Graphical, Algebraic.	

Research Questions: How does the use of MR influence students': (1) identification of PC; (2) transition from PC to CU; (3) transfer of their CU to other related concepts?

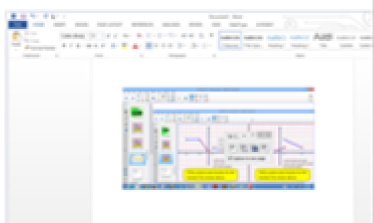
SMARTBOARD NOTES	TEACHER DISCUSSIONS	STUDENT DISCUSSIONS/ACTIVITY	COMMENTS/REFLECTION
Ref to SB notes Week 1 Day 6 What are Piece-wise functions? Families of Functions & Their Corresponding Models See slide # 25-#29 on SB See slide # 25 Displayed with questions below 1. Write a Piecewise function. 2. Find the domain & the range. 3. Write in both "inequality notation" and "interval notation". 4. Describe the	Lesson Objective: To Write the equation of a piecewise function given the graph. State the domain, range and end behavior piecewise. Teacher starts class by asking students to look at the graph of the following piecewise function. 9:40 Slide #26-#27  Write a Piecewise function P(x) shown in slide #26 above. RQ 1&3 Find the domain & the range. Write in both "inequality notation" and "interval notation". Describe the end behavior of this function. What is the significance of the open circle in the graph?	Students engaged in writing P(x) and responding to the questions (3min). Teacher invites 1.1 to demo on the document camera. 1.1 I think P(x) can be written as $P(x) = \begin{cases} 3, & if\ x < -2 \\ -\frac{3}{2}x, & if -2 < x \le 0 \\ 1, & if -1 < x \le 5 \end{cases}$ Domain: $-\infty < x \le 5$: $(-\infty, 5]$ Range: $0 \le y \le 3$: $[0, 3]$ 3.1End behavior: $as\ x \to -\infty, y = 3$ $as\ x \to \infty, y = 3$ 3.2: Open circle means that point is not a solution to the piece indicated. Closed mean that point is a solution to the piece indicated. 3.4: Name of the function P(x), 3 pieces, constant, linear and constant. Independent variable is x. Dependent is P(x)	Transition from Graphical to Algebraic RQ2 identify the process. How did use of MR influence student recognition that task involved several steps? How did not seeing where graph crosses y-intercept influence decision? RQ2 When you looked at the graph what ASAP crossed your mind? (processing) ID PROCESS involved!!!! How important is it for students to explain the terms and expressions they are using.
end behavior of this function. 5. What is the name of the function? 6. How many pieces does this function have? 7. What type of functions are the pieces of this piecewise function? 8. What is the name of the independent	What is the significance of the closed circle in the graph? -- 9:55 Look at slide #27 and respond to the same questions as above. T: what was different with this piece compared to the previous piece? RQ 2 Class: Unanimous response involved more calculation. There three linear functions and we had to find y-intercepts? How many recognized ASAP that it was going to be involving? (5 students 4.1, 3.4, 3.5, 1.1, & 1.5 raised hands). T: Why? 1.1: since line was linear and slanted mean find slope + could not tell y-intercept so had to calculate.	-- Students engaged in writing P(x) and responding to the questions (3min). Teacher invites 5.1 to demo to the class on the document camera. Teacher's choice dependent on how they responded. $P(x) = \begin{cases} -x-2, & if -5 < x < -2 \\ \frac{1}{2}x + \frac{2}{3}, & if -2 < x \le 1 \\ -\frac{1}{2}x + \frac{7}{2}, & if\ 1 < x \le 5 \end{cases}$ Domain: $-5 < x \le 5$: $(-5, -2] \cup (2.1] \cup (1, 5]$ or simply $(1, 5]$	Emphasize on identifying symbols and their meanings. Ask why and how certain symbols Slide # 26&27 RQ 1&2: investigate understanding of the end behavior. Interview Q: explain EB of piecewise function. -- . 1.1 demonstrate PC why? Symbolic meaning initiated by students. How does use of MR influence

Figure 8. Sample Observation Log Day 6. Includes; (a) header to record essential information, (b) space to record descriptive notes on particular activities, and (c) space to record reflective notes.

The classroom observations were supplemented by audiotaped recording of teacher-student and student-student interaction. Students' conversations during small group work were captured using audiotape recorders placed at two different locations in the classrooms. One in front to capture the teacher's communication with the class. The second audio recorder was with me the researcher located at the back of the class to capture students' responses and verbal communications. I later transcribed the audio recordings for further analysis. The field notes that I recorded and the transcriptions were analyzed immediately. I studied and analyzed the data immediately to ensure that: (a) I learned and understood the nuances of the students' verbal and written responses; and (b) I had a better understanding of the directions that the data was taking.

Rationale for Observation.

The rationale for using classroom observation as a data collection techniques was informed by: (a) potential advantage in collecting non-verbal data; (b) discernment of ongoing behaviors and taking appropriate notes of salient features in the study; (c) developing a greater rapport and better access to the students and their activities in an informal and a natural environment (Bailey, 1978). During classroom observations, I was able to explore and confirm some of the ideas that emerged in the other data collection techniques i.e., the task-based interviews and in the collection of artifacts. Data collected during the classroom observations supported the: improvement of the design of the other data collection techniques such as task-based interviews protocol; refinement of the interview questions to ensure they were appropriate and relevant; and discernment of subtleties within participants' responses (Mack, et. al, 2005).

Task-Based Interviews.

Four students volunteered to participate in five 20-minutes, individual task-based interviews (Goldin, 2003). Task-based interview is a form of clinical interview where the interviewee interacts with; the interviewer, and also the task environment that is carefully designed for the purposes of the interview (Goldin, 2003, and Maher & Sigley, 2014). Task-based interview technique was appropriate for this study because of its potential in eliciting estimates of students; existing knowledge, growth in knowledge, and representations of their mathematical ideas (Maher & Sigley, 2014). Informed by Goldin (2003) principles for designing task-based interview I used the following principles to design the task-based interview: (a) designed task-based interviews to addressed advance research questions; (b) chose the tasks that were accessible to the students as part of test reviews; (c) chose tasks that embodied a rich representational structure; (d) developed explicitly described interviews and establish criteria for major contingencies; (e) encouraged free problem solving; maximized interaction with the external learning environment which included graphs on the smartboard and TI calculator available; (f) decided what was to be recorded and recorded it as much as possible; and (g) compromised in term of time and schedules when it was appropriate.

I conducted all the interviews in an assigned classroom on Thursdays during the second period of the day from 9:30am to 10:30 am when the school has scheduled activity periods. In situations where the participants were not available in the time slot above we did reschedule the interview after school between 3:00pm and 4:00pm. To minimize class interruptions and interferences with other after school activities, the participants felt comfortable interviewing at that time because they did not have an assigned class to attend (study hall). The task questions were developed from the test reviews that the students were using to study for the unit tests. The students had the option of responding to the writing interview questions either on the white board or on paper and pencil. There was a graph projected on the top-right end of the white board in case they needed to utilize a graph.

During the interviews I followed the protocol (see Appendix C.) for a sample protocol, but was free to depart from it when necessary to investigate inferences about students' thinking (Clement, 2004; & Goldin, 2003). See appendices (D, E, F, G, and H) for copies of the tasks. Task 1 (see appendix D), Task 2 (see appendix E) and Task 3 (see appendix F) involved piecewise functions, absolute value functions, and applications of absolute value functions respectively. These three tasks this matched with test on unit focus 2. Task 4 (see appendix G) and Task 5 (see appendix H) involved the quadratic functions and their applications respectively. These two tasks matched with test on unit focus 3.

I prompted the students to read each problem aloud and explain their thinking when solving each task. For the purpose of understanding students' thinking and reasoning processes, the task-based interview approach was open-ended. Overall my goal was to understand the participants thinking (Steffe & Thompson, 2000) to the extent possible in the short period of time I had with them. When necessary, I rephrased questions to students when requested, but kept the rephrasing questions in the spirit of using language or ideas that might solicit further mathematical activities from the students (Steffe & Thompson, 2000).

As the participants were engaged in the mathematical activity, I observed their actions and their verbal responses to assigned tasks, audio recorded the conversations and took photos of the work on the board for accuracy in transcription and for more detailed future analysis. The audio recordings, accompanied transcripts, interview field notes, participants' work, the teacher's smart-notes, teacher's test reviews provided the data for the task-based interview analysis. I then coded and analyzed the data for emerging patterns that best described participants' responses to the assigned tasks.

Collection of Artifacts.

Documents and artifacts that I collected included; activity sheets, written tests and exams, written test and exam reviews, written quizzes, teacher's lesson plans, handouts, test and quizzes, end of semester examination, students note book, and photographic images of students' work on the board. Documents and artifact collection was a follow up to the interview process that was conducted. The objective of the artifact collection was to provide potential evidence of themes that emerged from the interviews and the classroom observations. The documents provided information about the students'; existing knowledge, and process of growth in their conceptual understanding of algebra II. Prior (2004) document analysis questionnaire tool was used to provide the structure for the examination of the different documents that I collected. The intention was to understand: (a) the context within which the various documents were assigned; (b) how the documents were created; (c) why the documents were created; (d) the influences and conditions under which the documents were created; and (e) how the documents related to the use of multiple representations. In addition, I used the documents like the digital Smart-Notes lessons from the teacher to identify further analytical categories that occurred. The documents were also utilized in finding suggested questions from the data for the participants that potentially pointed out at discrepancies between the interview data and the document analysis. (Stage & Manning, 2003).

Data Analysis Procedures

Data analysis involves examining, categorizing, tabulating, testing and recombining evidence to produce logically based findings (Yin, 2014). Huberman and Miles (1994) observe that the process of data collection, data analysis, and report writing are not distinct steps in the process but are interrelated and often go on simultaneously in a research project. In analyzing the massive amount of data that was generated, I engaged in the process of moving in analytic circles rather using a fixed linear approach (Creswell, 2013). Informed by this understanding I utilized Creswell (2013) data analysis spiral see figure 9 to organize my analysis process. I went through the analysis process by: (a) collecting the data using the three data collection techniques mentioned; (b) moving into the data management stage; (c) reading, writing, reflecting, and memoing;(d) describing, classifying, interpreting, categorizing and comparing the data; and (d) representing and visualizing the data.

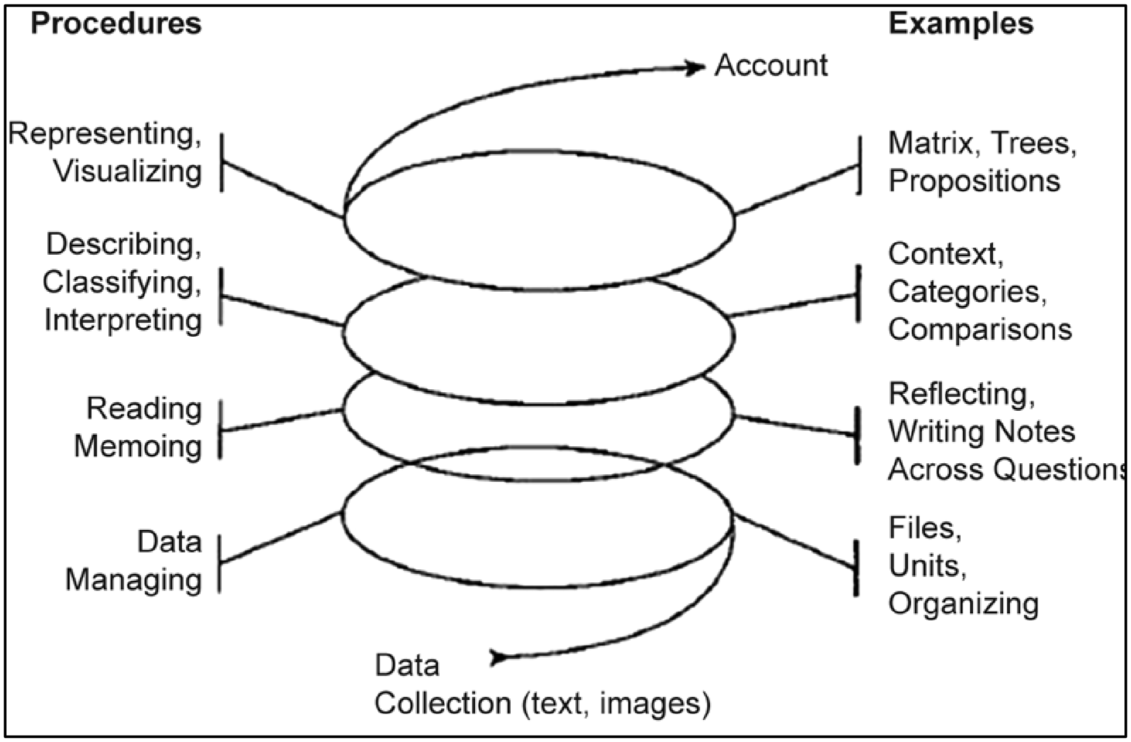


Figure 9. Data Analysis Spiral Model (Creswell, 2013) used to organize the data analysis process in the study.

Data that I collected from the class observation, task-based interview, and examination of documents was stored in a data base and organized into file folders, index cards, and computer files. After organizing the data I continued to analyze the data to get a sense of the whole data base (Creswell, 2013). Ager (1980) advocates that researchers should "…read the transcripts entirely several times. Immerse themselves in the details, trying to get a sense of the interview as a whole before breaking it into parts" (p.103). I scanned through the entire database and identified major organizing ideas. I engaged in repeated review of the audio recordings, and took detailed analytic notes for each participants' (Cobb & Gravemeijer, 2008) responses to task questions. This initial reading of the transcribed scripts and repeated review of the audio recording allowed for familiarity with the data and the identification of students' responses that were clustered in sub-categories Chahine (2013). I wrote memos in the margin of the field notes, interview transcripts, observation transcripts, observation protocols, interviews protocol and students written response documents to assist in the initial process of exploring the database. These memos were short phrases, ideas, and key concepts that occurred to me as I read the documents. To write the memos, I asked probing questions about each participant's data and made theoretical comparisons to Ernest theory of semiotics systems (Ernest, 2006) which provided the theoretical framework for the study.

Methodological Framework

The unit of analysis was students' responses (verbal and written) to the mathematical tasks assigned during; classroom observation, task-based interview, and the artifact collection phase of the study. The analysis was guided by: (a) Ernest Theory of Semiotic Systems (Ernest, 2006); and (b) Schwarz, Dreyfus & Hershkowitz (2009) Recognizing, Building –with and Constructing (RBC+C) Model which is located within Abstraction in Context (AiC) methodological framework. Ernest Semiotic Systems Theory provided a structure for describing and analyzing how students identified: a) the mathematical signs involved in the tasks; b) the rules of transformation of these mathematical signs; and c) the underlying meaning structure of the mathematical signs and the transformation rules. RBC+C model provided a structure for analyzing students' process of constructing abstract mathematical knowledge. In adapting ideas from multiple perspectives i.e., Ernest (2006) theory of Semiotic Systems and Schwarz, Dreyfus & Hershkowitz (2009) RBC+C model, I was attempting to deepen my fundamental understanding of the process of mathematics learning. This view has been advocated for in various studies and by several researchers (Lester, 2005; Kieren, 2000; Stinson, 2004; Lerman, 2006; and Cobb, 2007), where

Lester (2005) describes it as acting as *bricoleurs.* Kieren (2000) observes that the use of multiple theories provides "different lenses through which one can attain a more complete and embodied view of mathematics education" (p. 228).

In particular my objective was to observe the semiotic productions in the students'; written responses, oral response transcriptions, and the observation and interview protocols where I analyzed their semiotic productions. These productions included understanding the influence of multiple representations in the understanding and use of; mathematical signs, transformation of mathematical signs, and the underlying meaning structures. For example, in response to research sub question 1, I selected written and audio transcribed responses to assigned tasks that demonstrated pseudo-conceptual or pseudo-analytic thought process (Vinner, 1997; Moore and Carlson, 2012) and analyzed them using Ernest (2006) semiotic system theory to identify evidence of; random associations of mathematical signs, lack of validation efforts in the sets of transformational rules, and absence of inquiry about underlying meaning structure in the sets mathematical signs in the task. I finally employed a detailed; description of the task, interpretation of the students' responses and a semiotic analysis of the task based responses. The process of describing, interpreting, analyzing, and classifying the data then allowed me to develop initial codes or subcategories that I later used to develop the overarching categories. Guided by my research question and theoretical framework i.e., Ernest Semiotic System Theory (2006), I used categorical aggregations (Stake, 1995) to identify instances from the data that I used to categorize emerging themes. I first identified and selected salient features from the classroom observations and the interviews where various semiotic resources (e.g., written text, spoken words, algebraic symbols, and drawings) were evoked.

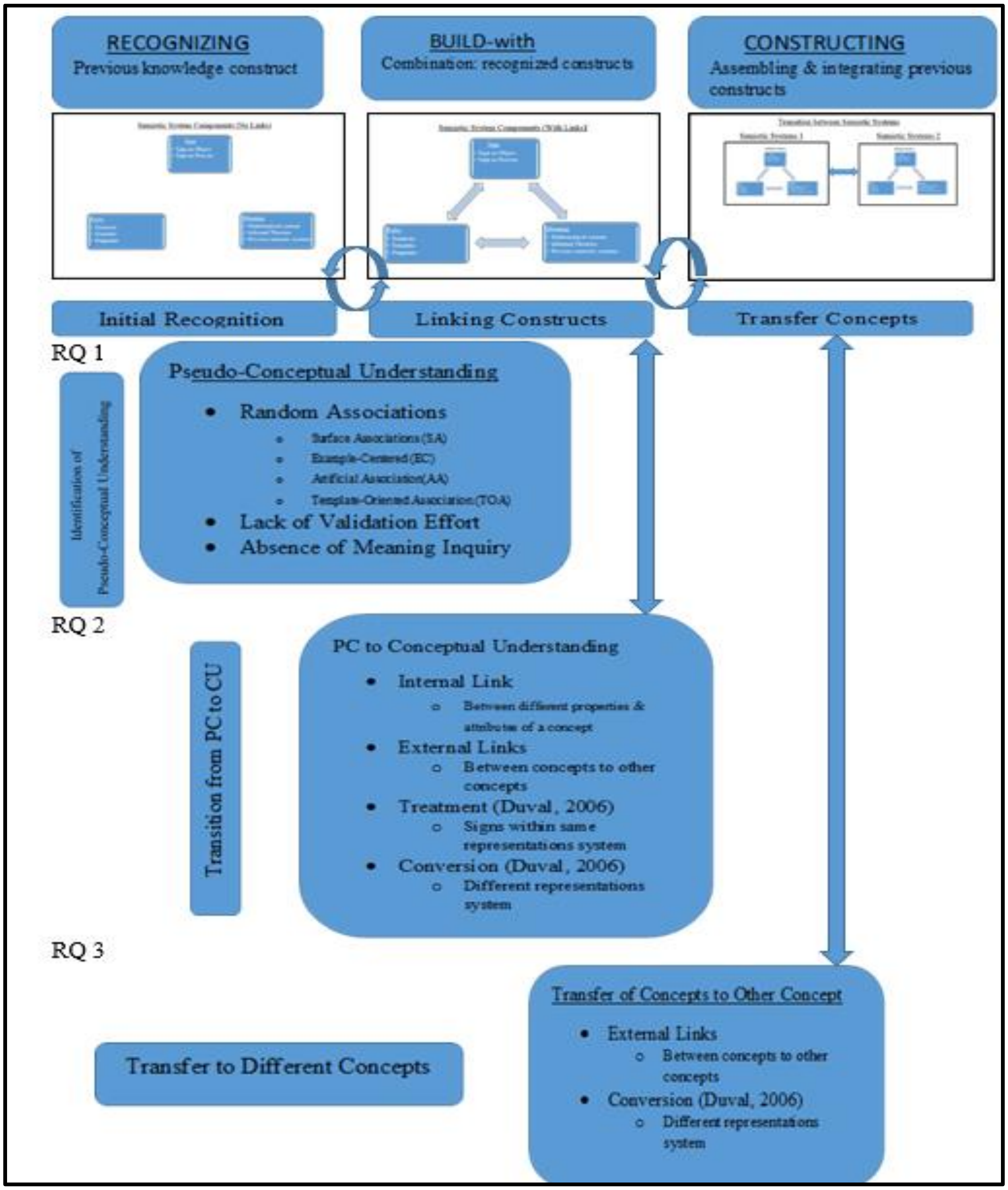


Figure 10.Methodological Framework Model. Using RBC+C Model Schwarz, Dreyfus & Herschkowitz (2009) and Ernest Semiotic Systems Theory (Ernest, 2006) to analyze the data for semiotic activities. Research sub-questions RQ 1, RQ 2, & RQ3 are addressed by analyzing students' responses using Recognizing, Building –with and Construction of knowledge (RBC+C) model and using Ernest (2006) semiotic systems theory to analyze mathematical signs, transformational rules and meaning structures in the signs.

