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Perfect-fluid cylinders and walls—sources for the Levi–Civita space–time

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Abstract

The diagonal metric tensor whose components are functions of one spatial coordinate is considered. Einstein's field equations for a perfect-fluid source are reduced to quadratures once a generating function, equal to the product of two of the metric components, is chosen. The solutions are either static fluid cylinders or walls depending on whether or not one of the spatial coordinates is periodic. Cylinder and wall sources are generated and matched to the vacuum (Levi–Civita) space–time. A match to a cylinder source is achieved for $-\frac{1}{2} < \sigma < \frac{1}{2}$, where σ is the mass per unit length in the Newtonian limit $\sigma \to 0$, and a match to a wall source is possible for $|\sigma| > \frac{1}{2}$, this case being without a Newtonian limit; the positive (negative) values of σ correspond to a positive (negative) fluid density. The range of σ for which a source has previously been matched to the Levi–Civita metric is $0 \leq \sigma < \frac{1}{2}$ for a cylinder source.

1 Introduction

Although a large number of vacuum solutions of Einstein's equations are known, the physical interpretation of many (if not most) of them remains unsettled (see, e.g., [1]). As stressed by Bonnor [1], the key to physical interpretation is to ascertain the nature of the sources which produce these vacuum space–times; even in black-hole solutions, where no matter source is needed, an understanding of how a matter distribution gives rise to the black-hole space–time is necessary to judge the physical significance (or lack of it) of the complete analytic extensions of these solutions.

As to how we gain an understanding of the sources, there is really no substitute for constructing an interior solution for a matter distribution which matches to the vacuum space–time in question. The coordinate freedom of general relativity can make attempts to discover the nature of the source for a vacuum field a hazardous affair: in

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the vacuum space–time the source appears as a singularity in the curvature which may look quite different after a coordinate transformation [2]. If we have an interior solution however such coordinate transformations will invariably *introduce* a singularity in the interior and so must be rejected. Although attempts are made to deduce properties of sources by analysis of the vacuum space–times which represent the exterior fields, this approach can never be conclusive; only a complete interior and exterior solution will afford confidence in the analysis.

In this paper we construct sources for the vacuum space–time described by a diagonal metric depending on one spatial coordinate; this vacuum solution, found by Tullio Levi–Civita [3], we shall write as

$$ds^2 = A^2(r-k)^{8\sigma^2-4\sigma}(dr^2+dz^2) + B^2(r-k)^{2-4\sigma}d\phi^2 - C^2(r-k)^{4\sigma}dt^2, \tag{1}$$

where σ, k, A, B and C are arbitrary constants. (In fact, by rescaling, (1) can be cast in the form

$$ds^2 = r^{8\sigma^2-4\sigma}(dr^2+dz^2) + D^2 r^{2-4\sigma}d\phi^2 - r^{4\sigma}dt^2,$$

but the constant D cannot be removed if ϕ is to be a periodic coordinate with period 2π [10]. Thus the Levi–Civita metric has two parameters when ϕ is periodic. The form (1) is needed in matching to the interior solutions.) This space–time has in general a curvature singularity at $r = k$ and is flat in the limit $r \to \infty$; the Riemann tensor vanishes everywhere only for $\sigma = 0$ and $\sigma = \frac{1}{2}$.

A test particle at rest in the coordinate system of the metric (1) will experience a proper acceleration

$$\ddot{r} = -\frac{2\sigma}{(r-k)^{1+8\sigma^2-4\sigma}}. \tag{2}$$

For small σ this is approximately

$$\ddot{r} = -\frac{2\sigma}{r-k},$$

which is of the same form as the Newtonian expression for the acceleration of a particle a distance $r-k$ from a line mass of mass per unit length σ. This well-defined Newtonian limit is the reason why the Levi-Civita space–time is usually said to represent the field outside an infinitely long static cylinder—thus the ϕ-coordinate is taken to be periodic—and cylinder sources have indeed been found for this space–time [4, 5, 6, 7]. The sources found are of two types: static dust cylinders (composed of equal amounts of dust rotating in opposite senses around the axis to produce zero net angular momentum) which were shown to match to the Levi–Civita space–time for values of σ in the range $0 \le \sigma < \frac{1}{4}$ [5, 7], and perfect-fluid cylinders, one example of which could be matched to the Levi–Civita exterior for $0 \le \sigma < \frac{1}{4}$ [4] while the other example was valid for $0 \le \sigma < \frac{1}{2}$ [6]. These results are consistent with the interpretation of the Levi-Civita space–time as the relativistic field outside a line mass, at least for σ in the range $0 \le \sigma < \frac{1}{2}$. [1]

Nevertheless, we can just as easily take all the coordinates in (1) to be Cartesian and the supposition then is that the Levi–Civita space–time represents the field outside an infinite static wall; although this possibility has been noticed before [17] an example of such a wall source has not been constructed. Indeed, the interpretation of ϕ as a Cartesian coordinate appears more tenable for certain values of σ. For example, when $\sigma = \frac{1}{2}$ the metric (1) describes flat space–time in the local coordinate system of an observer undergoing constant acceleration in the r-direction [11], and when $\sigma = -\frac{1}{2}$ the metric, in addition to the three Killing vectors ∂_t, ∂_ϕ and ∂_z, admits a fourth Killing vector $\phi\partial_z - z\partial_\phi$ which identifies it as Taub's plane-symmetric metric [12] (though in different coordinates); these two metrics are the favourite proposals for the exterior field of a plane mass (e.g. [13, 14, 15, 16]). There is thus a rich structure to the vacuum metric (1) (for more detail, see [2]) and we take the view that its physical significance can only be meaningfully explored by constructing sources which generate (1) for a wide range of σ.

We shall construct both cylinder and wall sources for the Levi–Civita space–time

[1] It is worth mentioning that for $\sigma \ge \frac{1}{4}$ the Levi–Civita space–time does not contain timelike circular geodesics [8] (when $\sigma = \frac{1}{4}$ the circular geodesics are null, when $\sigma > \frac{1}{4}$ they are spacelike). The explanation, suggested by the Newtonian case, that this is due to the gravitational attraction becoming strong enough to prevent any particle (including light) orbiting circularly [9] has been contrasted with the fact that a study of the curvature invariants suggests that the field gets weaker as σ increases from $\frac{1}{4}$ to $\frac{1}{2}$ (where the metric is flat but accelerated) [2, 7]. However, although it sounds strange, what matters here is not the field (i.e. the curvature) but the acceleration experienced by the test particle towards the source: this does not vanish at $\sigma = \frac{1}{2}$ although the field does, and indeed the proper acceleration towards the origin experienced by a test particle at rest in the coordinate system of (1) (or of (50) below) *increases* as σ goes from $\frac{1}{4}$ to $\frac{1}{2}$. Hence it is quite reasonable to conclude that the cylinder sources producing σ in the range $\frac{1}{4} < \sigma < \frac{1}{2}$ have a gravitational attraction large enough to disallow circular orbits.

