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Long Memory Conditional Volatility and Asset Allocation

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Abstract

In this paper, we evaluate the economic benefits that arise from allowing for long memory in forecasting the covariance matrix of returns over both short and long horizons, using the asset allocation framework of Engle and Colacito (2006). In particular, we compare the statistical and economic performance of four multivariate long memory volatility models (the long memory EWMA, long memory EWMA-DCC, FIGARCH-DCC and component GARCH-DCC models) with that of two short memory models (the short memory EWMA and GARCH-DCC models). We report two main findings. First, for longer horizon forecasts, long memory models produce forecasts of the covariance matrix that are statistically more accurate and informative, and economically more useful than those produced by short memory models. Second, the two parsimonious long memory EWMA models outperform the other models – both short memory and long memory – at all forecast horizons. These results apply to both low and high dimensional covariance matrices and both low and high correlation assets, and are robust to the choice of estimation window.

Keywords: Conditional variance-covariance matrix; Long memory; Asset allocation.

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1. Introduction

It is well established that the covariance matrix of short horizon financial asset returns is both time varying and highly persistent. A number of multivariate conditional volatility models, including the multivariate RiskMetrics EWMA model, multivariate GARCH models and multivariate Stochastic Volatility models, have been developed to capture these features. These models are now routinely used in many areas of applied finance, including asset allocation, risk management and option pricing. Recent evidence suggests that there are significant economic benefits to exploiting the forecasts of multivariate conditional volatility models relative to using the unconditional covariance matrix (see, for example, Engle and Colacito, 2006). In the vast majority of conditional volatility models used in practice, the elements of the conditional covariance matrix are specified as weighted averages of the squares and cross-products of past return innovations with weights that decline geometrically, so that shocks to individual variances and covariances dissipate rapidly. However, there is a mounting body of empirical evidence that suggests that although volatility is almost certainly stationary, the autocorrelation functions of the squares and cross-products of returns decline more slowly than the geometric decay rate of the EWMA, GARCH and Stochastic Volatility models, and hence volatility shocks are more persistent than these models imply (see, for example, Taylor, 1986; Ding et al., 1993; Andersen et al., 2001). This 'long memory' feature is important not only for the measurement of current volatility, but also for forecasts of future volatility, especially over longer horizons. In particular, in the GARCH and Stochastic Volatility frameworks, forecasts of future volatility converge to the unconditional volatility at an exponential rate as the forecast horizon increases. In the EWMA framework, in contrast, a volatility shock has a permanent effect on forecast volatility at all horizons, and so forecasts of future volatility do not converge at all despite the fact that it is a short memory model. If volatility is indeed a long memory process, as the empirical evidence suggests, the short memory EWMA, GARCH and Stochastic Volatility models are misspecified. Moreover, the errors in forecasting the elements of the covariance matrix that arise from this misspecification are compounded as the forecast horizon increases.

The empirical evidence on volatility dynamics has prompted the development of long memory models of conditional volatility, and in the univariate context a number of approaches have been proposed. The FIGARCH model of Baillie et al. (1996) introduces long memory through a fractional difference operator, which gives rise to a slow hyperbolic decay for the weights on lagged squared return innovations while still yielding a strictly stationary process. The Hyperbolic GARCH (HYGARCH) model of Davidson (2004) is a generalisation that nests the GARCH, FIGARCH and IGARCH (or EWMA) models, allowing for a more flexible dynamic structure than the FIGARCH model and facilitating tests of short versus long memory in volatility dynamics. The Stochastic Volatility framework has been extended to allow for long memory by Breidt et al. (1998), who incorporate an ARFIMA process in the standard discrete time Stochastic Volatility model. Long memory can also be induced using a component structure for volatility dynamics. For example, the Component GARCH (CGARCH) model of Engle and Lee (1999) assumes that volatility is the sum of a highly persistent long run component and a mean-reverting short run component, each of which follows a short memory GARCH process. Zumbach (2006) introduces a long memory model in which the dynamic process for volatility is defined as the logarithmically weighted sum of standard EWMA processes at different geometric time horizons. Like the short memory EWMA model of JP Morgan (1994) on which it is based, the long memory EWMA model has a highly parsimonious specification, which facilitates its implementation in practice.

In the multivariate context, long memory volatility modelling poses significant computational challenges, especially so for the high dimensional covariance matrices that are typically encountered in asset allocation and risk management. Indeed, so far the literature on long memory multivariate volatility modelling has generally restricted itself to the analysis of low dimensional covariance matrices, and has provided only limited evidence on the relative benefits from allowing for long memory in the multivariate setting. For example, Teyssiere (1998) estimates the covariance matrix for three foreign exchange return series using both an unrestricted multivariate FIGARCH model and a FIGARCH model implemented with the Constant Conditional Correlation (CCC) structure of Bollerslev (1990). Similarly, Niguez and Rubin (2006) model the covariance matrix of five foreign exchange series using an Orthogonal HYGARCH model, which combines the univariate HYGARCH long memory volatility model of Davidson (2004) with the multivariate Orthogonal GARCH framework of Alexander (2001). They show that the Orthogonal HYGARCH model outperforms the standard Orthogonal GARCH model in terms of 1-day forecasts of the covariance matrix. Zumbach (2009b) develops a multivariate version of the univariate long memory EWMA model, in which elements of the covariance matrix are estimated as the averages of the squares and cross products of past returns with predetermined logarithmically decaying weights.

In this paper, we evaluate the economic benefits that arise from allowing for long memory in forecasting the covariance matrix of returns over both short and long horizons, using the asset allocation framework of Engle and Colacito (2006). In so doing, we compare the performance of a number of long memory and short memory multivariate volatility models. While many alternative volatility models have been developed in the literature, our choice reflects the need for parsimonious models that can be used to forecast high dimensional covariance matrices. We employ four long memory volatility models: the multivariate long memory EWMA model of Zumbach (2009b), and three multivariate long memory implemented using the Dynamic Conditional Correlation (DCC) framework of Engle (2002). These are the univariate long memory univariate EWMA model of Zumbach (2006), the component GARCH model of Engle and Lee (1999) and the FIGARCH model of Baillie et al. (1996). We compare the four multivariate long memory models with two multivariate short memory models. These are the very widely used RiskMetrics EWMA model of JP Morgan (1994), and the DCC model implemented with the univariate GARCH model.

We use the six multivariate conditional volatility models to forecast the covariance matrices for the same three sets of assets employed by Engle and Colacito (2006). These comprise a high correlation bivariate system (the S&P500 and DJIA indices), a low correlation bivariate system (the S&P500 and 10-year Treasury bond futures), and a moderate correlation high dimensional system (21 international stock indices and 13 international bond indices). We additionally consider another high dimensional system, namely the components of the DJIA index. The analysis is conducted using data over the period 1 January 1988 to 31 December 2009, and considers forecast horizons up to three months. For the two bivariate systems, we first evaluate the forecasts of the models using a range of statistical criteria that measure the accuracy, bias and informational content of the models' forecasts over varying time horizons. For all four systems, we then employ Engle and Colacito's (2006) approach to assess the economic value of the forecast covariance matrices in an asset allocation setting. We report two main findings. The first is that for longer horizon forecasts, multivariate long memory models generally produce forecasts of the covariance matrix that are both statistically more accurate and informative, and economically more useful than those produced by short memory volatility models. The second is that the two long memory models that are based on the Zumbach (2006) univariate model outperform the other models – both short memory and long memory – at all forecast horizons. These results apply to all four datasets and are robust to the choice of estimation window.

The remainder of this paper is organised as follows. Section 2 provides details of the six multivariate conditional volatility models used in the empirical analysis. Section 3 describes the methods applied to evaluate forecast performance for the six models. The data are summarised in Section 4. In Section 5, we report the empirical results of our analysis, while Section 6 offers some concluding comments and some suggestions for future research.