The RBC+C methodological framework looks at the processes of abstraction i.e., emergence of a new construct that can be described and analyzed by means of three observable epistemic actions: (a) recognizing, (b) building –with and (c) constructing. Recognizing refers students realizing that a specific previous knowledge construct is relevant in the situation at hand. This construct could include a sign, a set of sign rules or algorithm, or a set of underlying meanings structure in a mathematical sign. Students tapped into their previous knowledge and attempted to adapt what they know to a task. Building-with represents a combination of recognized constructs, in order to achieve a localized goal such as the actualization of a strategy, a justification or solution to a problem. Students used their knowledge and strategies to; understand a task, and use rules, propositions and formulae to solve a problem. Constructing refers to the assembling and integrating previous construct to produce a new construct. Students use their knowledge to construct personal meanings and an understanding of institutional meaning. Observable actions would involve reorganization, and refinement of their understanding of a concept. Table 2 is a sample of how I analyzed participant Tyron's responses on task 4.

Sample of data analysis (see table 2): In task 4 question (i.), Tyron is comparing two outputs $Q(30)$ & $Q(40)$ of a quadratics function $Q(x) = -3x^2 + 2x + 5.$ He recognizes that the sign $Q(30)$ represents output (or y-coordinate) when the input is 30. Similarly sign $Q(40)$ represents output (or y-coordinate) when the input is 40. With that understanding he reaches out to the graph (strategy building-with). Points to the graph at the intersection of graph of $Q(x)$ and a vertical line drawn from x=30 and x=40. Confirms that y – coordinate of $Q(30)$ is above the y-coordinate of $Q(40)$ and concludes that $Q(30) > Q(40)$. He also realizes that the same information (constructing new construct) could be abstracted from the end behavior of $Q(x)$

i.e. $as\ x \to \infty, y \to -\infty;\ as\ x \to -\infty, y \to -\infty$. As the values of x increases to the right the values of y-decreases. He is able to make the link between the end behavior and the graph without necessarily involving any mathematical processing. He then concludes that " I think it also looks like from end behavior as x moves further away from the vertex the less the value of y".

Table 2. Sample of Tyron's Data Analysis Using RBC+C Model.

Epistemic Action	Description	MR	Sample Task Activity
Recognizing	Students realizing that a specific previous knowledge construct is relevant in the situation at hand. Students tapped into their previous knowledge and attempted to adapt what they know to the task.	VD AL G	Task 4: Tyron Tyron: Comparing $Q(30)$ & $Q(40)$ I think $Q(30) > Q(40)$. I: Why do you think $Q(30) > Q(40)$?
Building-with	A combination of recognized constructs, in order to achieve a localized goal such as the actualization of a strategy, a justification or solution to a problem. Students use their knowledge and strategies to; understand a task, and use rules, propositions and formulae to solve a problem.	VD AL G N	Tyron: I looked at the graph (*points to the graph*) at $x = 30$. $Q(30)$ is approximately located here (points on the graph the intersection of graph of Q(x) and the vertical line roughly drawn from x=30 and x = 40).
Constructing	Assembling and integrating previous construct to produce a new construct. Students use their knowledge to construct personal meanings and an understanding of institutional meaning. Observable actions would involve reorganization, and refinement of their understanding of a concept.	VD AL G	Tyron: I think it also looks like from end behavior $as\ x \to \infty, y \to -\infty;\ as\ x \to \infty, y \to -\infty$ as x moves further away from the vertex the less the value of y.

Key: MR: Multiple Representations observed; VD: Verbal Descriptions; AL: Algebraic Symbols; G: Graphical; N: Numerical
RBC+C Model describe three epistemic observable actions: Recognizing, Building-with, and Constructing drawn from Schwarz, Dreyfus & Hershkowitz (2009) Abstract in Context (AiC) methodology framework on process of constructing abstract mathematical knowledge.

I then used the RBC+C Model to analyze all the four participants' responses in task1 through task 5 and other transcribed written and oral responses from classroom observation and collection of artifacts. Referencing back to the research question and the data, I used this model and Ernest Semiotic Systems Theory (Ernest, 2006) to analyze: (a) identification of pseudo-conceptual understanding; (b) transition from pseudo-conceptual to conceptual understanding; and (c) transferring of conceptual understanding to other concepts.

Research question was: How does exposure to and use of multiple representations influence algebra II students' understanding and transfer of their algebraic concepts? Specifically the following sub-questions were examined:

1. How does exposure to and use of multiple representations influence students' identification of pseudo-conceptual understanding of algebraic concepts?
2. How does exposure to and use of multiple representations influence students' transition from pseudo-conceptual to conceptual understanding?
3. How does exposure to and use of multiple representations influence students' transfer of their conceptual understanding to other related concepts?

To address sub-question 1, I used the following constructs to identify observable pseudo-conceptual responses: (a) Random associations which include; surface associations, example centered associations, artificial associations, and template oriented associations; (b) lack of validation effort; and (c) absence of meaning effort (Vinner, 1997; and Berger 2006). Figure 10 gives a detail descriptions of the three constructs. Guided by the RBC+C model, the semiotic systems theory and the constructs in figure 10, I analyzed students' task responses for any *recognition* of previous knowledge and understanding of: set of signs produced with different actions e.g., speaking, writing, drawing, and handling their TI calculators; set of transformational rules or algorithms, and set of relationships among the signs and their underlying meaning structures (Arzarello, 2006, p. 279).

Table 3. Pseudo-Conceptual Understanding Indicators.

Pseudo-conceptual Understanding Indicators	Symbol	Explanation/Meaning
Random Associations	Surface Association (SA)	Superficial reading of a set of mathematical signs or statements. Involves the participants isolating particular aspect or part of a mathematical expression and associating those signs or words with a new sign (Berger, 2004).
	Example-Centered (EC)	Using an example as a nucleus round which to construct a concept: When appropriating a new mathematical object, the example is the nucleus/ core around which a new concept is built Vygotsky (1986). "Thinking of prototypical example rather than definition" (Sierpinska, 2000, p.222) where one particular example (perhaps the only available example) constitutes the core of the attributes of other mathematical examples or ideas.
	Artificial Association (AA)	Unfamiliar objects artificially associated with a familiar object. New mathematical objects/ expression, part of the object or expression (perhaps a sign or symbol) may remind learners of another epistemologically more accessible and familiar mathematical sign (Berger, 2004). The more familiar sign may then become the nucleus of the new concept, but the connection between the two signs may be artificial or not relevant.
	Template-Oriented Association (TOA)	Transferring the properties of a sign with which they are familiar to a new or unfamiliar sign which has the same template (Berger, 2006). Sfard (2000) argues that this usage of mathematical sign which is signifier-oriented is an indication of pseudo-conceptual understanding with the students focusing on the templates of the two functions instead of the meanings structures of the functions.
Validation Effort	Validation Effort (VE)	Lack of control mechanism where students reacts to their first associations without checking their thoughts about the solution to a problem (Vinner, 1997).
Meaning Inquiry	Meaning Inquiry (MI)	Absence of examining their responses to determine if their answers fit the context of a problem (Vinner, 1997).

To address sub-question 2, the RBC+C and the semiotic theory model, guided my analysis of identifying instances of transitions from pseudo-conceptual to conceptual understanding. I looked for evidence where students *build-with* by combining different elements of a sign to make meaning of the entire sign. This included students'; oral and written responses, explanations, drawings and inscriptions that reflected participants attempt to use and provide a logical internal links between different entities of a sign. For example in the following absolute-value functions $y = a|x - H| + K$, a statement like “in an absolute value function when the parameter ‘a’ and parameter ‘k’ have the same sign it means there is no x-intercept” was categorized as a potential indication of a conceptual understanding of the absolute value function.

To address sub-question 3, the RBC+C and the semiotic theory model, guided my analysis of how multiple representations influenced students’ transfer of conceptual understandings to other related concepts. In particular I analyzed: (a) students’ responses and logical explanations of external links related to a concept; and (b) students’ logical explanations of how those links related to and connected to other concepts. Example of an external logical link was in task 4. Certain mathematical entities and properties like the domain, range and end behavior are not explicit in the function $Q(x) = -3x^2 + 2x + 5$ but can be inferred from looking at the leading coefficient or graphing the function $Q(x)$. In task 5, the students were expected to tap on to their knowledge of quadratic functions (like in task 4) to respond to questions in this task. Making the consistent and logical link/connection between the previously learned mathematical entities like the domain, range and end-behavior and using it to respond to questions in task 5, I categorized them as *constructing* new construct and making a connection between two semiotic systems. Informed by Sfard (2000) notion that conceptual understanding is demonstrated when students attend to a mathematical object in its entirety (i.e., including its external link) and not just as a fragmented aspect of an object, I also categorized students responses that demonstrated logical understanding of the entire mathematical object as constructing new concept.

Limitation of the Study Design

This case study methodology was limited with respect to generalizability. Since the participants in the study represented a bounded system (Merriam, 2009; and Yin, 2003), this represented a unique example of students in real classroom situation. Bounded system refers to the temporal (in this case a twelve weeks semester as indicated in the time line), geographical, institutional (south east of the country in a middle to high school institution) and confined to a teacher and his algebra II class. The data collection, analysis and interpretation in this study apply only to individual/s with these similar characteristics rather than the general population. Equally important was the high likelihood of my subjectivity and biases affecting and limiting the generalizability of this method. These subjectivity and bias which included, my ideas, experiences (in a different educational system) and insights as an algebra teacher might have interfered with the data collection process, interpretations and analysis of the findings.

Trustworthiness of the Data

I believe that as a qualitative researcher I have the obligation of convincing my readers and myself that the findings in my study are genuine and dependable (Merriam, 2009). Within the constructivist paradigm of inquiry, the criteria for assessing the noteworthiness of an inquiry is its trustworthiness (Denzin & Lincoln 2005). In this study I adopted Schwandt, Lincoln and Guba (2007) criteria of trustworthiness that included; credibility, transferability, dependability, and confirmability.

Credibility which refers to the correspondence of the perspective of the participants with the description of their perspectives by the researcher was ensured using the following techniques; extended field work, persistent observation, reflexivity, member check, and data triangulation. Extended field work involved my prolonged engagement in the site where I spent significant amount of time (four days a week for twelve weeks) in the research site to overcome the effect of misinformation, and build the necessary trust to uncover local constructions. I was also persistent in my daily observations which involved regular reflection and in-depth study of the research problem to gain a detailed understanding of how the most relevant characteristic features of the situation for the problem under investigation were to be identified. Reflexivity involved my monitoring the biases that I brought to the study. This was accomplished by continually reflecting on how my personal experiences as a mathematics teacher had the potential of impacting my interpretation of the data, continuously discussing my interpretation of the data with the classroom teacher and the participants and use of memos to record those interpretations. Member check involved my inclusion of the research participants' input in the interpretation and reporting of the study. In using this technique, the participants' voice was heard in the findings, interpretation and conclusions of the study where I had to ask if my interpretation of their responses matched with what they had intended to respond. I used data triangulation which involves the use of multiple data sources to strengthen the claims and interpretations that I made in the study. These multiple data sources included; field notes from observation, observation protocol transcripts, audio tapes of the task-based interviews, transcripts from the audio recording of interview, interview protocol and analysis of documents (tests, written test, lesson plans, and end of term exams).

Transferability involves the extent to which the findings in my study can be applied in other contexts or with other respondents. This was assured by facilitating a "thick description" (Geertz, 1973). Thick descriptions included information that was; dense and rich in detail, and an interpretive description of the participants intentions. Dependability which refers to the stability of my research findings over time was ensured by keeping a detailed and comprehensive documentation of the research process and every methodological decision made (Bogdan and Bilken, 1998).

The other transferability criteria included confirmability. Confirmability refers to researcher's bias and prejudice in the data collection, interpretation and analysis of findings. This was ensured by my demonstration of reflexivity and openly discussing my epistemological and personal involvement in the study.

5 RESULTS

Overview

In this chapter I provide an analysis and the results of the four students' responses and thinking on the five different tasks during the task-based interview. These tasks were part of the tests review materials that they used to prepare for the tests. Tasks 1, 2, and 3 were used to review Unit I on Piecewise Functions and Absolute Value Functions including the application of absolute value function. Tasks 4, and 5 were used to review Unit II on Quadratic Functions including the application of quadratic functions. See appendix B for a sample of the lesson sequencing. The unit of analysis was students' responses (verbal and written) to the mathematical tasks assigned. The analysis was built around the four students' responses on the five task. I then triangulated the data from the interviews, classroom observations and the documents collected in interpreting the students' responses. To ensure accuracy in transcription each interview session was audio-taped. The data presented in this section elucidates the common findings from analyzing; the audio-taped transcripts, students' written data, students' documents, transcripts of the classroom observations specific to each units, teachers' notes, and written tests and test reviews of the four students. For each task I offer a discussion of the findings from analyzing the participants written, uttered and drawn material. Although I focus on the students' semiotic productions in each task, I contextualize the analysis by: (a) briefly describing the task; and (b) interpreting students' responses using the RBC+C model. I then utilize the semiotic system theory (Ernest, 2006) to analyze semiotic productions and identify the emerging themes in the analysis. I frame my analysis in terms of the way the use of multiple representations influences the identification and use of the three semiotic components of a mathematical sign. I selected excerpts of the original transcripts so as to illustrate any argument I make. The analysis that follow is an attempt to offer some insights into the students' understanding of the task problem and how that understanding influenced their solutions.

Task 1: Domain and Range in Piecewise Functions

Description: Task 1 was in two parts. In Part I, prior to interaction with multiple representations students were asked to use their understanding to describe the domain and the range of four functions: (a) constant function $G(x)$; (b) linear function $F(x)$; (c) a piecewise function $R(x)$ involving only constant functions; and (d) piecewise function $P(x)$ involving both the constant functions and linear functions see figure 12.

Task 1

I.

Describe the Domain and the Range of the following functions. Write the Domain and the Range in both, inequality notation and interval notation.

a. Constant function of the form $G(x) = c, where\ c \in \mathbb{R}$

b. Linear function of the form $F(x) = mx + b, where\ m\&\ b \in \mathbb{R}$

c. Piecewise function

$$R(x) = \begin{cases} 1, & if x < 0 \\ 2, if\ 0 < x \leq 2 \\ -3, if\ 2 < x < 5 \\ 0, & if x \geq 5 \end{cases}$$

d. Piecewise function

$$P(x) = \begin{cases} 0, & if\ x = 0 \\ 3x - 1, & if\ x > 0 \\ -x, if\ -4 < x < 0 \\ -4, & if\ x \leq -4 \end{cases}$$

Figure 11. Task 1 Part I. Domain & Range Questions prior to use of Multiple Representations.

Following is a table (see table 4) of the summary of the participants' part I responses to the domain and the range of each of the four functions named above. For the constant function $G(x) = 2$, all four students correctly identified the domain written in both inequality and interval notation as $\mathbb{R}\ and\ (-\infty, \infty)$ respectively. However, they all were quick to incorrectly respond to the range of the same constant function as $\mathbb{R}$ in inequality notation and $(-\infty, \infty)$ in interval notation. No efforts towards reflecting on the responses was evident in all the four students.

Table 4 . Ron's, Tyron's, Stacey's & Yolanda's Responses Task I part I questions.

Function	Ron Domain	Ron Range	Tyron Domain	Tyron Range	Stacey Domain	Stacey Range	Yolanda Domain	Yolanda Range
$G(x)$	$\mathbb{R}\ (-\infty, \infty)$	$\mathbb{R}\ (-\infty, \infty)$	$\mathbb{R}\ (-\infty, \infty)$	$\mathbb{R}\ (-\infty, \infty)$	$\mathbb{R}\ (-\infty, \infty)$	$\mathbb{R}\ (-\infty, \infty)$	$\mathbb{R}\ (-\infty, \infty)$	$\mathbb{R}\ (-\infty, \infty)$
$F(x)$	$\mathbb{R}\ (-\infty, \infty)$	$\mathbb{R}\ (-\infty, \infty)$	$\mathbb{R}\ (-\infty, \infty)$	$\mathbb{R}\ (-\infty, \infty)$	$\mathbb{R}\ (-\infty, \infty)$	$\mathbb{R}\ (-\infty, \infty)$	$\mathbb{R}\ (-\infty, \infty)$	$\mathbb{R}\ (-\infty, \infty)$
$R(x)$	a	$\mathbb{R}\ (-\infty, \infty)$	a	$\mathbb{R}\ (-\infty, \infty)$	a	$\mathbb{R}\ (-\infty, \infty)$	a	$\mathbb{R}\ (-\infty, \infty)$
$P(x)$	b	$\mathbb{R}\ (-\infty, \infty)$	b	$\mathbb{R}\ (-\infty, \infty)$	b	$\mathbb{R}\ (-\infty, \infty)$	b	$\mathbb{R}\ (-\infty, \infty)$

Key: a = (-∞, 0) ∪ (0, 2] ∪ (2, 5) ∪ [5,∞) b : (-∞, -4) ∪ (-4, 0) ∪ [0, ∞)

For the linear function $F(x) = mx + b$, all the four students had different linear functions: Ron used $F(x) = -2x + 4$; Tyron used $F(x) = -\frac{1}{2}x + 3$; Stacey used $F(x) = -3x + 2$; and Yolanda used $F(x) = 2x + 5$. They all correctly identified the domain of their respective function written in both inequality and interval notation as $\mathbb{R}\ and\ (-\infty, \infty)$. Similarly the range of their respective linear functions were correctly identified as $\mathbb{R}\ and\ (-\infty, \infty)$ respectively.

For the piecewise function $R(x)$, all the four students chose to first graph the function in order to support their understanding of the domain and the range of the function.

The four students had similar responses to the domain and identified the domain of the linear function $R(x)$ in interval notation as (-∞, 0) ∪ (0, 2] ∪ (2, 5) ∪ [5, ∞). Similarly they all incorrectly described the range of $R(x)$ in both inequality and interval notation as $\mathbb{R}\ and\ (-\infty, \infty)$ respectively. No efforts towards reflecting on their responses were evident.

Following is an excerpt of Ron's responses to questions on functions $R(x)$ & $P(x)$, discussion and explanation of his rationale.

Table 5. Ron's Response top Task I part I on function R(x) and P(x)

Ron:	To find the domain and the range of this function, I will have to…. graph it (*moves to the smartboard where a graph is projected and graphs function R(x)*).
Ron:	The domain will then be um…. (-∞, 0) ∪ (0, 2] ∪ (2, 5) ∪ [5, ∞) and the range will be $\mathbb{R}$ *and* $(-\infty, \infty)$
Int:	Is there another way you can express the domain?
Ron:	Probably also write it as $\mathbb{R}$ *and* $(-\infty, \infty)$
Int:	Do the two answers for the domain match?
Ron:	Yes I think so…because the domain starts from $-\infty$ *to* ∞.
Int:	What about the function $P(x)$?
Ron:	To find the domain and the also the range I think I will also have to um…. graph the function. It makes it easier for me to see the domain and also the range (*moves to the graph projected on the smart board and graphs P(x). He accurately graphs the function identifying all open and closed circles*).
Ron:	I think the answer is I mean the domain is… (-∞, -4) ∪ (-4, 0) ∪ [0, ∞) or in the other notation $\mathbb{R}$ *and* $(-\infty, \infty)$.

All four students incorrectly identified the range of $R(x)$ in interval notation as $\mathbb{R}\ (-\infty, \infty)$. They tapped their former Algebra I formal knowledge about the range of a linear function $\mathbb{R}\ (-\infty, \infty)$ to respond to the range of the; constant function $G(x) = 2$, and piecewise function $R(x)$. In algebra I following their linear function understanding of the notion of domain and range, they alluded to the understanding that if the domain was $\mathbb{R}\ (-\infty, \infty)$, then the range was inevitably $\mathbb{R}\ (-\infty, \infty)$. Tyron's explanation summarizes all the four students' explanation of the notion of range in a piecewise function involving constant function "if the domain representing the x-coordinate is from $-\infty\ to\ \infty$ then the range must also be from $-\infty\ to\ \infty$". This explanation was consistent with several verbal and written responses I logged in during week 5 class observation.

Interpretation: The pseudo conceptual understanding indicator evident in the students' response to the range of function $G(x)\ and\ R(x)$ is example-centered association. Function $G(x)$ was a constant function and $R(x)$ was a piecewise function involving only constant functions. The expected correct response for the range of $G(x) = 2$ was $\{2\}$, and the expected correct response to the range of the piecewise function $R(x)$ was$\{-3, 0, 1, 2\}$. The four students' responses to the range $G(x)\ and\ R(x)$ was $\mathbb{R}\ (-\infty, \infty)$ which was incorrect. Similar responses to the range in a constant function that I recorded during class observation in week 3 indicated that $\mathbb{R}\ (-\infty, \infty)$ was the common understanding among most students in the class. In an example-centered pseudo conceptual understanding, students use examples of a given notion (e.g. the range of the linear function) as the nucleus or the core around which they build a new notion (e.g. the range of a constant function). In this part of task I, students used the notion of the range of a linear function i.e. $\mathbb{R}\ (-\infty, \infty)$ which they were familiar with, as a prototype to describe the range of constant functions $G(x)\ and\ R(x)$ which was unfamiliar to them. The four participants in the interview and 10 out of the 17 students in the class made the wrong assumption and used the range of the linear function as a prototype in which to describe the range in G(x) and R(x). This was evidence that the class did not systematically reflect on the definitions of the range in a constant function.

In part I (d), the students were required to describe the domain and the range of a piecewise function $P(x)$ that involves both the linear and constant functions (see table 4). To find the domain and the range all four students confidently graphed the piecewise function P(x). Their responses to the domain as shown in table 3 was (-∞, 0) ∪ (0, 2] ∪ (2, 5) ∪ [5, ∞) in interval notation and $\mathbb{R}$ $(-\infty, \infty)$. Their responses to the range of P(x) was $\mathbb{R}$ $(-\infty, \infty)$ which was incorrect. This response was common in all the four students' work including the majority of the students in the class as indicated in their verbal and written responses to the range of a piecewise function.