(1), composed of perfect fluid. We work with the metric form

$$ds^2 = \alpha^2(r)(dr^2 + dz^2) + \beta^2(r)d\phi^2 - \gamma^2(r)dt^2, \tag{3}$$

the vacuum solution for which is given by (1). If ϕ is a periodic coordinate then this metric is cylindrically symmetric and static; if all the coordinates are Cartesian then the space–time is homogeneous on the spacelike (z, ϕ)-planes but not (in general) isotropic on those planes—we therefore call this latter type *plane-homogeneous* space–times and metric (3) is then static and plane-homogeneous. [2] In Sec. 2 the field equations for (3) with a perfect-fluid source are reduced to quadratures once the function $\beta\gamma$ is chosen; the difference between cylindrically symmetric solutions and plane-homogeneous solutions emerges in the behaviour of the metric tensor at $r = 0$ through the demand that the geometry be regular there. In Sec 3 we use this scheme to generate cylindrically symmetric and plane-homogeneous solutions with a boundary (that is, solutions in which the pressure falls to zero at a finite value of r) which are then matched to the exterior (Levi–Civita) space–time; a cylinder source can be matched for σ in the range $-\frac{1}{2} < \sigma < \frac{1}{2}$, and a wall source for $|\sigma| > \frac{1}{2}$, where the positive (negative) values of σ correspond to a positive (negative) density for the fluid. These results are discussed in Sec. 4.

2 The field equations

Einstein's equations for the metric (3) in geometrical units ($c = G = 1$) with a perfect fluid energy–momentum tensor ($T^a_{\ b} = \mathrm{diag}(p, p, p, -\rho)$) are

$$\frac{\alpha'}{\alpha}\left(\frac{\beta'}{\beta} + \frac{\gamma'}{\gamma}\right) + \frac{\beta'}{\beta}\frac{\gamma'}{\gamma} = 8\pi\alpha^2 p, \tag{4}$$

[2]The term plane-symmetric has already been used to denote space–times which are homogeneous *and* isotropic on spacelike planes [12], such as Taub's metric; in order to be plane-symmetric, (3) must have a third spacelike Killing vector $\phi\partial_z - z\partial_\phi$. Plane-homogeneous solutions have been considered in the literature, though not by this name, the usual practice being simply to state that there is a G_2 on S_2 or that there are two spacelike Killing vectors (see [18, 19] and references therein); the similarity to plane symmetry and some resulting confusion has been noted [19]. Li and Liang [20] have introduced a title for this symmetry; they found electrovac solutions where the metric is plane-symmetric but the electromagnetic field inherits only the two translational symmetries and not the rotational one and referred to these electromagnetic fields as having *semi-plane symmetry*. We prefer the term "plane-homogeneous" because it gives a clearer idea of the symmetry involved than "semi-plane-symmetric".

$$-\frac{\alpha'}{\alpha}\left(\frac{\beta'}{\beta}+\frac{\gamma'}{\gamma}\right)+\frac{\beta'}{\beta}\frac{\gamma'}{\gamma}+\frac{\beta''}{\beta}+\frac{\gamma''}{\gamma}=8\pi\alpha^2 p, \tag{5}$$

$$\left(\frac{\alpha'}{\alpha}\right)'+\frac{\gamma''}{\gamma}=8\pi\alpha^2 p, \tag{6}$$

$$\left(\frac{\alpha'}{\alpha}\right)'+\frac{\beta''}{\beta}=-8\pi\alpha^2 \rho. \tag{7}$$

The solutions to these equations may describe either cylindrically symmetric or plane-homogeneous space–times; the essential difference between the two types of solution manifests itself in the boundary conditions for the equations.

If the metric (3) is cylindrically symmetric with axis at $r = 0$ then the regularity condition [21]

$$\frac{X_{,a}X^{,a}}{4X}\longrightarrow 1 \quad \text{as } r\longrightarrow 0, \qquad X=g(\partial_\phi,\partial_\phi), \tag{8}$$

gives

$$\frac{\beta'^2}{\alpha^2}\longrightarrow 1 \qquad \text{as } r\longrightarrow 0. \tag{9}$$

In addition the requirement (indeed, the definition) of the axis, that $g(\partial_\phi, \partial_\phi) = 0$ there, gives

$$\beta(0)=0. \tag{10}$$

To simplify the discussion, we scale the coordinates so that the metric approaches flat space–time in cylindrical coordinates as $r \longrightarrow 0$, i.e.

$$ds = dr^2+dz^2+r^2d\phi^2-dt^2 \quad \text{as } r\longrightarrow 0. \tag{11}$$

It therefore follows from (9) and (10) that

$$\beta'(0)=1. \tag{12}$$

In addition, it is reasonably clear from (10), (12) and the field equations (4)–(7) that α', γ' and β'' must also vanish at $r = 0$ at least as quickly as β or else some terms in the Einstein tensor components diverge. In fact, Lake and Musgrave [22] have taken the set of fourteen independent second-order curvature invariants found by Carminati and McLenaghan [23] and worked out the necessary and sufficient conditions for them to be finite at the origin of certain static space–times; for the cylindrically symmetric case the conditions are, in our coordinate system, those deduced above, namely

$$\alpha'(0) = 0, \qquad \gamma'(0) = 0, \qquad \beta(0) = 0, \qquad \beta'(0) = 1, \qquad \beta''(0) = 0. \tag{13}$$

Equations (13) are, then, the necessary and sufficient conditions for the metric (3) to describe a cylindrically symmetric space–time which is regular on the axis. When the metric (3) describes a plane-homogeneous space–time however, β and the rest of the metric components must be non-zero constants at $r = 0$ and with rescaling the metric can be written

$$ds^2 = dr^2 + dz^2 + d\phi^2 - dt^2 \qquad \text{at } r = 0, \tag{14}$$

the coordinate ϕ now being a Cartesian coordinate. It is this difference in the behaviour of the metric components at $r = 0$ that distinguishes the cylindrically symmetric solutions from the plane-homogeneous ones.

