General Relativity in Electrical Engineering

Ulf Leonhardt and Thomas G. Philbin
School of Physics and Astronomy, University of St Andrews,
North Haugh, St Andrews KY16 9SS, Scotland

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Abstract

In electrical engineering metamaterials have been developed that offer unprecedented control over electromagnetic fields. Here we show that general relativity lends the theoretical tools for designing devices made of such versatile materials. Given a desired device function, the theory describes the electromagnetic properties that turn this function into fact. We consider media that facilitate space-time transformations and include negative refraction. Our theory unifies the concepts operating behind the scenes of perfect invisibility devices, perfect lenses, the optical Aharonov-Bohm effect and electromagnetic analogs of the event horizon, and may lead to further applications.

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1 Introduction

Modern metamaterials offer remarkable control over electromagnetic fields [1] with applications ranging from future invisibility devices [1]-[7] to existing perfect lenses [8]-[14]. Imagine there were no practical limits on the electromagnetic properties of materials. Given a desired function, how do we find the design of the device that turns this function into fact? In this paper, we show that general relativity provides clear recipes for calculating the required material properties. The practical use of general relativity in electrical engineering may seem surprising: relativity has been associated with the physics of gravitation [15] and cosmology [16] or, in engineering [17], has been considered a complication, not a simplification. But the design concept described here is rooted in a simple idea with a distinguished history: according to Fermat's Principle [4, 18, 19] light rays follow the shortest optical paths in media; they are effective geodesics, and general relativity has developed the theoretical tools for fields in curved geometries [15]. Of course, here we exploit only some of the aspects of general relativity: we use the physics of curved space, but not the physics [15] that creates space-time curvature in gravity.[1] In our case, electromagnetic metamaterials, not masses, create effective geometries. Our theory generalizes the concept behind the proposed perfect invisibility devices [1, 2] to magneto-electric or moving media and it incorporates negative refraction [8]-[14]. For example, using a simple pictorial argument we show how to design magnifying perfect lenses. Finally, metamaterials may be also applied for laboratory analogs of general relativity, in particular for artificial black holes [20]-[30]. In this way, we unify and generalize a range of physical phenomena that rely on the geometry of media and the recent opportunities of metamaterials.

Metamaterials are materials with designed properties that stem from structure, not substance; where man-made structures determine the electromagnetic properties, structures that are smaller than the electromagnetic wavelengths involved. Metamaterials have a long history: mediaeval ruby glass, for example, is a metamaterial. Ruby glass contains nano-scale gold colloids that render the glass neither golden nor transparent, but ruby, depending on the size and concentration of the gold droplets. The color originates from a resonance of the surface plasmons [31] on the metallic droplets. Metamaterials *per se* are nothing new: what is new is the degree of control over the structures in the material that generate the desired properties. For example, in the modern version of negatively-refracting ruby glass [32] nano-manufactured pairs of gold-pillars on a silicon substrate generate finely-tuned plasmon resonances where each pair acts like a designer atom with controllable properties.

Our starting point is not new either: in the early 1920's Gordon [33] noticed that moving isotropic media appear to electromagnetic fields as certain effective space-time geometries. Tamm [34, 35] generalized this geometric approach to anisotropic media and briefly applied this theory [35] to the propagation of light in curved geometries. In 1960 Plebanski [36] formulated the electromagnetic effect of curved

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[1]As in other analogue models of gravity [20], we use the kinematic aspects of general relativity, not the dynamic ones.

space-time or curved coordinates in concise constitutive equations. Dielectric media act on electromagnetic fields as geometries and geometries act as effective media. In 2000 it was understood [37] that the dipole forces of electromagnetic fields in media appear as the effects of geometries as well, electromagnetic fields act as geometries on media, a concept used to identify the conditions for the Abraham or the Minkowski momentum in media [38]. Only very recently [1]-[4] meta-material implementations of coordinate transformations were considered as engineering tools, ideas we take further here: we show that general relativity lends the most natural recipes for such engineering applications.

2   Theory

2.1   Electromagnetism in curved coordinates

A geometry is characterized by the space-time measure [39] $ds^2 = \sum_{\alpha\beta} g_{\alpha\beta} dx^\alpha dx^\beta$, the metric, where we denote the space-time coordinates by x^α with Greek indices running from 0 to 3. Latin indices indicate the spatial coordinates and run from 1 to 3, whereas $x^0 = ct$ describes time measured in spatial units with c being the speed of light in vacuum. The matrix $g_{\alpha\beta}$, the metric tensor, may vary as a function of the coordinates, because space-time may be curved or because curved coordinates are used in flat space. The determinant g of $g_{\alpha\beta}$ measures the 4D volume of an infinitesimal space-time element as $\sqrt{-g}$ times the infinitesimal coordinate volume [15, 39]. We denote the matrix inverse of $g_{\alpha\beta}$ by $g^{\alpha\beta}$; where, as customary in general relativity [15, 39], the position of the indices indicates that $g_{\alpha\beta}$ is co-variant and $g^{\alpha\beta}$ is contra-variant under coordinate transformations. For example, flat space in cylindrical coordinates ct, r, φ, z is described by the metric tensor $g_{\alpha\beta} = \mathrm{diag}(1, -1, -r^2, -1)$ with matrix inverse $g^{\alpha\beta} = \mathrm{diag}(1, -1, -r^{-2}, -1)$ and determinant $g = -r^2$.