2. Multivariate Long Memory Conditional Volatility Models

Motivated by the need for parsimonious models that can be used to forecast high dimensional covariance matrices, we first consider two simple multivariate long memory conditional volatility models based on the univariate long memory volatility model of Zumbach (2006). The first is the multivariate long memory EWMA (LM-EWMA) model of Zumbach (2009b), which is a simple multivariate extension of the univariate long memory EWMA model in which both the variances and covariances are governed by the same long memory process, and is thus the long memory analogue of the short memory multivariate RiskMetrics EWMA model of JP Morgan (1994). In the second, we employ the Dynamic Conditional Correlation framework of Engle (2002) to model the dynamic processes of the correlations directly, using the univariate long memory EWMA model for the individual variances. This is the long memory EWMA-DCC (LM-EWMA-DCC) model. We compare the two long memory EWMA models with the multivariate FIGARCH(1,*d*,1) and component GARCH(1,1) (CGARCH) long memory models, both implemented using the DCC framework. To evaluate the relative benefits of allowing for long memory in forecasting the covariance matrix, we compare the four long memory multivariate models with two short memory multivariate volatility models. These are the multivariate RiskMetrics EWMA model of JP Morgan (1994) and the GARCH(1,1) model implemented using the DCC framework. In this section, we give details of each of these six models.

2.1 The Multivariate LM-EWMA Model

Consider an *n*-dimensional vector of returns $\boldsymbol{r}_t = (r_{1t}, r_{2t}, \ldots, r_{nt})'$ with conditional mean zero and conditional covariance matrix $\mathbf{H}_t$:

$$\boldsymbol{r}_t = \mathbf{H}_t^{\frac{1}{2}} \boldsymbol{\varepsilon}_t \,, \tag{1}$$

where $\boldsymbol{\varepsilon}_t$ is i.i.d with $E\left(\boldsymbol{\varepsilon}_t\right) = 0$ and $\mathrm{var}\left(\boldsymbol{\varepsilon}_t\right) = \mathbf{I}_n$. Zumbach (2009b) considers the class of conditional covariance matrices that are the weighted sum of the cross products of past returns:

$$\mathbf{H}_{t+1} = \sum_{i=0}^{\infty} \lambda(i) \boldsymbol{r}_{t-i} \boldsymbol{r}_{t-i}^{'} \,, \tag{2}$$

with $\sum \lambda(i) = 1$. In the RiskMetrics EWMA model of JP Morgan (1994), the weights $\lambda(i)$ decay geometrically, yielding a short memory process for the elements of the variance-covariance matrix. The long memory conditional covariance matrix is defined as the weighted average of *K* standard (short memory) multivariate EWMA processes:

$$\mathbf{H}_{t+1} = \sum_{k=1}^{K} w_k \mathbf{H}_{k,t} \tag{3}$$

where

$$\mathbf{H}_{k,t} = \mu_k \mathbf{H}_{k,t-1} + \left(1 - \mu_k\right) \boldsymbol{r}_t \boldsymbol{r}_t' \,. \tag{4}$$

The decay factor μ_k of the k^{th} EWMA process is defined by a characteristic time τ_k such that $\mu_k = \exp\left(-1/\tau_k\right)$, with geometric time structure $\tau_k = \tau_1 \rho^{k-1}$ for $k = (1,...,K)$. Zumbach (2006) sets ρ to the value of $\sqrt{2}$. The memory of the volatility process is determined by the weights w_k, which are assumed to decay logarithmically:

$$w_k = \frac{1}{C}\left(1 - \frac{ln\left(\tau_k\right)}{ln\left(\tau_0\right)}\right) \tag{5}$$

with the normalization constant $C = K - \sum_k \frac{ln\left(\tau_k\right)}{ln\left(\tau_0\right)}$ such that $\sum_k w_k = 1$. The conditional covariance matrix is therefore parsimoniously defined as a process with just three parameters: τ_1 (the shortest time scale at which volatility is measured, i.e. the lower cut-off), τ_K (the upper cut-off, which increases exponentially with the number of components *K*), and τ_0 (the logarithmic decay factor). For the univariate case, Zumbach (2006) sets the optimal parameter values at $\tau_0 =$ 1560 days = 6 years, $\tau_1 =$ 4 days and $\tau_K =$ 512 days, which is equivalent to $K = 15$.

The EWMA process in (4) can also be expressed as

$$\mathbf{H}_{k,t} = (1-\mu_k)\sum_{i=0}^{\infty}\mu_k^i \boldsymbol{r}_{t-i}\boldsymbol{r}_{t-i}^{'} . \tag{6}$$

Hence the LM-EWMA model can be written in the form of (2):

$$\mathbf{H}_{t+1} = \sum_{i=0}^{\infty}\sum_{k=1}^{K} w_k (1-\mu_k)\mu_k^i \boldsymbol{r}_{t-i}\boldsymbol{r}_{t-i}^{'} = \sum_{i=0}^{\infty}\lambda(i)\boldsymbol{r}_{t-i}\boldsymbol{r}_{t-i}^{'} \tag{7}$$

with $\lambda(i) = \sum_k w_k (1-\mu_k)\mu_k^i$ and $\sum_i \lambda(i) = 1$. When $K = 1$, the LM-EWMA process reduces to the short memory RiskMetrics EWMA process. Note that since $\mathbf{H}_{k,t}$ is a positive definite matrix (see Riskmetrics, 1994), $\mathbf{H}_{t+1}$, which is a linear combination of $\mathbf{H}_{t,k}$ with positive weights, will also be positive definite. Since the LM-EWMA covariance matrix is the sum of EWMA processes over increasing time horizons, forecasts of the covariance matrix are straightforward to obtain using a recursive procedure (see Zumbach (2006) for details of the univariate case). The 1-step-ahead forecast of the covariance matrix is already given by (7). Under the assumption of serially uncorrelated returns, the *h*-step cumulative forecast of the covariance matrix given the information set $\mathcal{F}_t$ at time *t* is equal to:

$$\mathbf{H}_{t+1:t+h} = h\sum_{i=0}^{T}\lambda(h,i)\boldsymbol{r}_{t-i}\boldsymbol{r}_{t-i}^{'} \tag{8}$$

with the weights $\lambda(h,i)$ given by

$$\lambda(h,i) = \sum_{k=1}^{K}\frac{1}{h}\sum_{j=1}^{h-1} w_{j,k}\frac{1-\mu_k}{1-\mu_k^T}\mu_k^i \tag{9}$$

where *T* is the cut-off time, $w_{j,k}$ is the k^{th} element of vector $\boldsymbol{w}_j = \boldsymbol{w}'\left[\boldsymbol{M} + (\boldsymbol{\iota} - \boldsymbol{\mu})\boldsymbol{w}'\right]^j$, $\boldsymbol{\mu}$ is the vector of μ_k, $\boldsymbol{M}$ is the diagonal matrix consisting of μ_k, and $\boldsymbol{\iota}$ is the unit vector. Since $\sum_k w_k = 1$, we obtain $\sum \lambda(h,i) = 1$. Also note that when $K = 1$, then $w = 1$, and so the LM-EWMA forecast function reduces to a standard short memory EWMA forecast function with forecast weights $\lambda(h,i) = (1-\mu_k)\mu_k^i \big/ (1-\mu_k^T)$, independent of the forecast horizon. Since the weights $\lambda(h,i)$ are estimated a priori, without reference to the data, the forecast in (8) is straightforward to compute. As with the standard EWMA model, the LM-EWMA model circumvents the computational burden of other multivariate long memory models, and indeed can easily be implemented in a spreadsheet.