The content of the conservation equation $T^{ab}{}_{;b} = 0$ comes from the component orthogonal to the fluid four-velocity, i.e. $(g^{ab} + u^a u^b) T_b{}^c{}_{;c} = 0$, which yields

$$\frac{\gamma'}{\gamma} = -\frac{p'}{p + \rho}, \tag{15}$$

and this relation can in fact be derived from the field equations (4)–(7). Thus we have four independent equations for the five unknown functions α, β, γ, p and ρ; therefore once one further relation is imposed—be it an equation of state for the fluid, an explicit form for one of the unknowns, or any other independent equation—everything is determined up to arbitrary constants. What we shall now show is that if this additional relation is an explicit form for the quantity $\beta\gamma$, the metric components may be found by mere integration.

Adding (4) and (5) gives

$$\frac{(\beta\gamma)''}{\beta\gamma} = 16\pi\alpha^2 p. \tag{16}$$

Subtracting (5) from (6) gives

$$\left(\frac{\alpha'}{\alpha}\right)' + \frac{\alpha'}{\alpha}\left(\frac{\beta'}{\beta} + \frac{\gamma'}{\gamma}\right) - \frac{\beta''}{\beta} - \frac{\beta'\gamma'}{\beta\gamma} = 0,$$

which may be written

$$\frac{1}{\beta\gamma}\frac{d}{dr}\left(\frac{\alpha'}{\alpha}\beta\gamma\right) - \frac{1}{\beta\gamma}\frac{d}{dr}\left(\beta'\gamma\right) = 0.$$

Multiplying across by $\beta\gamma$ and integrating gives

$$\frac{\alpha'}{\alpha}\beta\gamma - \beta'\gamma = -c, \qquad c \text{ a constant.}$$

For a cylindrically symmetric solution we must have, from (10)–(13), $\beta(0) = \alpha'(0) = 0$ and $\beta'(0) = \gamma(0) = 1$, so that $c = 1$; c remains free if the solution is plane-homogeneous. We write the last formula as

$$\frac{\alpha'}{\alpha} = \frac{\beta'}{\beta} - \frac{c}{\beta\gamma}. \tag{17}$$

Subtracting (5) from (4) we get

$$\frac{\alpha'}{\alpha}\left(\frac{\beta'}{\beta} + \frac{\gamma'}{\gamma}\right) - \frac{1}{2}\frac{\beta''}{\beta} - \frac{1}{2}\frac{\gamma''}{\gamma} = 0$$

and substituting for $\frac{\alpha'}{\alpha}$ from (17) this becomes

$$\left(\frac{\beta'}{\beta}\right)^2 + \frac{\beta'\gamma'}{\beta\gamma} - c\frac{(\beta\gamma)'}{(\beta\gamma)^2} - \frac{1}{2}\frac{\beta''}{\beta} - \frac{1}{2}\frac{\gamma''}{\gamma} = 0,$$

or

$$\left(\frac{\beta'}{\beta}\right)^2 + 2\frac{\beta'\gamma'}{\beta\gamma} - c\frac{(\beta\gamma)'}{(\beta\gamma)^2} - \frac{1}{2}\frac{(\beta\gamma)''}{\beta\gamma} = 0.$$

Now

$$\frac{\gamma'}{\gamma} = \frac{(\beta\gamma)'}{\beta\gamma} - \frac{\beta'}{\beta}, \tag{18}$$

so the previous equation becomes

$$\left(\frac{\beta'}{\beta}\right)^2 - 2\frac{\beta'}{\beta}\frac{(\beta\gamma)'}{\beta\gamma} + c\frac{(\beta\gamma)'}{(\beta\gamma)^2} + \frac{1}{2}\frac{(\beta\gamma)''}{\beta\gamma} = 0.$$

This is a quadratic for $\frac{\beta'}{\beta}$ giving

$$\frac{\beta'}{\beta} = \frac{(\beta\gamma)'}{\beta\gamma} \mp \sqrt{\left(\frac{(\beta\gamma)'}{\beta\gamma}\right)^2 - c\frac{(\beta\gamma)'}{(\beta\gamma)^2} - \frac{1}{2}\frac{(\beta\gamma)''}{\beta\gamma}},$$

or

$$\frac{\gamma'}{\gamma} = \pm\sqrt{\left(\frac{(\beta\gamma)'}{\beta\gamma}\right)^2 - c\frac{(\beta\gamma)'}{(\beta\gamma)^2} - \frac{1}{2}\frac{(\beta\gamma)''}{\beta\gamma}}. \tag{19}$$

The procedure for generating solutions is now clear: choose some function of r as $\beta\gamma$; then γ is found by integrating (19), β is given by $\frac{\beta\gamma}{\gamma}$, and α is found by integrating (17). The choice of $\beta\gamma$ determines the type of solution which will result: for a cylindrically symmetric solution the function $\beta\gamma$ must satisfy, from (13),

$$(\beta\gamma)(0) = 0, \qquad (\beta\gamma)'(0) = 1, \qquad (\beta\gamma)''(0) = 0, \tag{20}$$

and, as explained above, in the previous equations we put $c = 1$; for a plane-homogeneous solution we must have, from (14), $(\beta\gamma)(0) = 1$, since the coordinates are now Cartesian, and the constant c remains arbitrary. By first substituting the chosen $\beta\gamma$ into (16) we can immediately obtain important physical information before deciding whether or not to proceed with generating the solution; for since $\alpha^2 > 0$ everywhere, (16) shows whether the pressure is positive or negative and whether it reaches zero at finite r, thus allowing this value of r to be taken as a boundary and a vacuum solution to be joined on. However the behaviour of the density ρ can only be judged after $\frac{\gamma'}{\gamma}$, $\frac{\beta'}{\beta}$ and $\frac{\alpha'}{\alpha}$ have been derived and substituted into (7).

Finally we show that plane-symmetric solutions are also produced by this procedure. We have remarked that the plane-homogeneous metric obtaining when ϕ is Cartesian in (3) is distinguished from the plane-symmetric metric by lacking a third spacelike Killing vector $\phi\partial_z - z\partial_\phi$. Suppose now that (3) has this Killing vector; then the Killing equation $\pounds_{\phi\partial_z - z\partial_\phi}\, g = 0$ gives

$$\alpha^2 = \beta^2.$$

Now this is implemented by putting $c = 0$ in (17); hence, after generating a plane-homogeneous solution it is simply a matter of setting $c = 0$ and a plane-symmetric solution results.