As our starting point, we use the result of general relativity [36, 40], derived in Appendix A, that the free-space Maxwell equations can be written in the macroscopic form [41]

$$\nabla \cdot \mathbf{D} = \rho\,, \quad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}\,, \quad \nabla \cdot \mathbf{B} = 0\,, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\,, \qquad (1)$$

or, in Cartesian components,

$$\sum_i \frac{\partial D^i}{\partial x^i} = \rho \quad, \qquad \sum_i \frac{\partial B^i}{\partial x^i} = 0\,, \qquad (2)$$

$$\sum_{jk} \epsilon^{ijk} \frac{\partial H_k}{\partial x^j} = \frac{\partial D^i}{\partial t} + j^i \quad, \qquad \sum_{jk} \epsilon^{ijk} \frac{\partial E_k}{\partial x^j} = -\frac{\partial B^i}{\partial t}\,, \qquad (3)$$

where ϵ^{ijk} is the completely antisymmetric Levi-Civita tensor [15]: in Cartesian components ϵ^{ijk} is 1 for all cyclic permutations of ϵ^{123}, -1 for all cyclic permutations of ϵ^{213} and 0 otherwise. The spatial indices indicate that in this representation E_i and H_i form the components of vectors that are co-variant under purely spatial

transformations, whereas D^i and B^i constitute contra-variant spatial vectors. The charge density ρ and the current density $\mathbf{j}$ are given by $\sqrt{-g}\,j^0$ and $c\sqrt{-g}\,j^i$ of the four-current j^α [39]. In empty but possibly curved space, the electromagnetic fields are connected by the constitutive equations in SI units [36],

$$\mathbf{D} = \varepsilon_0 \varepsilon\,\mathbf{E} + \frac{\mathbf{w}}{c} \times \mathbf{H}\,, \quad \mathbf{B} = \frac{\mu}{\varepsilon_0 c^2}\,\mathbf{H} - \frac{\mathbf{w}}{c} \times \mathbf{E}\,. \tag{4}$$

In dielectric media, the $\mathbf{E}$, $\mathbf{B}$ vectors generate electric polarizations and magnetizations that constitute the $\mathbf{D}$, $\mathbf{H}$ fields. The constitutive equations are described here in implicit form, but one can also express them as $\mathbf{D}$, $\mathbf{H}$ as functions of $\mathbf{E}$ and $\mathbf{B}$ [17]. In general, the electric permittivity ε and the magnetic permeability μ are symmetric matrices – space-time appears as an anisotropic medium – but, in empty space, ε and μ are equal, as in perfect impedance matching [41]. The $\mathbf{w}$ vector describes a magneto-electric coupling between the magnetic and the electric fields. Some materials are magneto-electric, but the simplest example is a moving medium [17, 37, 42], because a moving medium responds to the electromagnetic field in locally co-moving frames and Lorentz transformations from the laboratory frame naturally mix electric and magnetic fields [17, 42]. In explicit form [36],

$$\varepsilon = \mu = -\frac{\sqrt{-g}}{g_{00}}\,g^{ij}\,, \quad w_i = \frac{g_{0i}}{g_{00}}\,, \tag{5}$$

the ε and μ are constructed from the spatial components of the inverse metric tensor and from the determinant and the time-time component of the metric tensor, whereas the $\mathbf{w}$ vector is given in terms of the time-space components of the metric tensor. Empty space appears as an impedance-matched anisotropic magneto-electric or moving medium.

2.2   Transformation media

Since empty space appears as a medium, what happens if a medium appears as empty space? This is the idea behind the recent proposals for invisibility devices [1]-[4]. To be more precise, suppose that the medium appears as the result of a coordinate transformation from some fictitious space-time, say electromagnetic space-time, to physical space, see, for example, Fig. 1. Electromagnetic space-time could be flat with light traveling along straight lines, whereas to electromagnetic fields physical space appears to be curved, bending light. Of course, this apparent curvature is an illusion, the same type of illusion as straight lines appearing curved in curved coordinates, because in theory it is removable by the inverse coordinate transformation: but, in practice, one can exploit this apparent curvature to create illusions [1]-[4].

We use primes to denote the geometry and coordinates of electromagnetic space-time[2] and describe physical space in generalized coordinates x^i with spatial metric γ_{ij} and determinant γ, for keeping the theory as flexible as possible. For example, we may wish to use cylindrical coordinates r, φ, z with spatial metric tensor $\gamma_{ij} =$

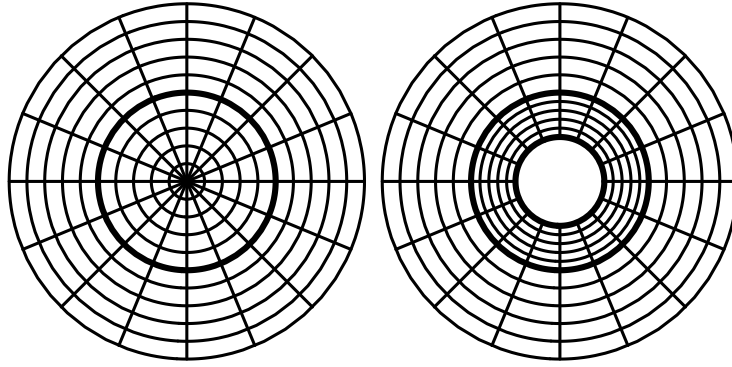[2]We use the opposite notation of Ref. [1], since, we believe, it is more logical.