2.2 The Multivariate LM-EWMA-DCC Model

In the Dynamic Conditional Correlation (DCC) model of Engle (2002), the conditional covariance matrix is decomposed as follows:

$$\mathbf{H}_t = \mathbf{D}_t \mathbf{R}_t \mathbf{D}_t \tag{10}$$

$$\mathbf{R}_t = diag(\mathbf{Q}_t)^{-\frac{1}{2}} \mathbf{Q}_t diag(\mathbf{Q}_t)^{-\frac{1}{2}} \tag{11}$$

$$\mathbf{Q}_t = \Omega + \alpha \boldsymbol{\varepsilon}_{t-1} \boldsymbol{\varepsilon}_{t-1}^{'} + \beta \mathbf{Q}_{t-1} \tag{12}$$

where $\mathbf{R}_t$ is the conditional correlation matrix, $\mathbf{D}_t$ is a diagonal matrix with the time varying standard deviations $\sqrt{h_{i,t}}$ on the i^{th} diagonal, i.e., $\mathbf{D}_t = diag\left(\sqrt{h_{i,t}}\right)$, and $\mathbf{Q}_t$ is the approximation of the conditional correlation matrix $\mathbf{R}_t$. In the DCC model, $\mathbf{Q}_t$ converges to the unconditional average correlation $\bar{\mathbf{R}} = \frac{1}{T}\sum \boldsymbol{\varepsilon}_{t-1}\boldsymbol{\varepsilon}_{t-1}^{'}$ and $\Omega = (1-\alpha-\beta)\bar{\mathbf{R}}$. The positive semi-definiteness of $\mathbf{Q}_t$ is guaranteed if α and β are positive with $\alpha + \beta < 1$ and the initial matrix $\mathbf{Q}_1$ is positive definite.

Here, we estimate the conditional volatility $\mathbf{D}_t$ employing the univariate long memory volatility model of Zumbach (2006). We divide returns by their conditional volatility and use the standardized, zero-mean residuals $\boldsymbol{\varepsilon}_t = \mathbf{D}_t^{-1}\mathbf{r}_t$ to compute the quasi-conditional correlation matrix $\mathbf{Q}_t$. As the diagonal elements of $\mathbf{Q}_t$ are equal to unity only on average, $\mathbf{Q}_t$ is rescaled to obtain the conditional correlation matrix $\mathbf{R}_t = diag(\mathbf{Q}_t)^{-1/2}\mathbf{Q}_t diag(\mathbf{Q}_t)^{-1/2}$. The conditional volatility $\mathbf{D}_t$ and conditional correlations $\mathbf{R}_t$ are then combined to estimate the conditional covariance matrix $\mathbf{H}_t$.

The h-step-ahead conditional covariance matrix is given by

$$\mathbf{H}_{t+h} = \mathbf{D}_{t+h}\mathbf{R}_{t+h}\mathbf{D}_{t+h}\,. \tag{13}$$

The forecast of each volatility in $\mathbf{D}_{t+h}$ is estimated using the recursive procedure as in (8) for the univariate case. Since $\mathbf{R}_t$ is a non-linear process, the h-step forecast of $\mathbf{R}_t$ cannot be computed using a recursive procedure. However, assuming for simplicity that $E_t\left(\boldsymbol{\varepsilon}_{t+1}\boldsymbol{\varepsilon}_{t+1}^{'}\right) \approx \mathbf{Q}_{t+1}$, Engle and Shephard (2001) show that the forecasts of $\mathbf{Q}_{t+h}$ and $\mathbf{R}_{t+h}$ are given by

$$\mathbf{Q}_{t+h} = \sum_{j=0}^{h-2}\left(1-\alpha-\beta\right)\overline{\mathbf{Q}}\left(\alpha+\beta\right)^{j} + \left(\alpha+\beta\right)^{h-1}\mathbf{Q}_{t+1}\,, \tag{14}$$

and

$$\mathbf{R}_{t+h} = diag\left(\mathbf{Q}_{t+h}\right)^{-\frac{1}{2}}\mathbf{Q}_{t+h}\,diag\left(\mathbf{Q}_{t+h}\right)^{-\frac{1}{2}}\,. \tag{15}$$

2.3 The FIGARCH(1,d,1)-DCC Model

Baillie et al. (1996) propose the Fractionally Integrated GARCH (FIGARCH) model, in which long memory is introduced through a fractional difference operator, d. This model incorporates a slow hyperbolic decay for lagged squared innovations in the conditional variance while still letting the cumulative impulse response weights tend to zero, thus yielding a strictly stationary process. In the FIGARCH(1,d,1) model, the conditional volatility is modelled as:

$$h_t = \omega + [1-\beta L - \left(1-\phi L\right)\left(1-L\right)^{d}]\varepsilon_t^2 + \beta h_{t-1}\,. \tag{16}$$

Baillie et al. (1996) show that for $0 < d \leq 1$, the FIGARCH process does not have finite unconditional variance, and is not weakly stationary, a feature shared with the IGARCH model. However, they show that the FIGARCH model is strictly stationary and ergodic by a direct extension of the corresponding proof for the IGARCH model.

The 1-step ahead forecast of the FIGARCH(1,d,1) model is given by

$$h_{t+1} = \omega\left(1-\beta\right)^{-1} + [1-\left(1-\beta L\right)^{-1}\left(1-\phi L\right)\left(1-L\right)^{d}]\varepsilon_t^2\,, \tag{17}$$

and the h-step ahead forecast by

$$h_{t+h} = \omega\left(1-\beta\right)^{-1} + [1-\left(1-\beta L\right)^{-1}\left(1-\phi L\right)\left(1-L\right)^{d}]\varepsilon_{t+h-1}^{2}. \tag{18}$$

To implement the FIGARCH(1,d,1) model in the multivariate context, we use the DCC approach described above, with the same forecast functions for $\mathbf{Q}_{t+h}$ and $\mathbf{R}_{t+h}$.

2.4 The CGARCH(1,1)-DCC Model

An alternative way to capture the long memory feature is through a component structure for volatility. Engle and Lee (1999) propose the component GARCH (CGARCH) model, in which the long memory volatility process h_t is modelled as the sum of a long term trend component, q_t, and a short term transitory component, s_t. The CGARCH(1,1) model has the following specification:

$$h_t - q_t = \alpha\left(\varepsilon_{t-1}^{2} - q_{t-1}\right) + \beta(h_{t-1} - q_{t-1}) \tag{19}$$

$$q_t = \omega + \rho q_{t-1} + \phi(\varepsilon_{t-1}^{2} - h_{t-1}) \tag{20}$$

where $s_t = h_t - q_t$ is the transitory volatility component. The volatility innovation $\varepsilon_{t-1}^{2} - h_{t-1}$ drives both the trend and the transitory components. The long run component evolves over time following an AR process with ρ close to 1, while the short run component mean reverts to zero at a geometric rate $\alpha + \beta$. It is assumed that $0 < \alpha + \beta < \rho < 1$ so that the long run component is more persistent than the short run component.

The 1-step ahead forecast of the CGARCH(1,1) model is given by

$$h_{t+1} = q_{t+1} + \alpha\left(\varepsilon_{t}^{2} - q_t\right) + \beta(h_t - q_t) \tag{21}$$

$$q_{t+1} = \omega + \rho q_t + \phi(\varepsilon_{t}^{2} - h_t), \tag{22}$$

and the h-step ahead forecast by

$$h_{t+h} = q_{t+h} + \left(\alpha + \beta\right)^{h-1}(h_t - q_t) \tag{23}$$

$$q_{t+h} = \frac{\omega}{1-\rho} + \rho^{h-1}\left(q_t - \frac{\omega}{1-\rho} \right). \tag{24}$$

As with the FIGARCH(1,*d*,1) model, in order to implement the CGARCH(1,1) model in the multivariate context, we use the DCC approach described above, with the same forecast functions for $\mathbf{Q}_{t+h}$ and $\mathbf{R}_{t+h}$.