Prior to this the only schemes for generating perfect-fluid solutions of the form (3) appear to be those of Evans [24] and Kramer [25]. Starting with different coordinate systems from that used here these authors also reduced the problem to the choosing of a generating function, after which everything else is determined. In these formulations however one must either solve a second-order linear homogeneous differential equation [24] or a pair of coupled first-order equations [25]. Our method has the advantage that the entire solution is reduced to quadratures once the generating function is chosen—there are no differential equations to solve; in addition, our method allows the immediate assessment of important physical information (the pressure), a feature lacking in the aforementioned schemes.

3 Solutions with a boundary

These solutions are sources for the Levi–Civita space–time (1), the physical interpretation of which will be accordingly illuminated. In this regard we shall be considering, in addition to solutions which are physically acceptable, solutions with negative mass so as clarify when the Levi–Civita metric represents a negative-mass source.

The only static perfect-fluid cylinders with boundary to be found in the literature are those of Evans [24] and Davidson [26, 27] and we give for the first time a plane-homogeneous solution representing an infinite wall of perfect fluid with a boundary.

The boundary occurs when the pressure of the fluid vanishes at some finite value of r, $r = s$ say. The matching condition between interior and exterior is that the first and second fundamental forms of the boundary hypersurface $r = s$ shall be the same when calculated in both regions [28]; this requires, in addition to continuity of the metric components at $r = s$, continuity of the first derivatives of g_{zz}, $g_{\phi\phi}$ and g_{tt}. We therefore have the conditions

$$\alpha^2(s) = A^2(s-k)^{8\sigma^2-4\sigma}, \qquad \beta^2(s) = B^2(s-k)^{2-4\sigma}, \qquad \gamma^2(s) = C^2(s-k)^{4\sigma}, \quad (21)$$

$$\frac{\alpha'}{\alpha}(s) = \frac{4\sigma^2-2\sigma}{s-k}, \qquad \frac{\beta'}{\beta}(s) = \frac{1-2\sigma}{s-k}, \qquad \frac{\gamma'}{\gamma}(s) = \frac{2\sigma}{s-k}. \qquad (22)$$

Equations (21) may be taken as defining the constants A, B and C; in (22) there are three conditions for the two constants σ and k, but since $\frac{\alpha'}{\alpha}(s)$, $\frac{\beta'}{\beta}(s)$ and $\frac{\gamma'}{\gamma}(s)$ satisfy (4) with $p=0$, only two of these are independent so they can always be satisfied. The simplest way to solve (22) is to focus on the second and third equations: adding them eliminates σ and so k is found; σ is then obtained from the third equation.

3.1 Perfect-fluid cylinder

A cylindrically symmetric solution with a boundary arises from the choice

$$\beta\gamma = r + ar^3 - a^2br^5, \qquad a, b \text{ constants}, \qquad (23)$$

which satisfies (20) and hence generates a cylindrically symmetric solution. Eqn (19) gives

$$\frac{\gamma'}{\gamma} = \pm\frac{ar\sqrt{6+5b-17abr^2+15a^2b^2r^4}}{1+ar^2-a^2br^4}. \qquad (24)$$

As to the choice of sign here, we continue initially with the positive sign as this leads to a positive density for the fluid. We then get from (18) and (17)

$$\frac{\beta'}{\beta} = \frac{1+3ar-5a^2br^4}{1+ar^2-a^2br^4} - \frac{\gamma'}{\gamma}, \qquad (25)$$

$$\frac{\alpha'}{\alpha} = \frac{\beta'}{\beta} - \frac{1}{\beta\gamma}. \qquad (26)$$

It is possible to integrate (24) analytically and the other metric components are found from (23) and (26); the results are [3]

$$\left.\begin{aligned} \gamma =& L\left(-17b+30ab^2r^2+2b\sqrt{15}f(ar^2)\right)^{-\sqrt{15}/2} \\ & \times\left(\frac{12+10b-17bn+(-17b+30b^2n)ar^2+2f(n)f(ar^2)}{ar^2-n}\right)^{-\frac{f(n)}{2\sqrt{1+4b}}} \\ & \times\left(\frac{12+10b-17bm+(-17b+30b^2m)ar^2+2f(m)f(ar^2)}{m-ar^2}\right)^{\frac{f(m)}{2\sqrt{1+4b}}} \\ \beta =& \gamma^{-1}r(1+ar^2-a^2br^4), \\ \alpha =& M\gamma^{-1}(1+ar^2-a^2br^4)^{5/4}\left(\frac{m-ar^2}{ar^2-n}\right)^{\frac{2b-1}{4\sqrt{1+4b}}}, \end{aligned}\right\} \tag{27}$$

where

$$m=\frac{1+\sqrt{1+4b}}{2b}, \qquad n=\frac{1-\sqrt{1+4b}}{2b},$$

$$f(x)=\sqrt{6+5b-17bx+15b^2x^2}.$$

The pressure and density of the fluid are

$$p=\frac{1}{8\pi\alpha^2}\left(\frac{3a-10a^2br^2}{1+ar^2-a^2br^4}\right), \tag{28}$$

$$\rho=\frac{1}{8\pi\alpha^2}\left(\frac{24a+20ab-102a^2br^2+120a^3b^2r^4+(30a^2br^2-9a)f(ar^2)}{(1+ar^2-a^2br^4)f(ar^2)}\right). \tag{29}$$

A physically reasonable solution with a boundary is obtained by taking the constants a and b to be positive; in this case the pressure falls to zero at $r=s=\sqrt{3/10ab}$. There are a few possibilities of misbehaviour which must be checked: in the range $0\leq r\leq s$ and with a and b positive (i) the expressions ar^2-n and $m-ar^2$ are positive since $n<0$ and $m>as^2$ (ii) the polynomial appearing in $f(r)$ is positive since it is

[3]We choose the constants of integration L and M so that the line element approaches the form

$$ds^2=dr^2+dz^2+r^2d\phi^2-dt^2$$

as $r\to 0$.

positive at $r = 0$ and has roots

$$\pm\frac{1}{\sqrt{30ab}}\sqrt{17 \pm \sqrt{-71 - 300b}}$$

which are all complex (iii) the polynomial $1 + ar^2 - a^2br^4$ is also positive since it is positive at $r = 0$ and the only one of its roots

$$\pm\frac{1}{\sqrt{2ab}}\sqrt{1 \pm \sqrt{1 + 4b}}$$

that is real and positive is greater than s.

Although the metric components (27) are unwieldy to say the least, the quantities $\frac{\alpha'}{\alpha}$, $\frac{\beta'}{\beta}$ and $\frac{\gamma'}{\gamma}$ are quite manageable; in particular the match (22) to the exterior yields the simple relations

$$\sigma = \frac{3}{2\sqrt{9 + 20b}}, \qquad k = \frac{12}{45 + 100b}\sqrt{\frac{6}{5ab}}. \tag{30}$$

In the exterior metric, which is valid for $r > s$, we must have $r - k > 0$ to avoid a singularity; this is indeed the case because

$$s - k = \sqrt{\frac{3}{10ab}}\left(\frac{21 + 100b}{45 + 100b}\right) \tag{31}$$

is positive.