Figure 1: Transformation media implement coordinate transformations. The left figure shows an orthogonal grid of coordinates in electromagnetic space (a slice of cylindrical coordinates at constant z). The right figure shows the transformed grid in physical space that corresponds to the invisibility device described by Eq. (11) and illustrated in Fig. 2.

diag$(1, r^2, 1)$ and $\gamma = r^2$. The metric γ_{ij} should differ from the effective geometry $g_{\alpha\beta}$ generated by the medium. The transformation rules of tensors in general relativity give rise to a simple recipe for the construction of media that facilitate space-time coordinate transformations. Contra-variant tensors are transformed as [17, 15, 39]

$$g^{\alpha\beta} = \sum_{\alpha'\beta'} \frac{\partial x^\alpha}{\partial x'^{\alpha'}} \frac{\partial x^\beta}{\partial x'^{\beta'}} g'^{\alpha'\beta'} \ . \tag{6}$$

The transformed inverse metric serves as the building block of the dielectric tensors (5) of the medium. If we wish to express physical space in generalized coordinates, we need to consider a subtlety that appears when we write the divergences in Maxwell's equations (2) in spatially co-variant form [39]

$$\frac{1}{\sqrt{\gamma}} \sum_i \frac{\partial \sqrt{\gamma} D^i}{\partial x^i} = \rho \, , \quad \frac{1}{\sqrt{\gamma}} \sum_i \frac{\partial \sqrt{\gamma} B^i}{\partial x^i} = 0 \, . \tag{7}$$

The ε^{ij} and μ^{ij} tensors are naturally contra-variant with respect to the background geometry of physical space, but γ differs from $-g$, in general. Consequently, for writing the medium as an active coordinate transformation, we need to multiply $\mathbf{D}$ and $\mathbf{B}$ by $\sqrt{\gamma}$ in the constitutive equations (4) and re-scale ρ and $\mathbf{j}$ accordingly, which is also consistent with the form of the Levi-Civita tensor in curved coordinates [15] in the curls in Maxwell's equations (3). However, a second subtlety arises from the curls: the coordinate transformation may turn a right-handed coordinate system into a locally left-handed one, but the curls (3) implicitly assume a right-handed system, because ϵ^{ijk} changes sign under coordinate transformations from right to left-handed systems. Consequently, we need to invert the sign of ε, μ and ρ, $\mathbf{j}$ wherever this is the case. The transformation to a left-handed coordinate system thus corresponds to negative refraction [13, 14] occurring in what has, for other reasons [14], been

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fittingly described as left-handed media.[3] Taking all this into account, we arrive at the simple recipe

$$\varepsilon^{ij} = \mu^{ij} = \mp \frac{\sqrt{-g}}{\sqrt{\gamma}} \frac{g^{ij}}{g_{00}}, \quad w_i = \frac{1}{\sqrt{\gamma}} \frac{g_{0i}}{g_{00}}. \tag{8}$$

The $\mp$ sign indicates the handedness: minus for right-handed transformations and plus for locally left-handed ones. Starting from the inverse metric $g'^{\alpha\beta}$ in electromagnetic space, the $g^{\alpha\beta}$ matrix is calculated according to the transformation rule (6). The matrix inverse of $g^{\alpha\beta}$ gives the metric tensor $g_{\alpha\beta}$ and the inverse of the determinant of $g^{\alpha\beta}$ gives g. Equations (8) specify the required electromagnetic properties that will turn the coordinate transformation into reality. In the special case of right-handed and purely spatial transformations our recipe agrees with the theory of Refs. [1, 45], see Appendix B, where however, instead of the contra-variant ε and μ mixed tensors occur with one of the indices lowered by the spatial metric,

$$\varepsilon^i_k = \mu^i_k = \sum_j \varepsilon^{ij} \gamma_{jk}. \tag{9}$$

Here we consider more general transformations that may mix space and time and that may be multivalued. As long as the matrix (6) is single-valued and not explicitly time-dependent such a device is physically allowed and stationary.

The coordinate transformation encodes the function of the device. Outside of it, the electromagnetic coordinates agree with the coordinates of physical space: the transformation is trivial; ε and μ are unity and $\mathbf{w}$ vanishes. Inside the device, electromagnetic fields are controlled as prescribed by the coordinate transformation. If the physical coordinates enclose a hole one obtains the blueprint of an invisibility device [1]-[4]. We show in the next section that perfect lensing [8]-[14], the optical Aharonov-Bohm effect [21, 22, 46, 47] and optical black holes [20]-[30] can be understood as applications of the same idea as well.

3 Examples

3.1 Perfect invisibility devices

Perfect invisibility devices [1, 2] facilitate coordinate transformations with holes in physical space. In this way, electromagnetic radiation is naturally guided around such excluded regions. Anything placed inside is hidden. Perfect invisibility devices [1, 2] must employ anisotropic media, because the inverse scattering problem for waves in isotropic media has unique solutions [48, 49]. They also require media where the phase velocity of light approaches infinity at the inside of the cloaking layer, because coordinate transformations with holes rip apart points of zero volume in electromagnetic space and tear them to finite volumes in physical space, unless

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[3]Negative refraction in left-handed media is not related to the case of negative phase velocities in gravity [43, 44], because transformations to left-handed or any other coordinates have no physical significance there, in contrast to transformations facilitated by media.

the coordinate transformation is singular. To prove this, consider the determinant of the ε and μ tensor (8). For an invisibility device [1, 2] only space is transformed, so $\mathbf{w}$ vanishes, g_{00} is unity and $-g$ is reduced to the inverse of the determinant of the spatial components g^{ij}. We obtain from Eqs. (6) and (8) in three-dimensional space

$$\det \varepsilon = (-g)^{1/2} \gamma^{-3/2} = (-g')^{1/2} \gamma^{-3/2} J, \quad J = \frac{\partial(x'^1, x'^2, x'^3)}{\partial(x^1, x^2, x^3)}, \tag{10}$$

in terms of the Jacobian J [39]. At a point of measure zero $\sqrt{-g'}$ vanishes and so does $\det \varepsilon$, the product of the eigenvalues of ε and μ. Consequently, at least one of the eigenvalues of ε and μ are zero; therefore in at least one direction of the anisotropic medium the phase-velocity of light diverges near the inside of the cloak. The speed of light in media can reach high values in narrow frequency ranges or, naturally, for static fields. On the other hand, invisibility devices that are only perfect within the limits of geometrical optics [3, 4, 5] are not subject to such constraints [5].