2.5 The RiskMetrics EWMA Model

The short memory RiskMetrics EWMA covariance matrix is defined by

$$\mathbf{H}_t = \lambda \mathbf{H}_{t-1} + (1-\lambda)\, \boldsymbol{r}_{t-1}\boldsymbol{r}_{t-1}' \tag{25}$$

where λ is the decay factor $0 < \lambda < 1$. The larger the value of λ, the higher the persistence of the covariance matrix process and the lower the response of volatility to return shocks. It is straightforward to show that the *h*-step cumulative forecast of the EWMA model is given by

$$\mathbf{H}_{t+1:t+h} = h \times \mathbf{H}_{t+1} \tag{26}$$

(See, for example, JP Morgan, 1994). In the empirical analysis, we set λ to the values suggested by JP Morgan (1994) of 0.94 and 0.97 for daily and weekly forecasts, respectively.

2.6 The GARCH(1,1)-DCC Model

The short memory GARCH(1,1) model of Bollerslev (1990) is given by

$$h_t = \omega + \alpha \varepsilon_{t-1}^2 + \beta h_{t-1}. \tag{27}$$

The parameter α determines the speed at which the conditional variance responds to new information, while the parameter $\alpha + \beta$ determines how fast the conditional variance reverts to its long run average. In the GARCH(1,1) model, the weights on past squared errors decline at an exponential rate. The 1-step ahead forecast of the GARCH(1,1) model is given by

$$h_{t+1} = \omega + \alpha \varepsilon_t^2 + \beta h_t, \tag{28}$$

and the *h*-step ahead forecast by

$$h_{t+h} = \sigma^2 + (\alpha + \beta)^{h-1} (h_{t+1} - \sigma^2) \tag{29}$$

where σ^2 is the unconditional variance. In order to implement the GARCH(1,1) model in the multivariate context, we again use the DCC approach described above, with the same forecast functions for $\mathbf{Q}_{t+h}$ and $\mathbf{R}_{t+h}$.

3. Forecast Performance Measurement

We evaluate the forecast performance of the six conditional volatility models using a range of statistical and economic measures. We first measure the accuracy, bias and information content of the models' forecasts for each element of the covariance matrix using the squares and cross-products of daily returns as proxies for the actual variances and covariances. Forecast accuracy is evaluated using the Root Mean Squared Error (RMSE), the Mean Absolute Error (MAE) and the Heteroscedasticity-adjusted MSE (HMSE) of Bollerslev and Ghysels (1996). These are given by

$$RMSE = \sqrt{\frac{1}{T}\sum_{t=1}^{T} r_{i,t}r_{j,t} - \hat{\sigma}_{ij,t}{}^{2}} \tag{30}$$

$$MAE = \frac{1}{T}\sum_{t=1}^{T}\left|r_{i,t}r_{j,t} - \hat{\sigma}_{ij,t}\right| \tag{31}$$

$$HMSE = \frac{1}{T}\sum_{t=1}^{T}\left(\frac{r_{i,t}r_{j,t}}{\hat{\sigma}_{ij,t}} - 1\right)^2 . \tag{32}$$

The Heteroscedasticity-adjusted MSE (HMSE) of Bollerslev and Ghysels (1996) penalises underpredictions more heavily than overpredictions, and hence may better match the user's actual loss function. Forecast bias and information content are measured using the Mincer-Zarnowitz regression, given by

$$r_{i,t}r_{j,t} = \alpha_{ij} + \beta_{ij}\hat{\sigma}_{ij,t} + \varepsilon_{ij} . \tag{33}$$

A forecast is conditionally unbiased (i.e. weak-form efficient) if and only if $\alpha_{ij} = 0$ and $\beta_{ij} = 1$.

As noted by Engle and Colacito (2006), the statistical evaluation of covariance matrix forecasts on an element-by-element basis has a number of drawbacks, particularly for high dimensional systems. In particular, direct comparisons between two covariance matrices are difficult because the distance between them is not well specified. Indeed, the statistical approaches described above implicitly assume that all elements of the covariance matrix are equally important (in the sense that the same error in each element is equally costly in economic terms), but there is no a priori reason why this should necessarily be the case. Moreover, the use of low frequency realized volatility as a proxy for true volatility introduces considerable noise that inflates the forecast errors of the conditional volatility forecasts, substantially reducing their explanatory power. This has prompted tests of covariance matrix forecast performance based instead on economic loss criteria. Such tests have shown that conditional volatility models perform better when performance is measured using an economic loss function than when based on traditional statistical measures (see, for example, West et al., 1993, Engle et al., 1996).

In this paper, we employ the economic loss function developed by Engle and Colacito (2006), who study the usefulness of forecasts of the conditional covariance matrix in an asset allocation framework. Assume that an investor allocates a fraction $\mathbf{w}_t$ of his wealth to *n* risky assets and the remainder $1-\mathbf{w}_t^{'}\boldsymbol{\iota}$ to the risk-free asset, where $\boldsymbol{\iota}$ is the $n\times 1$ unit vector. In the mean-variance optimization framework, the investor solves the following optimization problem at time *t*:

$$\min_{\mathbf{w}_t} \mathbf{w}_t^{'}\mathbf{H}_{t+1}\mathbf{w}_t \tag{34}$$

$$\text{subject to } \mathbf{w}_t^{'}\boldsymbol{\mu} + \left(1-\mathbf{w}_t^{'}\boldsymbol{\iota}\right) r_t^f = \mu_p^* \tag{35}$$

where $\mathbf{H}_{t+1}$ is the covariance matrix at time t+1, $\boldsymbol{\mu}$ is the vector of expected returns, r_t^f is the risk-free rate and μ_p^* is the target return. As $\boldsymbol{\mu}$ is assumed to be constant, the optimal weight of each asset changes over time as a result of changes in the covariance matrix. Since the true covariance matrix $\mathbf{H}_{t+1}$ is unobserved, the optimisation problem is solved using a forecast of $\mathbf{H}_{t+1}$ obtained from a multivariate conditional volatility model, to yield an approximation to the true optimal portfolio. The investor chooses among competing forecasts of the conditional covariance matrix on the basis of the volatility of the resulting portfolio. Engle and Colacito (2006) show that the lowest volatility of the investor's portfolio is obtained when the forecast covariance matrix is equal to the true covariance matrix, irrespective of both the expected excess return vector $\boldsymbol{\mu}$ and the target return μ_p^*. This then yields a straightforward economic test of the relative performance of competing covariance matrix forecasts based on the volatility of the optimal portfolio.

Engle and Colacito (2006) also note that in the bivariate context, the relative volatilities of portfolios depend on the relative returns of the n risky assets, and not on their absolute returns. Using polar coordinates, all possible pairs of relative expected returns can be expressed in the form $\mu = \left[\sin\frac{\pi j}{20}, \cos\frac{\pi j}{20}\right]$, for $j \in \{0,\ldots,10\}$. When $j = 5$, for example, the expected returns are identical, which yields the global minimum variance portfolio. To obtain a single summary vector of expected returns, we construct prior probabilities for different vectors of expected returns using the sample data and the quasi-Bayesian approach introduced by Engle and Colacito (2006). We use these probabilities as weights to estimate a single weighted average vector of expected returns. In the empirical study, we assume a target excess return equal to 1.[1]

For each vector of expected returns, and for each pair of covariance matrix forecasts, we test whether the portfolio variances are equal using the Diebold and Mariano (1995) test. In particular, we consider the loss differential $u_t^k = (\sigma_t^{1,k})^2 - (\sigma_t^{2,k})^2$, where $(\sigma_t^{1,k})^2$ and $(\sigma_t^{2,k})^2$ are the variances of portfolios 1 and 2, respectively, for the expected return vector μ^k. By regressing u_t^k on a constant, and using the Newey and West (1987) adjusted covariance matrix, the null hypothesis of equal variances is simply a test that the mean of u is equal to zero. Engle and Colacito note that because u_t^k is itself heteroscedastic, a more efficient estimator can be obtained by dividing u by the true variance. Since the true covariance matrix is unknown and there are two estimators being compared, they suggest using the geometric mean of the two variance estimators as the denominator. The improved loss differential is given by

$$v_t^k = u_t^k \left[2 \left(\mu^{k'} \left(\mathbf{H}_t^1\right)^{-1} \mu^k\right) \left(\mu^{k'} \left(\mathbf{H}_t^2\right)^{-1} \mu^k\right)\right]^{-1}. \quad (36)$$

We apply the Diebold and Mariano tests to both the u and v series. We also conduct joint tests for all vectors of expected returns.

