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Maximum Likelihood Estimation of Parameters in Exponential Power Distribution with Upper Record Values

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MAXIMUM LIKELIHOOD ESTIMATION OF PARAMETERS IN EXPONENTIAL POWER DISTRIBUTION WITH UPPER RECORD VALUES

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the requirements for the degree of

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in

STATISTICS

by

Tianchen Zhi

2017

To: Dean Michael R. Heithaus
College of Arts, Sciences and Education

This thesis, written by Tianchen Zhi, and entitled Maximum Likelihood Estimation of Parameters in Exponential Power Distribution with Upper Record Values, having been approved in respect to style and intellectual content, is referred to you for judgment.

We have read this thesis and recommend that it be approved.

Florence George

Kai Huang

Jie Mi, Major Professor

Date of Defense: March 27, 2017

The thesis of Tianchen Zhi is approved.

Dean Michael R. Heithaus
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Andrés G. Gil
Vice President for Research and Economic Development
and Dean of the University Graduate School

Florida International University, 2017

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ABSTRACT OF THE THESIS

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by

Tianchen Zhi

Florida International University, 2017

Miami, Florida

Professor Jie Mi, Major Professor

The exponential power (EP) distribution is a very important distribution that was used by survival analysis and related with asymmetrical EP distribution. Many researchers have discussed statistical inference about the parameters in EP distribution using i.i.d random samples. However, sometimes available data might contain only record values, or it is more convenient for researchers to collect record values. We aim to resolve this problem.

We estimated two parameters of the EP distribution by MLE using upper record values. According to simulation study, we used the Bias and MSE of the estimators for studying the efficiency of the proposed estimation method. Then, we discussed the prediction on the next upper record value by known upper record values. The study concluded that MLEs of EP distribution parameters by upper record values has satisfactory performance. Also, prediction of the next upper record value performed well.

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1. INTRODUCTION

The exponential power (EP) distribution was firstly introduced as a lifetime model by Smith & Bain (1975). The EP distribution has been discussed by many authors, for examples, Leemis (1986), Rajarshi & Rajarshi (1988), Hanagal & Dabade (2015), among others. Moreover, exponential power distribution is not only used by survival analysis but is also related with asymmetrical exponential power distributions in statistics as mentioned in Hazan et al. (2003) and Delicado & Goria (2008).

A random variable is said to have an exponential power distribution with shape parameter $\theta > 0$ and scale parameter $\lambda > 0$ if its probability density function is given by

$$f(t) = \lambda\theta t^{\theta-1}\exp(\lambda t^{\theta})\exp(1-\exp(\lambda t^{\theta})), \qquad t > 0.$$

The corresponding survival and hazard rate functions are given by

$$S(t) = P(T > t) = \exp(1-\exp(\lambda t^{\theta})), \qquad t > 0$$

and

$$h(t) = \frac{f(x)}{S(x)} = \lambda\theta t^{\theta-1}\exp(\lambda t^{\theta}), \qquad t > 0.$$

In recent years, many researchers have discussed statistical inference about the parameter in the EP model from different perspectives. These works include Xie et al. (2002), Barriga et al. (2010), Lemonte (2013), etc. All of their estimators of the parameters are estimated by i.i.d. random samples.

However, in many practical situations, either the available data contain only record values, or it is more convenient for researchers to collect record values. We will take care of this concern.

Definition and real example of record values:

Let $X_1, X_2, \ldots$ be an infinite sequence i.i.d. random variables having the same distribution as the population described by random variable X. An observation X_j will be called an upper record value (or simply record) if its value exceeds those of all previous observations (Arnolde et al (1998)). Thus, X_j is a record if $X_j > X_q$, $\forall$ $1 \le q \le j-1$. An analogous definition deals with lower record values. Then we assume that X_j is observed at times j. The record time sequence $\{T_i, i \ge 0\}$ is defined in the following manner: $T_0 = 1$ with probability 1 and for $i \ge 1$, $T_i = \min\{j : X_j > X_{T_{i-1}}\}$. The record value sequence $\{R_i\}$ is then defined by

$$R_i = X_{T_i}, \qquad i = 0,1,2,\ldots .$$

Here R_0 is referred to as the reference or the trivial record (Arnolde et al. (1998)).

There are many real examples related with record values. Air quality researchers estimated and obtained confidence interval of parameters of the model for air quality by upper record values (Wu & Tseng (2006), Jafari & Zakerzadeh (2015)). Meanwhile, new joint confidence region for the parameters was obtained using records.

Engineering consideration of the breakdown time of electrical insulating fluid at constant voltage level (Nelson (1982)). Huang & Mi (2015) used records data to compute MLEs of parameters for the model and estimated the prediction interval of the next record value.

The incandescent lamp failure data presented in Davis (1952) was consisted of lifetimes of 417 40-W 110-V incandescent lamps taken from 42 weekly quality control forced-life test samples. Cramer &Naehrig (2012) prompted two versions of generating the record data by the complete sample. They researched MLEs of parameters for the model of incandescent lamp lifetimes using record data.

Therefore, using record values to estimate the parameters of EP distributions will be meaningful and important in those situations. We will investigate the existence and uniqueness of the maximum likelihood estimators of the two parameters λ and θ in the EP distribution using the upper record values. The performance of the MLEs will be explored with simulated data.

The notations, frequently used in the paper are given below for easy reference.

Notation

$EPD(\lambda,\theta)$	Exponential power distribution with parameters λ and θ.
λ	Scale parameter in exponential power distribution, $\lambda > 0$
θ	Shape parameter in exponential power distribution, $\theta > 0$
$\{X_i, i \geq 1\}$	i.i.d random variables with $EPD(\lambda,\theta)$
R_i	$0 \leq i \leq n$ The first (n+1) upper record values associated with $\{X_i, i \geq 1\}$
$\hat{\lambda}$	Maximum likelihood estimator of λ
$\hat{\theta}$	Maximum likelihood estimator of θ

2. MODEL AND LIKELIHOOD FUNCTION

Consider exponential power distribution with parameters $\lambda > 0$ and $\theta > 0$. The CDF and pdf are given as

$$F(t) = 1 - \exp(1 - \exp(\lambda t^{\theta})), \qquad t > 0 \tag{1}$$

and

$$f(t) = \lambda \theta t^{\theta - 1} \exp(\lambda t^{\theta}) \exp(1 - \exp(\lambda t^{\theta})), \qquad t > 0. \tag{2}$$

The fail rate function of this distribution is this

$$h(t) = \frac{f(x)}{S(x)} = \lambda \theta t^{\theta - 1} \exp(\lambda t^{\theta}), \qquad t > 0. \tag{3}$$

Let $R_0, R_1, R_2, ..., R_n$ be the first (n+1) upper record values of the exponential power population described by (1)-(3). It is well-known that the joint pdf of ($R_0, R_1, R_2, ..., R_n$) is given by

$$f_{R_0, R_1, ..., R_n}(r_0, r_1, ..., r_n) = f(r_n) \prod_{i=0}^{n-1} h(r_i), \qquad 0 < r_0 < r_1 < ... < r_n < \infty. \tag{4}$$

For details we refer to Arnold (1998) with the help of (4) the likelihood function of parameter (λ, θ) is clearly. Given as

$$L(\theta, \lambda) = (\lambda \theta)^{n+1} \exp(1 - e^{\lambda r_n^{\theta}}) \prod_{i=0}^{n-1} r_i^{\theta - 1} e^{\lambda r_i^{\theta}}. \tag{5}$$

Hence the log-likelihood function of (λ, θ) is

$$l(\theta, \lambda) = \ln L(\theta, \lambda) = (n+1)\ln(\lambda) + (n+1)\ln(\theta) + (1 - e^{\lambda r_n^{\theta}}) + (\theta - 1)\sum_{i=0}^{n} \ln(r_i) + \lambda \sum_{i=0}^{n} r_i^{\theta}. \tag{6}$$

3. MLE OF SCALE PARAMETER λ WITH KNOWN SHAPE PARAMETER

In this section we study the MLE of parameter λ. Assume $\theta = \theta_0$ is known.