Figure 2 illustrates a cylindrical invisibility device that stretches the z axis out in the radial direction to a full cylinder of radius R_1, compressed within a cylindrical volume of radius R_2, as shown in Fig. 1. This device is an invisibility cloak of thickness $R_2 - R_1$ where anything placed inside the inner radius R_1 is hidden. We obtain from the theory

$$r = R_1 + r' \frac{R_2 - R_1}{R_2} \quad \Longrightarrow \quad \varepsilon^i_j = \frac{R_2}{R_2 - R_1} \frac{r'}{r} \operatorname{diag}\left(\frac{(R_2 - R_1)^2}{R_2^2}, \frac{r^2}{r'^2}, 1\right) \tag{11}$$

within $R_1 \leq r \leq R_2$ or, equivalently, $0 \leq r' \leq R_2$. Clearly, close to the lining of the cloak at $r \to R_1$ where $r' \to 0$ the speed of light in the r and z directions diverges, whereas in φ direction the phase velocity tends to zero.

3.2 Perfect lenses

In perfect invisibility devices, electromagnetic space does not cover the entire physical space: here we show that perfect lenses correspond to multi-valued electromagnetic space. Consider in Cartesian coordinates the multi-valued transformation $x(x')$ illustrated in Fig. 3, whereas all other coordinates are not changed. In the fold of the function $x(x')$ a point x' in electromagnetic space has three faithful images in physical space. Obviously, electromagnetic fields at one of those points are perfectly imaged onto the other: the device is a perfect lens [8]. This simple pictorial argument may contribute to settling the debate on perfect lensing [50]-[54] in addition to the experimental proof for enhanced imaging in existing perfect lenses [9]-[12]. Our theory also shows why perfect lensing requires left-handed media with negative ε and μ [13, 14]: inside the device, *i.e.* inside of the x' fold, the derivative of $x(x')$ becomes negative and the coordinate system changes handedness. We obtain from our recipe (8) the compact result

$$\varepsilon = \mu = \operatorname{diag}\left(\frac{\mathrm{d}x'}{\mathrm{d}x}, \frac{\mathrm{d}x}{\mathrm{d}x'}, \frac{\mathrm{d}x}{\mathrm{d}x'}\right). \tag{12}$$

If, for example, $\mathrm{d}x'/\mathrm{d}x$ is -1 inside the device and $+1$ outside, we obtain the standard perfect lens [8]-[14] based on an isotropic left-handed material with $\varepsilon = \mu = -1$
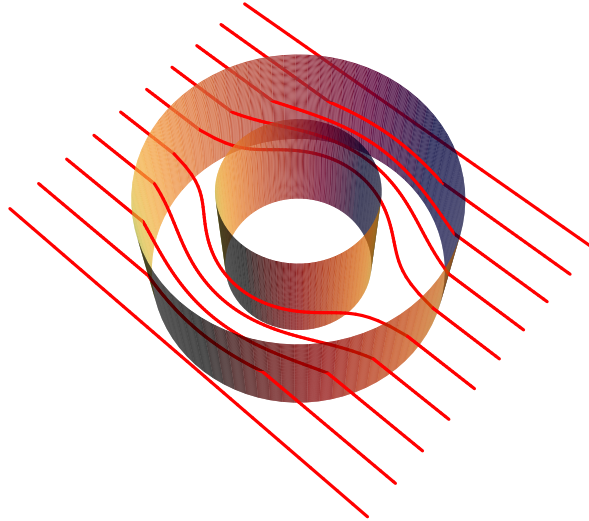
Figure 2: Invisibility device. The transformation medium (11) acts as an invisibility cloak, guiding light around the interior of the cloak without causing any distortion. The device facilitates the coordinate transformation illustrated in Fig. 1.

inside; but this is not the most general choice. One could use an anisotropic medium to magnify perfect images, in contrast to the existing examples [9]-[12], by embedding the source or the image in transformation media with $|dx'/dx| \neq 1$.

3.3 Optical Aharonov-Bohm effect

Perfect invisibility devices and perfect lenses exploit transformation media with non-trivial topology, with excluded regions in physical space or folds in electromagnetic space. Here we consider an example where physical space is multi-valued, but the medium is single-valued and hence physically allowed: the optical Aharonov-Bohm effect [21, 22, 46, 47]. One can demonstrate this effect with light passing through a water vortex [21] or with slow light in Bose-Einstein condensates [22]. Note that the effect is related to the Aharonov-Bohm effect with surface waves [55]-[57] and to the gravitational Aharonov-Bohm effect [58]. The optical Aharonov-Bohm effect is an example of a transformation medium that mixes space and time, and yet the medium is stationary. Consider in cylindrical coordinates the transformation

$$ ct = ct' + a\varphi', \quad r = \frac{r'}{n}, \quad \varphi = \varphi', \quad z = \frac{z'}{n} \tag{13} $$