[1] The choice of the target return is immaterial in the sense that it does not affect the relative volatilities of portfolios.

4. Data Description

The empirical analysis employs the same datasets as those in Engle and Colacito (2006). We first study the forecast performance of the six conditional volatility models in two bivariate systems. The low correlation system uses daily data for the S&P500 and 10-year Treasury bond futures, while the high correlation system uses daily data for the S&P500 and Dow Jones Industrial Average (DJIA) indices. All data are from Datastream and cover the period 01 January 1988 to 31 December 2009. Returns are calculated as the log price difference over consecutive days. We exclude from the sample all days on which any of the markets was closed, yielding 5548 observations for each dataset. As the futures contracts require no initial investment, the futures returns can be interpreted as excess spot returns. The returns of the S&P500 and DJIA indices are converted to excess returns by subtracting the daily 1-month T-Bill rate.[2] Table I reports descriptive statistics of the four return series. The sample correlation of the stock and bond futures is very close to zero, while for the S&P500 and DJIA indices, it is close to one. For all four series, returns are negatively skewed and leptokurtic.

[Insert Table I here]

Figure 1 plots the sample autocorrelations for returns, absolute returns and squared returns for the four series. While the autocorrelations of returns are not significantly different from zero at any lag, the autocorrelations of absolute returns and squared returns are highly persistent and still significant at up to 100 lags. The autocorrelations of absolute returns are also consistently higher than those of squared returns, a feature first identified by Taylor (1986). The slowly decaying autocorrelation functions of absolute returns and squared returns suggest the presence of long memory in volatility.

[Insert Figure 1 here]

Formal tests are conducted to confirm the visual evidence on long memory, the results of which are also reported in Table I. The parametric FIGARCH model is estimated for the whole sample, and the estimated fractional difference operators range from 0.35 to 0.49. We also apply two semi-parametric tests of long memory. These are the narrow band log periodogram (GPH) estimator of Geweke and Porter-Hudak (1983) and the broad band log

[2] This is the simple daily rate that, over the number of trading days in the month, compounds to the 1-month T-Bill rate from Ibbotson and Associates, Inc.

periodogram (MS) estimator of Moulines and Soulier (1999). To estimate the GPH and MS operators, we use the recommended bandwidth m equal to the square root of the sample size $(m = 77)$ and the Fourier term p equal to the log of the sample size $(p = 4)$, respectively. Following Hurvich and Soulier (2002), we report results for both squared returns and log squared returns. The tests suggest long memory in volatility for all four series, and that stock return volatility has longer memory than bond return volatility. We conduct a one-sided test for the hypothesis $d = 0.5$, against the alternative $d < 0.5$. Rejecting this hypothesis, we confirm that the volatility processes of all four series are characterised by long memory, but are nevertheless stationary.

Following Engle and Colacito (2006), we also consider a moderate correlation high dimensional system. An international stock and bond portfolio is constructed from 34 assets, comprising 21 stock indices from the FTSE All-World indices and 13 5-year average maturity bond indices. The 21 stock indices and 13 bond indices include all of the major world stock and government bond markets. All data are taken from Datastream and converted to US dollar denominated prices. Following Engle and Colacito (2006), we use weekly returns to avoid the problem of non-synchronous trading. Weekly returns are calculated as the log price difference using Friday to Friday closing prices. The dataset comprises 22 years of weekly returns, yielding a total of 1147 observations from 01 January 1988 to 31 December 2009. Descriptive statistics for the international dataset are given in Table II. Returns are, again, leptokurtic and, in most cases, negatively skewed. The international stock markets are relatively highly correlated, as are the international bond markets. The average correlation coefficient among the 21 stock market return series is 0.54, while among the bond market return series it is 0.61. However, the stock and bond markets as a whole have an average correlation coefficient of only 0.20. All 34 return series show evidence of long memory in volatility. For all countries in which both stock and bond indices are present, stock index volatility is, again, more persistent than bond index volatility. The average fractional difference operator for the stock indices is 0.44 with the parametric FIGARCH test, and 0.32 with the semiparametric GPH tests. The corresponding values for the international bond indices are 0.30 and 0.25.

[Insert Table II here]

We additionally consider a higher frequency high dimensional system, comprising the components of the Dow Jones Industrial Average (DJIA) index as of 31 December 2009.

Daily data are collected from the Center for Research in Security Prices from 01 March 1990 to 31 December 2009. We exclude Kraft, which was listed only in June 2001. Returns are calculated as the log price difference over consecutive days. All days on which the market was closed are excluded from the sample, yielding 5001 observations. Table III provides summary statistics for the 29 DJIA stocks. The return series are again highly non-normal, with very high leptokurtosis. The average correlation coefficient of the DJIA components is 0.34. The system also exhibits long memory in volatility. The average estimated fractional difference orders are 0.37 with the FIGARCH test and 0.42 with the GPH test. Some return volatilities are even non-stationary with $d \geq 0.5$.

[Insert Table III here]

For each series, the whole sample is divided into an initial estimation period of 252 observations (one year for the daily return series and five years for the weekly return series), and a forecast period of 5296, 895 and 4749 observations for the two bivariate portfolios, the international stock and bond portfolio and the DJIA component portfolio, respectively. The initial estimation period is used to estimate each model to generate out-of-sample forecasts of the covariance matrix for observation 253. The estimation window is then rolled forward one observation, the models re-estimated, and forecasts made for observation 254, and so on until the end of the sample is reached. We initially estimate the conditional covariance matrix using all of the multivariate models described in Section 2, except the FIGARCH(1,d,1)-DCC model. This model is excluded owing to the prohibitively short estimation period. In Section 5.4, we employ longer estimation periods and consider all six models.

5. Empirical Results

5.1 Low Dimensional Systems: The Stock-Bond and S&P500-DJIA Portfolios

Statistical Evaluation

Table IV reports the statistical evaluation of the accuracy of the five conditional volatility models using the RMSE, MAE, and HMSE measures for the two bivariate systems, namely the Stock-Bond and S&P500-DJIA portfolios. The LM-EWMA and LM-EWMA-DCC models yield identical RMSE, MAE and HMSE measures for the variances since in both models, the variance forecasts are based on the univariate long memory EWMA model. However, the LM-EWMA model performs better with respect to the covariance forecasts.

The LM-EWMA model also yields the lowest RMSE and MAE for all elements in the Stock-Bond covariance matrix, while the short memory EWMA model performs best in the S&P500-DJIA case, although the difference between the EWMA and LM-EWMA models is small. Among the DCC models, the LM-EWMA-DCC model dominates, suggesting that there are potential benefits from allowing for long memory in volatility. The short memory GARCH-DCC model is the worst model in terms of forecast accuracy under the symmetric RMSE and MAE measures. The HMSE measure, which accounts for asymmetry in the treatment of under- and over-predictions, however, chooses the models least favoured by the RMSE and MAE measures, with the GARCH-DCC and CGARCH-DCC models producing the lowest forecast errors. We do not report the results of HMSE for the low correlation Stock-Bond covariance because the conditional correlation for some individual observations is very close to zero, leading to very high values of $r_{i,t}r_{j,t}/\hat{\sigma}_{ij,t}$, which severely distorts the reported statistics.

[Insert Table IV here]

The results of the Mincer-Zarnowitz regressions for the two bivariate systems are summarised in Table V. The table reports the estimated coefficients of the regression, the R-squared statistic and the *p*-value for each element of the covariance matrix for the null hypothesis of conditional unbiasedness. The unbiasedness hypothesis cannot be rejected at conventional significance levels for any of the stock variance forecasts, nor for the covariance forecasts in the S&P500-DJIA system for the LM-EWMA and LM-EWMA-DCC models, but it is rejected in all other cases. In the cases that the unbiasedness hypothesis cannot be rejected, the LM-EWMA and LM-EWMA-DCC models have slope coefficients that are very close to unity. The EWMA model, though evidently not as efficient, performs slightly better in terms of explanatory power, as measured by the R-squared statistic. The CGARCH-DCC model performs rather badly, indeed only marginally better than the GARCH-DCC model.