From (6) we donate

$$l_1(\lambda) = (n+1)\ln(\lambda) + (n+1)\ln(\theta_0) + (1 - e^{\lambda r_n^{\theta_0}}) + (\theta_0 - 1)\sum_{i=0}^{n} \ln(r_i) + \lambda \sum_{i=0}^{n} r_i^{\theta_0} \tag{7}$$

We have

$$l_1'(\lambda) = \tfrac{dl_1(\lambda)}{d\lambda} = \frac{n+1}{\lambda} - e^{\lambda r_n^{\theta_0}} r_n^{\theta_0} + \sum_{i=0}^{n} r_i^{\theta_0} \,. \tag{8}$$

It is easy to see that

$$\lim_{\lambda \to 0^+} l_1'(\lambda) = +\infty \tag{9}$$

and

$$\lim_{\lambda \to +\infty} l_1'(\lambda) = -\infty \,. \tag{10}$$

That imply that the likelihood equation $l_1'(\lambda) = 0$ has at least one solution in $(0, \infty)$.

We further have

$$l_1''(\lambda) = \tfrac{d^2 l_1(\lambda)}{d\lambda^2} = -\frac{n+1}{\lambda^2} - e^{\lambda r_n^{\theta_0}} r_n^{2\theta_0} < 0 \,.$$

Therefore, $l_1'(\lambda)$ is a concave function of $\lambda \in (0, \infty)$. The above results show that the log-likelihood function $l_1(\lambda)$ attain its maximum over $(0, \infty)$ at a unique point $\hat{\lambda}$.

Therefore, the MLE of λ uniquely exacts and is the unique solution of the equation:

$$\frac{n+1}{\lambda} - e^{\lambda r_n^{\theta_0}} r_n^{\theta_0} + \sum_{i=0}^{n} r_i^{\theta_0} = 0 \,. \tag{11}$$

4. MLE OF SHAPE PARAMETER θ WITH KNOWN SCALE PARAMETER

Let the scale parameter $\lambda = \lambda_0$ is known. In this case, we express the log-likelihood function of θ as:

$$l_2(\theta) = (n+1)\ln(\lambda_0) + (n+1)\ln(\theta) + (1 - e^{\lambda_0 r_n^\theta}) + (\theta - 1)\sum_{i=0}^{n} \ln(r_i) + \lambda_0 \sum_{i=0}^{n} r_i^\theta . \qquad (12)$$

We have

$$l_2'(\theta) = \tfrac{dl_2(\theta)}{d\theta} = \frac{n+1}{\theta} - \lambda_0 e^{\lambda_0 r_n^\theta} r_n^\theta \ln(r_n) + \sum_{i=0}^{n} \ln(r_i) + \lambda_0 \sum_{i=0}^{n} r_i^\theta \ln(r_i) . \qquad (13)$$

Below we explore the limiting behavior of $l_2'(\theta)$ as $\theta \to 0^+$ and $\theta \to +\infty$.

Claim 1.

$$\lim_{\theta \to 0^+} l_2'(\theta) = +\infty .$$

Since when $\theta \to 0^+$,

$$\frac{n+1}{\theta} \to +\infty$$

and

$$\lambda_0 e^{\lambda_0 r_n^\theta} r_n^\theta \ln(r_n) + \sum_{i=0}^{n} \ln(r_i) + \lambda_0 \sum_{i=0}^{n} r_i^\theta \ln(r_i) \to \lambda_0 e^{\lambda_0} \ln(r_n) + \sum_{i=0}^{n} \ln(r_i) + \lambda_0 \sum_{i=0}^{n} \ln(r_i) > -\infty$$

Claim 2.

$$\lim_{\theta \to +\infty} l_2'(\theta) < 0 .$$

As $\theta \to +\infty$ it holds that

$$\lim_{\theta \to +\infty} l_2'(\theta) = \sum_{i=0}^{n} \ln(r_i) - \lambda_0 \lim_{\theta \to +\infty} [r_n^\theta \ln(r_n) e^{\lambda_0 r_n^\theta} - \sum_{i=0}^{n} r_i^\theta \ln(r_i)] . \qquad (14)$$

Since $\frac{n+1}{\theta} \to 0$, and $\sum_{i=0}^{n} \ln(r_i) - \lambda_0 r_n^{\theta} \ln(r_n)[e^{\lambda_0 r_n^{\theta}} - \sum_{i=0}^{n} \frac{r_i^{\theta} \ln(r_i)}{r_n^{\theta} \ln(r_n)}]$

when $\theta \to +\infty$,

$$\frac{r_i^{\theta} \ln(r_i)}{r_n^{\theta} \ln(r_n)} = \left(\frac{r_i}{r_n}\right)^{\theta} \left(\frac{\ln(r_i)}{\ln(r_n)}\right) \to 0, \tag{15}$$

and here

$$\lim_{\theta \to +\infty} \sum_{i=0}^{n} \frac{r_i^{\theta} \ln(r_i)}{r_n^{\theta} \ln(r_n)} = 0 + 0 + \ldots + \left(\frac{r_n}{r_n}\right)^{\theta} \left(\frac{\ln(r_n)}{\ln(r_n)}\right) = 1.$$

Thus,

$$\lim_{\theta \to +\infty} l_2'(\theta) = \sum_{i=0}^{n} \ln(r_i) - \lambda_0 \lim_{\theta \to +\infty} r_n^{\theta} \ln(r_n)[e^{\lambda_0 r_n^{\theta}} - 1]. \tag{16}$$

Now we study 3 case separately.

Case 1.

$$r_n > 1.$$

In this case, clearly

$$\lim_{\theta \to +\infty} l_2'(\theta) = -\infty.$$

Case 2.

$$r_n < 1.$$

We have $r_n^{\theta} \to 0$ and hence $r_n^{\theta} \ln(r_n)[e^{\lambda_0 r_n^{\theta}} - 1] \to 0$.

There functions imply

$$\lim_{\theta \to +\infty} l_2'(\theta) = \sum_{i=0}^{n} \ln(r_i) < 0 \text{ since } 0 < r_0 < r_1 < \ldots < r_n < 1.$$

Case 3.

$$r_n = 1.$$

We have $r_n^{\theta} = 1$.