with the constants n and a. We obtain from our theory (8) that the medium is isotropic with refractive index n and is magneto-electric with $w_i = (0, a/r, 0)$, which corresponds to a fluid vortex with velocity profile $\mathbf{u}/c = \mathbf{w}/(n^2 - 1)$, as we see comparing Eq. (4) with the constitutive equations of moving media [17, 42] in lowest relativistic order. Normally, a moving medium Fresnel-drags light [21], but
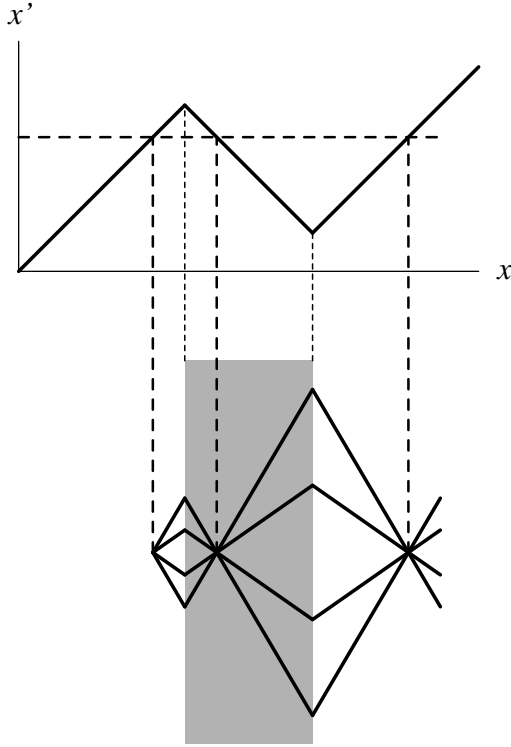
Figure 3: Perfect lens. Negatively refracting perfect lenses employ transformation media. The top figure shows a suitable coordinate transformation from the physical x axis to the electromagnetic x', the lower figure illustrates the corresponding device. The inverse transformation from x' to x is either triple or single-valued. The triple-valued segment on the physical x axis corresponds to the focal region of the lens: any source point has two images, one inside the lens and one on the other side. Since the device facilitates an exact coordinate transformation, the images are perfect with a resolution below the normal diffraction limit: the lens is perfect [8]. In the device, the transformation changes right-handed into left-handed coordinates. Consequently, the medium employed here is left-handed, with negative refraction [14].

in the case of a vortex, light rays follow straight lines, because the transformation (13) changes only time, such that straight rays in electromagnetic space are mapped onto straight lines in physical space. Similarly, in the original Aharonov-Bohm effect [59, 60] electron rays are not bent by a magnetic vortex, but they develop a phase slip in the direction of incidence. In our case, when the light has passed the vortex, the time change in the transformation (13) results in a phase slip of $\pm\pi ak$, where k is the wave number, depending on whether the light propagates with or against the current, see Fig. 4. Physical space-time is multi-valued, with a branch cut in the direction of incidence, resembling the infinitely sheeted Riemann surfaces around a logarithmic branch point [61], but the medium is single-valued and has the simple physical interpretation of a moving fluid forming a vortex.
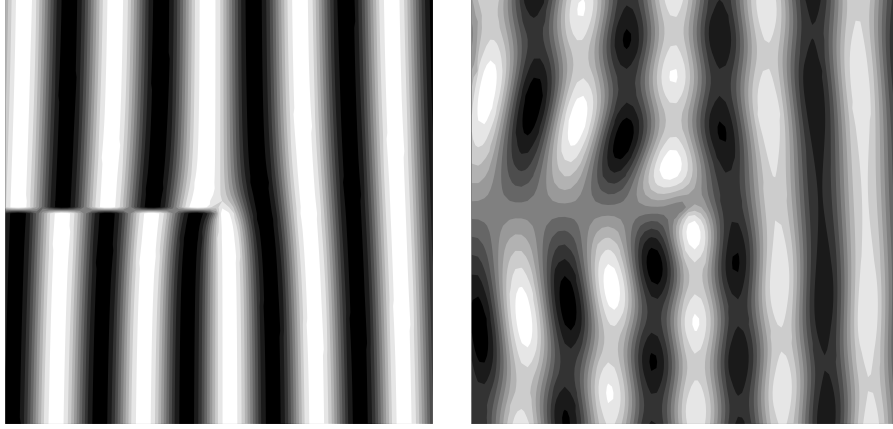
Figure 4: Aharonov-Bohm effect. A fluid vortex generates the optical Aharonov-Bohm effect described by the coordinate transformation (13). Light, incident from the right, is Fresnel-dragged by the moving medium: light propagating with the flow is advanced, whereas light propagating against the current is retarded. The wave should develop the phase slip shown in the left figure. However, although the transformation (13) is exact, physical space-time is multi-valued here. Instead of the simple phase slip, the light turns out to exhibit the characteristic interference pattern illustrated in the right figure [59, 60].

3.4 Artificial black holes

Moving isotropic media generate the effective space-time geometry discovered by Gordon in 1923 [33]. Consider an effectively one-dimensional situation where the medium is moving in x direction and the electromagnetic field varies along the x axis with field vectors pointing orthogonal to x. An impedance-matched medium of refractive index n and velocity u is described by the inverse metric tensor [33, 37, 21]

$$g^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + \frac{n^2 - 1}{1 - u^2/c^2} \begin{pmatrix} 1 & u/c & 0 & 0 \\ u/c & u^2/c^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{14}$$

Both the velocity u and the index n may vary. We show that this effective geometry is generated from empty space by the coordinate transformation

$$ct' = \frac{c}{2}(t_- + t_+) , \quad x' = \frac{c}{2}(t_- - t_+) , \quad t_\pm = t - \int \frac{n \pm u/c}{nu \pm c} \, \mathrm{d}x , \tag{15}$$

as long as we restrict our attention to fields varying in x direction. For this, we use the transformation rule (6), but from un-primed to primed space-time, and find that

$$g'^{\alpha\beta} = \mathrm{diag}\left(\frac{n^2(c^2 - u^2)}{c^2 - n^2 u^2}, -\frac{n^2(c^2 - u^2)}{c^2 - n^2 u^2}, -1, -1 \right) . \tag{16}$$

In electromagnetic space-time, we are free to apply the recipe (8) for assigning the medium properties as well as in physical space. We find $\varepsilon' = \mu' = 1$ in the directions