Interpretation: As in the previous functions $G(x)$ *and* $R(x)$ all the four students described the range of the function $P(x)$ as $\mathbb{R}$ $(-\infty, \infty)$. Similar pattern was observed in the verbal and oral responses of the other students during class observation. Their responses were incorrect. Two pseudo-conceptual behavior in response to the range of $P(x)$ were evident. These were consistent with example-centered association indicator, and lack of validation effort on the correctness of their responses. The rationale provided by Yolanda "since the domain means the values of x so I think the x-values will start from um...$-\infty$ *to* ∞. If the x's are from $-\infty$ *to* ∞ the y's meaning the range will also be from $-\infty$ *to* ∞" sums up a consistent pattern of understanding of this notion in the class. The class used the definition of the range of a linear function as the prototype example of the range of all the other functions they came across. As in the previous responses to the ranges functions $G(x)$ *and* $R(x)$, the four students and majority of the students in the class still tapped on to their formal knowledge of range in a linear function acquired in algebra I to describe the range of P(x). It is evident that they did not systematically think through the definition of the piecewise function $P(x)$ as a function whose definition changes as the independent variable changes. My interpretation of students' response in part I according to RBC+C model is that all the four students easily recognized the domain of all the four functions. They equally recognized the task requirement for the range. However general pattern in students' responses to the range of the four functions $G(x), R(x)$ *and* $P(x)$, were incorrect. Analysis in Part

II of the task continues my interpretation of the students' responses after interaction with multiple representation.

In part II of Task I, the four students were required to refer to the Smart-Notes "Domain Truck and Range Car" graphical animation (see figure 15.) class room activity that they had previously discussed briefly in class. The questions in Part II of the task were administered separately after the students completed part I. This was part of the classroom observation that I had noted and I had an electronics copy of the activity from the teacher as part of my data collection artifacts. This being an electronic activity, it was projected on the Smart-board in front of the class during the interview. In the animation there was a picture of a truck and a car. The truck was referred to as the "Domain Truck" and the car was referred to as the "Range Car". They were supposed to assume that the graph of the function they drew represented a "road" and they were driving the domain car from left to right. In the car was a passenger with the responsibility of collecting all the x-coordinates along the way (The Domain). Similarly in the "Range Car" was a passenger responsible for collecting all the y-coordinates (The Range) along the way from left to right. At the end of the "trip" they had to do an inventory of what they had in the "Domain Truck" and in the "Range Car". This dynamic animation representation offered visualization of multiple representations where students drew inferences from pictures and animations (Ainsworth & Van Labake, 2004). Students were then supposed to respond to the same questions they had in part I of this task (see figure 12).

II.

Describe the Domain and the Range of the following functions in the light of the Smart Notes "Domain truck" and the "Range Car" discussion. Write the Domain and the Range in both, inequality notation and interval notation.

a. Constant function of the form $G(x) = c, where\ c \in \mathbb{R}$
b. Linear function of the form $F(x) = mx + b, where\ m\&\ b \in \mathbb{R}$
c. Piecewise function

$$R(x) = \begin{cases} 1, & if x < 0 \\ 2, if\ 0 < x \leq 2 \\ -3, if\ 2 < x < 5 \\ 0, & if x \geq 5 \end{cases}$$

d. Piecewise function

$$R(x) = \begin{cases} 0, & if\ x = 0 \\ 3x - 1, & if\ x > 0 \\ -x, if - 4 < x < 0 \\ -4, & if\ x \leq -4 \end{cases}$$

Figure 12. Task I part II questions after exposure to Domain and Range Dynamic Multiple Representations.

II

Refer to the Smart Notes "Domain truck & Range car" graphical animations discussed in class. Answer the following questions.

a). Multiple representations of a Constant function of the form $G(x) = c, where\ c \in \mathbb{R}$ using graphical animation.

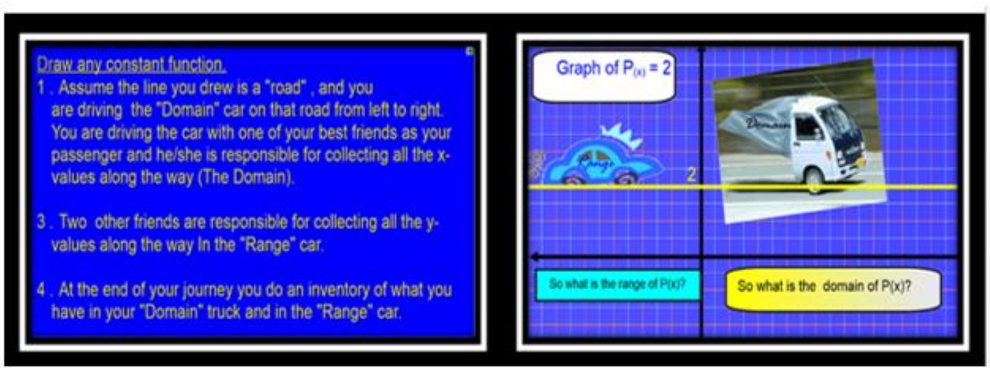


Figure 13. Smart Notes Graphical Animation of Domain & Range of a Constant Function

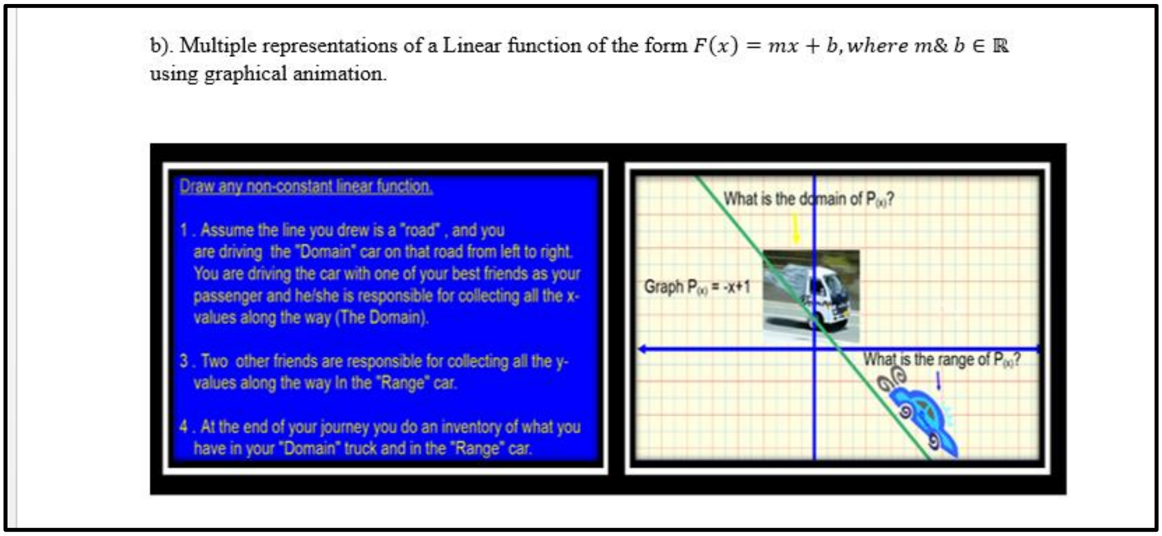


Figure 14. Smart Note Graphical Animation of a Linear Function.

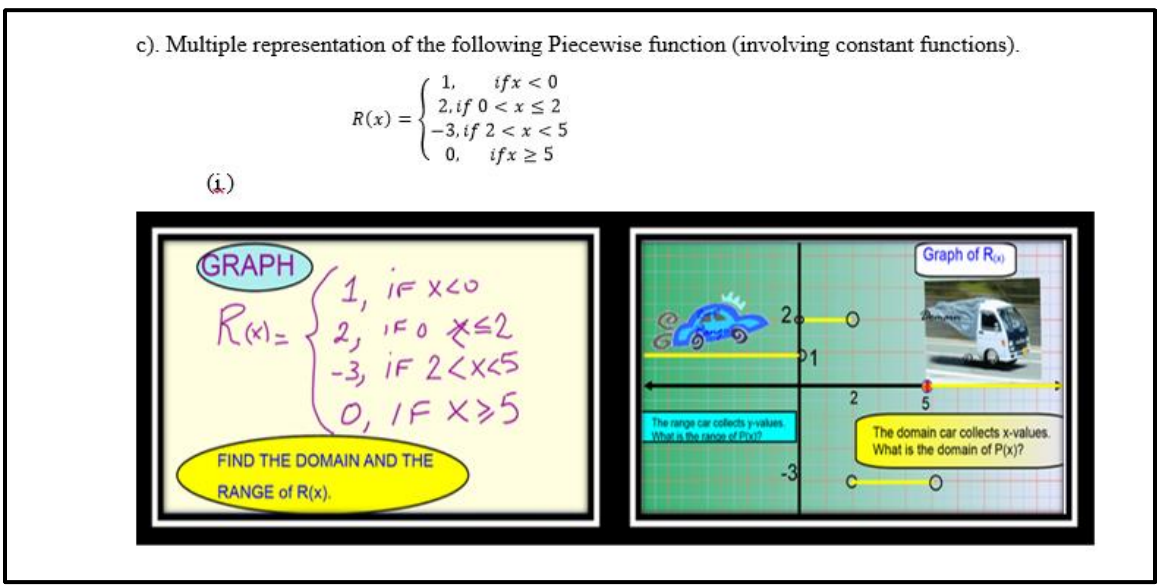


Figure 15. Smart Notes Graphical Animation of a Piecewise Function involving only Constant Functions

Description

The activity in part II of task I represents the students' interaction with multiple representations involving graphical, algebraic symbols, and verbal representations (uttered and written words). Graphical animations supports the development of mental representations skills which

Chahine (2013) describes as “skills developed and appropriated by students as they engage in learning amidst a range of modalities and resources” (p. 445).

Table 6 is a summary of the four students’ responses to the domain and the range of the four functions: (a) constant function $G(x)$; (b) linear function $F(x)$; (c) piecewise function $R(x)$ involving only constant functions; and (d) piecewise function $P(x)$ involving both linear and constant functions. Included in the summary is: the multiple representations used or referred to in the task; the pseudo-conceptual understanding indicators identified in the task; and the semiotic systems components used to analyze the participants responses to part II of task 1

During the Smart-Notes “Domain Truck and Range Car” discussion, the four students had an opportunity to systematically reflect and re-image their understanding of the notion of domain and the range of the four functions but more so in the piecewise functions like $R(x)$ and $P(x)$ which was something new to them at that point. The “Domain Truck” represented an inventory of all the x coordinates from left to right. The “Range Car” represented an inventory of all the y-coordinates of the functions from left to right. The four students’ responses to the domain of all the four functions were $\mathbb{R}$ $(-\infty, \infty)$. For the constant function $G(x) = 2$ all the four students’ response to the range was $\{2\}$. This is an insight into how they potentially re-imaged their understanding. This was a significant shift from their previous responses of the range of $G(x)$ being $\mathbb{R}$ $(-\infty, \infty)$. In their explanation to this response, Tyron explained that “Range is 2 because the word is constant and the range car can only collect the y-coordinate 2”. Yolanda echoed same explanation “range car is only collecting the numbers 2 so the range is $\{2\}$”. Stacey also observed the same “range car only collects the y-coordinate $\{2\}$ and now I see the difference between this and my previous answer”.

Table 6. Summary of Task I Part II Responses.

Participant	Question/ Function	Response: Domain	Response: Range	Multiple Representations Referred	Pseudo-Conceptual Understanding Indicator/s	Semiotic System Component/s
Ron	$G(x)$	$\mathbb{R}\ (-\infty,\infty)$	$\{2\}$	VD, AL, G, N Dynamic representations	Surface Association	S (Process) M(informal theory)
	$F(x)$	$\mathbb{R}\ (-\infty,\infty)$	$\mathbb{R}\ (-\infty,\infty)$	VD, AL, G, N Dynamic representations	----------------	S (object)
	$R(x)$	a	$\{-3,0,1,2\}$	VD, AL, G, N Dynamic representations	Example-centered Association	M(mathematical content) M(previous semiotic system)
	$P(x)$	b	$[-4,\infty)$	VD, AL, G, N Dynamic representations	Example-centered Association	M(mathematical content) M(previous semiotic system)
Tyron	$G(x)$	$\mathbb{R}\ (-\infty,\infty)$	$\{2\}$	VD, AL, G, N Dynamic representations	Surface Association	M(mathematical content)
	$F(x)$	$\mathbb{R}\ (-\infty,\infty)$	$\mathbb{R}\ (-\infty,\infty)$	VD, AL, G, N Dynamic representations	Surface Association	S (Process) M(informal theory)
	$R(x)$	a	$\{-3,0,1,2\}$	VD, AL, G, N Dynamic representations	Example-centered Association	M(mathematical content) M(informal theory)
	$P(x)$	b	$[-4,\infty)$	VD, AL, G, N Dynamic representations	Example-centered Association	M(mathematical content) M(previous semiotic system)
Stacey	$G(x)$	$\mathbb{R}\ (-\infty,\infty)$	$\{2\}$	VD, AL, G, N Dynamic representations	Surface Association	S (Process) M(informal theory)
	$F(x)$	$\mathbb{R}\ (-\infty,\infty)$	$\mathbb{R}\ (-\infty,\infty)$	VD, AL, G, N Dynamic representations	----------------	S (Process) M(informal theory)
	$R(x)$	a	$\{-3,0,1,2\}$	VD, AL, G, N Dynamic representations	Example-centered Association	M(mathematical content) M(informal theory)
	$P(x)$	c	$[-4,\infty)$	VD, AL, G, N Dynamic representations	Example-centered Association	M(mathematical content) M(informal theory) M(previous semiotic system)
Yolanda	$G(x)$	$\mathbb{R}\ (-\infty,\infty)$	$\{2\}$	VD, AL, G, N Dynamic representations	Surface Association	S (Process) M(informal theory)
	$F(x)$	$\mathbb{R}\ (-\infty,\infty)$	$\mathbb{R}\ (-\infty,\infty)$	VD, AL, G, N Dynamic representations	--------------	S (Process) M(informal theory)
	$R(x)$	a	$\{-3,0,1,2\}$	VD, AL, G, N Dynamic representations	Example-centered Association	M(mathematical content) M(informal theory) M(previous semiotic system)
	$P(x)$	c	$[-4,\infty)$	VD, AL, G, N Dynamic representations	Example-centered Association	M(mathematical content) M(informal theory) M(previous semiotic system)

Key: a: $(-\infty, 0) \cup (0, 2] \cup (2, 5) \cup [5, \infty)$ b: $(-\infty, -4) \cup (-4, 0) \cup [0, \infty)$ c: $\{-4\} \cup [-1, \infty)$

All four students had similar responses to the domain and range of the linear function $F(x)$. Domain was $\mathbb{R}\ (-\infty, \infty)$ and the range $\mathbb{R}\ (-\infty, \infty)$. Their responses to the domain and the range of $F(x)$ did not deviate from their previous response pre-multiple representation activity in part I. For the piecewise function $R(x)$ that only involved linear functions, their domain responses was the same $\mathbb{R}\ (-\infty, \infty)$. Ron's response summed up their similar responses "Doman Truck will collect all real numbers $\mathbb{R}\ from\ -\infty\ to\ \infty$, but can also be written as (-∞, 0) ∪ (0, 2] ∪ (2, 5) ∪ [5, ∞)". All their responses for the range of $R(x)$ were $\{$-3, 0, 1, 2$\}$. Their rationale as to why their answers to the range was different from before the "Domain Truck and Range Car" activity, pointed out to the students taking their time to carefully look at the graph of the function, being reflective in understanding the graph, and trying to relate specifically to what the "Range Car" inventory had or meant. They all expressed care and concern while responding to function $R(x)$. They explained that the range of the function $R(x)$ was 'tricky' and they had to carefully analyze the motion of the "Range Car". For the function $P(x)$ involving both linear and constant functions, the following represents their responses to the domain and the range of the function P(x). This pattern of response was consistent with the recorded and analyzed class verbal and written responses during the class observation.

Table 7. Ron's, Tyron's, Stacey's, & Yolanda's response to function P(x) question.

Function	Ron		Tyron		Stacey		Yolanda	
	Domain	Range	Domain	Range	Domain	Range	Domain	Range
$P(x)$	$\mathbb{R}\ (-\infty, \infty)$ or b	$[-4, \infty)$	$\mathbb{R}\ (-\infty, \infty)$ or b	$[-4, \infty)$	$\mathbb{R}\ (-\infty, \infty)$ or b	c	$\mathbb{R}\ (-\infty, \infty)$ or b	c

Key: b: (-∞, -4) ∪ (-4, 0) ∪ [0, ∞) c: {-4} ∪ [-1, ∞)

Interpretation: I employed the RBC+C model to analyze the four students' responses to Task I. Exposure to and use of multiple representations in part II of the task enabled the four students to *recognize* the domain and the range in all the four functions. They realized that a previous knowledge construct i.e., range and domain was relevant in responding to part I & II of the task. Ron and Tyron described the domain of the function $P(x)$ as $\mathbb{R}$ $(-\infty, \infty)$ and even wrote it in interval notation as (-∞, -4) ∪ (-4, 0) ∪ [0, ∞). Ron attributed better understanding of the domain to a reflective and thorough look at the "Domain Truck" inventory. *Building –with* involves students attempt to justify their understanding of the task. Pointing and inscribing on the graph was means of communicating his understanding of the domain and range of each piece of the piecewise function. For example starting from the left to the right and inscribing on the graph he explains the following "the domain will be from $-\infty$ $to - 4$ (*pointing at the graph of the first piece and moving his figure horizontally from left to right*), the second piece domain will be from here to here -4 to 0 (*pointing at the graph of the second piece from (-4, 4) to (0, 0)*). For the third piece it will be it only be 0 because this is a point. Um… for the forth piece domain will be from-1 to ∞" Ron's range was$[-4, \infty)$ or $(y \geq -4)$. This was incorrect. The expected correct response to the range of $P(x)$ was {-4} ∪ [-1, ∞). Ron did not notice the gap between $(0, -4) and$ $(0, -1)$ on the y-axis. Similarly Tyron described domain of $P(x)$ as $\mathbb{R}$ $(-\infty, \infty)$ and range as $(y \geq -4)$. As much as the "Domain Truck and Range Car" activity made him have a deeper reexaminations of the inventory in the range, he still missed out an essential small portion of the range where there was a gap between$(0, -4) and$ $(0, -1)$ on the y-axis. However a thorough look at the "domain truck and range car" inventory gave them a better understanding and response to the range of $P(x)$. Use of multiple representations allowed Ron and Tyron to reorganize, refine and *construct* a better understanding of the range construct in a piecewise function.

I did infer from his responses that the use of and exposure to multiple representations; afforded Ron and the other participants an opportunity to re-image their understanding, self-reflect on their response, and evoked some internal configurations (Chahine, 2013) that students used to provide a meaningful response.

Stacey and Yolanda had similar responses to the domain of $P(x)$ as ℝ $(-\infty, \infty)$. However for the range their responses was {-4} ∪ [-1, ∞), “observe that at $x = 0$ umm… I will skip from -4 to -1 (*pointing at the gap between (0, -4) and (0, -1) on the y-axis*)”. Stacey and Yolanda also did reflect, re-imaged and reconstructed their understanding of the range construct in a piecewise function.

Analysis of Task 1.

Example centered associations which are indicators pseudo-conceptual understanding were evident in this task. Prior to the introduction of the multiple representation “Domain Truck and Range Car” activity, all four students responses in the task-based interview and 10 out of 17 oral and verbal responses from the rest of the class incorrectly used the example of the notion of the range in linear functions which is ℝ $(-\infty, \infty)$ as the nucleus around which they constructed the notion of the range of the other three functions; constant $G(x)$, piecewise involving only constant functions$R(x)$, and piecewise wise involving both linear and constant function $P(x)$. In the constant function $G(x)$, or the piecewise function$R(x)$, that involved only constant functions, it was evident that the use of the word ‘constant’ to represent the outcome that does not change given any input, was not well understood. This represented a pseudo-conceptual understanding where the four students used a word like ‘constant’ without knowing exactly what it means or represents in a particular function. “Domain Truck and Range Car” multiple representation activity afforded the students access to an alternative representation of the notion of the range in piecewise functions which was a new semiotic system at that point. They explored the notion of range in piecewise function that was different from their previous fixed understanding of range as $\mathbb{R}$ $(-\infty, \infty)$ that was informed by their previous understanding of range in a linear function. As they collected the “Range Car” inventory of the y- coordinates in the piecewise function they repeatedly pointed at each piece of the piecewise function and explored the “Car” movement from left to right. This was an indication of; re-imaging, and reflecting on the concepts as their attention was shifting back and forth between the graph and the algebraic symbol of the functions $R(x)$ and $P(x)$. For each piece in the functions $R(x)$ and $P(x)$ written in algebraic form, students focused on its graphical representation and the meaning of the range i.e. collection of the y-coordinates for that particular piece. They each made inscriptions on the graph and verbal descriptions of their understanding of the graph as they explored it from left to right and proposed correct responses to the range of the functions.