$\lim\limits_{\theta\to+\infty} l_2'(\theta) = \sum_{i=0}^{n} \ln(r_i) - \lambda_0 \ln(1)(e^{\lambda_0} - 1) < 0$ since $0 < r_0 < r_1 < \ldots < r_{n-1} < 1$ and $\lambda_0 > 0$.

The above argument show that at any case

$$\lim_{\theta\to+\infty} l_2'(\theta) < 0.$$

We have seem that $\lim\limits_{\theta\to 0^+} l_2'(\theta) = +\infty$ and $\lim\limits_{\theta\to+\infty} l_2'(\theta) < 0$ so that equation $l_2'(\theta) = 0$ has at least one solution in the interval $(0, \infty)$.

Recall that $l_2(\theta)$ is given in (12) we have

$$\lim_{\theta\to 0^+} l_2(\theta) = -\infty.$$

In order to investigate the sign of $\lim\limits_{\theta\to+\infty} l_2(\theta)$, we reconsider the above three case.

Case 1.

$$r_n > 1.$$

In this case, we have three facts:

Fact 1: $\lim\limits_{\theta\to+\infty} \dfrac{r_i^{\theta}}{e^{\lambda_0 r_n^{\theta}}} = 0$.

Fact 2: $\lim\limits_{\theta\to+\infty} \dfrac{\theta - 1}{e^{\lambda_0 r_n^{\theta}}} = 0$.

Fact 3: $\lim\limits_{\theta\to+\infty} \dfrac{\ln(\theta)}{e^{\lambda_0 r_n^{\theta}}} = 0$.

Then

$$\lim_{\theta\to+\infty} l_2(\theta) = 1+(n+1)\ln(\lambda_0) + \lim_{\theta\to+\infty} e^{\lambda_0 r_n^\theta}\left[\frac{(n+1)\ln(\theta)}{e^{\lambda_0 r_n^\theta}} + \frac{\theta-1}{e^{\lambda_0 r_n^\theta}}\sum_{i=0}^{n}\ln(r_i) + \sum_{i=0}^{n}\frac{r_i^\theta}{e^{\lambda_0 r_n^\theta}} - 1\right]$$

According above three facts, we have

$$\lim_{\theta\to+\infty} l_2(\theta) = 1+(n+1)\ln(\lambda_0) + \lim_{\theta\to+\infty} e^{\lambda_0 r_n^\theta}(-1) = -\infty \ .$$

Case 2.

$$0 < r_n < 1 .$$

In this case,

$$\lim_{\theta\to+\infty} l_2(\theta) = (n+1)\ln(\lambda_0) + (1-e^{\lambda_0 r_n^\theta}) + \lim_{\theta\to+\infty}(n+1)\ln(\theta)\left[1+\frac{(\theta-1)\sum_{i=0}^{n}\ln(r_i)}{(n+1)\ln(\theta)}\right]$$

$$= -\infty$$

Case 3.

$$r_n = 1 .$$

$$\lim_{\theta\to+\infty} l_2(\theta) = (n+1)\ln(\lambda_0) + (1-e^{\lambda_0}) + \lim_{\theta\to+\infty}(n+1)\ln(\theta)\left[1+\frac{(\theta-1)\sum_{i=0}^{n}\ln(r_i)}{(n+1)\ln(\theta)}\right] + \lambda_0 \lim_{\theta\to+\infty}\sum_{i=1}^{n} r_i^\theta$$

$$= -\infty .$$

Summarizing the above, we have shown that $\lim_{\theta\to 0^+} l_2(\theta) = \lim_{\theta\to+\infty} l_2(\theta) = -\infty$ and the equation:

$$l_2'(\theta) = 0 \tag{17}$$

has at least one solution in $(0,\infty)$. Therefore, $l_2(\theta)$ a continuous function of θ on $(0,\infty)$ must attain it maximum at some interior point of $(0,\infty)$.

Unfortunately the uniqueness of the MLE of θ is still an open question. However, we can compute the MLE $\hat{\theta}$ of θ.

All in all, in order to estimate θ and λ we need to solve the system of equations

$$\frac{n+1}{\lambda} - e^{\lambda r_n^\theta} r_n^\theta + \sum_{i=0}^{n} r_i^\theta = 0$$

$$\frac{n+1}{\theta} - \lambda e^{\lambda r_n^\theta} r_n^\theta \ln(r_n) + \sum_{i=0}^{n} \ln(r_i) + \lambda \sum_{i=0}^{n} r_i^\theta \ln(r_i) = 0 . \qquad (18)$$

Since this non-linear equation set cannot be solved directly, a numerical root finding technique must be used. There are a lot of ways can be used to find the roots. We will use the well-known Newton-Raphson method.

5. SIMULATIONS FOR MAXIMUM LIKELIHOOD ESTIMATIONE

We describe the computer simulations and discuss the behavior of the maximum likelihood estimators of the exponential power distribution parameters from upper record data. The analytical work has been done by using R.

Step 1: Generated $x_0, x_1, x_2, \ldots$ from exponential power distribution with parameters λ and θ.

To this purpose, the probability integrate transform is employed. Thus, we first generate $u_0, u_1, u_2, \ldots$ from Uniform distribution $(0,1)$. Then, solve the following equation for x.

$$u = F(x) = 1 - \exp(1 - e^{\lambda x^{\theta}}) . \tag{19}$$

We can get

$$x = F^{-1}(u) = \left(\frac{\ln(1 - \ln(1-u))}{\lambda} \right)^{1/\theta} . \tag{20}$$

By this way, we can produce random observation $x_0, x_1, x_2, \ldots$ from R project easily.

Step 2: Obtain a sample of upper record value.

A record sample $r_0, r_1, r_2, \ldots$ was produced form sequence $x_0, x_1, x_2, \ldots$ obtained in step 1. R code upper.record.value can help us to do it.

Step 3: Calculate the MLEs $\hat{\lambda}$ and $\hat{\theta}$.

The MLEs of $\hat{\lambda}$ and $\hat{\theta}$ for the record sample are found by Newton-Raphson method using equation (18). Name the resulting estimates as $\hat{\lambda}_i$ and $\hat{\theta}_i$.

Step 4: Repeat step 1-3 1000 times

The above process repeated 1000 times. Consequently, we have a set of 1000 parameter estimates using method.

Step 5: Evaluate the performance of $\hat{\lambda}$ and $\hat{\theta}$.

For each pair of given (λ, θ). Compute the mean, bias and mean squared error (MSE) for each of estimates of the parameters using the 1000 values of $\hat{\lambda}$ and $\hat{\theta}$.

The procedure described above was repeated for different complete sample with sizes m=50, 100, 200, 500, 1000, 2000, 5000, 10000. Then the mean values $\bar{\lambda}$ and $\bar{\theta}$ and mean squared errors values MSE_λ and MSE_θ were computed using:

$$\bar{\lambda} = \frac{1}{1000}\sum_{i=1}^{1000}\hat{\lambda}_i , \quad \bar{\theta} = \frac{1}{1000}\sum_{i=1}^{1000}\hat{\theta}_i ,$$

$$MSE_\lambda = \frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{\lambda}_i - \lambda_0\right)^2 , \quad MSE_\theta = \frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{\theta}_i - \theta_0\right)^2$$

Firstly we study the behavior of $\hat{\lambda}$. We assume $\lambda_0 = 2$ and $\theta_0 = 0.5$. Table 1 and Figure 1 show the simulation results of $\hat{\lambda}$.