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orthogonal to x and $w' = 0$: vacuum, which proves our point. Here wave packets are superpositions of waves propagating in positive or negative direction as functions of either t_+ or t_-. Substituting for $t_\pm$ the representations (15) as functions of ct and x, we obtain the electromagnetic waves in physical space. The transformation (15) describes the relativistic addition theorem of velocities [17, 39] for the medium and light propagating in positive or negative direction. Interesting phenomena occur when the velocity of the medium, u, reaches the speed of light in the medium, c/n, which is possible in theory and perhaps also in practice. In this case, the integral in $t_\pm$ develops a logarithmic singularity. Light propagating against the current freezes with exponentially increasing oscillations at a horizon [62], see Fig. 5. This horizon is completely analogous to the event horizon of the black hole if the light is escaping from a superluminal region and to a white hole if the light is attempting to enter a counter-propagating superluminal medium [29]. Horizons cut physical space-time into distinct regions without possible communication. They correspond to disconnected branches of multi-valued electromagnetic space, covering it multiple times, in contrast to the case of perfect lensing where the folds are connected. Horizons are predicted to exhibit remarkable quantum effects [62], in particular Hawking radiation [63], that are extremely difficult to observe in astronomy, but may possibly be demonstrated in laboratory analogs [20]-[30]. Our method suggests a magneto-electric analog of the event horizon, perhaps with metamaterials, if we interpret the effective geometry (14) as generating the constitutive equations (4) expressed in terms of the D and H fields as

$$D_y = \varepsilon_0 \frac{(n^2 - u^2/c^2)E_y - (n^2 - 1)uB_z}{n(1 - u^2/c^2)}, \quad H_y = \varepsilon_0 \frac{(c^2 - n^2 u^2)B_y - (n^2 - 1)uE_z}{n(1 - u^2/c^2)}.$$
(17)

Here the functions n and u of the moving medium appear as parameters of a magneto-electric material at rest. The material establishes horizons at $u = c/n$ without singular dielectric properties, in contrast to the previous proposal [23]. general relativity can be put to practical use in electrical engineering, but electrical engineering may be also used for demonstrating some elusive effects of general relativity.

4   Conclusions

Perfect invisibility devices [1, 2], perfect lenses [8]-[14], the optical Aharonov-Bohm effect [21, 22, 46, 47] and artificial event horizons [20]-[30] are all examples of one unifying concept: electromagnetic media that facilitate coordinate transformations. Adopting ideas from general relativity [33]-[40] we developed a concise formalism for finding the properties of meta-materials that turn a desired function into fact. We extended the previously reported method [1, 2] to media that act as space-time transformations and that may exhibit negative refraction and signs of multi-valued physical space. The most interesting properties of such transformation media seem to stem from non-trivial topologies: spaces with holes in the case of invisibility devices, coordinate folds for negative refraction, multi-valued physical space for the
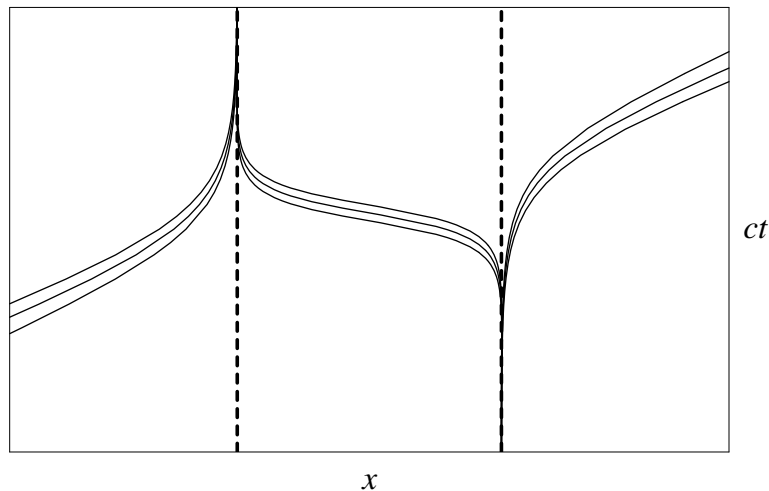
Figure 5: Event horizons. Space-time diagram showing light propagation near two artificial event horizons, a white-hole horizon at the left and a black-hole horizon at the right. The horizons could be made by a medium moving from the right side to the left. The places x where the medium velocity reaches the local speed of light in the medium are the horizons. At the black-hole horizon the medium turns from subluminal to superluminal velocity; at the white-hole horizon it returns to subluminal speed. The exponential confinement near the horizons translates into exponential red shifts of optical oscillations at the black hole and exponential blue shifts at the white hole. Magneto-electric media could mimic the moving medium generating the horizons.

Aharonov-Bohm effect and multiple sheets of electromagnetic space-time in the case of artificial event horizons. Our theory can be extended in at least two directions. So far we assumed that electromagnetic space-time is empty, but one could easily incorporate media with non-trivial ε' and μ' here. We also assumed that the mapping between electromagnetic fields in real and fictitious space is exact. In practice, the accuracy of geometrical optics [18] is often completely sufficient; one could further generalize our method to include transformations that are only exact within the validity range of geometrical optics. Optical conformal mapping [3, 4] is such an example. These more general transformation media act as local transformations of the dispersion relation of light waves. They all share the same spirit: electromagnetic media act as effective geometries.

Acknowledgments

Appendix A

In this appendix we show that the free-space Maxwell equations in generally covariant form are equivalent to Maxwell's equations in a material medium with constitutive equations (4),(5). The generally covariant Maxwell equations are [15, 39]

$$F_{[\mu\nu;\lambda]} = \partial_{[\lambda}F_{\mu\nu]} = 0\,, \quad \varepsilon_0 F^{\mu\nu}{}_{;\nu} = \frac{\varepsilon_0}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}F^{\mu\nu}\right) = j^\mu\,, \tag{A1}$$

where $F_{\mu\nu}$ denotes the electromagnetic field tensor, j^μ is the four-current, the square brackets denote antisymmetrization and the semi-colon indicates covariant differentiation. We employ Einstein's summation convention over repeated indices. The covariant tensor $F_{\mu\nu}$ contains the $\mathbf{E}$ and $\mathbf{B}$ fields in the usual manner:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -cB_z & cB_y \\ -E_y & cB_z & 0 & -cB_x \\ -E_z & -cB_y & cB_x & 0 \end{pmatrix}. \tag{A2}$$