As $b \to \infty$, $s \to 0$ so the cylinder disappears, and $\sigma \to 0$; the exterior metric approaches flat space–time in cylindrical coordinates. As $b \to 0$, $s \to \infty$ and we have a space-filling perfect fluid without a boundary; in this limit $\sigma \to \frac{1}{2}$ so the Levi–Civita metric can be matched to the cylinder for values of σ up to, but not including, $\frac{1}{2}$. If the minus sign is taken in (24) then the resulting solution still has pressure given by (28), so s is unchanged, but the density is now

$$\rho = \frac{1}{8\pi\alpha^2}\left(\frac{-24a - 20ab + 102a^2br^2 + 120a^3b^2r^4 + (30a^2br^2 - 9a)f(ar^2)}{(1 + ar^2 - a^2br^4)f(ar^2)}\right),$$

which (for positive a and b) is negative in the range $0 \le r \le s$; also, in the match (22)

k is unchanged from (30) but $\sigma \to -\sigma$:

$$\sigma = -\frac{3}{2\sqrt{9+20b}}, \qquad k = \frac{12}{45+100b}\sqrt{\frac{6}{5ab}}. \tag{32}$$

Hence for this negative-mass solution the exterior metric may have σ in the range 0 (where the interior vanishes) down to, but not including, $-\frac{1}{2}$. We have then a cylinder source for the Levi–Civita metric for the range

$$-\tfrac{1}{2} < \sigma < \tfrac{1}{2}, \tag{33}$$

where we include the 'no source' case $\sigma = 0$. As is clear from (2) the cylinder is attractive for positive values of σ, which correspond to a positive density for the fluid, and repulsive for negative values of σ, corresponding to a negative density.

These results support the interpretation of the Levi–Civita metric in the range (33) as the relativistic line-mass field. As we remarked in Sec. 1 the Newtonian limit is given by $\sigma \to 0$, with σ approaching the Newtonian mass per unit length. As is well known, there is in general relativity no unambiguous measure of the mass-energy of a system which is not asymptotically flat in all spatial directions; this we will demonstrate by using the Tolman mass formula [29] to give another measure of the mass per unit length. The formula is

$$M = \int (T^{\alpha}_{\ \alpha} - T^{4}_{\ 4})\sqrt{-g}\, d^3x,$$

where the integral is over all space. It is, of course, intended to apply to finite systems (but see [30]); the intention here is not to suggest that the Tolman formula can provide a correct measure of the mass per unit length of a cylinder, but simply to give another example of a calculation of this quantity besides the acceleration of a test particle (2) (which suggests σ as the mass per unit length). From the Tolman formula we take as a measure of the mass per unit length

$$\begin{aligned} m_T &= \int_0^{2\pi}\int_0^{1}\int_0^{s} (T^{\alpha}_{\ \alpha} - T^{4}_{\ 4})\sqrt{-g}\, dr\, dz\, d\phi, \\ &= 2\pi \int_0^{s} (3p+\rho)\sqrt{-g}\, dr, \end{aligned} \tag{34}$$

which for our solution gives the simple relation

$$m_T = \pm \frac{3}{40b}\sqrt{9+20b}, \tag{35}$$

where the plus and minus signs correspond, respectively, to choosing plus or minus in (24). Comparison of (35) with (30) and (32) shows that for small σ (large b) $m_T \approx \sigma$ since, in the limit as $\sigma \to 0$, i.e. $b \to \infty$, σ and m_T behave like

$$\sigma \approx \pm \frac{3}{2\sqrt{20b}}, \qquad m_T \approx \pm \frac{3}{2\sqrt{20b}}.$$

Thus, as indicators of the gravitational mass, both σ and m_T agree in the Newtonian limit; however in the extreme relativistic regime, occurring when b is small, σ approaches $\pm\frac{1}{2}$ whereas m_T goes to infinitely positive or negative values. In fact, far from the Newtonian limit, there are no cogent physical reasons for taking either of these as a measure of a putative 'mass per unit length' since such an idea has no well-defined meaning.

3.2 Perfect-fluid wall

We now present a plane-homogeneous solution with boundary. The ϕ-coordinate is now Cartesian and hence we are constructing an infinite wall of perfect fluid which we shall join to the exterior (Levi–Civita) space–time. The solution follows from the choice

$$\beta\gamma = 1 + ar + a^2r^2 - a^4br^4, \qquad a, b \text{ constants}, \tag{36}$$

which is suitable for a plane-homogeneous space–time. [4] Equation (19) then gives

$$\frac{\gamma'}{\gamma} = \pm \frac{\sqrt{-ac + (3a^3 - 2a^2c)r + (3a^4 + 6a^4b)r^2 + (4a^4bc - 2a^5b)r^3 - 9a^6br^4 + 10a^8b^2r^6}}{1 + ar + a^2r^2 - a^4br^4}. \tag{37}$$

[4] We need only consider the $r \geq 0$-half of the system as the other half is identical, i.e. we could replace r by $|r|$ in what follows.

As with the cylinder, the positive sign in (37) leads to a positive fluid density and we first consider this case. From (18) and (17) we get

$$\frac{\beta'}{\beta} = \frac{a + 2a^2 r - 4a^4 b r^3}{1 + ar + a^2 r^2 - a^4 b r^4} - \frac{\gamma'}{\gamma}, \tag{38}$$

$$\frac{\alpha'}{\alpha} = \frac{\beta'}{\beta} - \frac{c}{\beta\gamma}. \tag{39}$$

The constant c is now arbitrary and is a parameter for the solution. The pressure and density are found to be

$$p = \frac{1}{8\pi\alpha^2}\left(\frac{a^2 - 6a^2 b r^2}{1 + ar + a^2 r^2 - a^4 b r^4}\right), \tag{40}$$

$$\rho = \frac{1}{8\pi\alpha^2}\left(\frac{3a^3 - 2a^2 c + (6a^4 + 12a^4 b)r + (12a^4 bc - 6a^5 b)r^2 - 36a^6 b r^3 + 60a^8 b^2 r^5 + (18a^4 b r^2 - 3a^2)g(r)}{(1 + ar + a^2 r^2 - a^4 b r^4)g(r)}\right) \tag{41}$$

where

$$g(r) = \sqrt{-ac + (3a^3 - 2a^2 c)r + (3a^4 + 6a^4 b)r^2 + (4a^4 bc - 2a^5 b)r^3 - 9a^6 b r^4 + 10a^8 b^2 r^6}.$$