Table 1. Bias and MSE from simulations of $\hat{\lambda}$ when $\lambda_0 = 2$ and $\theta_0 = 0.5$.

m	Bias of $\hat{\lambda}$	MSE of $\hat{\lambda}$
50	59.0072	108027.6159
100	31.1752	67270.6156
200	6.9452	10458.8708
500	0.0313	0.2179
1000	-0.0458	0.1553
2000	-0.0679	0.1272
5000	-0.0968	0.1351
10000	-0.0845	0.1137

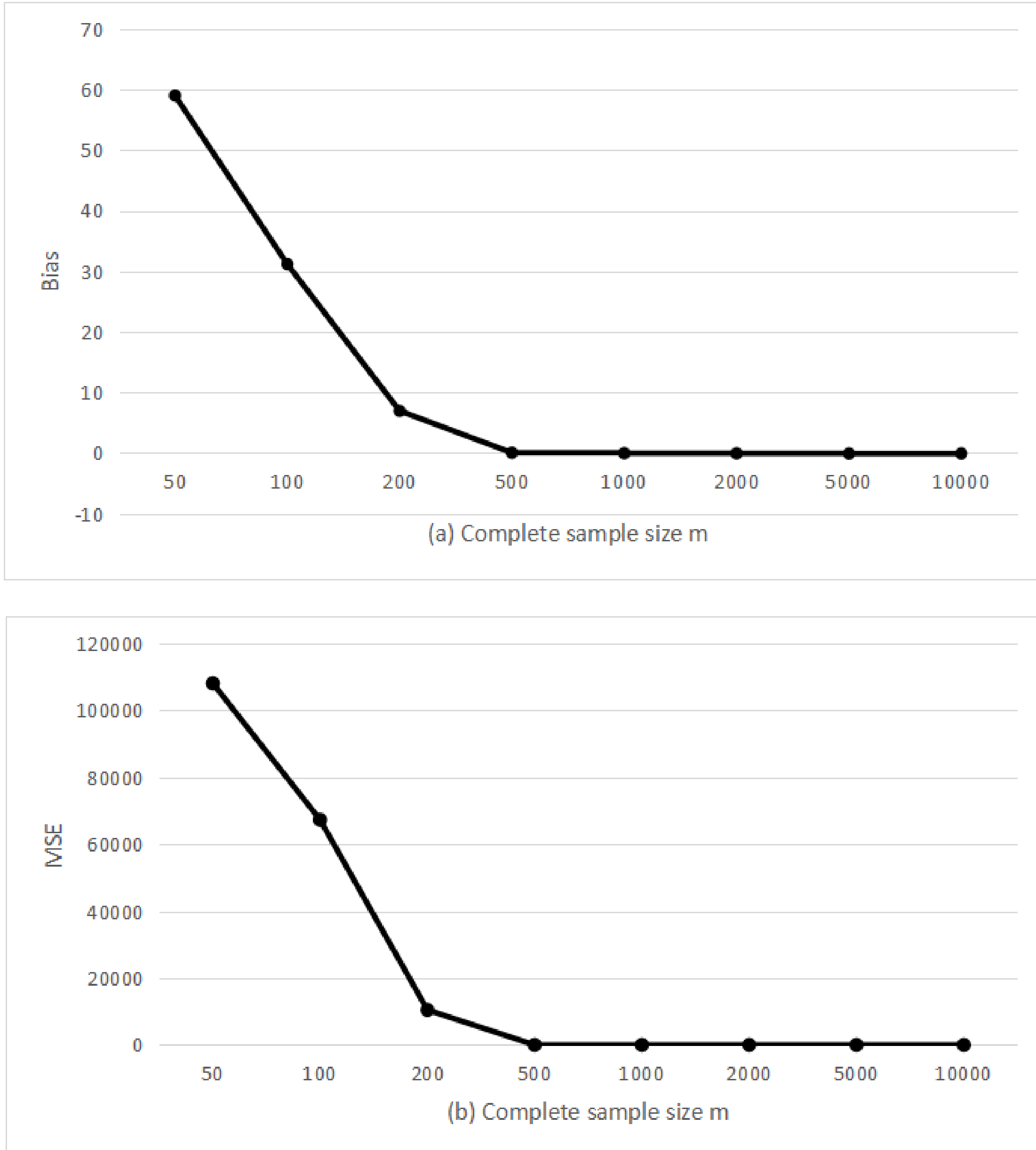
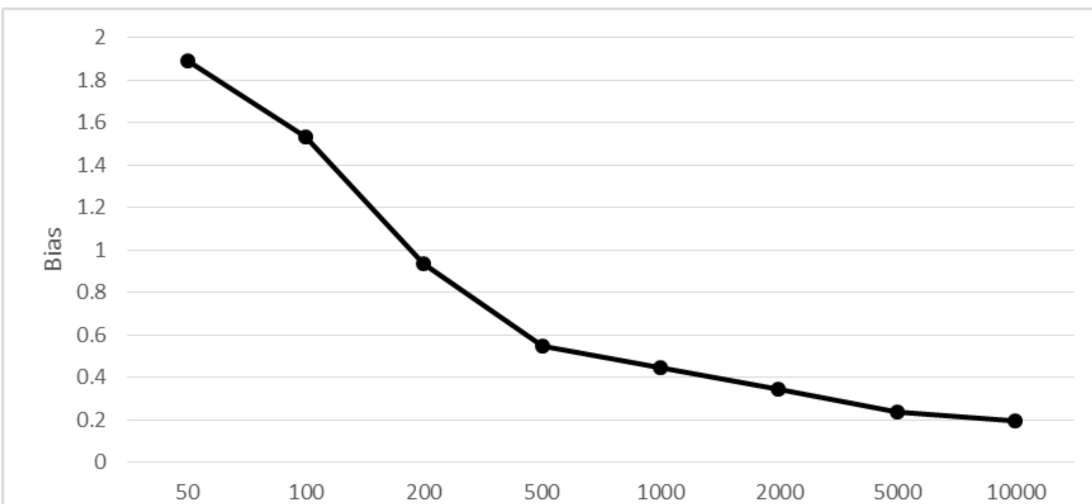


Figure 1 (a) Bias of $\hat{\lambda}$ (b) MSE of $\hat{\lambda}$ when $\lambda_0 = 2$ and $\theta_0 = 0.5$.

According to Table 1 and Figure 1, we could see that the absolute value of the Bias and MSE of $\hat{\lambda}$ decreases from m=50 to m=10000 clearly. When m is larger than 500, both bias and MSE decrease quickly. Since we could get enough number of upper record values from random sample when complete sample sizes are greater than 500. In this situation, estimating λ by record values is reasonable. Secondly, we study the behavior of $\hat{\theta}$. We assume $\lambda_0 = 2$ and $\theta_0 = 0.5$. Table 2 and Figure 2 show the simulation results of $\hat{\theta}$.