We define a quantity $H^{\mu\nu}$ with contravariant indices by

$$H^{\mu\nu} = \varepsilon_0\sqrt{-g}F^{\mu\nu} = \varepsilon_0\sqrt{-g}g^{\mu\lambda}g^{\nu\rho}F_{\lambda\rho} \tag{A3}$$

$$\implies \quad F_{\mu\nu} = \frac{1}{\varepsilon_0\sqrt{-g}}g_{\mu\lambda}g_{\nu\rho}H^{\lambda\rho} \tag{A4}$$

and regard $H^{\mu\nu}$ as being constructed from $\mathbf{D}$ and $\mathbf{H}$ fields as follows:

$$H^{\mu\nu} = \begin{pmatrix} 0 & -D_x & -D_y & -D_z \\ D_x & 0 & -H_z/c & H_y/c \\ D_y & H_z/c & 0 & -H_x/c \\ D_z & -H_y/c & H_x/c & 0 \end{pmatrix}. \tag{A5}$$

Then, introducing a new four-current $J^\mu = \sqrt{-g}j^\mu$, Maxwell's equations (A1) can be written

$$\partial_{[\lambda}F_{\mu\nu]} = 0, \qquad \partial_\nu H^{\mu\nu} = J^\mu\,, \tag{A6}$$

which are the electromagnetic equations in a material medium, described by the constitutive equations (A2)-(A5), with free charge and current densities J^μ.

To obtain relations between the vector fields $\mathbf{D}$, $\mathbf{H}$ and $\mathbf{E}$, $\mathbf{B}$ consider first the components F_{0i}; from Eqs. (A2), (A4) and (A5) one obtains

$$E_i = \frac{1}{\varepsilon_0\sqrt{-g}}\left(g_{i0}g_{j0} - g_{ij}g_{00}\right)D_j - \frac{c}{\varepsilon_0\sqrt{-g}}[jkl]g_{0j}g_{ik}\,H_l\,, \tag{A7}$$

where $[jkl]$ denotes the completely antisymmetric permutation symbol [15] with $[xyz] = +1$. We simplify this result as follows. The identity

$$g_{\mu\lambda}g^{\lambda\nu} = \delta^\nu_\mu$$

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gives

$$g_{i\lambda}g^{\lambda 0} = 0 \quad \Longrightarrow \quad g_{i0} = -\frac{1}{g^{00}}\,g_{ij}g^{j0}\,, \tag{A8}$$

$$g_{0\lambda}g^{\lambda i} = 0 \quad \Longrightarrow \quad g^{i0} = -\frac{1}{g_{00}}\,g^{ij}g_{j0}\,, \tag{A9}$$

$$g_{j\lambda}g^{\lambda i} = g_{j0}g^{0i} + g_{jk}g^{ki} = \delta_j^i\,. \tag{A10}$$

Use of Eqs. (A8) or (A9) in Eq. (A10) produces the two relations

$$\left(g^{ij} - \frac{1}{g^{00}}g^{i0}g^{k0}\right)g_{kj} = \delta_j^i\,, \quad g^{ik}\left(g_{kj} - \frac{1}{g_{00}}g_{k0}g_{j0}\right) = \delta_j^i\,, \tag{A11}$$

which reveal inverse-related 3×3 matrices. In view of Eqs. (A10) and (A11), transvection of (A7) by g^{li} yields

$$D_i = -\frac{\varepsilon_0\sqrt{-g}}{g_{00}}g^{ij}E_j + \frac{\varepsilon_0 c}{g_{00}}[ijk]g_{j0}H_k\,, \tag{A12}$$

the first of the constitutive equations (4) with (5).

To obtain the second constitutive relation, we employ the tensors dual to $F_{\mu\nu}$ and $H^{\mu\nu}$. This requires use of the 4D Levi-Civita tensor [15], given by

$$\epsilon_{\mu\nu\lambda\rho} = \sqrt{-g}\,[\mu\nu\lambda\rho]\,, \quad \epsilon^{\mu\nu\lambda\rho} = -\frac{1}{\sqrt{-g}}\,[\mu\nu\lambda\rho]\,, \quad [0123] = +1\,. \tag{A13}$$

The dual tensors $^*F^{\mu\nu}$ and $^*H_{\mu\nu}$ are defined by [15]

$$^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho} \quad \Longrightarrow \quad F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}\,^*F^{\lambda\rho}, \tag{A14}$$

$$^*H_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}H^{\lambda\rho} \quad \Longrightarrow \quad H^{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}\,^*H^{\lambda\rho}, \tag{A15}$$

so they have components

$$^*F^{\mu\nu} = \frac{1}{\sqrt{-g}}\begin{pmatrix} 0 & -cB_x & -cB_y & -cB_z \\ cB_x & 0 & E_z & -E_y \\ cB_y & -E_z & 0 & E_x \\ cB_z & E_y & -E_x & 0 \end{pmatrix}, \tag{A16}$$

$$^*H_{\mu\nu} = \sqrt{-g}\begin{pmatrix} 0 & H_x/c & H_y/c & H_z/c \\ -H_x/c & 0 & D_z & -D_y \\ -H_y/c & -D_z & 0 & D_x \\ -H_z/c & D_y & -D_x & 0 \end{pmatrix}. \tag{A17}$$

Re-expressed in terms of the dual tensors, the constitutive equations (A3)-(A4) read

$$^*H_{\mu\nu} = \varepsilon_0\sqrt{-g}g_{\mu\lambda}g_{\nu\rho}\,^*F^{\lambda\rho}, \tag{A18}$$

$$^*F^{\mu\nu} = \frac{1}{\varepsilon_0\sqrt{-g}}g^{\mu\lambda}g^{\nu\rho}\,^*H_{\lambda\rho}. \tag{A19}$$

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Writing out $^*H_{0i}$ using Eqs. (A16)-(A18) one finds

$$H_i = -\frac{\varepsilon_0 c^2}{\sqrt{-g}} \left(g_{00}g_{ij} - g_{i0}g_{j0}\right) B_j + \frac{\varepsilon_0 c}{\sqrt{-g}}[jkl]g_{j0}g_{ik}E_l \,. \qquad (A20)$$

Comparison of this with Eqs. (A7) and (A12) shows that

$$B_i = -\frac{\sqrt{-g}}{\varepsilon_0 c^2 g_{00}}g^{ij}H_j - \frac{1}{\varepsilon_0 c g_{00}}[ijk]g_{j0}E_k, \qquad (A21)$$

which is the second of the constitutive equations (4) with (5).