[Insert Table V here]

Economic Evaluation

We use the forecasts of the covariance matrix to construct the minimum variance portfolios subject to a target excess return of 1. The relative conditional volatilities of portfolios constructed using the different conditional covariance matrix estimators and all possible vectors of expected returns are compared in Table VI. The pairs of Bayesian prior weighted returns are obtained from non-overlapping consecutive subsamples of 63 days (3 months) from the full datasets. Engle and Colacito (2006) show that by considering unconditional mean-adjusted returns, one can obtain a consistent estimator of the true conditional portfolio variance. The lowest conditional volatility, corresponding to the best covariance matrix estimate, is normalised to 100. The 'Const' portfolio is the fixed weight portfolio constructed with the ex-post constant unconditional covariance matrix. It is clear that the conditional covariance matrices generally outperform the unconditional covariance matrix, highlighting the economic value of volatility timing strategies. The results are favourable for the two LM-EWMA models. For both the low correlation Stock-Bond portfolio and the high correlation S&P500-DJIA portfolio, the LM-EWMA model consistently yields the lowest portfolio volatility. Incorporating long memory into the EWMA structure therefore appears to improve the forecasts of the conditional covariance matrix in a way that is economically valuable. Among the DCC models, the LM-EWMA-DCC model again dominates. Although the CGARCH model is designed to capture long memory volatility, its high degree of parameterization evidently hinders its performance. It is also interesting to note that the simple EWMA model outperforms more sophisticated models such as the GARCH-DCC and CGARCH-DCC models, and is even superior to the LM-EWMA-DCC models in most cases.

[Insert Table VI here]

In practice, investors may be more concerned with out-of-sample realized volatility than conditional volatility. This is reported in Table VII for each model for the two bivariate portfolios. Here, the results are similar, with the LM-EWMA model consistently yielding the lowest out-of-sample portfolio volatility.

[Insert Table VII here]

Next, Diebold-Mariano tests are applied to test for the equality of different models with each vector of expected returns. Joint tests are also carried out for all vectors of expected returns applying the GMM method with a robust HAC covariance matrix. Instead of reporting all of the results, we focus on those with expected returns close to the sample mean. Table VIII shows the results of both the standard and the improved tests for the Stock-Bond portfolio with $\mu_{Stock}, \mu_{Bond} = [0.95, 0.31]$ and $[0.99, 0.16]$, and for the joint tests. Each cell in the table corresponds to the test of the hypothesis that the two models in the row and column are equal in terms of volatility forecasting against the alternative that the model in the row is better or worse than the model in the column. A positive sign indicates that the model in the row is better than the model in the column, and vice-versa. The Diebold-Mariano tests confirm our earlier results. The standard Diebold-Mariano test shows that the LM-EWMA model significantly dominates all other conditional volatility models, both short memory and long memory. With the improved version of the Diebold-Mariano test, the difference between each pair of models is less clearly marked and the outperformance of the LM-EWMA model is not significant in some cases. However, the Diebold-Mariano statistics are still uniformly positive. The S&P500-DJIA portfolio yields similar results. To save space, only the results of the joint tests are reported in Table IX.

[Insert Tables VIII and IX here]

5.2 High Dimensional Systems: The International Stock and Bond and the DJIA Portfolios

Economic Evaluation

In practice, a portfolio may comprise hundreds of assets and consequently an investor may want to examine the forecast performance of different conditional volatility models in a higher dimensional framework. In an asset allocation problem, the investor needs to estimate both the expected returns and the covariance matrix. However, since there are a prohibitively large number of possible expected return vectors for the high dimensional portfolios, we study the value of covariance matrix forecasts in two restricted cases. First, we form global minimum variance portfolios, where all expected returns are assumed to be equal. Note that the correctly specified covariance matrix will produce portfolios with the lowest volatility for any particular vector of expected returns, including the case that they are all equal. The results are reported in Table X. For the multivariate portfolios, we assume a risk free rate of 4%. Consistent with previous findings, in the international stock and bond portfolio, the LM-EWMA model yields the lowest conditional and out of sample volatilities. Owing to its simplicity, the simple EWMA model also performs very well, indeed better than the long memory LM-EWMA-DCC and CGARCH-DCC models. The short memory GARCH-DCC model is the least successful model. However, the results for the DJIA portfolio are markedly different in that the DCC models tend to outperform the non-DCC models. Indeed, the superiority of the LM-EWMA model deteriorates significantly, although it still renders better forecasts than the EWMA model. Consistent with the results for the bivariate portfolios, the LM-EWMA-DCC model always produces the best portfolios among the DCC models.

[Insert Table X here]

In the second experiment, we form hedging portfolios in which one asset is hedged against all other assets in the portfolio. In so doing, we select the expected return vectors such that one entry is equal to one and all others are set to zero. With this strategy, the LM-EWMA-DCC model is the best performing model in 33 of the 34 hedging portfolios of international stocks and bonds, and 24 of the 29 portfolios of DJIA components. The LM-EWMA model, though still dominating the EWMA model, is generally inferior to the GARCH-DCC and CGARCH-DCC models. The Diebold-Mariano joint tests for all hedging expected returns are applied and the findings are consistent with those of the relative volatilities (Table XI). The LM-EWMA-DCC model significantly outperforms all other models in both versions of the Diebold-Mariano tests. The LM-EWMA performs badly, significantly outperforming only the EWMA model. In the DJIA portfolio, the LM-EWMA model is even dominated by the unconditional estimator.

[Insert Table XI here]

These results show consistently that incorporating long memory in volatility dynamics improves the forecasts of the covariance matrix. The LM-EWMA model generally outperforms the EWMA model, while the LM-EWMA-DCC model always yields the best results among the DCC models. Our results also reveal an important difference in the relative forecasting power of the DCC and non-DCC models in low dimensional and high dimensional systems, respectively. In particular, the greater flexibility that arises from separately estimating volatility and correlation is evidently beneficial in the high dimensional case. This deserves attention for future research.

5.3 Longer Horizon Forecasts

Practical problems often require forecasts over longer horizons than the 1-step ahead forecasts considered above. In this section, we evaluate the forecast performance of different conditional volatility models, both statistically and economically, for horizons of up to three months. Table XII reports the RMSE of different conditional volatility models for 1-week, 1-month and 1-quarter ahead forecasts. The benchmarks are the true variances and covariances, proxied by the sum of squares and cross products of daily returns over the forecast horizons. The long memory volatility models generally outperform the short memory models, with the LM-EWMA and LM-EWMA-DCC models consistently yielding the smallest forecast error, although the standard EWMA model again proves itself a simple yet statistically accurate model. The MAE results are similar and are hence not reported.

[Insert Table XII here]

The Mincer-Zarnowitz regression is implemented for the longer horizons in Table XIII. Compared to the 1-step ahead forecasts, the forecasts for longer horizons have higher information content, which may be attributable to the use of more accurate proxies of the true variances and covariances. Again, the two LM-EWMA models dominate the other short and long memory conditional volatility models at all forecast horizons. They are the only two models that generally yield conditionally unbiased forecasts for the elements of the covariance matrix. To save space, only results for the LM-EWMA model and the two short memory EWMA and GARCH-DCC models are reported in Table XIII.