Table 2. Bias and MSE from simulations of $\hat{\theta}$ when $\lambda_0 = 2$ and $\theta_0 = 0.5$.

m	Bias of $\hat{\theta}$	MSE of $\hat{\theta}$
50	1.892	24.6591
100	1.5346	24
200	0.9354	13.3547
500	0.5441	3.1677
1000	0.4482	2.7563
2000	0.3425	1.4081
5000	0.2332	0.3074
10000	0.1963	0.2692



(a) Complete sample size m

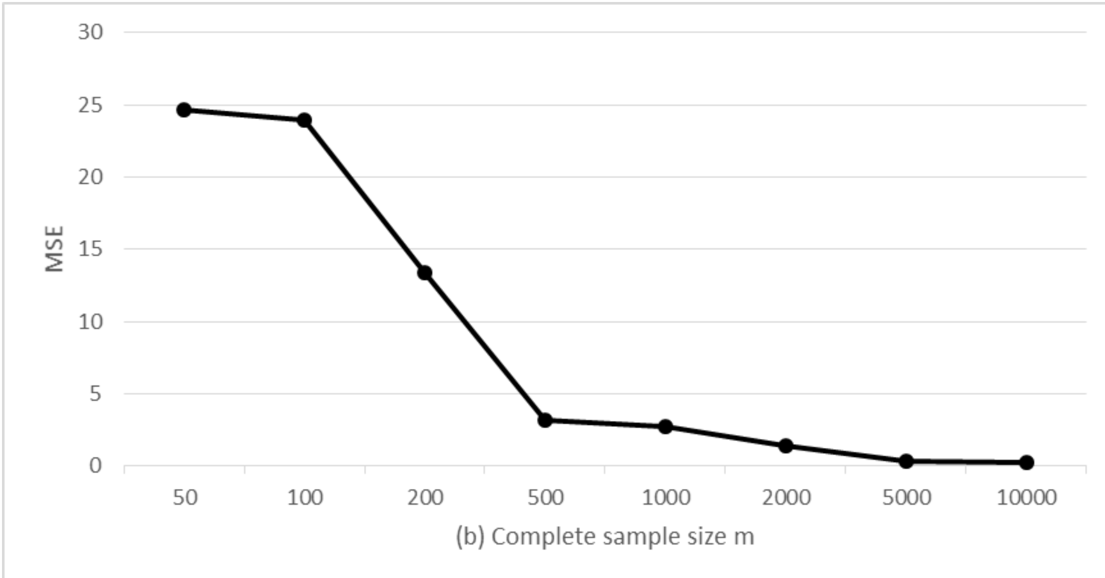


Figure 2 (a) Bias of $\hat{\theta}$ (b) MSE of $\hat{\theta}$ when $\lambda_0 = 2$ and $\theta_0 = 0.5$.

We observe that bias and MSE of $\hat{\theta}$ decreases from m=50 to m=10000 by Table 2 and Figure 2 explicitly. When m is larger than 500, both bias and MSE decrease quickly. The reason is as similar as the case of $\hat{\lambda}$.

Next the procedure described above was repeated for different sample sizes of upper record values k=5, 6, 7, 8, 9, 10. We assume $\lambda_0 = 2$ and $\theta_0 = 0.5$. We have got Table 3 and Figure 3.

Table 3. Bias and MSE from simulations of $\hat{\lambda}$ and $\hat{\theta}$ for $\lambda_0 = 2$ and $\theta_0 = 0.5$.

k	Bias of $\hat{\lambda}$	MSE of $\hat{\lambda}$	Bias of $\hat{\theta}$	MSE of $\hat{\theta}$
4	0.4664	91.3631	1.0556	2.5865
5	-0.6896	0.5673	0.7084	1.1334
6	-0.5081	0.3178	0.4855	0.5172
7	-0.3857	0.185	0.339	0.2836
8	-0.2545	0.0947	0.247	0.1864
9	-0.1475	0.0575	0.1803	0.0989
10	-0.0399	0.0244	0.1457	0.0822

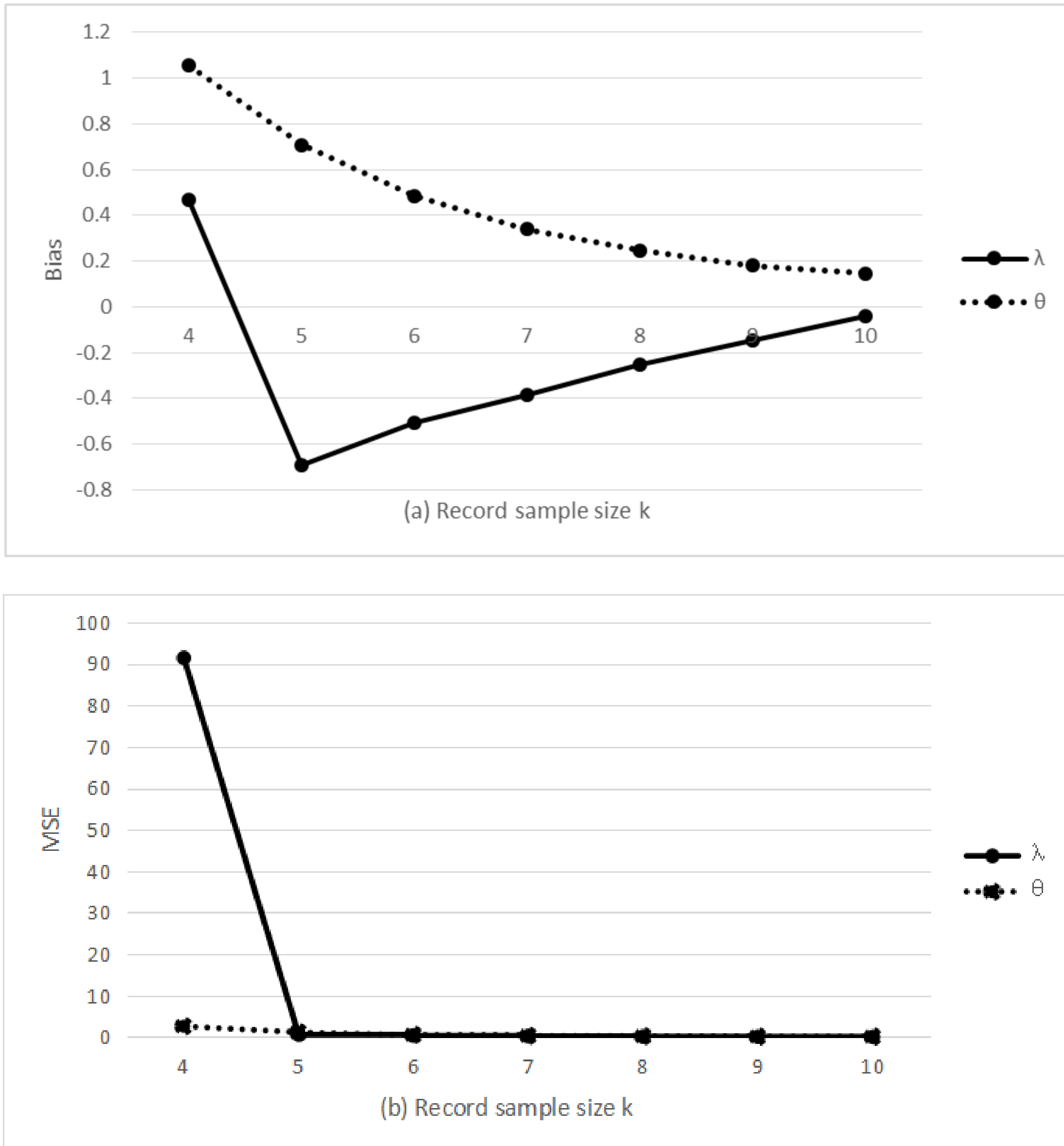
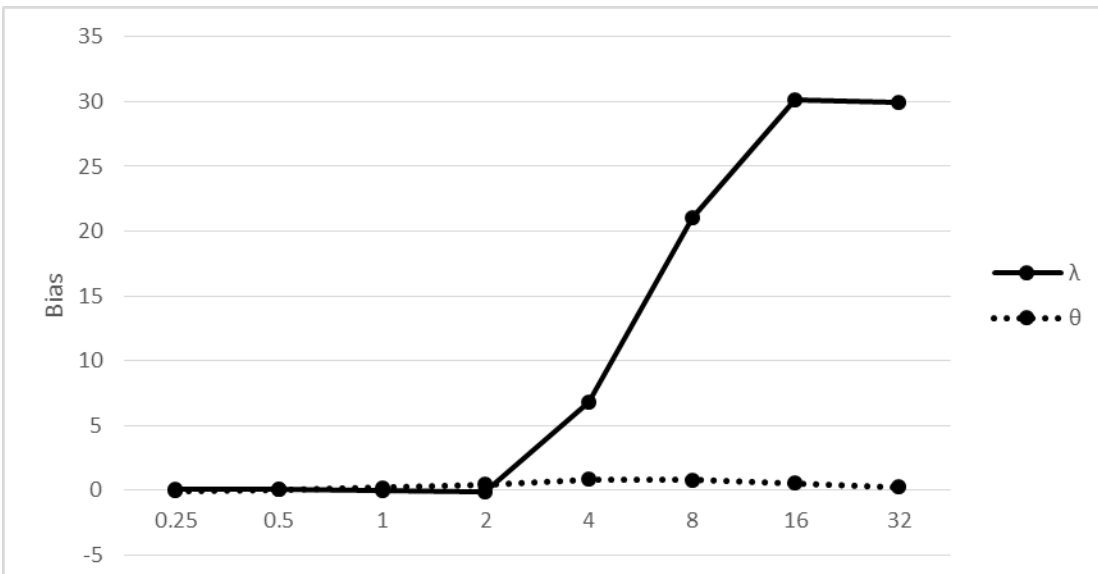