In order to have a positive pressure and a boundary we must have $a > 0$ and $b > 0$. At $r = 0$ the density is

$$\rho(0) = \frac{1}{8\pi\alpha^2}\left[-3a^2 + \left(2a - \frac{3a^2}{c}\right)\sqrt{-(ac)}\right],$$

so for ρ to be real and finite on the axis we require $c < 0$. [5] Thus we confine the ranges of a, b and c as follows:

$$a > 0, \qquad b > 0, \qquad c < 0. \tag{42}$$

From (40) we see that the boundary is at $r = s = \sqrt{\frac{1}{6a^2 b}}$. The function on the right of (37) cannot be integrated analytically, however plots of γ for values of the constants satisfying (42) show that it is well-behaved in its range of validity $0 \le r \le s$. The functions α, β, p and ρ are also well-behaved in the range $0 \le r \le s$ and moreover p (up to the boundary) and ρ are positive.

[5] The plane-symmetric case $c = 0$ is therefore unphysical.

The match (22) to the exterior at $r = s$ gives

$$\sigma = \frac{\sqrt{3\left(8a^2/b + 27a^2 + 12a^3\sqrt{\frac{6}{a^2b}} - 27ac - 6a^2c\sqrt{\frac{6}{a^2b}}\right)}}{18a + 4a^2\sqrt{\frac{6}{a^2b}}}, \tag{43}$$

$$k = \frac{3 - 36b}{36ab + 8a^2b\sqrt{\frac{6}{a^2b}}}. \tag{44}$$

Since these give

$$s - k = \frac{5 + 36b + 6ab\sqrt{\frac{6}{a^2b}}}{36a + 8a^2b\sqrt{\frac{6}{a^2b}}} > 0$$

the exterior metric is free from singularities.

What range of σ does this source produce? As either b or c goes to zero σ approaches $\frac{1}{2}$; in the limit $b \to \infty$, σ approaches a constant value, but as $c \to -\infty$, σ increases without bound. Hence, from (42), we can match to the Levi–Civita metric for all σ greater than $\frac{1}{2}$.

We still have the option of taking the negative sign in (37). This leads to the same pressure but we now have the density

$$\rho = \frac{1}{8\pi\alpha^2}\left(\frac{-3a^3 + 2a^2c - (6a^4 + 12a^4b)r - (12a^4bc - 6a^5b)r^2 + 36a^6br^3 - 60a^8b^2r^5 + (18a^4br^2 - 3a^2)g(r)}{(1 + ar + a^2r^2 - a^4br^4)g(r)}\right)$$

which is negative in the range $0 \leq r \leq s$. The match to the exterior only differs from (43) and (44) in the sign of σ:

$$\sigma = -\frac{\sqrt{3\left(8a^2/b + 27a^2 + 12a^3\sqrt{\frac{6}{a^2b}} - 27ac - 6a^2c\sqrt{\frac{6}{a^2b}}\right)}}{18a + 4a^2\sqrt{\frac{6}{a^2b}}}, \tag{45}$$

$$k = \frac{3 - 36b}{36ab + 8a^2b\sqrt{\frac{6}{a^2b}}}. \tag{46}$$

With the restriction (42) we have a well-behaved solution in the range $0 \leq r \leq s$, and the exterior now has any value of σ less than $-\frac{1}{2}$. Thus we have a wall source for the Levi–Civita metric for

$$|\sigma| > \tfrac{1}{2}. \tag{47}$$

The Tolman mass formula can again be used, this time to calculate a mass per unit area for the source. For this we take

$$m_T = \int_0^1 \int_0^1 \int_0^s (T^\alpha_{\ \alpha} - T^4_{\ 4})\sqrt{-g}\, dr\, dz\, d\phi,$$

which gives

$$m_T = \pm \frac{\sqrt{8a^2/b + 27a^2 + 12a^3\sqrt{\frac{6}{a^2 b}} - 27ac - 6a^2 c\sqrt{\frac{6}{a^2 b}}}}{12\pi\sqrt{3}},$$

where the plus and minus signs correspond to positive and negative σ respectively. Unlike the density ρ, m_T is well-behaved when $c = 0$; as $c \to -\infty$, m_T, like σ, goes to $\pm\infty$.

4 Discussion

In the previous section we constructed a perfect-fluid cylinder with positive density and pressure and found that matching to the Levi–Civita exterior was possible for $0 \leq \sigma < \frac{1}{2}$; this is the same range of σ as Bonnor and Davidson [6] found for their cylinder source. In addition, we matched a negative-density, perfect-fluid cylinder to the Levi–Civita metric for which σ could take values in the range $-\frac{1}{2} < \sigma \leq 0$. We also showed that the Levi–Civita space–time represents the exterior field of a plane mass: we constructed a perfect-fluid wall with positive density and pressure which matches to the Levi–Civita metric for $\sigma > \frac{1}{2}$, and also a negative-density, perfect-fluid wall for which this match gives $\sigma < -\frac{1}{2}$.

The previous work on static cylinder sources in conjunction with the results obtained here leads us to suspect that a perfect-fluid cylinder source for the Levi–Civita space–time does not exist outside the range $-\frac{1}{2} < \sigma < \frac{1}{2}$. This exterior field provides a relativistic analogue of the Newtonian line-mass field; it has a clear Newtonian limit, given by $\sigma \to 0$, wherein σ approaches the Newtonian mass per unit length. There is however no justification for calling σ the mass per unit length far from this limit and hence concluding that there is an upper limit on the mass per unit length of a relativistic perfect-fluid cylinder; such an idea has no well-defined meaning in general relativity as we have illustrated by using the Tolman mass formula to calculate the quantity m_T

given by (35), which has as much a claim to be the "mass per unit length" as σ in that it gives the correct Newtonian limit, but which takes the range $-\infty < m_T < \infty$ for our cylinder sources.