[Insert Table XIII here]

The economic usefulness of alternative covariance matrix estimators is assessed for both low and high dimensional portfolios over longer investment horizons. We let the investor rebalance his portfolios weekly, monthly and quarterly. These rebalancing frequencies would cover the situations of most investors in practice, at least approximately, from a day trader to a mutual fund. Table XIV gives the out-of-sample performance of the weekly rebalanced bivariate portfolios. Results for the conditional volatilities are similar. The gains from using the conditional volatility models for a trader who rebalances weekly, as compared to those for a day trader, are smaller. The two LM-EWMA models still outperform both the short memory models and the long memory CGARCH-DCC, though the gains, again, are lower. Among the two LM-EWMA models, neither dominates. The LM-EWMA model tends to perform better when the hypothetical vectors of expected returns are close to the unconditional mean and in the overall returns (which use the Bayesian priors as the weighting factors).

[Insert Table XIV here]

For the monthly and quarterly rebalanced portfolios, the results are similar. The two long memory EWMA models consistently produce better forecasts than the short memory and constant volatility models. The short memory conditional volatility models either rapidly revert to the unconditional volatility at an exponential rate or, in the case of the EWMA model, do not converge at all, and consequently have relatively uninteresting long-run forecasts. With slowly decaying autocorrelations, the long memory volatility models are able to better exploit past information and consequently yield more accurate forecasts over longer horizons. The outperformance of the two long memory EWMA models in the monthly and quarterly rebalanced portfolios confirms this intuition. To save space, only the out-of-sample results for the quarterly rebalanced portfolios are reported.

[Insert Table XV here]

Results for the two high dimensional portfolios are consistent with those for the two low dimensional portfolios. Under the global minimum variance strategy, the LM-EWMA and LM-EWMA-DCC models generally yield the most favourable results over horizons of up to three months (Table XVI). The CGARCH-DCC model also consistently outperforms the GARCH-DCC model. Similar results are obtained in favour of the long memory volatility models under the hedging strategy. However, as with the daily rebalanced portfolios, the DCC models outperform the non-DCC models. For all rebalancing frequencies, the LM-EWMA-DCC consistently yields the most economically useful forecasts in both high dimensional portfolios. A similar conclusion, though not reported here, follows from the Diebold-Mariano joint tests for the equality of the different models' forecasts at different forecast horizons.

[Insert Table XVI here]

5.4 Additional Robustness Tests

Forecast performance is potentially affected by the size of the rolling window used to estimate the conditional volatility models. Therefore, we re-evaluate the forecast performance of the multivariate conditional volatility models using estimation windows of two years, five years and ten years of daily returns. In the cases of 5-year and 10-year rolling windows, we also estimate the conditional covariance matrix using the FIGARCH-DCC model. We do not estimate the FIGARCH-DCC model with 1-year and 2-year rolling windows since the estimation of the FIGARCH model requires a prohibitively high upper lag cut-off. Following standard practice in the literature, we set the truncation lag for the FIGARCH model equal to 1000.

The outperformance of the two parsimonious long memory EWMA models reported above is found to be insensitive to the choice of estimation window length, in the both low dimension and high dimension cases. To save space, Table XVII reports only the economic evaluation for the two bivariate portfolios with a 5-year estimation window. The two long memory LM-EWMA and LM-EWMA-DCC models consistently produce forecasts that are more accurate and informative, and more economically useful than other short and long memory models. The simple EWMA model, although not as good as the LM-EWMA model, generally outperforms the more sophisticated GARCH model. The long memory FIGARCH model is the worst performing model, which may be attributable to the complexity of its specification. Although not reported, the use of longer forecast horizons (one week, one month and one quarter) yields very similar conclusions.

[Insert Table XVII here]

6. Conclusion

In this paper, we evaluate the economic benefits that arise from allowing for long memory in forecasting the covariance matrix of returns over both short and long horizons, using the asset allocation framework of Engle and Colacito (2006). In so doing, we compare the performance of a number of long memory and short memory multivariate volatility models. Incorporating long memory property improves forecasts of the conditional covariance matrix. In particular, we find that long memory volatility models dominate short memory and unconditional models on the basis of both statistical and economic criteria, especially at longer horizons. Moreover, the relatively parsimonious long memory EWMA models outperform the more complex multivariate long memory GARCH models. The high degree of parameterization of the Component GARCH and FIGARCH models evidently generates large estimation errors that are detrimental to their performance. The results are consistent across different datasets, and are robust to different investment horizons and estimation windows. The findings of the paper are consistent with those in the univariate volatility literature.

The non-DCC conditional covariance matrix estimators (such as the EWMA model with exponential weights and the LM-EWMA model with logarithmic weights) impose the same dynamic structure on all elements of the covariance matrix, which facilitates their implementation in high dimensional systems, but it comes at a cost in terms of estimation error. In a high dimensional system, employing a potentially less correctly specified but more flexible DCC structure may yield better results. Also, some of the eigenvalues of the high dimensional covariance matrix are inevitably very small, and so the inverse of the covariance matrix used in the asset allocation is likely to be ill-conditioned (see, for example, Zumbach, 2009a). This may partly explain the poor performance of the LM-EWMA model in large systems. It would be interesting to investigate this issue in greater detail.

The use of the long memory conditional covariance matrix produces optimal portfolios with lower realised volatility than the static unconditional covariance matrix. However, since our aim is simply to evaluate the forecasts of alternative conditional covariance matrices, and to choose the estimator that produces the lowest portfolio volatility, we do not explicitly consider realised portfolio returns. In particular, it does not follow that the portfolio with the lowest volatility is necessarily the best portfolio in terms of portfolio performance measures such as the Sharpe ratio. Thus it would also be of interest to investigate further the economic value of long memory volatility timing in the asset allocation framework, allowing for differences in return as well as risk, and for the effect of transaction costs.

References

ALEXANDER, C. 2001. Orthogonal Garch. *Mastering risk,* 2**,** 21--38.

ANDERSEN, T. G., BOLLERSLEV, T., DIEBOLD, F. X. & EBENS, H. 2001. The distribution of realized stock return volatility. *Journal of Financial Economics,* 61**,** 43--76.

BAILLIE, R. T., BOLLERSLEV, T. & MIKKELSEN, H. O. 1996. Fractionally integrated generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics,* 74**,** 3--30.

BOLLERSLEV, T. 1990. Modelling the Coherence in Short-run Nominal Exchange Rates: A Multivariate Generalized ARCH Model. *The Review of Economics and Statistics,* 72**,** 498--505.

BOLLERSLEV, T. & GHYSELS, E. 1996. Periodic Autoregressive Conditional Heteroscedasticity. *Journal of Business & Economic Statistics,* 14**,** 139--151.

BREIDT, F. J., CRATO, N. & DE LIMA, P. 1998. The detection and estimation of long memory in stochastic volatility. *Journal of Econometrics,* 83**,** 325--348.

DAVIDSON, J. 2004. Moment and memory properties of linear conditional heteroscedasticity models, and a new model. *Journal of Business & Economic Statistics,* 22**,** 16--29.

DIEBOLD, F. X. & MARIANO, R. S. 1995. Comparing Predictive Accuracy. *Journal of Business & Economic Statistics,* 13**,** 253--263.

DING, Z., GRANGER, C. W. J. & ENGLE, R. F. 1993. A long memory property of stock market return and a new model. *J. Empirical Finance,* 1**,** 83--106.

ENGLE, R. 2002. Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models. *Journal of Business & Economic Statistics,* 20**,** 339--350.

ENGLE, R. & COLACITO, R. 2006. Testing and valuing dynamic correlations for asset allocation. *Journal of Business and Economic Statistics,* 24**,** 238--253.

ENGLE, R. & LEE, G. 1999. A Permanent and Transitory Component Model of Stock Return Volatility, in ed. R. Engle and H. White Cointegration, Causality, and Forecasting: A Festschrift in Honor of Clive WJ Granger.

ENGLE, R. F., KANE, A. & NOH, J. 1996. Index-option pricing with stochastic volatility and the value of accurate variance forecasts. *Review of Derivatives Research,* 1**,** 139--157.

ENGLE, R. F. & SHEPPARD, K. 2001. Theoretical and empirical properties of dynamic conditional correlation multivariate GARCH. *NBER Working Paper*.