Figure 3 (a) Bias of $\hat{\lambda}$ and $\hat{\theta}$ (b) MSE of $\hat{\lambda}$ and $\hat{\theta}$ when $\lambda_0 = 2$ and $\theta_0 = 0.5$.

According to Table 3 and Figure 3, We can see that there are good behavior of $\hat{\lambda}$ and $\hat{\theta}$ when $k \geq 5$.

Then we have researched behavior of $\hat{\lambda}$ when complete sample size m=10000 and $\lambda_0 = 0.25, 0.5, 1, 2, 4, 8, 16, 32$ and $\theta_0 = 1$. We have get Table 4 and Figure 4.

Table 4. Bias and MSE from simulations of $\hat{\lambda}$ and $\hat{\theta}$ for different λ_0 and $\theta_0 = 1$.				
λ_0	Bias of $\hat{\lambda}$	MSE of $\hat{\lambda}$	Bias of $\hat{\theta}$	MSE of $\hat{\theta}$
0.25	0.1008	0.0438	-0.0732	0.9378
0.5	0.07	0.4239	0.0282	0.8841
1	-0.0705	0.1155	0.1871	0.2545
2	-0.1083	0.1425	0.4101	0.8475
4	6.7628	668.7936	0.835	3.8634
8	21.0445	1231.2698	0.7886	0.7549
16	30.0993	1455.9522	0.502	0.9023
32	29.9711	1342.8429	0.2458	0.9763



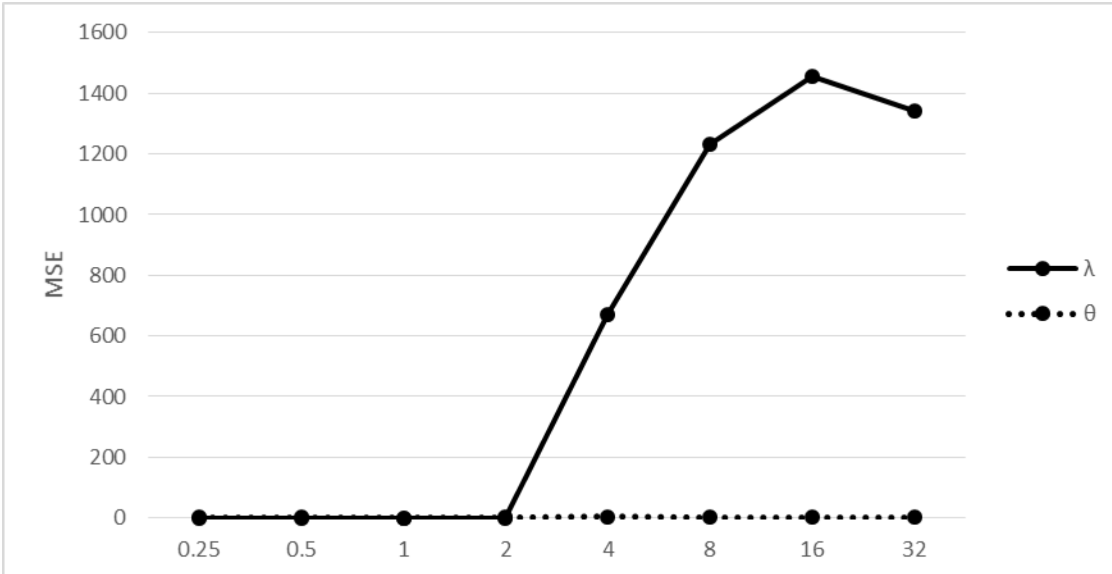


Figure 4 (a) Bias of $\hat{\lambda}$ and $\hat{\theta}$ (b) MSE of $\hat{\lambda}$ and $\hat{\theta}$ for different λ_0 and $\theta_0 = 1$.

According to Table 4 and Figure 4. We can conclude that with the increase of λ_0 bias and MSE of $\hat{\lambda}$ increase. Bias and MSE of $\hat{\theta}$ do not have great change. The unexpected tendency of $\hat{\lambda}$ suggests that maximum likelihood estimation of parameter λ of the exponential power distribution from record samples is not effective when λ is too large. However, as λ_0 decreases, the MSE of $\hat{\lambda}$ decreases.