Clearly, several other relations between $\mathbf{D}$, $\mathbf{H}$, $\mathbf{E}$ and $\mathbf{B}$ are contained in (A3)-(A4) and (A18)-(A19). For example, to express $\mathbf{D}$ and $\mathbf{H}$ in terms of $\mathbf{E}$ and $\mathbf{B}$, as is done in Eq. (17), we need only take the time-space components of (A3), obtaining

$$D_i = \varepsilon_0\sqrt{-g}\left(g^{i0}g^{j0} - g^{ij}g^{00}\right)E_j - \varepsilon_0 c\sqrt{-g}[jkl]g^{k0}g^{ij}B_l \,, \qquad (A22)$$

and the required formulae are Eqs. (A20) and (A22).

Appendix B

In this appendix we show that the expressions for ε^{ij} and μ^{ij} derived in Ref. [45] and utilized in Ref. [1] are a special case of the formalism used in this paper. Consider the effective ε^{ij} and μ^{ij} resulting from a transformation of the *spatial* coordinates of a Minkowski system. From Eq. (5) we obtain

$$\varepsilon^{ij} = -\sqrt{-g}g^{ij}\varepsilon' \,, \quad \mu^{ij} = -\sqrt{-g}g^{ij}\mu' \,, \qquad (B1)$$

where we have allowed for general isotropic permittivity and permeability $\varepsilon' \neq 1$, $\mu' \neq 1$ in electromagnetic space.

Rescale the spatial coordinate basis vectors ∂_i so that they form a (in general non-coordinate) basis $\mathbf{u}_i$ of vectors of unit length:

$$\mathbf{u}_i = \frac{1}{\sqrt{-g_{ii}}}\partial_i \,, \quad \mathbf{g}(\mathbf{u}_i, \mathbf{u}_i) = 1 \,. \qquad \text{(No summation.)} \qquad (B2)$$

Let $\bar{g}_{ij}$ be the components of the metric tensor in the basis $\mathbf{u}_i$:

$$\bar{g}_{ij} = \mathbf{g}(\mathbf{u}_i, \mathbf{u}_j) = \frac{1}{\sqrt{g_{ii}g_{jj}}}g_{ij} \,. \qquad \text{(No summation.)} \qquad (B3)$$

We need to compute the triple product $\mathbf{u}_1 \cdot (\mathbf{u}_2 \times \mathbf{u}_3)$ in the coordinate basis ∂_i; for this we require the 3D Levi-Civita tensor in this basis:

$$\varepsilon_{ijk} = \sqrt{-g}\,[ijk] \,, \quad \varepsilon^{ijk} = -\frac{1}{\sqrt{-g}}\,[ijk] \,, \qquad (B4)$$

where g is both the determinant of the space-time metric and the negative determinant of the spatial metric g_{ij} since $g_{00} = 1$, $g_{i0} = 0$. Using Eqs. (B2)–(B4) we find

$$
\begin{aligned}
\mathbf{u}_1 \cdot (\mathbf{u}_2 \times \mathbf{u}_3) &= -g_{ij} u_1^i \varepsilon^{jkl} u_{2k} u_{3l} = -g_{ij} u_1^i \varepsilon^{jkl} g_{km} u_2^m g_{ln} u_3^n \\[2mm]
&= -g_{1j} \frac{1}{\sqrt{-g_{11}}} \varepsilon^{jkl} g_{k2} \frac{1}{\sqrt{-g_{22}}} g_{l3} \frac{1}{\sqrt{-g_{33}}} \\[2mm]
&= \frac{1}{\sqrt{-g_{11} g_{22} g_{33}}} \frac{1}{\sqrt{-g}} [jkl] g_{1j} g_{2k} g_{3l} = \frac{1}{\sqrt{-g_{11} g_{22} g_{33}}} \frac{1}{\sqrt{-g}} g \\[2mm]
&= \frac{\sqrt{-g}}{\sqrt{-g_{11} g_{22} g_{33}}} \,.
\end{aligned}
\tag{B5}
$$

Using Eqs. (B3) and (B5) we write (B1) as

$$
\varepsilon^{ij} = -\varepsilon' \mathbf{u}_1 \cdot (\mathbf{u}_2 \times \mathbf{u}_3) \sqrt{-g_{11} g_{22} g_{33}} \frac{1}{\sqrt{g_{ii} g_{jj}}} \bar{g}^{ij}\,, \qquad \text{(No summation.)} \qquad \text{(B6)}
$$

$$
\mu^{ij} = -\mu' \mathbf{u}_1 \cdot (\mathbf{u}_2 \times \mathbf{u}_3) \sqrt{-g_{11} g_{22} g_{33}} \frac{1}{\sqrt{g_{ii} g_{jj}}} \bar{g}^{ij}\,, \qquad \text{(No summation.)} \qquad \text{(B7)}
$$

which are the expressions derived in Ref. [45]. Note that this form of the theory implicitly contains the case of negative refraction, because for a transition to a left-handed system the cross products change sign, but this fact was never mentioned nor used so far.

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