In terms of semiotic systems theory, this task involved: (a) identification of the mathematical sign ‘range’ and; (b) its underlying meaning structure. Students understanding of the notion of range as a collection of y-coordinates was challenged. Piecewise functions represents example of unique functions where the definition of the function changes as the domain (independent variable) changes. Superficial reading of the mathematical signs and statements in the task represented or lead to a pseudo-conceptual understanding and response to the task. Though the mathematical sign i.e. the word range, triggered an awareness of what was required in this task, a careful conscious action but systematically reflected upon approach to responding to the question of range in piecewise function was also needed. The participants’ use of multiple representations in this task and careful reflection of the y-coordinate 'inventory' as directed in the activity enabled them to access alternative representations of the range and support a deeper focus and understanding of the underlying meaning of the notion of range in piecewise functions. This represents a shift from pseudo-conceptual to conceptual understanding of the notion of range. From the mathematical sign i.e. word 'range' being the primary focus of attention to the meaning or the idea that the word 'range' represents.

I also observed and analyzed students' statements that indicated an understanding of the underlying meaning structure of the two piecewise functions $R(x)$ and $P(x)$. This was after their interaction with the "Domain Truck and Range Car" activity. For example Stacey made following observation "Oh I see that the domain can also be found from here (*pointing at and making inscriptions indicating conditions of the piecewise functions*). She observed that the conditions in the piecewise functions also matched the domain of the function "I think these conditions match up with the answers to the domain" Sfard (2000) describe the meaning structure of a semiotic system as a reservoir of meanings that can be drawn upon in formulating, developing, and operating a semiotic system. In an attempt to better understand the meaning structure of the domain of the piecewise functions $R(x)$ and $P(x)$, students used their informal theory "the conditions of the piecewise function matches with the domain of the piecewise function". The use of multiple representations activity added to the reservoir of meanings that supported the students' understanding of semiotic system piecewise function. This informal theory served as the meaning structure from which they drew their understanding of the notion of domain and range in piecewise function.

Task 2: Absolute Value Function

Description: In his absolute value function task, the four students were given a set of three conditions written in algebraic symbols. These conditions were: (a) the vertex of $P(x)$ which was$(5, 3)$; (b) a set of ordered pairs written in functional notation, $P(-3) = 15$, $P(18) = 22.5$, and $P(0) = 10.5$; and (c) the end behavior of $P(x)$ given as: $As\ x \to \infty, y \to \infty$, $As\ x \to -\infty, y \to \infty$. They were then supposed to use these conditions to determine the absolute value functions named $P(x)$ and then respond to ten questions related to the absolute value function that followed see figure 16. The following figures show a summary of each students' responses to the questions in task 2 followed by a description, interpretation and an analysis of their responses.

Task 2

Use the following information to determine the absolute value function named $P(x)$.

I. The vertex of $P(x)$ is $(5, 3)$

II. $P(-3) = 15$, $P(18) = 22.5$, $P(0) = 10.5$

III. The end behavior: $As\ x \to \infty, y \to \infty$, $As\ x \to -\infty, y \to \infty$.

Using the information above

a. Find two formulas for $P(x)$, one with absolute value symbol and one without the absolute value symbol.

b. Find y-intercept of $P(x)$.

c. Find the equation of the axis of symmetry. What is the significance of the axis of symmetry?

d. Graph $P(x)$. Use a ruler and be neat.

e. Explain using "sophisticated language" how the graph of $P(x)$ was could be obtained from the parent function $y = |\ x\ |$.

f. Find absolute max/min of $P(x)$. How can you tell if it is absolute max or min?

g. Determine the x-intercept/s of $P(x)$. Show your work analytically or (JYA)

h. Find the domain and the range of $P(x)$. Write answer in interval notion.

i. Find $P(12)$. Compare to $P(20)$. Use symbols $<, =\ or\ >$.

j. Compare $P(2)$ and $P(8)$. Use symbols $<, =\ or\ >$. Explain your response.

Figure 16. Task 2 Absolute-Value Function questions.

Interpretation of Task 2 Responses.

All four students correctly identified the absolute value function as $P(x) = \frac{3}{2}|x - 5| + 3$ Using the RBC+C model, they did so by correctly *recognizing* and identifying the signs in the given task i.e. vertex (5,3), $P(-3) = 15,$ and $P(18) = 22.5$. They then *build-with* and used that information to substitute in the absolute value function model. $P(x) = a|x - H| + K$. *Building-with* involved correctly identifying the different algebraic symbols that were used to represent $P(x)$ i.e. as absolute value function and as piecewise function. However the process by which they used to identify the piecewise function varied from one participant to the other.

Ron (see table 8.) utilized the point-slope model to find both pieces written in slope-intercept form $y = mx + b$. $y = -\frac{3}{2}x + 10.5$, $x \leq 5$ and $y = \frac{3}{2}x - \frac{9}{2}$ for $x > 5$.

In identifying the piece with negative slope, evidence of pseudo-conceptual understanding was apparent. Surface association pseudo-conceptual understanding indicator was evident in Ron's, Stacey's and Yolanda's identification of the y-intercept (see tables 8, 10, and 11). Ron was focused on processing the slope intercept form of $P(x)$ and did not attend to the significance of the sign $P(0) = 10.5$ which represents the y-intercept. In surface association type of pseudo-conceptual indicator, students tend to give undue focus on a set of symbol or words and consequently do not attend to significant information in the mathematical signs in a task. Identifying and using the sign $P(0) = 10.5$ would have reduced that amount of cognitive effort required to find the two slope-intercept form of function $P(x) = \begin{cases} -\frac{3}{2}x + 10.5, & x \leq 5 \\ \frac{3}{2}x - \frac{9}{2}, & x > 5 \end{cases}$. In this particular process of finding the slope–intercept form of the function $P(x)$, Ron latched on to a particular process of finding the slope-intercept form.

Table 8. Ron's Responses to Task 2 Questions.

Question	Response	Multiple Representations Referred	Pseudo-Conceptual Understanding Indicator/s	Semiotic System Component
Function	$P(x) = \frac{3}{2}\lvert x-5\rvert + 3$	VD, AL, G, N	Surface Association	S(object) (R semantic)
a) Find two formulas for $P(x)$,	A	VD, AL, G	Surface Association	R(semantic) M(mathematical content)
b) Find y-intercept of $P(x)$.	$(0, \frac{21}{2})$	VD, AL, G	Artificial Association	R(semantic)
c) Find the equation of the axis of symmetry	$x = 5$	VD, AL, N, G	Artificial Association	R(semantic)
d) Graph $P(x)$.	Ref. fig.	G	------------------	R(semantic)
e) Transformation from parent function	H-shift 5 unit right V-shift 3 units up V-stretch factor $\frac{3}{2}$ Upright	VD, AL, G	------------------	R(semantic) S(process) M(mathematical content)
f) absolute max/min of $P(x)$	Absolute min 3	VD, AL, G	Surface Association	M(informal theory) S(object)
g) x-intercept/s of $P(x)$	$x\ int = None$	VD, AL, G	Surface Association	M(informal theory) R(semantic)
h) domain and the range of $P(x)$	D: $\mathbb{R}\ (-\infty, \infty)$ R: (∞, ∞)	VD, AL, G	Template oriented Association	S (Process) M(informal theory)
i) Find $P(12)$. Compare to $P(20)$.	$P(12) = \frac{27}{2}$ $P(12) < P(20)$	VD, AL, N, G	Surface Association	M(informal theory) S (Process)
j) Compare $P(2)$ and $P(8)$.	$P(2) = P(8)$.	VD, AL, N, G	Surface Association	M(informal theory) S(object)

Key: VD: Verbal Description; N: Numerical; AL: Algebraic; G: Graphing

A: $P(x) = \frac{3}{2}\lvert x-5\rvert + 3$ and $P(x) = \begin{cases} -\frac{3}{2}x + \frac{21}{2}, & x \le 5 \\ \frac{3}{2}x - \frac{9}{2}, & x > 5 \end{cases}$

Table 9. Tyron's Responses to Task 2 Questions.

Question	Response	Multiple Representations Referred	Pseudo-Conceptual Understanding Indicator/s	Semiotic System Component
Function	$P(x) = \frac{3}{2}\lvert x-5\rvert + 3$	VD, AL, G, N	Surface Association	S (object) R (semantic)
a) Find two formulas for $P(x)$,	B	VD, AL, N, G	Surface Association	R(semantic) M(mathematical content)
b) Find y-intercept of $P(x)$.	$(0, \frac{21}{2})$	VD, AL, N, G	Artificial Association	R(semantic)
c) Find the equation of the axis of symmetry	$x = 5$	VD, AL, N, G	Artificial Association	R(semantic)
d) Graph $P(x)$.	Ref. fig.	G	------------------	R(semantic) & S(object)
e) Transformation from parent function	H-shift 5 unit right V-shift 3 units up V-stretch factor $\frac{3}{2}$ Upright	VD, AL, G	------------------	R(semantic), S(process) M(mathematical content)
f) absolute max/min of $P(x)$	Absolute min $y = 3$	VD, AL, N, G	Surface Association	M(informal theory) S(object)
g) x-intercept/s of $P(x)$	$x\ int = None$	VD, AL, G	Artificial Association	M(informal theory) R(semantic)
h) domain and the range of $P(x)$	D: $\mathbb{R}\ (-\infty, \infty)$ R: (∞, ∞)	VD, AL, G	--------------------------	S (Process) M(informal theory)
i) Find $P(12)$. Compare to $P(20)$.	$P(12) = \frac{27}{2}$ $P(12) < P(20)$	VD, AL, N, G	Surface Association	M(informal theory) S (Process)
j) Compare $P(2)$ and $P(8)$.	$P(2) = P(8)$.	VD, AL, N, G	Surface Association	M(informal theory) S(object)

Key: VD: Verbal Description; N: Numerical; AL: Algebraic; G: Graphing

B: $P(x) = \frac{3}{2}\lvert x-5\rvert + 3$ and $P(x) = \begin{cases} -\frac{3}{2}x + \frac{21}{2}, & x \leq 3 \\ \frac{3}{2}x - \frac{9}{2}, & x > 3 \end{cases}$

Table 10. Stacey's Responses to Task 2 Questions

Question	Response	Multiple Representations Referred	Pseudo-Conceptual Understanding Indicator/s	Semiotic System Component
Function	$P(x) = \frac{3}{2}\|x-5\|+3$	VD, AL, G, N	Surface Association	S (object) R (semantic)
a) Find two formulas for $P(x)$,	C	VD, AL, N, G	Surface Association	R(semantic) M(mathematical content)
b) Find y-intercept of $P(x)$.	$(0, \frac{21}{2})$	VD, AL, N, G	Artificial Association	R(semantic)
c) Find the equation of the axis of symmetry	$x = 5$	VD, AL, N, G	Artificial Association	R(semantic)
d) Graph $P(x)$.	Ref. fig.	G	------------------	R(semantic) & S(object)
e) Transformation from parent function	H-shift 5 unit right V-shift 3 units up V-stretch factor $\frac{3}{2}$ Upright	VD, AL, G	------------------	R(semantic), S(process) M(mathematical content)
f) absolute max/min of $P(x)$	Absolute min 3	VD, AL, N, G	Surface Association	M(informal theory) S(object)
g) x-intercept/s of $P(x)$	$x\ int = None$	VD, AL, G	Artificial Association	M(informal theory) R(semantic)
h) domain and the range of $P(x)$	D: $\mathbb{R}\ (-\infty, \infty)$ R: (∞, ∞)	VD, AL, G	--------------------------	S (Process) M(informal theory)
i) Find $P(12)$. Compare to $P(20)$.	$P(12) = \frac{27}{2}$ $P(12) < P(20)$	VD, AL, N, G	Surface Association	M(informal theory) S (Process)
j) Compare $P(2)$ and $P(8)$.	$P(2) = P(8)$.	VD, AL, N, G	Surface Association	M(informal theory) S(process)

Key: VD: Verbal Description; N: Numerical; AL: Algebraic; G: Graphing

C: $P(x) = \frac{3}{2}|x-5|+3$ and $P(x) = \begin{cases} -\frac{3}{2}x + \frac{21}{2}, & x \le 5 \\ \frac{3}{2}x - \frac{9}{2}, & x > 5 \end{cases}$

When I prompted Ron to closely look at the sign $P(0) = 10.5$ and the graph, he was able to see a close link between the sign and the y-intercept. He was able to re-image his understanding of y-intercept "wait a minute this is the same point that was given to us. I think $P(0) = 10.5$ which is $(0, 10.5)$ is the y-intercept". Ron was able to assemble and integrate previous *construct* of y-intercept and produce a new construct or understanding of functional notation of y-intercept.

Table 11. Yolanda Responses to Task 2 Questions.

Question	Response	Multiple Representations Referred	Pseudo-Conceptual Understanding Indicator/s	Semiotic System Component
Function	$P(x) = \frac{3}{2}\|x-5\|+3$	VD, AL, G, N	Surface Association	S (object) R (semantic)
a) Find two formulas for $P(x)$,	D	VD, AL, N, G	Surface Association	R(semantic) M(mathematical content)
b) Find y-intercept of $P(x)$.	$(0, \frac{21}{2})$	VD, AL, N, G	Artificial Association	R(semantic)
c) Find the equation of the axis of symmetry	$x = 5$	VD, AL, N, G	Artificial Association	R(semantic)
d) Graph $P(x)$.	Ref. fig.	G	------------------	R(semantic) & S(object)
e) Transformation from parent function	H-shift 5 unit right V-shift 3 units up V-stretch factor $\frac{3}{2}$ Upright	VD, AL, G	------------------	R(semantic), S(process) M(mathematical content)
f) absolute max/min of $P(x)$	Absolute min 3	VD, AL, N, G	Surface Association	M(informal theory)
g) x-intercept/s of $P(x)$	$x\ int = None$	VD, AL, G	Surface Association	M(informal theory) R(semantic)
h) domain and the range of $P(x)$	D: $\mathbb{R}\ (-\infty, \infty)$ R: (∞, ∞)	VD, AL, G	Template Oriented Association	S (Process) M(informal theory)
i) Find $P(12)$. Compare to $P(20)$.	$P(12) = \frac{27}{2}$ $P(12) < P(20)$	VD, AL, N, G	Surface Association	M(informal theory) S (Process)
j) Compare $P(2)$ and $P(8)$.	$P(2) = P(8)$.	VD, AL, N, G	Surface Association	M(informal theory) S(process)

Key: VD: Verbal Description; N: Numerical; AL: Algebraic; G: Graphing

D: $P(x) = \frac{3}{2}|x-5|+3$ and $P(x) = \begin{cases} -\frac{3}{2}x + \frac{21}{2}, & x \le 5 \\ \frac{3}{2}x - \frac{9}{2}, & x > 5 \end{cases}$

All four students correctly *recognized* and identified the axis of symmetry as x $= 5$ using the x-coordinate of the vertex. They correctly recognized and identified the transformation of $P(x)$ from the parent function $y = |x\ |$ using 'sophisticated' verbal description and accurately drew the graph of $P(x)$. Except for Tyron, the other three participants described the absolute min as equals to 3. Tyron described the absolute min as y = 3. In responding to absolute min as 3 they did not pay attention in the meaning of or the difference between absolute min equal to 3 and absolute min as $y = 3$. Absolute min represents the lowest value of the y coordinate. Their responses were indicative of surface association pseudo-conceptual understanding that involved superficial reading of a set of mathematical signs. Revisiting the definition of absolute max and looking at the graph again, Ron, Stacey, and Yolanda were able to re-image and redefine their responses to absolute min as $y = 3$. They were able to *build –with* a new understanding of the correct presentation of the absolute max. They had different explanations as to why $y = 3$ represented absolute min. Tyron reasoned that "since $a = \frac{3}{2}$ was positive the graph can only have absolute min" line 22 &23. Stacey had a different approach and used the end behavior symbol and the fact that the vertex was above the x-axis to describe the absolute min "$As\ x \to \infty, y \to \infty$, $As\ x \to -\infty, y \to \infty$." Line 24. Students managed to construct a new understanding of the link between the end behavior symbols and the graph. This in turn enabled them to produce a logical explanation of this link. I inferred from Stacey's response that she tapped on to her informal theory semiotic system component to understand the underlying meaning structure of the sign absolute min and the link to the end behavior sign.

In determining the x-intercepts of the function $P(x)$, Ron and Tyron used the analytical method substituting zero for y and calculating for x. Stacey used the same end behavior symbol and the fact that the vertex was above the x-axis to confirm that there was no x-intercept. The four participants correctly used analytical methods to show that there was no x-intercept. The final answer step in the analysis $|x - 5| = -2$ implied that there was no solution (i.e.$x = \emptyset$). This was indicative of surface association of pseudo-conceptual understanding. It involved superficial reading of mathematical signs and not focusing on the meaning of those symbols. After I prompted Ron and Tyron to look closely and see if they could come up with any conjectures about the x-intercepts, Ron observed that "in absolute value functions if parameters 'a' and 'k' have the same sign it implied that the function had no x-intercepts" line 30. Tyron observed the same but also concluded that "if 'a' was positive and 'k' was positive it implies an upright absolute value function above the x-axis" line 25.

Ron, Stacey, Yolanda identified the domain as D: $\mathbb{R}\ (-\infty, \infty)$ and range as $\mathbb{R}\ (-\infty, \infty)$. Though the domain response was correct the range response was incorrect. The expected correct range response was $\mathbb{R}\ [3, \infty)$. The three students tapped on to their previous knowledge of range in a linear function to respond to the question of range in absolute value function. Template –oriented pseudo-conceptual understanding indicator was evident. Since $P(x)$ appeared and could be written as two linear functions in part (a) of this task, the three students transferred their understanding of the properties of range in a linear function to the range in an absolute value function $P(x)$. Using the graph moving back and forth between the graph and the absolute value function $P(x)$ the three students reflected, re-imaged and re-evaluated their understanding of range in an absolute value function. Referring to their 'domain truck' and 'range car' activity in task 1 that emphasized conscious and systematic reflection of the domain and range of any function, the three students were able to correctly determine the range in $P(x)$.

In comparing $P(12)$ and $P(20)$, all the four participants showed that $P(12) < P(20)$. Ron, Stacey, and Yolanda used the analytical method to show that $P(12) < P(20)$. They each calculated the individual value of $P(12)\ and\ P(20)$ and compared their values. They were quickly drawn and focused on the processing of $P(12)$ and $P(20)$. They did not focus much on the meanings of the signs. Tyron used analytical method also, but proceeded to explain how he compared $P(12)$ and $P(20)$ using end-behavior "from the vertex, the further I move to the right the higher the point is $As\ x \to \infty, y \to \infty,\ As\ x \to -\infty, y \to \infty$." Line 35. In comparing $P(2)$ and $P(8)$, all four students correctly determined that $P(2) = P(8)$. Processing methods varied.

Tyron drew a vertical line at $x = 2$ and $x = 8$ and observed that $P(2)$ and $P(8)$ were on the same horizontal level on the implying that they were equal. Stacey and Yolanda used the reasoning that $x = 2$ and $x = 8$ were at the same horizontal distance from axis of symmetry $x = 5$ on both sides their height above the x-axis was same. Informal theory that for absolute value function the points whose x-coordinates are at the same horizontal distance from the axis of symmetry on either side of the vertex have same y-coordinate.

Analysis of Task 2 Responses.

In several of the responses to the following questions; comparing $P(2) = P(8)$, comparing $P(12) = P(20)$, determining the x-intercepts, and determining the y-intercept, the participants were quick to algebraically process the solutions to these questions. Pre using multiple representation or prompted to look at the meaning of the mathematical signs in the questions, students did not reflect through the underlying meaning structure of the signs. They relied on the second component of the semiotic system i.e. they focused on the transformational rule. All four students correctly used given information to process and determine the function $P(x)$ both as absolute value function and as piecewise function. Mathematical sign like $P(0) = 10.5$, are expected to act as trigger for identifying the y-intercept so that Ron, Stacey and Yolanda would have been aware of the meaning of the sign $P(0) = 10.5$ as y-intercept and reduced amount of cognitive effort required to calculate and use y-intercept to respond to the question. The three students focused on processing the y-intercept more than using the ‘clues’ embedded in the sign $P(0) = 10.5$. From a semiotic perspective they focused on the transformational rules demonstrating some conscious effort and action to solve the problem, but did not systematically reflect on the meanings of the various elements of the absolute value function $P(x)$ like the y-intercept. Exposure to and use of multiple representations i.e. graphing the function, inscribing on the graph, verbal description, and use of algebraic sign $P(0) = 10.5$, served as a means of semiotic mediation and redirected their understanding of the meaning of the mathematical sign. The multiple representations enabled the participants to connect the attributes of the functions $P(x) = \frac{3}{2}|x - 5| + 3$ with the mathematical features represented in the graph. I inferred that use of multiple representations allowed students a reified understanding of absolute value functions. Reified understanding (Sfard, 2000) is a component of re-imaging of conceptual understanding and refers to when students' perception of a mathematical entity transitions from a process at one level to an object at another level. For example a statement like "when parameters 'a' and 'k' in an absolute value function are both positive it implies no x-intercepts and the graph is above the x-axis" indicates an understanding of the absolute-value function as an object as well as process of transformation.. There were evidence of students transitioning from static perception of the signs i.e. sign as object approach, to signs as process approach. For example the end behavior sign i.e. $As\ x \to \infty, y \to \infty,\ \ As\ x \to -\infty, y \to \infty$ was a trigger to Stacey and Yolanda who used it compare $P(12)$ and $P(20)$ and predict that the further away the x coordinate is from the axis of symmetry the greater the y-coordinate. In this example, the participants transitioned from looking at the end behavior sign as mathematical object to sign as a process representing a dynamic approach. The mediation role of the multiple representations (graph, verbal descriptions, numeric, and algebraic notations) facilitated the participants': (a) re-imaging of their understanding to make connections between the graph and the algebraic notation; (b) representational versatility, where students responses transitioned seamlessly between different representations. In this task the students were expected to make deduction of the various external properties of the absolute value function $P(x)$ e.g. end behavior. Conceptual understanding requires the ability to make the logical link between the various elements of the function like $P(x)$. For example Tyron's use of the end behavior sign to make comparisons of between $P(12)$ and $P(20)$ demonstrate a transition from identifying end behavior sign as a mathematical object and an end in itself to identifying the sign as representative of an idea. It represented a shift in the students understanding from sign as a signifier (object) to sign as representation of an idea. Multiple representations facilitated: (a) the identification of the mathematical sign/ideas in the sign; (b) the identification of the correct transformational rule; (c) the identification of the underlying meaning structure necessary to make a logical links between various elements of a function like $P(x)$.