Finally, we study behavior of $\hat{\theta}$ when complete sample size m=10000 and λ_0 =1 and $\theta_0 = 0.25, 0.5, 1, 2, 4, 8, 16, 32$. We have get Table 5 and Figure 5.

Table 5. Bias and MSE from simulations of $\hat{\lambda}$ and $\hat{\theta}$ for different θ_0 and $\lambda_0 = 1$.

θ_0	Bias of $\hat{\lambda}$	MSE of $\hat{\lambda}$	Bias of $\hat{\theta}$	MSE of $\hat{\theta}$
0.25	-0.4367	0.3757	0.3006	0.2122
0.5	-0.0621	0.2021	0.0856	0.1054
1	-0.0705	0.1155	0.1871	0.2545
2	-0.4405	0.2528	0.5508	0.433
4	-0.0637	0.1223	0.7096	3.9918
8	-0.0569	0.1168	0.6428	3.7206

16	-0.0532	0.1272	1.3168	15.492
32	-0.0365	0.1085	4.4169	186.62

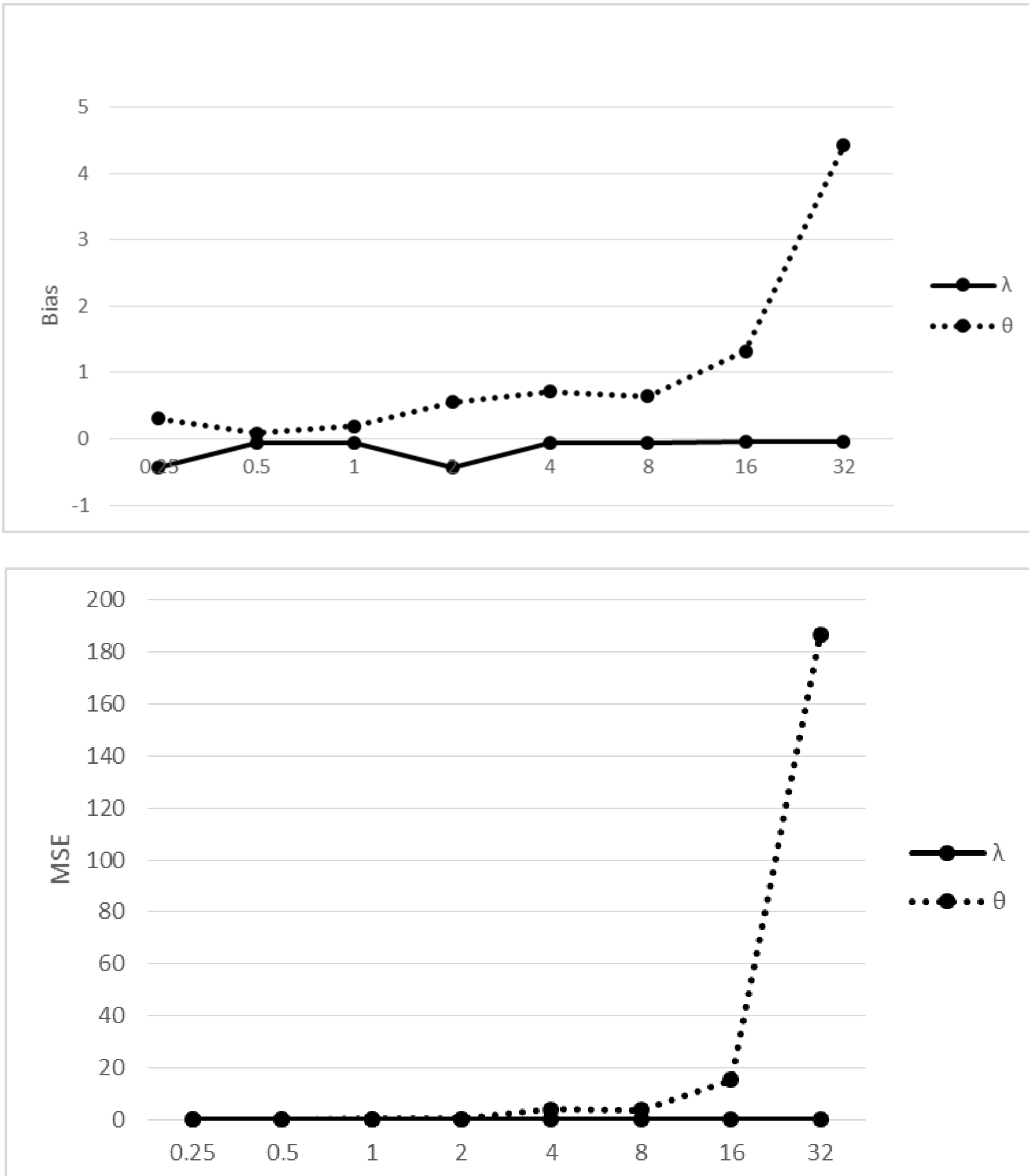


Figure 5 (a) Bias of $\hat{\lambda}$ and $\hat{\theta}$ (b) MSE of $\hat{\lambda}$ and $\hat{\theta}$ for different θ_0 and $\lambda_0 = 1$.

For samples of records, the MSEs of $\hat{\lambda}$ do not have great change with increase of θ_0. In the general, MSEs of $\hat{\theta}$ increase slightly.

All in all, when λ_0 and θ_0 are not every large, the estimates of λ and θ have a good performance. Meanwhile, with the increase of the complete sample size m or the record sample size k, bias and MSE of $\hat{\lambda}$ and $\hat{\theta}$ decrease quickly.

6. PREDICTION INTERVAL OF THE NEXT RECORD VALUE

Assume that we have observed the first (n+1) upper record values $r_0, r_1, r_2, \ldots, r_n$ from the exponential power distribution. Then we can make certain prediction on the next record value R_{n+1}.

In order to be able to make prediction on R_{n+1} using $\mathbf{r} \equiv (r_0, r_1, r_2, \ldots, r_n)$ we need the conditional density of R_{n+1} given $\mathbf{r}$. The conditional distribution of R_{n+1} given $\mathbf{r}$ is obtained as

$$f_{R_{n+1}|\mathbf{R}}(r_{n+1} \mid \mathbf{r}) = \frac{f(r_0, \ldots, r_n, r_{n+1})}{f(r_0, \ldots, r_n)} = \frac{f(r_{n+1}) \prod_{i=0}^{n} h(r_i)}{f(r_n) \prod_{i=0}^{n-1} h(r_i)}$$

$$= \frac{f(r_{n+1}) h(r_n)}{f(r_n)} = \frac{f(r_{n+1}) h(r_n)}{\overline{F}(r_n) h(r_n)}$$

$$= \frac{f(r_{n+1})}{\overline{F}(r_n)} \qquad r_{n+1} > r_n . \tag{21}$$

Let the density function of random variable W be given by (21). Obviously, W has survival function

$$\overline{G}(w) \equiv P(W > w) = P(R_{n+1} > w \mid \mathbf{r}) = \int_w^\infty f_{R_{n+1}}(r_{n+1} \mid \mathbf{r}) dr_{n+1}$$

$$= \int_w^\infty \frac{f(r_{n+1})}{\overline{F}(r_n)} dr_{n+1} = \frac{\overline{F}(w)}{\overline{F}(r_n)}$$

$$= \frac{\exp(1 - \exp(\lambda w^\theta))}{\exp(1 - \exp(\lambda r_n^\theta))}$$

$$= \frac{\exp(-\exp(\lambda w^\theta))}{\exp(-\exp(\lambda r_n^\theta))} . \tag{22}$$