What about cylinder sources other than perfect-fluid? If we allow arbitrary energy–momentum tensors then there *do* exist cylinder sources for the Levi–Civita space–time outside the range $-\frac{1}{2} < \sigma < \frac{1}{2}$; for example, the metric given by

$$\alpha = \gamma = \left(1 + \frac{1+as^2}{s^2} r^2\right)^{\frac{as^2}{1+as^2}}, \qquad \beta = \frac{r}{1 + \frac{1+as^2}{s^2} r^2}, \qquad a, s \text{ constants}, \tag{48}$$

describes a static cylinder with energy–momentum tensor

$$T^r_{\ r} = T^\phi_{\ \phi} = \frac{a}{2\pi\alpha^2} \frac{\beta^2}{r^2} \left(1 - \frac{r^2}{s^2}\right), \qquad T^t_{\ t} = T^z_{\ z} = \frac{1}{4\pi\alpha^2} \frac{\beta^2}{r^2} \left(\frac{1+as^2}{s^2} r^2 - 3 - 2as^2\right). \tag{49}$$

At $r = s$, $T^r_{\ r} = T^\phi_{\ \phi} = 0$ and a correct match may be made to the Levi–Civita metric (1), giving $\sigma = 1$, $k = -\frac{2}{as}$. For positive a and s the complete solution is well-behaved everywhere and the cylinder has positive density and positive radial and azimuthal pressures, with a longitudinal stress equal to the density. This last property, $T^t_{\ t} = T^z_{\ z}$ (familiar from cosmic string theory), represents an exotic relativistic situation and so although for $\sigma = 1$ at least, one can find both a cylinder and a wall source, we are inclined to the view that any cylinder source valid for σ outside $-\frac{1}{2} < \sigma < \frac{1}{2}$ will be composed of rather bizarre relativistic material, at least in comparison to a perfect fluid.

What are we to conclude regarding the Levi–Civita space–time as the field outside a plane mass? It is more difficult to find a firm basis for the analysis here because of the lack of a Newtonian limit. Such a limit could be identified by considering the proper acceleration of a test particle initially at rest with respect to the wall; in the Newtonian limit this acceleration would approach a constant value throughout the exterior and this constant is then $2\pi G$ times the Newtonian mass per unit area. But the acceleration in question is given by (2) and this is not a constant for any real value of σ. This result is not so surprising when we consider that in this putative Newtonian limit test particles at rest anywhere in the coordinate system of metric (1), that is at rest relative to the wall, would have to experience the same acceleration away from the wall (to counteract the uniform force directed towards the wall). But a coordinate frame in which all points experience the same proper acceleration cannot remain rigid in the sense that the proper distances between spacelike-separated points must change with time; hence the components of the metric tensor in this coordinate frame must depend on the time coordinate and therefore the Levi–Civita metric cannot achieve this limit.

Nevertheless we have found that the Levi–Civita space–time can represent the exterior field of a wall of perfect fluid for $|\sigma| > \frac{1}{2}$. As remarked above, the usual suspects for the exterior field of a plane mass are the Taub metric ($\sigma = -\frac{1}{2}$) and the flat accelerated metric ($\sigma = \frac{1}{2}$). These two appear as the bounding values of the exterior fields of the perfect-fluid wall (and cylinder) sources we have constructed, in the cases of negative and positive mass density respectively. It therefore appears from this work that if perfect-fluid sources for these two metrics exist they are not continuously related to perfect-fluid sources for neighbouring values of σ. At any rate, the only perfect-fluid walls with boundary the author has found using the procedure of Sec. 2 and which upon matching to the Levi–Civita metric produce a range of σ including $-\frac{1}{2}, \frac{1}{2}$ are such that $s - k < 0$, that is the exterior space–time has a plane singularity. In fact these solutions are valid for *all* values of σ (including $\sigma = 0$, where however the source does not vanish). Although an attempt has been made to give some meaning to this type of plane singularity [31], in this case we regard only singularity-free solutions as being of physical interest.

If we again allow for more unorthodox energy–momentum tensors, wall sources for $\sigma = -\frac{1}{2}, \frac{1}{2}$ can certainly be found. A wall source for the Taub metric ($\sigma = -\frac{1}{2}$) is given by

$$\alpha = \beta = e^{ar}, \qquad \gamma = e^{-\frac{ar^2}{4s}}, \qquad a, s \text{ constants}$$

$$T^r{}_r = \frac{a^2}{8\pi\alpha^2}\left(1 - \frac{r}{s}\right) \qquad T^z{}_z = T^\phi{}_\phi = -\frac{a}{16\pi s\alpha^2}\left(1 - \frac{ar^2}{2s}\right), \qquad T^t{}_t = \frac{a^2}{8\pi\alpha^2},$$

which matches to the Levi–Civita metric at $r = s$ with $\sigma = -\frac{1}{2}$, $k = s - \frac{2}{a}$. If a and s are positive the solution is well-behaved everywhere and the wall has negative density (which tallies with its repulsive exterior field); the radial pressure is positive up to the boundary and $T^z{}_z = T^\phi{}_\phi$ may be negative in some regions and positive in others, depending on the value of a. A wall source for the accelerated metric $\sigma = \frac{1}{2}$ is

$$\alpha = \beta = e^{ar\left(1-\frac{r}{2s}\right)}, \qquad \gamma = 1 + \frac{r}{s}, \qquad a, s \text{ constants}$$

$$T^r_{\ r} = \frac{a}{8\pi s^2\alpha^2\gamma}\left(1 - \frac{r}{s}\right)\left(2s + as^2 - ar^2\right), \qquad T^z_{\ z} = T^\phi_{\ \phi} = -\frac{a}{8\pi s\alpha^2},$$

$$T^t_{\ t} = -\frac{a}{8\pi\alpha^2}\left[\frac{2}{s} - a\left(1 - \frac{r}{s}\right)^2\right].$$

This matches to the Levi–Civita metric at $r = s$ with $\sigma = \frac{1}{2}$, $k = -s$. The space–time is well-behaved everywhere if a and s are positive and moreover the density is positive throughout the interior if $0 < a < \frac{2}{s}$; then $T^z_{\ z} = T^\phi_{\ \phi} < 0$ and $T^r_{\ r} \geq 0$. These two sources are physically unappealing and are presented here simply to show that wall sources for $\sigma = -\frac{1}{2}, \frac{1}{2}$ exist; there is nothing in this work to support the interpretation of either of these metrics as the general-relativistic plane-mass field.