Task 3: Absolute-Value Functions Application (Pool Table)

In this task (see figure 17) students were required to write an equation of the path of a ball in a miniature golf and determine whether they make a hole in one or not. This task was adopted from Larson (2004) Algebra II class textbook. They were given the ordered pairs for the starting point, the banking position, and the hole. The task required their application of absolute value function understanding.

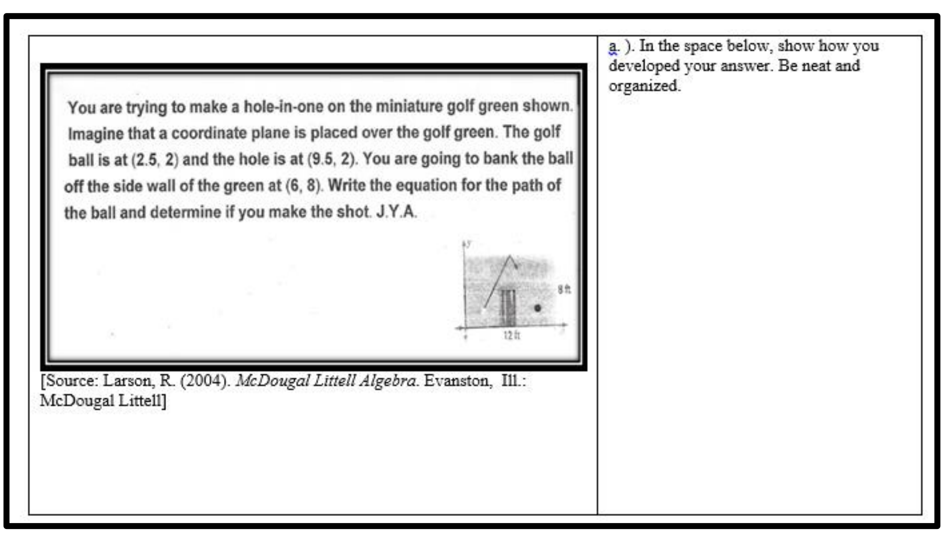


Figure 17. Task 3 Application of Absolute Value Functions

Following is a summary and a description of their responses to the task 3 questions. Interpretation of the responses and their analysis using RBC+C model and Ernest (2006) semiotic systems theory will follow.

Table 12. Participants' Responses to Task 3 Application of Absolute-value Functions

Participant	Question	Response	Multiple Representations Referred	Pseudo-Conceptual Understanding Indicator/s	Semiotic System Component
Ron	Make the shot?	*Yes*	VD, AL, G, N	Surface Association	S(object) (R semantic)
	Justification	$y = -\frac{12}{7}\lvert x-6\rvert + 8$ *Subs ordered pairs* $(6{,}8), (2.5, 2)$& $(9.5, 2)$	VD, AL, G		S(object) M(mathematical content)
	Alternative option	Graphing absolute value function	VD, AL, G	Surface Association	R(semantic) S(process) M(informal theory)
Tyron	Make the shot?	*Yes*	VD, AL, N, G	Surface Association	S(object) (R semantic)
	Justification	$y = -\frac{12}{7}\lvert x-6\rvert + 8$ *Sub ordered pairs* $(6{,}8), (2.5, 2)$& $(9.5, 2)$	VD, AL, G		S(object) (R semantic)
	Alternative option	Graphing absolute value function	VD, AL, G		R(semantic) S(process) M(informal theory)
Stacey	Make the shot?	*Yes*	VD, AL, G	Surface Association	S(object) (R semantic)
	Justification	$y = -\frac{12}{7}\lvert x-6\rvert + 8$ *Sub ordered pairs* $(6{,}8), (2.5, 2)$& $(9.5, 2)$	VD, AL, G	Surface Association	M(informal theory) S(object) (R semantic)
	Alternative option	Graphing absolute value function	VD, AL, G	Surface Association	M(informal theory) S(object) (R semantic)
Yolanda	Make the shot?	*yes*	VD, AL, N, G	Surface Association	R(semantic) S(process)
	Justification	$y = -\frac{12}{7}\lvert x-6\rvert + 8$ *Sub ordered pairs* $(6{,}8), (2.5, 2)$& $(9.5, 2)$	VD, AL, N, G	Surface Association	M(informal theory) S (Process) M(mathematical content) R(semantic)
	Alternative option	Graphing absolute value function	VD, AL, N, G	Surface Association	M(informal theory) S (Process) M(mathematical content) R(semantic)

Interpretation of Task 3 Responses.

All four students started off responding to the question by *recognizing* that the task involves an absolute value function but more importantly identify that the vertex of the absolute value function as $(6, 8)$. Ron, Stacey, and Yolanda started off not sure that they would make the shot. Stacey explicitly stated that she was not sure is she will make it "I am not sure if I will make it but I will try I think I might make the shot" line 1. All four students correctly *recognized* ordered pair $(6, 8)$ as the vertex of the absolute value function. They then *build-with* a strategy and substituted the coordinates of the vertex $(6, 8)$ and the point $(2.5, 2)$ and determined the absolute value function of the path of the ball as y $=-\frac{12}{7}|x-6|+8$. To determine if they made the shot or not, Ron, Stacey, and Yolanda substituted the values of the ordered pair$(9.5, 2)$ and ended up with a true statement 2$=2$. New understanding of the absolute value function was *constructed* when they used the true statement to infer that the ordered pair $(9.5,\ 2)$ which represents the hole was on the path of the ball hence they would make the shot. Tyron adopted a different position to determine if he made the shot or not. He *recognized* that the axis of symmetry split the function into two equal parts. Tyron *build-with* a strategy when he drew the axis of symmetry $x=6$ *recognized* and determined that from the y-coordinate, both points $(2.5, 2)$ and $(9.5, 2)$ had the same co-ordinate. Also he determined that the absolute value function $|x-6|$ signified that points on both sides of axis of symmetry that were at the same horizontal distance from the x-coordinate x = 6 had same y-coordinates. Since the ordered pairs $(2.5, 2)$ and $(9.5, 2)$ were both 3.5 units horizontally from axis of symmetry on both sides of the axis it implied that the hole was on the path of the absolute value function on the opposite side across from starting point and hence he would be able to make the shot.

Analysis of Task 3 Responses.

All four students recognized immediately what the goal of the task was. Semiotic system component that students gravitated to was the transformation of rules. Ron, Stacey, and Yolanda correctly identified the procedures to apply by using algebraic manipulation of the absolute value function to determine whether they made the shot or not. They processed the absolute value function in the path of the ball and determined it to be y $=-\frac{12}{7}|x-6|+8$. They then used ordered pair of the hole $(9.5, 2)$ in the absolute value function and determined if they made the shot of not. True statement $2=2$ implied that they made the shot. Use of multiple representations i.e., graph and algebraic symbol triggered and supported a symbol sense where the vertex $(6, 8)$ and the axis of symmetry were correctly identified. Tyron's demonstrates a deeper conceptual understanding of the underlying meaning structure of the signs in the absolute value function. He makes a logical link between the different elements of the function and uses them to respond to the task question. He makes a connection between the axis of symmetry and the absolute sign $|x-6|$ and infers that the initial golf ball position $(2.5, 2)$ and the hole are horizontally at equal distance of 3.5 from axis of symmetry $x=6$ on both sides. In absolute value function, "except for the vertex for every y-coordinate there are two x-coordinates". Tyron then infers that he will make the shot because these two ordered pairs $(2.5, 2)$ and $(9.5, 2)$ are same distance horizontally from axis of symmetry. Use of multiple representations allowed for the students to re-image their conceptual understanding of the absolute value functions, develop a reflective approach to understanding each sign in the function and make a logical external link between the parameters of the functions and the problem content in this task. Representational versatility i.e. ability to work seamlessly between representations was demonstrated when Tyron made the connections between the significance of the axis of symmetry and the two ordered pairs.

Task 4: Quadratic Functions

In task 4, the students were given a set of three quadratic function conditions. They were then required to use the given information to determine the quadratic function $Q(x)$. Using the information given they were also required to answer twelve questions (see figure 18). These questions involved: determining the y-intercepts given in different representations; determining the absolute max; and determining the meanings and transformations of various symbols in the quadratic function.

TASK 4

Use the following information to determine a quadratic function named $Q(x)$. Write $Q(x)$ in Standard Form.

I. The linear coefficient is "2"

II. The constant is "5"

III. The leading coefficient is an integer between -4 and -2.

Use this information to answer the following questions. .

i. Find the concavity of $Q(x)$. How can you tell?

ii. Find the y-intercept of $Q(x)$.

iii. Find $Q(0)$.

iv. Find the value/s of x for which $Q(x) = 0$. JYA

v. Find the absolute max or min.

vi. Find the domain for $Q(x) \geq 0$. What is the meaning of this symbol?

vii. Find the domain for $Q(x) < 0$.

viii. What is the meaning of this symbol?

ix. Find $Q\left(-\frac{2}{2(-3)}\right)$. What is the significance of this symbol?

x. Find the domain if the range is given as $-\frac{2^2}{4(-3)} + 5$.

xi. Compare $Q(30)$ & $Q(40)$. Use $<, =, >$

xii. Given that $Q(-1) = 0$ and $Q\left(\frac{5}{3}\right) = 0$, write the equation of the axis of symmetry.

Figure 18. Task 4 Questions.

Table 13 represents a summary of Ron's, Tyron's, Stacey's and Yolanda's responses to task 4 questions. The summary includes the multiple representations used or referred in the problem, pseudo-conceptual understanding indicators, and semiotic systems components identified in the problem.

Table 13. Ron's, Tyron's, Stacey's, & Yolanda's task 4 responses

Question	Response	Multiple Representations Referred	Pseudo-Conceptual Understanding Indicator/s	Semiotic System Component
Quadratic Function	$Q(x) = -3x^2 + 2x + 5$	VD, AL, G, N	Surface Association	S(object) (R semantic)
i. Concavity	Opens down	VD, AL, G		S(object) M(mathematical content)
ii. y-intercept of $Q(x)$.	$(0, 5)$	VD, AL, G	Artificial Association Surface Association	R(semantic) S(process) M(informal theory)
iii. Find $Q(0)$	$(0, 5)$	VD, AL, N, G	Artificial Association Surface Association	R(semantic) S(object) M(informal theory)
iv. value of x for which $Q(x) = 0$	$x = -1$ & $x = \frac{5}{3}$	G	Surface Association	S(object) (R semantic)
v. Absolute max/min	$y = \frac{16}{3}$	VD, AL, G	----------------------	R(semantic) S(process) M(mathematical content)
vi. Domain for $Q(x) \geq 0$	$[-1, \frac{5}{3}]$	VD, AL, G	Surface Association	M(informal theory) S(object) S(process)
vii Domain for $Q(x) < 0$	$(-\infty, -1) \cup (\frac{5}{3}, \infty)$	VD, AL, G	Surface Association	M(informal theory) S(object) S(process)
viii. Find $Q\left(-\frac{2}{2(-3)}\right)$	$y = \frac{16}{3}$	VD, AL, G	Template oriented Association	S (Process) M(informal theory) M(mathematical content)
ix. Find domain given range as $\left(-\frac{2^2}{4(-3)} + 5\right)$.	$\frac{1}{3}$	VD, AL, N, G	Surface Association	M(informal theory) S (Process) M(mathematical content) R(semantic)
x. Compare $Q(30)$ and $Q(40)$	$Q(30) > Q(40)$	VD, AL, N, G	Surface Association Template oriented Association	M(informal theory) S (Process) M(mathematical content) R(semantic)
xi. Compare $Q(-1)$ and $Q\left(\frac{5}{3}\right)$.	$x = \frac{1}{3}$	VD, AL, N, G	Surface Association	M(informal theory) S (Process) M(mathematical content) R(semantic)

Interpretation of Participants' Task 4 Responses.

Applying the RBC+C model, all four participants *recognized* and correctly named and identified the parameters: $a = -3$; $b = 2$; and $c = -3$. They then *build-with* and used these parameters to determine the quadratic function $Q(x)$ written in standard form $Q(x) = -3x^2 + 2x + 5$. The four participants then correctly identified the concavity of $Q(x)$ and referenced the leading coefficient 'a' being a negative as the justification for the graph opening down i.e. concave down. They all confirmed this using the graphing calculator. The y-intercept of $Q(x)$ was correctly identified by all the four participants though they had varied methods of finding the y-intercept as well as $Q(0)$. Prior to referring to the graph on their calculator that was projected on the smart board from the calculator, all the four students gravitated towards processing the y-intercept by substituting 0 in the function $Q(x)$. They then determined that the y-intercept was 5 i.e.$(0, 5)$. In *building –with* epistemic action I also observed the significance of the sign $Q(0)$ becoming apparent when Stacey for example moved back and forth (inscribing on the board the position of the y-intercept on the graph) between the graph, the quadratic function $Q(x) = -3x^2 + 2x + 5$ and *recognizing* that y-intercept was equivalent to the constant (5) in the standard form function i.e. $Q(0) = c$. Tyron also *recognized* and declared that the y-intercept was "$Q(0) = c$ and y = c" line 8 (see appendix I). To find the value of x for which $Q(x) = 0$, they *build-with* previous knowledge and used the factoring method to find the x-intercepts $(-1,0)$ & $(\frac{5}{3}, 0)$. Stacey immediately recognized that she was looking for the x-intercepts and produced a new *construct* by inferring that $Q(x) = 0$ represented the output or the y-coordinate that was equal to zero. They transformed the standard form into intercepts form $(-3x + 5)(x + 1) = 0$ facilitate the identification of the x-intercepts. Students' response question (iv.) indicated that the significance of sign $Q(x) = 0$ was not immediately *recognized* prior to using the graph. Using multiple representation that included; the graph on calculator, verbal description, and algebraic expressions of $Q(x)$, supported the *building-with,* and facilitated the students' ability to re-image and recognize that there were several form of representations of the same mathematical object i.e. the x-intercept. Evidence of superficial associations when students showed familiarity with the signs $Q(x) = 0$, yet their responses indicated unfamiliarity with its significance. The four students' undue focus was on the symbols in this task, and non- attendance to their significance was indication of pseudo-conceptual understanding. They transitioned from surface association type of pseudo-conceptual understanding where students demonstrated superficial non-reflective approach to reading the signs $Q(0)$ & $Q(x) = 0$, to a more reflective approach to understanding the meaning of the signs $Q(0)$ & $Q(x) = 0$. This transition represented students' *construction* of new meaning of the signs $Q(0)$ & $Q(x) = 0$. Ron's response "it looks like the answer to the $Q(x) = 0$ represents the x-intercepts" line 17 see appendix I. This pattern of unreflective and non-attendance to the significance or the meaning of the sign, followed by multiple representation use facilitating new insights into the meaning was evident in the other questions in this task. Unreflective response was demonstrated by students gravitating towards and privileging algebraic processing before establishing the underlying meaning structure of the mathematical signs in the task. For example in responding to question x. i.e., comparing $Q(30)$ and $Q(40)$ all the four students used the algebraic processing of the two symbol to compare the value. However examining their responses and reflecting on; the graph, end-behavior and the symbols they *recognized* that the end behavior written algebraically $as\ \ x \to \infty, y \to -\infty\ and\ as\ x \to -\infty, y \to \infty$ had clues and ideas that would have reduced their cognitive effort in determining the comparing $Q(30)$ and $Q(40)$. Use of multiple representations afforded the students with an opportunity to *construct* new meaning for the end behavior symbol. In another example, in question viii, the students were required to find $Q\left(-\frac{2}{2(-3)}\right)$ where the values of the leading coefficient $a = -3$ and linear coefficient $b = 2$ have been explicitly included in the input of the function. Prior to looking at the three representations i.e., graph, the sign $Q(x) = -3x^2 + 2x + 5$, and the sign $Q\left(-\frac{b}{2(a)}\right)$ together, all four students gravitated to responding to the question by algebraic processing. The later recognized that the sign $Q\left(-\frac{b}{2(a)}\right)$ represented the output at the absolute max which was equal to $\frac{16}{3}$.

Analysis of Task 4 Responses.

Pseudo-conceptual understanding associated with surface and template association was evident in all the four students' responses prior to utilizing multiple representations. In addition their responses to questions like x., demonstrated a conscious and genuine effort to action but did not systematically reflect upon the underlying meaning structures of the signs in the task. They all demonstrated ability to use the mathematical sign correctly in processing algebraically. However in questions that required some understanding of the meaning structure in order to reduce the cognitive effort required to solve the task, challenges were evident prior to use of multiple representations. In semiotic theory the students privileged the second component of the semiotic sign i.e. the transformational rules and paid little attention to the other two components. Use of multiple representations (i.e. inscribing on the graph, comparing graph to symbols, and reevaluating verbal descriptions) afforded the students opportunity to re-image their understanding and create new insights into the meanings of the signs used thus allowing the four students to generate new interpretations of the signs used. Students re-imaging of their understanding of the mathematical object $Q(x)$ was demonstrated by their reified understanding of the function. Reified understanding is what results when mathematical entities are perceived as an object at one level are also reconceived as process at another level (Sfard, 1991). Students transitioned their understanding of y-intercept written in functional form $Q(0)$ from a mathematical object to a process as a position on a graph (y-axis) with particular significance.

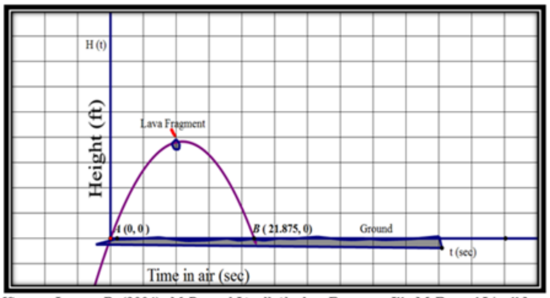
Task 5: Quadratic Function Application Kilauea Iki Crater Hawaii

Task 5 involved application of quadratic functions. The task was adapted from Larson (2004) Algebra II textbook that the students were using. The students were required to examine a picture that showed the path of a typical lava fragment while the fragment was in the air. They were then required to use the information on the graph to respond to eight questions that followed. Figure 19 shows the graph of path of the lava fragment including the required task 5 questions.

TASK 5

Task 5

The volcanic cinder cone Puu Puai in Hawaii was formed in 1959 when a massive "lava fountain" erupted at Kilauea Iki Crater, shooting lava hundreds of feet into the air. When the eruption was intense, the height H (in feet) of a typical lava fragment t seconds after being ejected from the ground could be modeled using a quadratic function H (t).



[Source: Larson, R. (2004). *McDougal Littell Algebra*. Evanston, Ill.: McDougal Littell.]

Examine the picture (graph) above which models the path of a typical lava fragment in the fountain while the fragment was in the air. Use the information available from the picture to answer the following questions. Be sure to show how you developed your answers.

I. Given that the height of the lava fragment after 20 s is 600ft, use the information from the graph to write the quadratic function $H(t)$. Write the function in

a. Intercept form

b. Vertex form

c. Standard form.

II. Use this information to answer the following questions.

i. For how long was the lava fragment in the air? How did you use the graph to get your answer?

ii. Estimate the lava fragment's maximum height above the ground.

iii. Does $H(t)$ have a stretch/shrink factor? How can you tell?

iv. Find the y-intercept of $H(t)$ what is the significance in this graph?

v. Find the x-intercept/s of $H(t)$.

vi. Find the domain for which $H(t) \geq 0$. What is the significance of this symbol?

vii. What was the height of the lava fragment after 10 seconds? What mathematical symbol would you use to show this height after 10s?

viii. Estimate after how long it took the lava fragment to be at height of 1000 feet above ground? What mathematical symbol would you use to describe this height after a given time?

Figure 19. Task 5 Application of Quadratic Functions: Lava fragments from Volcanic cinder Pua Puai in Hawaii.

Description: This task was part of the Unit II on Application of Quadratic Function and Piecewise Function test review. The students were required to examine the picture (graph) which models a path of a typical lava fragment while in the air. This lava fragment was from a volcanic eruption in Pua Puai in Hawaii. They were then required to use the information from the picture to answer the questions that followed.

Table 14. Summary of Ron's, Tyron's, Stacey's, & Yolanda's responses to task 5.