For any given $\delta \in (0,1)$ let c_δ be the δth quantile of $G(w)$. That imply $\overline{G}(c_\delta) = \delta$. The value c_δ can be obtained by solving the equation

$$\frac{\exp(-\exp(\lambda c_\delta^\theta))}{\exp(-\exp(\lambda r_n^\theta))} = \delta .$$

We obtained

$$c_\delta = \left(\frac{1}{\lambda} \ln[\exp(\lambda r_n^\theta) - \ln(\delta)] \right)^{\frac{1}{\theta}} . \tag{23}$$

Assuming we have θ_0. Substituting $\hat{\lambda}$ and θ_0 for λ and θ in (23) respectively, we can have the approximate value of c_δ.

$$c_\delta \approx \hat{c}_\delta \equiv \left(\frac{1}{\hat{\lambda}} \ln[\exp(\hat{\lambda} r_n^{\theta_0}) - \ln(\delta)] \right)^{\frac{1}{\theta_0}} . \tag{24}$$

Similarly, we assume we have λ_0. Then substituting λ_0 and $\hat{\theta}$ for λ and θ in (23) respectively, we can have the approximate value of c_δ.

$$c_\delta \approx \hat{c}_\delta \equiv \left(\frac{1}{\lambda_0} \ln[\exp(\lambda_0 r_n^{\hat{\theta}}) - \ln(\delta)] \right)^{\frac{1}{\hat{\theta}}} . \tag{25}$$

Now, for any given $\alpha \in (0,1)$, a $(1-\alpha)$ prediction interval of R_{n+1} given $\mathbf{r}$ can expressed as

$$(\hat{c}_{1-\alpha/2}, \hat{c}_{\alpha/2}) . \tag{26}$$

Simulation Study

We first generated $x_0, x_1, x_2, \ldots$ from exponential power distribution with parameters λ_0 and θ_0. We took 10 upper record values $\mathbf{r} = (r_0, r_1, \ldots r_8)$ and r_9 from the sequence

$x_0, x_1, x_2, \ldots$. Then we computed 90% prediction interval of R_9 by $\mathbf{r}$ from two situations. With λ_0 known and θ_0 unknown, we calculated $\hat{c}_\delta$ from (24). With λ_0 unknown and θ_0 known, we calculated $\hat{c}_\delta$ from (25). Repeating the above steps 1000 times, we got the average width of the prediction interval and the coverage probability $P(r_9 \in (\hat{c}_{1-\alpha/2}, \hat{c}_{\alpha/2}))$. The results were shown as Table 6 and Table 7.

Table 6. Average width of prediction interval and coverage probability when λ_0 is known and θ_0 is unknown

λ_0	θ_0	Average Width	Probability
0.5	0.5	4.3346	0.747
0.5	0.75	1.0385	0.766
0.5	1	0.4636	0.745
1	0.5	1.0915	0.739
1	0.75	0.4116	0.735
1	1	0.2332	0.762
2	0.5	0.2718	0.741
2	0.75	0.1633	0.763
2	1	0.1158	0.722

Table 7. Average width of prediction interval and coverage probability when λ_0 is unknown and θ_0 is known

λ_0	θ_0	Average Width	Probability
0.5	0.5	4.3343	0.734
0.5	0.75	1.0293	0.736
0.5	1	0.4633	0.725
1	0.5	1.0493	0.72
1	0.75	0.3987	0.733
1	1	0.2253	0.706
2	0.5	0.2554	0.683
2	0.75	0.1489	0.713
2	1	0.1059	0.705

According to Table 6 and Table 7, the performance of the prediction intervals R_9 by $\mathbf{r}$ is effective. Thus, it is reasonable for us to apply the derived prediction interval for predicting the value of the (n+1)th upper record by $\mathbf{r} = (r_0, r_1, \ldots r_n)$ from (24) , (25) and (26).

7. CONCLUSION

In the study presented above, we researched MLEs of exponential power distribution parameters by upper record values and discussed the uniqueness of MLEs. We then used simulation study to research the performance of $\hat{\lambda}$ and $\hat{\theta}$ and concluded that they have good performance in most situations. Finally, we studied the prediction of R_{n+1} from $\mathbf{r}$. According to the simulation study, we concluded that using $\mathbf{r}$ to estimate R_{n+1} is reasonable.

REFERENCE

Arnold, B.C, Balakrishnan, N., and Nagaraja, H.N. (1998). Records, Wiley.

Barriga, G.C.D., Louzada-Neto, F., and Cancho, V.G. (2010). The complementary exponential power lifetime model. Computational Statistics and Data Analysis, 55:1250–1259.

Cramer, E. and Naehrig, G. (2012). Laplace record data. Journal of Statistical Planning and Inference, 142:2019-2189.

Davis, D.J., (1952). An analysis of some failure data. Journal of the American Statistical Association, 47:113–150.

Delicado, P. and Goria, M.N. (2008). A small sample comparison of maximum likelihood, moments and l-moments methods for the asymmetric exponential power distribution. Computational Statistics & Data Analysis, 52.

Hanagal, D. D. and Dabade, A.D. (2015). Comparison of Shared Frailty Models for Kidney Infection Data under Exponential Power Baseline Distribution. Communications in Statistics—Theory and Methods, 44:5091–5108.

Hazan, A., Landsman, Z. and Makov, U.E. (2003). Robustness via a mixture of exponential power distributions. Computational Statistics & Data Analysis, 42 (1–2):111–121.

Jafari,A. A. and Zakerzadeh, H. (2015). Inference on the Parameters of the Weibull Distribution Using Records. Statistics and Operations Research Transactions, 39(1).

Leemis, L.M. (1986). Lifetime distribution identities. IEEE Transactions on Reliability, 35 (2).

Lemonte, A.J. (2013). A new exponential-type distribution with constant, decreasing, increasing, upside-down bathtub and bathtub-shaped failure rate function. Computational Statistics and Data Analysis, 62:149—170.

Nelson,W. B. (1982). Applied Life Data Analysis, Wiley.

Rajarshi, S. and Rajarshi, M.B. (1988). Bathtub distributions: a review. Communications in Statistics—Theory and Methods, 17:2597–2621.

Smith, R.M. and Bain, L.J. (1975). An exponential power life-testing distribution. Communications in Statistics—Theory and Methods, 58:469–481.

Wu, J. W. and Tseng, H. C. (2006). Statistical inference about the shape parameter of the Weibull distribution by upper record values. Statistical Papers, 48(1):95–129.

Xie, M., Tang, Y. and Goh, T.N. (2002). A modified Weibull extension with bathtub-shaped failure rate function. Reliability Engineering & System Safety, 76:279–285.