There is a further oddity of the wall sources, to be seen in the variation of the proper acceleration of a test particle (2) with σ. We first rewrite (2) in Gaussian normal coordinates $(\bar{r}, z, \phi, t)$, wherein $g_{\bar{r}\bar{r}} = 1$; the Levi–Civita metric in these coordinates is

$$ds^2 = d\bar{r}^2 + A^2(\bar{r} - \bar{k})^{\frac{8\sigma^2-4\sigma}{4\sigma^2-2\sigma+1}}dz^2 + B^2(\bar{r} - \bar{k})^{\frac{2-4\sigma}{4\sigma^2-2\sigma+1}}d\phi^2 - C^2(\bar{r} - \bar{k})^{\frac{4\sigma}{4\sigma^2-2\sigma+1}}dt^2, \tag{50}$$

and (2) becomes

$$\ddot{\bar{r}} = -\frac{2\sigma}{4\sigma^2 - 2\sigma + 1}\left(\frac{1}{\bar{r} - \bar{k}}\right). \tag{51}$$

This form is preferable to (2) because it isolates the σ-dependence. As a result $\ddot{\bar{r}}$ as a function of σ has the same form regardless of the value of $\bar{r}$; this form is shown in Figure 1.

The proper acceleration increases as σ goes from 0 to $\frac{1}{2}$ (0 to $-\frac{1}{2}$) but falls to zero as $\sigma \to \infty$ ($\sigma \to -\infty$)! Although the lack of a Newtonian limit means there is nothing we can term the mass per unit area for even a limited range of σ, zero acceleration would be expected to occur only when the source vanishes or when a combination of negative density and positive pressure (or vice versa) conspires to produce zero gravitational mass. This is certainly not the case for the wall with positive density and pressure for which the limit $\sigma \to \infty$ is produced by $c \to -\infty$: the total gravitational mass cannot

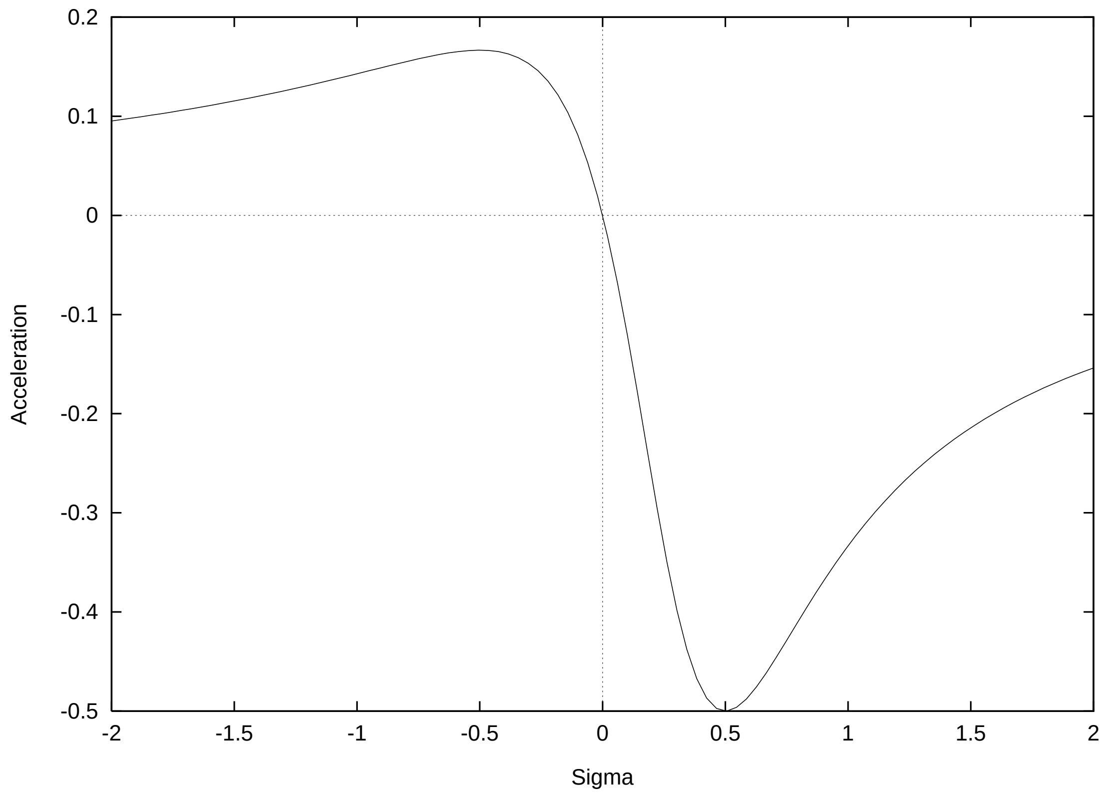

Figure 1: Plot of $\ddot{\bar{r}}$ against σ (see (51)) for $\bar{r} - \bar{k} = 2$.

be zero and in no way does the limit $c \to -\infty$ correspond to the wall vanishing—the position of the boundary is unaffected and the density becomes infinite.

Some insight into the nature of the system in this $\sigma \to \pm\infty$ limit is afforded by the Levi–Civita line element in Gaussian normal coordinates (50). This form has a well-defined $\sigma \to \pm\infty$ limit which is

$$ds^2 = d\bar{r}^2 + A^2(\bar{r} - \bar{k})^2 dz^2 + B^2 d\phi^2 - C^2 dt^2. \tag{52}$$

This metric is *flat*, being transformable to a manifestly Minkowskian form by

$$z' = (\bar{r} - \bar{k}) \sin Az, \qquad r' = (\bar{r} - \bar{k}) \cos Az. \tag{53}$$

Metric (52) has the general *appearance* of flat space–time in cylindrical polar coordinates, with z in the role of the angular coordinate, and (53) that of the usual transformation from cylindrical polars to Cartesian coordinates. But z is *not* a periodic coordinate and if (53) were enforced globally it would have the effect of changing the topology of the exterior: (53) assumes that the points z and $z + 2\pi/A$ are identified and consequently this transformation is only valid in a local region covered by a range of z smaller than $2\pi/A$. (Were we to take z in the exterior as periodic we would have to do so in the interior also, thus producing a singularity in the source.) The geometry of the space part of (52) is flat but rather bizarre: if we take two points on the boundary of the wall with different z-coordinate values and extend a straight (we are in flat space) line perpendicular to the boundary from each point, then the z-separation of these two straight lines is proportional to the distance from the wall, whereas the ϕ-separation remains constant.

Although the $\sigma \to \pm\infty$ limit is not physically realizable, corresponding to a diverging density, and although we have nothing to call the mass per unit area, it is still curious that the acceleration in Figure 1 due to the wall should approach zero as the density diverges. But, as discussed above, we are dealing here with a completely relativistic system; wall sources for the Levi–Civita space–time have no Newtonian limit to accommodate along with our Newtonian-based intuition about how a plane mass affects matter. An interesting question is: can a wall source be found in general relativity which *does* possess the Newtonian plane-mass limit? As remarked above, the exterior line element, in which the source is at rest, must depend on the time coordinate, so evidently this source will not be static.

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