Question	Response	Multiple Representations Referred	Pseudo-Conceptual Understanding Indicator/s	Semiotic System Component
Intercept Form	$H(t) = -16t(t - \frac{175}{16})$	VD, AL, G, N	Surface Association	S(object) R(semantic)
Vertex Form	$H(t) = -16(t - \frac{175}{16})^2 + 1914.0625$	VD, AL, N, G		S(object) (R semantic)
Standard Form	$H(t) = -16t^2 + 350t.$	VD, AL, N, G		S(object) (R semantic)
i) time in air	21.875s	VD, AL, G	Surface Association	R(semantic) M(mathematical content) M(informal theory)
ii) max height	$\frac{175}{16}$	VD, AL, N, G	Surface Association	M(informal theory) R(semantic)
iii) stretch/shrink factor	Vertical stretch $a = -16$	VD, AL, N, G	----------------------------	M(informal theory) S(object)
iv) y-intercept of H(t)	$(0, 0)$	VD, AL, N, G	Surface Association	S(process) M(mathematical content)
v) x-intercept/ of H(t)	$(0, 0)$ & (21.875)	VD, AL, G	Artificial Association	R(semantic) S(process) M(mathematical content) M(informal theory)
vi) Domain $H(t) \geq 0$	$[0\ , 21.875]$	VD, AL, N, G	Artificial Association	R(semantic) S(process) M(mathematical content) M(informal theory)
vii) Height after 10s.	$1900ft$	VD, AL, N, G	Surface Association	R(semantic) S(process) M(mathematical content)
viii) time taken to reach 1000ft	3.379s & at 18.5s	VD, AL, N, G	Surface Association	R(semantic) S(process) M(mathematical content)

Key: VD: Verbal Description; N: Numerical; AL: Algebraic; G: Graphing;
S: Mathematical sign; R: Transformational rule; M: Meaning structure of the sign.

Each student started the task by reading it out loud. In the first part of the task, each student was required to develop a quadratic function in three forms; a) intercept, b) vertex, and c) standard form using the given information. All four students initially approached solving task 5 questions by writing the three quadratic function models and substituting the ordered pairs $(0, 0)$ and $(21.875, 0)$. Students were free to use a calculator (TI 84 Plus) if they so wished. Ron, Stacey, and Yolanda had the correct response for the intercept form of the quadratic function

$H(t) = -16t(t - \frac{175}{16})$ that the lava modeled. They developed the correct intercept function by substituting $(20, 600)$ in the quadratic intercept form model $H(x) = a(x - p)(x - q)$.

Tyron: Uh, I wonder why this graph [*pointing at the graph with concavity opens up*] came out different from the rest of the graphs. I think I might have missed out something. The table of values for the first intercept form function is also different. [*He reexamines his work again*]. Oh I now see, I missed out the "a" values completely. Let me go over it once again. Think my" a" values should match the other two forms think it should be -16 but let me work it out.

I: Why do you think it should be -16?

Tyron: I think it makes sense since negative means opens down and 16 mean vertical stretch factor. I will work it out again. I will use the ordered pair (20, 600) in this umm model $H(x) = a(x - p)(x - q)$.

Tyron: This is my new intercept form answer $H(t) = -16t(t - \frac{175}{16})$ and it matches the other two including the table of values.

I: What did you do differently to come to that answer?

Tyron: I realized that without -16x, I was getting this graph [*pointing at the graph opens up*]. So after re-calculating my intercept form marched the other two.

Figure 20. Portion of Tyron's Transcripts of Task 5 Responses.

Tyron's intercept model was incorrect. He used the formula $H(x) = (x - p)(x - q)$ resulted in the intercept function $H(t) = t(t - \frac{175}{16})$. To confirm their responses, students were encouraged to graph all the three forms of the quadratic functions and compare the results. Tyron's intercept form graph did not match the graph of the other two models. As he reworked the intercept form again, he discussed the problem context (see figure 20).

Interpretation of Task 5 Tyron's Responses.

In applying the RBC+C model, Tyron *recognizes* that his intercept model does not match the given lava fragment path. Tyron's intercept model of the quadratic function generates an unexpected graph with concavity that does not march the model in the original problem or the graphs of the other two (vertex, and standard form). Tyron *builds-with* when he understands the task and develops a strategy to solve the problem he just discovered. The graph (representation) oriented differently from the rest causes Tyron to re-examine, reflect and reimage his understanding of the intercept form: "I think I might have missed out something" (line 2). He reexamines his tables of value generated by the three quadratic form. He realizes that the table of values from the intercept form also differs from the rest. He then compares certain values in the table of value to the picture (graph) which models a path of a typical lava fragment. His parameter 'a' value is missing and recalculates and corrects his intercept form to $H(t) = -16t(t - \frac{175}{8})$. Pseudo-conceptual behavior (surface association) is evident in Tyron's undue focus on other parts of the intercept form and non-attendance to the parameter 'a' which is a significant part of the mathematical object $H(x) = a(x-p)(x-q)$. Use of multiple representation i.e. graphical, verbal descriptions, table of values, and algebraic symbol influences the identification of this pseudo-conceptual understanding and a construction of a renewed insight of the significance of the parameter 'a' in the model.

Analysis of Task 5 Tyron's Responses.

Pseudo conceptual behavior identified involved surface association. Tyron did not pay attention to a significant attribute of the mathematical sign $H(t) = a(t-p)(t-q)$ i.e. the vertical stretch factor "a". In this quadratic function semiotic system, the mathematical sign $H(t) = t\left(t - \frac{175}{8}\right)$ used by Tyron is missing a significant feature 'a'. The use of multiple representations (graph, verbal description, algebraic symbol) reoriented and re-imaged Tyron's understanding of the significance of the parameter 'a' symbol in the quadratic function intercept form $H(x) = a(t-p)(t-q)$. Tyron repeatedly pointing at his initial intercept form graphs indicated that his attention moved back and forth between the meaning of 'a' sign and the graphical representation. This exposure and use of multiple representations supported the identification of the significance of an important feature 'a' in a mathematical sign intercept form $H(t) = a(t-p)(t-q)$. The use of multiple representations provided Tyron an opportunity to reflect and re-image his understanding of the underlying meaning structure of the mathematical sign. Transition from pseudo conceptual understanding to conceptual understanding is evident when he recalculates, refines and rewrites his intercept form function correctly $H(t) = -16t\left(t - \frac{175}{8}\right)$. He establishes the logical link between the variables (sign components) in the quadratic function and the meaning of the parameter as demonstrated in the graph. He confirms his refined intercepts form function by graphing it on TI calculator. This graph matches the graphs of the other two quadratic form models. Refining, rewriting and confirming his answer represents a transition from pseudo-conceptual to conceptual understanding.

Description of Task 5 Part (ii).

The second part of task 5 students were required to respond to question part (i.-viii). In part (i) students were asked to determine the time it took for the lava fragment to be in the air. All students correctly identified the length of time the lava fragment was in the air (i.e. 21.875 seconds). Ron, Tyron, and Yolanda used the ordered pair $(0,0)$ initial time when lava fragment was released and $(21.875, 0)$ end of the lava fragment path time on the graph to identify this length of time. However Stacey gravitated to solving the problem algebraically by solving the equation $0 = -16t\left(t - \frac{175}{8}\right)$ and obtaining the answer $(0,0)$ & $(21.875,0)$.

Interpretation Task 5 Part (ii) Responses.

When asked how she could have used the graph to easily identify the time taken by the lava fragment in the air, Stacey appeared surprised. Her response "I think it is just easier to substitute zero for y in this function (pointing at the intercept function form) and calculate for t." line 10. Stacey had not made a connection between the path of the graph and the algebraic sign intercept form $H(t) = -16t\left(t - \frac{175}{8}\right)$. Stacey moved back and forth between the information on the graph and the intercept form function by retracing the path of the fragment and highlighting the ordered pairs $(0, 0)$& $(21.875,0)$. She looked at her solution $\frac{175}{8}$ and correctly *recognized* and identified it on the graph by making an inscription. Making inscriptions on the graph was Stacey's attempt to *build-with* by relating and making a connection between the symbolic representation and the graphical representation of the mathematical object quadratic function in intercept form. She then *constructed* a new understanding when she realized that she could have easily determined the same answer by understanding the meaning of each component in the quadratic intercept form function that she used i.e. $H(t) = -16t\left(t - \frac{175}{8}\right)$. The inscriptions she made and the verbal statement "Now I see that this point (*pointing at the end of the graph with ordered pair* $(21.875,0)$ *inscribed*) represents same thing as this (*pointing at the* $\frac{175}{8}$) in the intercept form function" line 15 is evidence of her generating a personal meaning of the mathematical sign.

Analysis Task 5 Part (ii) Responses.

The pseudo conceptual understanding involved is surface attention. This was evident in Stacey's nonattendance to the meaning of the mathematical sign; a) $\left(t - \frac{175}{8}\right)$ and b) ordered pair $(21.875, 0)$ in the graph. The semiotic system component involved is the identification of the underlying meaning structure in the mathematical sign $H(t) = -16t\left(t - \frac{175}{8}\right)$. Though her algebraic transformational rules involving the procedure for calculating the intercepts were correct, there was evidence of not taking advantage of computational offloading a significant indicator of conceptual understanding. Pre-use of multiple representation evidence of absence of representational versatility. Computational offloading refers to "the extent to which different external representations reduce the amount of cognitive effort required to solve equivalent problems" (Ainsworth, 2006. P.185). Simultaneous use of multiple representations: (a) the graph that involved making inscriptions on the picture to indicate the time when the lava hit the ground; (b) algebraic symbol$\left(t - \frac{175}{8}\right)$; and (c) verbal description of the intercepts, was an indication of Stacey's attempt to make sense and *construct* a personal interpretation and meanings of the various attributes of the mathematical sign $H(t) = -16t\left(t - \frac{175}{8}\right)$.

Interpretation Task 5 Part (iii) Responses.

In part (iii) students were required to determine whether the function $H(t)$ had a vertical stretch or shrink factor and describe its significance in the context of the lava fragment problem. All four students correctly *recognized* and identified that the quadratic function $H(t)$had a vertical stretch factor 'a' of 16. They were able to correctly tell that the number 16 represented the stretch factor and the negative (-16) represented the concavity (concave down) of the graph of the quadratic function. Ron *build-with* and offered an addition explanation or insight in explaining that the vertical stretch 'a = 16' implied that the lava fragment path was "slim and a higher maximum height". "I think -16 also implies that the curve is thinner and has a higher maximum height than if it had a number like -1"

Interpretation: Ron had a grasp of the significance of the vertical stretch factor ‘a = -16’. He then was able to *construct* a connection between the algebraic function $H(t) = -16t^2 + 350t$ to the graph by suggesting that concavity was “thinner” because the absolute value of ‘a’ was 16. He further suggested that the path of the lava fragment would be ‘wider’ if the absolute value of the vertical stretch factor was less than 16.

Analysis Task 5 Part (iii) Responses.

The semiotic system component identified in this analysis is the identification of the underlying meaning structure of the mathematical sign $H(t) = -16t^2 + 350t$. Using multiple representations: (a) the graph (picture); (b) the generated algebraic symbol $H(t) = -16t^2 + 350t$; and (c) the verbal description in the informal ‘theory’. Third semiotic system component was evident when Ron *constructed* and developed an informal statement ‘theory’ i.e. “the greater the absolute value of ‘a’ the thinner the path of the lava fragment”. This informal ‘theory’ served as the meaning structure that Ron used in order to *construct* and make sense or generate a personal interpretation of the symbol ‘a’ in the quadratic function written in standard form. It is this understanding that Ron used to later to explain the significance of ‘a’ in the context of the problem. This informal theory laid the groundwork for a more formal understanding and explanation of the various component of the semiotic system $H(t) = -16t^2 + 350t$. I inferred that multiple representations afforded Ron to be reflective about the response and the interpretation of his understanding of concavity. Ron was able to transfer his understanding of ‘a’ to offer an explanation of the nature and structure of the concavity of the path of the lava fragment. The graph (picture) supported Ron’s understanding of the significance of parameter ‘a’ by refining his image of how the path of the lava fragment is related to the mathematical sign $H(t) = -16t^2 + 350t$.

Description Task 5 Part (iv.), (v) and (vi) Responses.

In part (iv.) students were to determine the y-intercept of the function $H(t)$ and describe the significance of the y-intercept in the context of the lava fragment problem. All four students correctly *recognized* and identified the y-intercept as$(0, 0)$. They all used the ordered (0, 0) in the graph (picture) and offered the explanation that at y-intercept the x-coordinate of the ordered pair is zero. The significance of the y-intercept was also correctly identified as implying the starting point (i.e. time) of the lava fragment release from the ground.

In part (v.) students were asked to determine the x-intercepts of the function $H(t)$ and describe the significance of the x-intercept in the context of the lava fragment problem. The four students *recognized* and offered correct responses to the x-intercepts as $(0,0)\ and\ (21.875, 0)$ or $\left(\frac{175}{8}, 0\right)$. Ron, Tyron, and Stacey further *build- with* their understanding and offered correct explanations to the significance of the intercepts as the time when the lava fragment leaves the ground and 21.875 as the time it takes for the fragment to hit the ground again. However in Yolanda explanation, she used the term “distance it takes for the lava fragment to hit the ground again”.

Interpretation Task 5 Part (iv.), (v), and (vi) Responses.

Yolanda incorrectly responded to the explanation of the significance of the number 21.875. The unit of measurements were time independent variable and height the dependent variable. Yolanda did not make immediate connection between $H(t)$ and the problem context. To Yolanda the symbol $H(t)$ still represented the horizontal distance that the lava fragment covered. Pseudo-conceptual understanding demonstrated was artificial association where she associated the unfamiliar sign $H(t)$ with the familiar sign $H(x)$ which represents distance covered. Yolanda demonstrated a conscious action i.e. verbal description of $H(t)$, but did not systematically reflect upon her statement. In using the symbol $H(t)$, this reminded her of the mathematical sign $H(x)$ which she was familiar with and was epistemologically more accessible to her. The connection between the two sign was artificial within the context of the problem. In part (vi.), the students were required to find the domain for which $H(t) \geq 0$ and explain the significance of the mathematical sign $H(t) \geq 0$ within the context of the problem. All four students responded correctly to the domain as $[0, 21.875]$. However Ron's solution method involved the use of the intercept function $H(t) = -16t(t - \frac{175}{8})$ to find the domain.

Interpretation: In retracing the graph and inscribing on the x-intercepts, Ron was attempting to explore the connection among; the term domain (verbal description), the symbols ordered pairs $(0, 0\ )\ and\ (21.875, 0\ )$, and their inscription on the graph. His attention moved between his domain response $[0, 21.875]$ and the graph. He constructed a new understanding and realized that the symbol was also represented in the graph and offers the following "I don't know why I took the long route calculating the domain the answer was right here on the graph". By engaging Ron in further reflection and re-examination of the significance of the term domain in this problem context, Ron was able to make a *build-with* and *construct* a personal connection of the meaning of the term domain as the time taken by the lava fragment in the air.

Analysis Task 5 Part (iv), (v) and (vi) Responses.

The component of the semiotic system that was referenced by Yolanda was the mathematical sign (both the verbal and the algebraic sign $H(t)$) that was used to represent the x-intercepts. A reflective approach and a re-examination of the symbol $H(t)$ and looking at the independent variable (t) within the context if the problem, Yolanda was able to refine and construct a new understanding of the significance of the sign $H(t)$ and the interpretation within the context of the problem. The semiotic systems components invoked in Ron's response were: (a) the transformational rule; and (b) the underlying meaning structures (informal theory). He demonstrated competence in processing the quadratic function in intercept form in order to determine the domain$[0, 21.875]$. This was evidence of his understanding of the transformational rule semiotic system component. Though he was comfortable processing the domain algebraically, the understanding of the underlying meaning structure of the system was missing. By referencing the graph (picture) Ron should have realized that the ordered pair $(21.875, 0)$ represented the domain, hence no need to engage in algebraic manipulation. Computational offloading opportunity which refers to the extent to which multiple representations reduce the amount of cognitive effort required to solve the problem was missed. Loose associations between the verbal, algebraic and the graphical representation of the concept of domain. Ron later reflected and re-examined the end of the path of the lava and realized that the solution to the domain was in the picture.

Description Task 5 Part (viii) and (ix).

In Part (viii.) the students were required to determine the height of the lava fragment after 10 seconds. All the four students *recognized* and correctly determined the height as 1900 feet by substituting 10 in the intercept form $H(10) = -16t(t - \frac{175}{8})$. They all responded to $H(10) = 1900\ and\ (10, 1900)$ as the correct mathematical sign they would use to indicate height after 10 seconds.

In part (ix) students were required to determine the amount of time it took the lava fragment to reach a height of 1000 feet above the ground. All four students used the standard form $H(t) = -16t^2 + 350t$ to algebraically determine the time at height 1000feet. They then used a graphing calculator to confirm their answers. In describing the mathematical symbol that they would use to show the height at a given time, they all used both the symbol $H(3.379) = 1000$ and $(3.379, 1000)$.

Ron: I think there are two times when the height is at 1000feet. One here (*points at tin horizontal axis as 3.379s*). This is the time we calculated. I think the other time is here (*time on the horizontal axis where there is 'a' symbol).* See at 1000feet starting from her *at 1000feet on the vertical axis and drawing a horizontal line*) it hits the parabola twice two points (*inscribes on the graph with circles*).

I: What is the value of the question mark?

Ron: Looks like approximately 18.5s. Oh um …same as my other solution in the quad function that I used to calculate 3.379s.

Ron: Also looking at the graph and since the graph is a parabola it appears that except vertex here (*points at the vertex*), it looks like for every height H there are two x coordir Maybe that is why the quadratic symbol $H(t) = -16t^2 + 350t$ has a degree two.

Figure 21. Portion of Transcript Ron's Response to Question viii.

In describing if this was the only time that the lava fragment was at 1000feet, Ron, Tyron and Yolanda responded that there could be another time that the lava fragment was at a height of 1000feet. Ron explanation which summarizes the general response follows.

Interpretation Task 5 Part (viii) and (ix).

Using algebraic manipulation of the standard quadratic function i.e. the quadratic formula, all the four students were able to determine the correct time the lava fragment was at a height 1000feet. Loose association between the graphing and algebraic representation (standard quadratic function) was evident in all the four students' explanations. Looking at the standard quadratic function that they all referenced, the expectation was that they would make deduction about the various properties of the function from the graph. However they only focused on the algebraic manipulation. Ron used an extra approach. Upon further re-examination and reimaging he constructed a new understanding of the path of the lava fragment evidenced by his inscribing on the graph and making a statement that suggested some indication of an association between the graph and the algebraic representation. “I think there are two times when the fragment is at a height 1000feet” line 1. A closer and reflective look at his responses and the graph which included; several pauses, inscription on the graph and reference to his algebraic manipulation, resulted in his making the following conjecture “except at the vertex, for every height there are two x-values” line 10.

Analysis Task 5 Part (viii) and (ix).

The semiotic system component of understanding the underlying meaning structure of the quadratic function in standard form was missing from three students; Tyron, Stacey, and Yolanda. The link between the various components of the mathematical sign $H(t) = -16t^2 + 350t$ were missing. These included: (a) relationship between the graph and the function; and (b) the meaning of the various signs that make up the quadratic function. Using the multiple representations i.e., the graph, verbal descriptions and algebraic symbols invoked in the students the need to re-image, re-examine and reflect on the underlying meaning structure in the transformational rule (processing the solutions). This included; supporting Ron’s understanding of why the quadratic equation $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ yielded two solutions, significance of the symbol $\pm$ in the quadratic equation formula. By; rewriting the quadratic function next to the graph, inscribing on the graph and marking approximately the position on the lava fragment at 1000feet and drawing a horizontal line from the 1000feet y axis position, Ron was trying to make sense of the meaning and the connection between the two representations (graph and algebraic symbol). The movement between the graphical representation and the mathematical sign was an attempt by Ron to make sense of the mathematical object $H(t) = -16t^2 + 350t.$ Reified abstraction Sfard (1991) which is the results of mathematical entity e.g. quadratic function $H(t)$ is perceived as a process at one level and as an object at another level was evident in Ron's work and explanation of two x-values for every y coordinate in a quadratic function. From a semiotic theory perspective, by Ron generating an informal statement "except at the vertex, it looks like for every height y, there are two X maybe that is why this sign has a degree 2" line 12, is an attempt to understand and construct the underlying meaning structure of the mathematical sign $H(t) = -16t^2 + 350t.$ I can infer that the use of multiple representations supported the student's construction of new insight of the quadratic function by providing an opportunity for participants: (a) re-image their understanding of the quadratic functions; (b) develop a reflective approach to understanding the quadratic function; and (c) develop representational versatility in understanding the internal and external links in the quadratic function.

6 SUMMARY AND DISCUSSION

"The transition from signifier-as-an-object-in-itself to signifier-as-a-representation of an-other object is a quantum leap in a subject's consciousness." (Sfard, 2000, p. 79)

Summary and Discussion

Following is a discussion of the research study findings including a discussion of how the emerging themes addressed the three sub-questions. Utilizing semiotic systems theory (Ernest, 2006), the focus of this study was to understand how algebra II students' use and exposure to multiple representations influenced their conceptual understandings of the following families of functions; piecewise, absolute value and quadratics. In particular the focus was to better under-stand how identification and use of the different semiotic systems components i.e., mathematical signs, mathematical transformational rules, and the underlying meaning structures embedded in the mathematical signs, interweaved and facilitated their understanding of the three families of functions.

The research question in this study was: How does exposure to and use of multiple repre-sentations influence algebra II students' understanding and transfer of their algebraic concepts? Specifically the following sub-questions were examined:

1. How does the use of multiple representations influence students' identification of pseudo-conceptual understanding of algebraic concepts?
2. How does the use of multiple representations influence students' transition from pseudo-conceptual to conceptual understanding?
3. How does the use of multiple representations influence students' transfer of their conceptual understanding to other related concepts?

In response to these three sub-questions, analysis of students' written and verbal responses revealed three emerging themes in regard to student's conceptual understanding when using or and exposed to multiple representations. These themes were: (a) re-imaging of conceptual understanding; (b) reflective approach to sign receptions and productions; (c) representational versatility of mathematical signs.

Re-imaging of Conceptual Understanding.

Exposure to and use of multiple representations promoted students' re-imaging of their conceptual understanding of piecewise, absolute value and quadratics families of functions. Re-imaging themes emerged through observing how multiple representations promoted the following constructs; abstraction, generalizations, and transparency between representations. Abstraction refers to process of creating mental entities that served as basis for new action (Kaput, 1989). Participants being involved and exposed to different means of representing same concepts, provided them with a rich and diverse source of representations (Ainsworth, 2006), which they then used to translate and construct references across representations. For example in task 3 (Pool table activity), Tyron used; algebraic symbol $|x-6|$, ordered pairs $(2.5, 2)$ & $(9.5, 2)$ and axis of symmetry to infer that points at the same horizontal distance from axis of symmetry on either sides have the same y-coordinate. He later generalized that for absolute value function, except at the vertex, for every y-coordinate there exists two x-coordinates. From a semiotic perspective exposure to multiple representations facilitated understanding that exposed the underlying meaning structures in absolute value function in task 3. The transparency between representations construct was evident through reified abstraction (Sfard, 1991). Reified abstraction is what results when mathematical entities are perceived as an object at one level are also reconceived as process at another level (Sfard, 1991). For example in task 5 (Lava Fragment Task), the domain symbol $Q(t) \geq 0$ was perceived by Stacey as an object $[0, 21.875]$. Through multiple representation i.e., verbal description, graph, and algebraic symbols in the task, Stacey was able to translate meaning of the symbol $Q(t) \geq 0$ to represent the process time taken by the lava fragment to hit the ground again.

With exposure to different representations of the same concept, participants developed clearer mental images of these families of functions. These mental images later facilitated the participants' conceptual understanding and transfer of constructs like; domain and range in task 1, significance of end-behavior in comparing two outputs in task 2, significance axis of symmetry in task 2 & 3, and the parameters absolute-value and quadratics functions. They potentially were able to visualize for example the link between the end-behavior and comparison of two outputs without necessarily engaging in long algebraic manipulations. Consistent with Chahine (2013), finding in the study on the impact of using multiple modalities, I also observed that the participants mental images that were developed during exposure and interaction with multiple representations later emerged as skills and not as mere recall activities. This was demonstrated in their verbal and written responses. For example in task 2, Ron had a clearer perspective of the y-intercepts presented in; algebraic form $P(0) = 10.5$ & $(10.5, 0)$, graphical form as point on the y-axis, and algebraic form as a symbol/number 10.5 in the function $y = -\frac{3}{2}x + 10.5$.

Reflective Approach to Sign Receptions and Sign Productions.

Use of multiple representations created an opportunity for the participants to shift their approach to understanding and processing of the absolute value functions, piecewise functions and quadratic functions from unreflective exploration of mathematical tasks to one that tapped into their regulatory powers (Ernest, 2008b). Unreflective exploration of mathematical tasks included reliance on; typical examples (Schoenfield, 1992), and social cues (Ernest, 2006) to provide the required responses and answers to the mathematical tasks. Tapping on to regulatory power the participants shifted their sign receptions and productions approach to a careful and systematically reflected upon approach to responding to mathematical tasks that included planning, monitoring, and controlling their responses to assigned tasks. Prior to engaging with multiple representations (verbal descriptions, algebraic symbols, numerical, and graphing) the participants conceptual understanding was characterized by: (a) reaction to their first associations to mathematical tasks without reflecting on the mathematical signs, mathematical transformational rules, and underlying meaning structures in a task; (b) lack of control mechanisms that triggered their sense making mechanisms; and (c) absence of meaning effort (Vinner, 1997; and Berger 2006). Equally important was an observation that prior to using multiple representations students continued to tap on to their previous formal knowledge and understanding of semiotic systems like the linear functions to respond to tasks involving the three functions mentioned above. For example the notion of domain and range in task 2 involved a careful transition in their understanding of domain and range in linear function to piecewise functions where the definition of the functions varies with the independent variable.

In the study I observed the influence of exposure to multiple representations on tapping into the participants' ability to regulate their mathematical activities during the two phases of understanding/ solving a mathematical task. These was during: (a) sign reception phase; and (b) sign production phase. Regulating mathematical activities in the sign reception phase involved spending time and making effort to explore and create the meaning of the mathematical signs in a given task. It included a reflective and controlled approach to the reading of mathematical signs and making and attempt to make sense of their meaning. It also included accessing a repertoire of mathematical transformation rules and a reservoir of meaning structure and applying them in completing an assigned task.

Regulating sign production phase involved self-monitoring and self-reflection of the mathematical signs that they produced. It also involved participants' judgement on whether the mathematical signs they produced followed the conventionalized and institutionalized structuring of mathematical rules of formal production. This observation was consistent with Montiel, et.al. (2009) observation in their study of the use of onto-semiotic approach to identify and analyze mathematical meanings. In their study, discrepancy between institutional and personal meanings of mathematical concepts was detected in their participants' verbal expressions and interpretations of properties of single and multivariate functions. In this study the regulatory power was evident when the participants consistently asked reflective questions privately or loudly as they processed the solutions to the assigned tasks. This was suggestive of self-monitoring and self-regulation of their problem solving problem (Ernest, 2006) an essential characteristic in conceptual understanding.

Representational Versatility of Mathematical Signs.

The third theme that emerged in the findings was the potential representational versatility of mathematical signs in students' verbal and written responses. Representational versatility is described as the ability to seamlessly work with and between representations as students engage in procedural and conceptual interaction with representations (Thomas, 2008). Noticeable behavior that indicated this theme was the identification of computational offloading opportunities with justification. I identified three construct that formed these theme: (a) activation retrieval processes; (b) essential relation focus; and (c) efficient and enhanced analysis.

Activation or triggering of retrieval process allowed students to conveniently search for information that otherwise would not have been obvious using one representation. For example, use of multiple representations refocused the manner in which students dealt with and used the notion of end behavior in a manner that could not be inferred or used in the object itself. In task 4, comparing $Q(30)$ and $Q(40)$ of a quadratic function in which the output were out of the graph (10 by 10) range required computational effort to compare. Essential relation focus allowed students to see the applicability conditions of a representation. This involved identifying the representations that were appropriate for the assigned task. Efficient and enhanced analysis construct involved students' ability to focus on more creative forms of generalizations in addition to standard generalization. For example, in the quadratic function task 5 (Lava Fragment Activity), students were able to generalize that for every y-value there are two values of x except at the vertex. They then extended this generalization to determine the values of times at a given height of 1000ft.

I inferred that exposure to and use of multiple representations supported students' representational versatility ability which in turn facilitated the identification of computational offloading opportunities. These they did by; activating the retrieval process, focusing on the essential relations in multiple representations, and analyzing and generating effective generalizations. Computational offloading refers to the extent to which an activity reduces the amount of cognitive effort required to solve a problem. I argue that identification of computational offloading opportunity that is justified by a high quality self-explanation (Roy & Chi, 2005) is an indication of conceptual understanding. High quality self-explanation refers to statements that demonstrate the generation of inferences, integrating phrases, and various comments that reflect deep analysis of a concept (Roy, & Chi, 2005). This argument follows from the position that in conceptual understanding; the internal links i.e., the link between the different properties and attributes of a concept or a mathematical object are consistent and logical, and the external link i.e., the link between that concept and other concepts are also consistent and logical. (Vygotsky, 1994). Further informed by Sfard (2000) notion that conceptual understanding is demonstrated when students attend to a mathematical object in its entirety and not just as a fragmented aspect of an object, I inferred that the use of multiple representations influenced and supported students explanations of the connections and the links between the different attributes of the three functions; absolute-value function, piecewise function, and quadratic function. Students' oral and written explanations after an interaction with multiple representations, reflected a logical link: (a) internal links between the different explicit elements of a mathematical objects e.g. the parameters in the absolute- value functions $y = a|x - H| + K$; and (b) external links of implicit elements of the same objects like the domain, range, and end behavior.

For example in task 2 use of multiple representations provided participants with an opportunity to make logical link between the end behavior sign $As\ x \to \infty, y \to \infty, as\ x \to -\infty, y \to \infty$, the verbal description of the end-behavior, and the ordered pairs $\mathrm{P}(12)\ and\ P(20)$. The participants were able to compare the values of $P(12)\ and\ P(20)$ and draw conclusion that $P(12) < and\ P(20)$ based on the information from the end behavior. This logical link reduced the amount of cognitive effort that was required to calculate and process both $P(12)\ and\ P(20)$ for comparison purposes. For example in task 2, the internal link of parameters of the graph of the function began to make more sense to the participants when they had an opportunity to interact with the graph, verbal descriptions, table of value and the algebraic expression of the function.

Use of multiple representations supported students' logical link of the parameters of the absolute value functions and the x-intercept concept reducing the amount of cognitive effort required to calculate for example the x-intercepts in examples where the concavity symbol 'a' and the vertical shift symbol 'k' had the same sign. Identification of the y-intercept written in function notation $P(0) = 10.5$ reduced the amount of cognitive effort required to process (i.e. write in slope-intercept form) and solve. In task 3 participants identified that in a quadratic function, points at the same horizontal distances from the axis of symmetry have the same y-coordinate. This clearly explained informal understanding reduced the amount of cognitive effort required by the participants to determine that they will be able to make the shot in the game of pool assigned in the task. Further in task 5 that involved the lava fragment eruption, information given on the graph (21.875, 0) and (0, 0) was easily used to determine the x-intercepts of the graph hence the amount of time that the lava fragment was in the air. Participant further made a logical link of the x-intercepts and the domain for which $H(t) \geq 0$. Participants were able to determine that $H(t) \geq 0$ represented the time with which the lava was in the air. Extending concept of external logical link also included students understanding in task 5 that for in a quadratic function, for every y-value there are two values of x except at the vertex. They were able to extend that understanding to responding to the values of time at a given height 1000ft and reasoned that there were two values of time at height 1000ft. In task 6 emphasis was on understanding and making logical link between parameters of a quadratic function

$Q(x) = -3x^2 + 2x + 5$. Participants easily identified that $Q(0)$ represented the y-intercept and concluded that for a quadratic function written in standard form, $Q(0) = c$ the parameter c represented the y-intercept. In addition $Q\left(-\frac{b}{2(a)}\right)$ represented the absolute max or minimum because the axis of symmetry is defined as. $x = -\frac{b}{2(a)}$. This understanding reduced the amount of cognitive effort required to determine $Q\left(-\frac{2}{2(a)}\right)$. Students also used the logical link of end behavior $As\ x \to \infty, y \to -\infty,\ \ As\ x \to -\infty, y \to -\infty$ to compare $Q(30)\ and\ P(40)$. They concluded that $Q(30) > P(40)$ based on the logical connection they made between end behavior sign and the graph of the function $Q(x)$.

Discussion in the Context of the Research Question

Following is a discussion of the findings and the emerging themes in the context of the research questions.

Research Sub-Question 1.

Research sub-question 1 was; how does exposure to and use of multiple representations influence students' identification of pseudo-conceptual understanding of algebraic concepts? Specifically the study looked at how the exposure to and use of multiple representations influenced the identification of pseudo-conceptual understandings in the following functions; piecewise, absolute-value, and quadratic. To address this sub-question, I used the following construct to identify the observable pseudo-conceptual verbal and written responses: (a) Random associations of mathematical signs which included surface associations, example centered associations, artificial associations, and template oriented associations; (b) Lack of validation effort during sign reception and sign production; and (c) Absence of effort to understand the meaning of the signs, transformational rule, and underlying meaning structures in the signs (Vinner, 1997; and Berger 2006).

From the findings in the analysis of the participants' verbal and written responses, I inferred that the exposure to and the use of multiple representations potentially influenced how students interacted and responded to the task questions assigned. Specifically this exposure provided an opportunity for participants to: (a) re-image their conceptual understanding of the three functions; and (b) reflect on their sign receptions and productions. By participants being involved and exposed to different means of representing the same concept, they were provided with a rich and diverse source of representations (Ainsworth, 2006) which allowed the students to re-image their conceptual understandings. This re-imaging of their conceptual understandings in turn made it easier for the students to construct references across representations providing the participants with a reservoir of resources to validate their understanding of the mathematical signs that they received and produced.

I also inferred that exposure to and use of multiple representations also allowed for participants to be reflective about how they received and produced the mathematical signs. They demonstrated a careful and systematically reflected upon approach to understanding the mathematical signs and in turn responding to the assigned tasks. Reflective approach to understanding their signs created a platform for students to inquire about the reasonableness of their responses including efforts to understand the meaning of the signs, transformational rule, and underlying meaning structures in the signs.

Research Sub-question 2.

Research sub-question 2 was; how does the exposure to and use of multiple representations influence students' transition from pseudo-conceptual to conceptual understanding? Exposure to and use of multiple representations afforded participants with an opportunity to shift their understanding from a procedural dependent to a more reflective approach to understanding and solving the assigned tasks. This approach also demonstrated potential representational transparency between different representations of the same concept. Guided by the RBC+C model and the semiotic system theory model in analyzing instances of transition from pseudo-conceptual to conceptual understanding in the students' verbal and written responses, the following themes emerged in the analysis: (a) reflective approach to understanding their sign receptions and productions; and (b) representational versatility. Pseudo-conceptual understanding of algebraic concept in general and of; piecewise, absolute value, and quadratic functions in particular can be potentially attributed to: (a) randomness of mathematical sign associations; (b) lack of validation effort; and (c) lack of effort in understanding the underlying meaning structure of the mathematical signs. On the contrary conceptual understanding of the algebraic functions named above can be attributed to students' ability to attend to a mathematical object in its entirety (i.e., including its external link) and not just as a fragmented aspect of an object (Sfard, 2000). This includes the ability to make consistent and logical link between the mathematical object e.g., absolute value function and previously learned mathematical entities like end-behaviors and the domain and range applications.

To make these logical links participants in the study tapped on to their regulatory powers (Ernest, 2008) of self-monitoring and self-reflection afforded by exposure to and use of multiple representations to control their responses to the assigned tasks. In addition participants also demonstrated ability to be reflective and control their mathematical sign receptions and sign production activities. Self-monitoring and self- reflections of mathematical signs emerged as skills that participants called upon in developing tools for problem solving the assigned task questions.

Research Sub-question 3.

Research sub-question 3 was; how does exposure to and use of multiple representations influence students' transfer of their conceptual understanding to other related concepts? The three emergent themes in the study were: (a) re-image their conceptual understanding of the three functions; (b) reflective approach to understanding their sign receptions and productions; and (c) representational versatility of mathematical signs. Analysis of students' verbal and written responses to task questions revealed potential improvement on their logical explanations of; external links related to a concept, and how those links related to and connected to other concepts. This involved how the students processed and generalized important attributes in one semiotic system (e.g., absolute-value system) and applied to other semiotic systems (e.g., quadratic function). Afforded by exposure to and use of multiple representations participants in the study demonstrated representational versatility which in turn enabled them to attend to mathematical objects (e.g., absolute value functions) in their entirety and not as a fragmented aspect of an object. Attending to mathematical objects implied that the participant extended their range of flexibility in thinking about the use and application of the piecewise functions, absolute value functions, and quadratic functions.

Recommendations and Implications

This study was inspired by theory of semiotic system (Ernest, 2006). The three themes that emerged in the analysis of students' verbal and written responses were: (a) re-imaging of conceptual understanding; (b) reflective approach to sign receptions and productions; (c) representational versatility during sign reception and production. These themes supported the notion that exposure to and use of multiple representations potentially facilitated the participants' conceptual understanding of the; piecewise, absolute value, and quadratic functions. In particular participants had a rich and diverse source of representations (Ainsworth, 2006), which in turn allowed them to translate and construct references across representations by re-imaging their conceptual understandings. In an attempt to understand and process the three functions, exposure to and use of multiple representations created opportunities for the participants to be more reflective about their sign receptions and sign productions. Further consistent with Sfard (1994), this exposure contribute to the reservoir of meanings that students can draw upon in formulating, developing and operating a semiotic system.

This study recommends that future studies should consider exploring instructional and assessment models that promote a reflective approach to understanding and processing mathematical concepts in various algebraic families of functions. In particular the focus should be exploring how students' exposure to and use of multiple representations can potentially promote the identifying and mastery of; mathematical signs, mathematical transformational rules, and underlying meaning structures embodied in the sign.

Further inspired by the concerns that: (a) there is a need to extend algebra students' conceptual understanding of mathematics beyond procedural competence; (b) students still memorize facts or procedures without understanding the underlying meaning structures in the mathematical concepts and in their procedures; and (c) the primarily procedural orientation to using functions to solve specific problems has led to absence of meaning and coherence for students (Carlson, 1998), this study endeavors to raise practical recommendations to call for change in instructional and assessment models to include exposure to and use of multiple representations from a semiotic perspectives. The findings in this study provide an account of how exposure to and use of multiple representations can support; re-imaging of conceptual understandings, reflective approach to sign receptions and sign productions, and representational versatility of mathematical sign use, which can in turn can play a significant role in potentially providing depth and mastery of algebraic conceptual understanding.

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APPENDICES

Appendix A

Georgia State University
Department of **Middle-Secondary Education and Instructional Technology**
Parental Permission Form

Title: Use of Multiple Representations in Algebra II Instruction

Principal Investigator: Christine Thomas, PhD.
Student Investigator: Isaac Gitonga

Sponsor: None

I. Purpose:

Your child is invited to participate in a research study. The purpose of the study is to investigate the use of multiple representations in mathematics learning instructions. The research will occur during students' regular algebra II class period this term. The study will focus on algebra II students' ability to understand and make translations within and between graphing, numeric, algebraic and verbal modalities of mathematics concepts. As educators the need to improve the teaching and learning of mathematics is our central goal. One of the most reliable sources of information on our teaching practices is the students themselves. As students attempt to make sense of mathematics, their talks and interaction in the classroom provide an important resource for meaning construction. Feedback and students voices are a significant part of collecting information on how educators can improve their teaching practices. Your child is invited to participate in this study because she/he is an algebra II student and a member of a class where the teacher utilizes multiple representations in his classroom instruction. A total of twenty one students in this class will be in the study. The research will occur during the regular algebra II class period this term.

II. Procedures:

I will be observing the teacher during instruction as well as the students. For accuracy in gathering information on the teacher's instruction I will utilize audio recording with a recorder placed at the teacher's desk. I will also be observing students either working individually or in groups. I may ask students to provide brief explanations on how they solve a problem. The work that students will do including the solutions to a given task that they might put up on the board or in writing will be analyzed for mathematical thinking. For accuracy in gathering student's responses, I will utilize brief audio recording without class disruption. Communication during this study will be informal as normal practice. Performance will be anonymous.

III. Risks:

In this study, participants will not have any more risks than you would in a normal day of life. Your child will not be subjected to risk or discomfort physically, psychologically, socially, or academically because of participation in this study.

IV. Benefits:

Participation in this study may benefit your child by providing opportunity to reason and solve mathematical problems and to demonstrate understanding of key algebraic concepts through use of multiple representations. Overall, we hope to gain information about how students make sense of mathematics

V. Voluntary Participation and Withdrawal:

Participation in research is voluntary. Your child does not have to be in this study. If he/she decides to be in the study and change their mind, they have the right to drop out at any time. They may skip responding to any questions or stop participating at any time. Whatever they decide, they will not be penalized in any way.

VI. Confidentiality:

We will keep participant's records private to the extent allowed by law. Your child's name and other facts that might point to your child will not appear when we present this study or publish the results. We will use pseudonym rather than the student's name on the study records. Only Isaac Gitonga will have access to the information that students provide. Information regarding the study will be kept no more than five years and then shredded. The audio recordings will be destroyed immediately after they are transcribed. The transcriptions and other research data will be stored in a locked file cabinet in the researcher's home. The computer used will be password and firewall protected.

VII. Contact Persons:

Call Dr. Christine Thomas (404) 651-2515 or cthomas11@gsu.edu, or Isaac Gitonga at (770) 457-7201 or gitongai@marist.com if you have questions about this study. If you have questions or concerns about your child's rights as a participant in this research study, you may contact Susan Vogtner in the Office of Research Integrity at 404 463- 0674 or svogtner1@gsu.edu.

VIII. Copy of Consent Form to Subject:

We will give you a copy of this consent form to keep.
If you are willing to volunteer for this research, please sign below.

____ ________________
Participant Date

____ ________________
Principal Investigator or Researcher Obtaining Consent Date

Appendix B

Summary of Instructional Sequence for the Advanced Algebra II Class

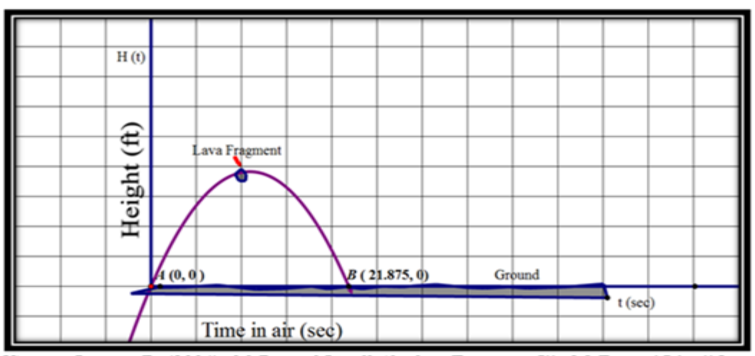
Topic (Lesson Focus)	Number of Lessons	Lessons Objectives	Resources
I. Matrices and Applications	8	• Organize information in a matrix • Add and subtract matrices • Multiply a matrix by scalar • Multiply matrices • Use matrices to model data • Use matrices to solve "real world" problems • Model events using linear equations and systems of linear equations • Use matrices to solve systems of linear equations • Solve systems of linear equations in three variables using linear combination • Solve systems of linear equations in three variables using substitution • Solve systems of linear equations in three variables using matrices with the graphing calculator(GC)	Teachers Handouts Smart board presentations notes (available in class website) Graphing Calculator (GC) Textbook Cha. 2, 3, & 4 Larson, R. (2004). *McDougal Littell Algebra*. Evanston, Ill.: McDougal Littell.
II. Absolute Value & Piecewise Functions	9	• Graph piecewise functions, including step functions • Write equations of piecewise functions given the graph • State the domain & the range of piecewise functions • Recognize and draw graphs of Absolute Value functions. • Find the vertex and axis of symmetry of absolute value functions. • Graph and interpret absolute value functions in the form $y = a\|x - H\| + K$ • Understand and use verbal descriptions of the effect of the parameters; "a", "h", and "k". • Find the equation of absolute value functions given the vertex & a point. • Maximize and minimize absolute value functions, describe end behavior and domain and range • Write the equation of an absolute value function as piecewise	Teachers Handouts Smart board presentations notes (available in class website) Textbook Chapter 2, 3, & 4
III. Quadratic Functions	19	• Recognize and draw graphs of quadratic functions. • Find the vertex and axis of symmetry of quadratic functions • Graph and interpret quadratic functions in the form $y = a(x - H)^2 + K$ • Graph and interpret quadratic functions in the form $y = a(x - p)(x - q)$ • Understand and describe in words the effect of the parameters; "a", "h", and "k" on a quadratics function. • Find the equation of a quadratic function given the vertex and a point • Maximize and minimize quadratic functions, describe end behavior, domain and range. • Use the method of "completing the square" to write quadratic functions in vertex form. • Solve quadratics equations using square roots when appropriate • Solve quadratic equations using the quadratic formula. • Use the discriminant to determine how many solutions a quadratic equations has. • Factor quadratic expressions • Solve quadratic equations by factoring when appropriate • Solve quadratic equations with complex solutions and perform operations with complex numbers. • Find the roots of quadratic functions • Find the end-behavior of quadratic functions • Find the domain & the range of quadratic functions • Find the point of intersection of two quadratic functions	Teachers Handouts Smart board presentations notes (available in class website) Textbook Chapter 5
IV. Polynomial Functions	5	• Evaluate, add, subtract, and multiply polynomial functions • Recognize graphs of polynomial functions and describe their important features • Find the zeros of higher order polynomial functions using the GC • Divide polynomial functions and relate the results to the remainder and the factor theorem	Teachers Handouts Smart board presentations notes (available in class website) Textbook Chapter 6
V. Exam Review	3 days	• Comprehensive Exam Review	Teachers Handouts Exam Review

Appendix C

Task-Based Interview Protocol

Session #: 5 Participant: ________	M T W TH F	Date:
Location: Researcher Classroom	Time: 9:30-10:30	Resources: Textbook/ Smart note /work-sheets
Unit/Topic: QUADRATIC FUNCTION REVIEW	Type of MR: Verbal, Graphical, Algebraic & Table of Values.	

Research Questions: How does the use of MR influence students': (1) identification of PC; (2) transition from PC to CU; (3) transfer of their CU to other related concepts?

TASK	STU-DENT'S RE-SPONSE DISCUS-SIONS/ ACTIVITY	COMMENTS/REFLECTION
Task 5: Task 5 The volcanic cinder cone Puu Puai in Hawaii was formed in 1959 when a massive "lava fountain" erupted at Kilauea Iki Crater, shooting lava hundreds of feet into the air. When the eruption was intense, the height H (in feet) of a typical lava fragment t seconds after being ejected from the ground could be modeled using a quadratic function H (t).  [Source: Larson, R. (2004). *McDougal Littell Algebra*. Evanston, Ill.: McDougal Littell.] Examine the picture (graph) above which models the path of a typical lava fragment in the fountain while the fragment was in the air. Use the information available from the picture to answer the following questions. Be sure to show how you developed your answers. I. Given that the height of the lava fragment after 20 s is 600ft, use the information from the graph to write the quadratic function $H(t)$. Write the function in a. Intercept form b. Vertex form c. Standard form. II. Use this information to answer the following questions. i. For how long was the lava fragment in the air? How did you use the graph to get your answer? ii. Estimate the lava fragment's maximum height above the ground. iii. Does $H(t)$ have a stretch/shrink factor? How can you tell? iv. Find the y-intercept of $H(t)$ what is the significance in this graph?		Sample comments/reflection RQ2 identify the process. How did use of MR influence student recognition of sign (21.875, 0) & (0,0) How did participants use info 20s and 600ft to relate to the three functions? How did the students make connec-tions between the 3 functions, the graph and the verbal descriptions? How did students use info given to identify the flight time? What clues did they utilize? What signs/symbols supported their id of max height? Is there symbol they can use to ASAP id the y-int? Is there symbol they can utilize to id x-int/s? How did they use previous knowledge to find domain. What clues to find sig-nificance of symbol? Symbol H(10)? Find H (t) =1000. Did students realize 2 values of x? Is there an emphasis on identifying symbols and their meanings? Ask why and how certain symbols Did this task questions provokes a closer examination of analytical-graphical translation of the concept function. PC evident! How does the symbols in the quad-ratic function influence the transition

v. Find the x-intercept/s of $H(t)$. vi. Find the domain for which $H(t) \geq 0$. What is the significance of this symbol? vii. What was the height of the lava fragment after 10 seconds? What mathematical symbol would you use to show this height after 10s? viii. Estimate after how long it took the lava fragment to be at height of 1000 feet above ground? What mathematical symbol would you use to describe this height after a given time?		from Analytic & Verbal to graphing do all students see the same interpretation? How important is it for students to explain the terms and expressions they are using. Emphasize on identifying symbols and their meanings. Ask why and how certain symbols
Probing Questions In answering the question above, how did the use of the following influence how you responded to the question? i. Graphical representations (Cartesian graphs) ii. Numerical (Correspondence in a table of values), iii. Algebraic or symbolic representations (Equations expressing the relationship between two or more quantities) iv. Verbal descriptions (Natural language) *Any other questions* ____ *Any other questions*		Any PC evident in the response? Why? Symbolic meaning initiated by students. How does use of MR influence the understanding of this symbol? ______________________ Teacher emphasized meaning of each quad function H(t). Why? How do the students interpret this emphasis? How did the use of graph of H(t) influence conceptual understanding of various symbols in the graph? Did the graph influence understanding of the clues in H(t)? ______________________ How does the use of MR result in different preference of the use of representations? ______________________ Why is it significant that students understand the type of questions/language that is expected? What is the relation with verbal descriptions? Have a verbal description language?

Appendix D

Task-Based Interview Task1

TASK 1

I.

Describe the <u>Domain</u> and the <u>Range</u> of the following functions. Write the Domain and the Range in both, inequality notation and interval notation.

a. Constant function of the form $G(x) = c, where\ c \in \mathbb{R}$

b. Linear function of the form
$F(x) = mx + b, where\ m\&\ b \in \mathbb{R}$

c. Piecewise function

$$R(x) = \begin{cases} 1, & if x < 0 \\ 2, if\ 0 < x \leq 2 \\ -3, if\ 2 < x < 5 \\ 0, & if x \geq 5 \end{cases}$$

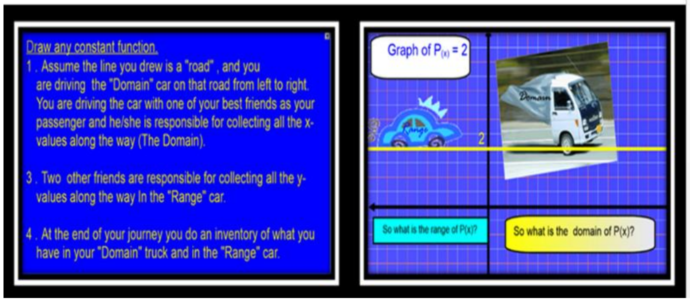
d. Piecewise function

$$P(x) = \begin{cases} 0, & if\ x = 0 \\ 3x - 1, & if\ x > 0 \\ -x, if - 4 < x < 0 \\ -4, & if\ x \leq -4 \end{cases}$$

II

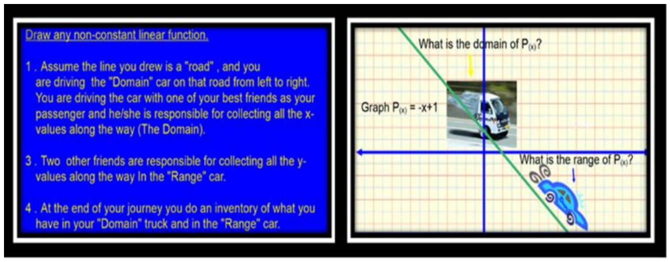
Refer to the <u>Smart Notes "Domain truck & Range car"</u> graphical animations discussed in class. Answer the following questions.

Describe the Domain and the Range of each of the following functions in the light of the Smart Notes <u>"Domain truck" and the "Range Car"</u> discussion. Write the Domain and the Range in both, inequality notation and interval notation.



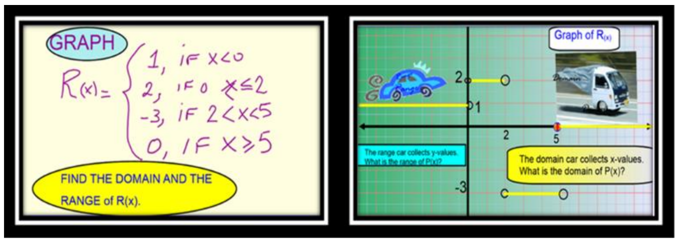
a) Multiple representations of a Constant function of the form $G(x) = c, where\ c \in \mathbb{R}$ using graphical animation.

b). Multiple representations of a Linear function of the form $F(x) = mx + b, where\ m\&\ b \in \mathbb{R}$ using graphical animation.



c) Multiple representation of the following Piecewise function (involving constant functions).

$$R(x) = \begin{cases} 1, & if x < 0 \\ 2, if\ 0 < x \leq 2 \\ -3, if\ 2 < x < 5 \\ 0, & if x \geq 5 \end{cases}$$



d) Multiple representation of the following Piecewise function Piecewise function

$$R(x) = \begin{cases} 0, & if\ x = 0 \\ 3x - 1, & if\ x > 0 \\ -x, if - 4 < x < 0 \\ -4, & if\ x \leq -4 \end{cases}$$

Appendix E

Task Based Interview Task 2

TASK 2

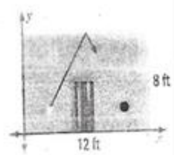
Use the following information to determine the absolute value function named $P(x)$. I. The vertex of $P(x)$ is $(5, 3)$ II. $P(-3) = 15,\ \ P(18) = 22.5,\ \ P(0) = 10.5$ III. The end behavior: $As\ x \to \infty, y \to \infty,\ \ As\ x \to -\infty, y \to \infty.$ Using the information above a. Find two formulas for $P(x)$, one with absolute value symbol and one without the absolute value symbol. b. Find y-intercept of $P(x)$. c. Find the equation of the axis of symmetry. What is the significance of the axis of symmetry? d. Graph $P(x)$. Use a ruler and be neat. e. Explain using "sophisticated language" how the graph of $P(x)$ was could be obtained from the parent function $y = \| x \|$.	f. Find absolute max/min of $P(x)$. How can you tell if it is absolute max or min? g. Determine the x-intercept/s of $P(x)$. Show your work analytically or (JYA) h. Find the domain and the range of $P(x)$. Write answer in interval notion. i. Find $P(12)$. Compare to $P(20)$. Use symbols $<, =$ $or >$. j. Compare $P(2)$ and $P(8)$. Use symbols $<, = or >$. Explain your response.

Appendix F

Task-Based Interview Task 3

TASK 3

<table>
<tr>
<td>
You are trying to make a hole-in-one on the miniature golf green shown. Imagine that a coordinate plane is placed over the golf green. The golf ball is at (2.5, 2) and the hole is at (9.5, 2). You are going to bank the ball off the side wall of the green at (6, 8). Write the equation for the path of the ball and determine if you make the shot. J.Y.A.



[Source: Larson, R. (2004). *McDougal Littell Algebra*. Evanston, Ill.: McDougal Littell]
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<td>
a). In the space below, show how you developed your answer. Be neat and organized.
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Appendix G

Task-Based Interview Task 4

TASK 4

Use the following information to determine a quadratic function named $Q(x)$. Write $Q(x)$ in Standard Form.

I. The linear coefficient is "2"

II. The constant is "5"

III. The leading coefficient is an integer between -4 and -2.

Use this information to answer the following questions. .

i. Find the concavity of $Q(x)$. How can you tell?

ii. Find the y-intercept of $Q(x)$.

iii. Find $Q(0)$.

iv. Find the value/s of x for which $Q(x) = 0$. JYA

v. Find the absolute max or min.

vi. Find the domain for $Q(x) \geq 0$. What is the meaning of this symbol?

vii. Find the domain for $Q(x) < 0$.

viii. What is the meaning of this symbol?

ix. Find $Q\left(-\frac{2}{2(-3)}\right)$. What is the significance of this symbol?

x. Find the domain if the range is given as $-\frac{2^2}{4(-3)} + 5$.

xi. Compare $Q(30)$ & $Q(40)$. Use $<, =, >$

xii. Given that $Q(-1) = 0 \; and \; Q\left(\frac{5}{3}\right) = 0$, write the equation of the axis of symmetry.

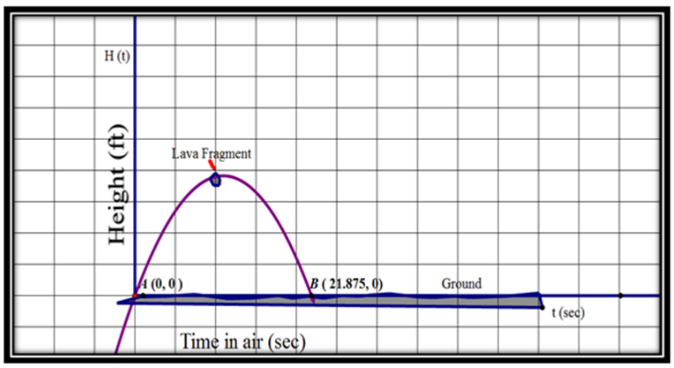
Appendix H

Task-Based Interview Task 5

TASK 5

Task 5

The volcanic cinder cone Puu Puai in Hawaii was formed in 1959 when a massive "lava fountain" erupted at Kilauea Iki Crater, shooting lava hundreds of feet into the air. When the eruption was intense, the height H (in feet) of a typical lava fragment t seconds after being ejected from the ground could be modeled using a quadratic function H (t).



[Source: Larson, R. (2004). *McDougal Littell Algebra*. Evanston, Ill.: McDougal Littell.]

Examine the picture (graph) above which models the path of a typical lava fragment in the fountain while the fragment was in the air. Use the information available from the picture to answer the following questions. Be sure to show how you developed your answers.

I. Given that the height of the lava fragment after 20 s is 600ft, use the information from the graph to write the quadratic function $H(t)$. Write the function in
a. Intercept form
b. Vertex form
c. Standard form.

II. Use this information to answer the following questions.
i. For how long was the lava fragment in the air? How did you use the graph to get your answer?

ii. Estimate the lava fragment's maximum height above the ground.

iii. Does $H(t)$ have a stretch/shrink factor? How can you tell?

iv. Find the y-intercept of $H(t)$ what is the significance in this graph?

v. Find the x-intercept/s of $H(t)$.

vi. Find the domain for which $H(t) \geq 0$. What is the significance of this symbol?

vii. What was the height of the lava fragment after 10 seconds? What mathematical symbol would you use to show this height after 10s?

viii. Estimate after how long it took the lava fragment to be at height of 1000 feet above ground? What mathematical symbol would you use to describe this height after a given time?

Appendix I

Ron's task-based interview transcripts: Task 4

Ron: Umm… $Q(x) = -3x^2 + 2x + 5$. I substituted $a = -3$, $b = 2$, and
$c = 5$ in the standard form model $Q(x) = ax^2 + bx + c$.
Ron: Concavity of $Q(x)$ is opens down. Since the leading coefficient $a = -3$
The y-intercept of $Q(x)$ is $(0, 5)$ (*proceeds to substitute* 0 *for x in the function*
$Q(x) = -3x^2 + 2x + 5$). Y-intercept = 5.
Ron: Concavity of $Q(x)$ is opens down. Since the leading coefficient $a = -3$
The y-intercept of $Q(x)$ is $(0, 5)$ (*proceeds to substitute* 0 *for x in the function*
$Q(x) = -3x^2 + 2x + 5$). Y-intercept = 5. $Q(0) = 5$ Since it represents the
y-intercept.
When x is zero, y is the y-intercept. For the value/s of x for which $Q(x) = 0$. I
will substitute 0 for y and solve the equation *(proceeds to solve equation*
$0 = -3x^2 + 2x + 5$ *by factoring* $0 = (-3x + 5)(x + 1)$*)* the x-values are
then $x = -1$ & $x = \frac{5}{3}$.
I: What is the significance of the answer in part (iv)?
Ron: You can use the GC to find the answer (*Entering the equation*
$y = -3x^2 + 2x + 5$ *in the GC and graphing and accessing the*
table of value mode) It looks like the answers to the $Q(x) = 0$ represents
the x-intercepts. Because at the x-intercepts the y-coordinate is zero and this
(*points at the x-intercepts*) is where the points are found. To find the abs max
I will use the equation $x = -\frac{b}{2a}$ to identify the x-coordinate at vertex.

The x-coordinate is $x = -\frac{2}{2(-3)}$ which is equal to $\frac{1}{3}$. For the vertex coordinate is $(-\frac{b}{2a}, Q(-\frac{b}{2a}))$ (*proceeds to calculate the y-coordinate and writes*) vertex is $(\frac{1}{3}, \frac{16}{3})$. The absolute max is $y = \frac{16}{3}$ since the graph opens down.

The domain for which $Q(x) \geq 0$ is from here to here (*on the whiteboard there is a projected graph of Q (x) he inscribes circles at both x-intercepts. He points at the parabolic portion of the graph above the x-axis)*. This is the area that covers the domain. The domain of $Q(x) \geq 0$ in interval notation$[-1, \frac{5}{3}]$. The symbol $Q(x) \geq 0$ represents the part of the graph that is above the x-axis. The domain of $Q(x) < 0$ in interval notation $(-\infty, -1) \cup (\frac{5}{3}, \infty)$. The symbol $Q(x) < 0$ and represents the part of the graph that is below the x-axis.

To find $Q\left(-\frac{2}{2(-3)}\right)$ umm… I think this is same as the y-coordinate of the vertex. Looking at $x = -\frac{b}{2a}$ a is -3, and b = 2. So $Q\left(-\frac{2}{2(-3)}\right)$ is equals to $y = \frac{16}{3}$. It is also same as the absolute max in part (v) (*proceeds to compare the answer in part (v), the function* $Q(x) = -3x^2 + 2x + 5$ *and the answer to this question*). If the range is $-\frac{2^2}{4(-3)} + 5$ the domain will be $\frac{1}{3}$.

I: How did you determine that?

Ron: This number $-\frac{2^2}{4(-3)} + 5$ represents the y-coordinate. Since at the vertex the x-coordinate is $x = -\frac{b}{2a}$ and the y-coordinate is $-\frac{b^2}{4a} + c$, I think $-\frac{2^2}{4(-3)} + 5$ is the absolute max umm… the domain will be the $x = -\frac{b}{2a}$ which is

$-\frac{2}{2(-3)} = \frac{1}{3}$

In comparing $Q(30)$ & $Q(40)$ I think $Q(30) > Q(40)$.

I: Why do you think $Q(30) > Q(40)$?

Ron: Because from end behavior as $x \to \infty,\ y \to -\infty$ it means that the further the value of x is away from the vertex the lower the value of y. $Q(-1) = 0$ & $Q\left(\frac{5}{3}\right) = 0$. Since these two (*pointing at* $Q(-1) = 0$ & $Q\left(\frac{5}{3}\right)$) are the x-intercept the axis of symmetry is halfway between the two numbers. So the answer is $x = \frac{1}{3}$. (*he shows the following calculations on the board* $x = \frac{\frac{5}{3}-1}{2}$ *as the calculation of the axis of symmetry*).