GEWEKE, J. & PORTER-HUDAK, S. 1983. The estimation and application of long memory time series models. *Journal of Time Series Analysis,* 4**,** 221--238.

HURVICH, C. M. & SOULIER, P. 2002. Testing for long memory in volatility. *Econometric Theory,* 18**,** 1291--1308.

JPMORGAN 1994. RiskMetrics™, Technical Document. JP Morgan.

MOULINES, E. & SOULIER, P. 1999. Broadband log-periodogram regression of time series with long-range dependence. *Annals of Statistics***,** 1415--1439.

NIGUEZ, T. M. & RUBIA, A. 2006. Forecasting the conditional covariance matrix of a portfolio under long-run temporal dependence. *Journal of Forecasting,* 25**,** 439--458.

TAYLOR, S. J. 1986. *Modelling financial time series,* Chichester, UK: John Wiley and Sons.

TEYSSIERE, G. 1998. Multivariate long-memory ARCH modelling for high frequency foreign exchange rates. Citeseer.

WEST, K., EDISON, H. & CHO, D. 1993. A Utility Based Evaluation of Some Models of Exchange Rate Variability. *Journal of International Economics,* 35**,** 23--46.

ZUMBACH, G. 2006. The Riskmetrics 2006 Methodology. *Technical Report, RiskMetrics Group.*

ZUMBACH, G. 2009a. The Empirical Properties of Large Covariance Matrices. *RiskMetrics Journal,* 9**,** 31-54.

ZUMBACH, G. 2009b. The statistical properties of the innovations in multivariate ARCH processes with large sizes. *To appear in Quantative Finance*.

Table I. Summary Statistics for the Two Bivariate Systems

The table reports descriptive statistics for the daily returns on Stock and Bond futures, and the daily excess returns on the S&P500 and DJIA indices. Means and standard deviations are annualised. The sample period is from 01 January 1988 to 31 December 2009. The table also reports the fractional difference operator, *d*, estimated using the FIGARCH, Geweke-Porter-Hudak (GPH) and Moulines-Soulier (MS) tests. The GPH and MS estimators are applied to both squared returns and log squared returns

Return series	***Mean (%)***	***Std. Dev. (%)***	***Skewness***	***Kurtosis***	***Min (%)***	***Max (%)***	***Normality test***	***Corr.***	$\hat{d}_{FIGARCH}$	***Squared returns***		***Log squared returns***	
										$\hat{d}_{GPH}$	$\hat{d}_{MS}$	$\hat{d}_{GPH}$	$\hat{d}_{MS}$
Stock	6.83	19.06	-0.19	14.18	-10.40	13.20	28936	-0.038	0.403	0.357	0.373	0.280	0.387
Bond	1.48	6.53	-0.28	6.63	-2.86	3.57	3123		0.355	0.410	0.190	0.169	0.189
S&P500	2.80	18.34	-0.25	12.32	-9.47	10.95	20117	0.960	0.492	0.441	0.461	0.449	0.534
DJIA	3.59	17.72	-0.20	11.62	-8.20	10.51	17194		0.487	0.396	0.417	0.462	0.294

Table II. Summary Statistics for the International Stock and Bond Returns

The table reports summary statistics for the weekly returns on 21 international stock indices and 13 government bond indices. Means and standard deviations are annualised. The sample period is from 01 January 1988 to 31 December 2009.

Return series	*Mean (%)*	*Std. Dev. (%)*	*Skewness*	*Kurtosis*	*Min (%)*	*Max (%)*	*Normality test*
Panel A. International stocks							
Australia	7.30	21.64	-1.77	21.33	-34.86	14.52	16657
Austria	6.03	25.81	-1.52	18.70	-38.22	20.94	12223
Belgium	5.45	20.98	-1.21	12.68	-26.88	12.53	4757
Canada	7.58	20.68	-1.13	13.91	-25.92	17.61	5930
Denmark	9.89	21.00	-1.31	13.35	-26.39	13.66	5446
France	7.44	21.25	-0.90	10.94	-27.16	13.76	3167
Germany	6.69	23.49	-0.80	8.93	-26.11	15.00	1800
Hongkong	9.22	25.37	-0.62	6.57	-21.08	13.85	682
Ireland	3.44	25.49	-1.72	19.88	-39.31	16.18	14184
Italy	2.91	24.82	-0.60	8.85	-26.71	19.04	1705
Japan	-1.29	22.56	0.07	4.67	-16.02	11.75	134
Mexico	19.21	33.81	-0.33	7.66	-30.20	23.23	1060
Netherland	6.88	20.89	-1.44	17.48	-31.48	14.85	10416
New Zealand	-0.08	22.04	-0.63	7.44	-23.06	12.07	1017
Norway	8.93	26.82	-0.84	10.37	-28.54	19.82	2733
Singapore	6.99	26.29	-0.69	13.21	-33.13	23.02	5071
Spain	6.92	22.28	-0.90	10.21	-26.22	13.76	2641
Sweden	9.79	26.88	-0.52	7.73	-25.12	19.05	1123
Switzerland	8.44	19.42	-0.70	11.14	-24.01	13.96	3263
UK	4.44	19.13	-1.05	16.81	-27.73	16.30	9324
US	7.03	16.81	-0.81	10.54	-20.19	11.45	2845
Panel B. International bonds							
Austria	0.92	10.58	-0.03	3.64	-5.85	5.72	20
Belgium	0.95	10.68	-0.02	3.47	-5.16	5.55	11
Canada	2.36	8.71	-0.51	6.53	-8.38	5.34	647
Denmark	1.60	10.92	0.00	3.84	-5.82	5.67	33
France	1.81	10.54	-0.02	3.47	-4.88	5.79	11
Germany	0.73	10.62	0.01	3.37	-4.52	5.77	7
Ireland	1.83	10.89	-0.25	4.19	-7.52	5.94	79
Japan	1.67	12.11	0.89	8.33	-6.05	14.30	1509
Netherland	0.55	10.64	-0.02	3.36	-4.82	5.45	6
Sweden	0.06	12.06	-0.18	3.84	-7.85	5.93	40
Switzerland	0.95	12.05	0.11	3.72	-6.28	6.89	27
UK	0.13	10.60	-0.24	4.93	-7.12	6.48	188
US	1.23	4.43	-0.19	3.82	-2.61	2.06	39

Table III. Summary Statistics for the DJIA Components

The table reports summary statistics for the daily returns on the 29 components of the DJIA index. Means and standard deviations are annualised. The sample period is from 01 March 1990 to 31 December 2009.

Return series	*Mean (%)*	*Std. Dev. (%)*	*Skewness*	*Kurtosis*	*Min (%)*	*Max (%)*	*Normality test*
AA	3.47	39.12	-0.02	11.23	-17.50	20.87	14102
AXP	7.22	38.76	0.03	9.94	-19.35	18.77	10043
BA	4.61	31.89	-0.33	9.73	-19.39	14.38	9525
BAC	1.51	45.21	-0.29	30.90	-34.21	30.21	162245
CAT	10.19	33.62	-0.08	7.18	-15.69	13.74	3652
C	28.74	46.95	0.00	7.48	-22.10	21.82	4175
CVX	7.62	25.60	0.13	12.63	-13.34	18.94	19331
DD	2.76	29.39	-0.09	7.10	-12.03	10.86	3513
DIS	6.43	32.11	0.00	10.40	-20.29	14.82	11410
GE	5.44	29.93	0.01	11.17	-13.68	17.98	13916
GM	13.54	35.10	-0.67	16.81	-33.88	13.16	40119
HD	11.51	40.37	-0.08	9.21	-20.70	18.99	8044
HPQ	8.14	30.53	0.04	9.76	-16.89	12.37	9537
IBM	13.99	42.72	-0.38	8.26	-24.89	18.33	5884
INTC	11.37	23.70	-0.19	9.75	-17.25	11.54	9510
JNJ	8.01	42.10	0.26	13.11	-23.23	22.39	21336
JPM	9.39	24.79	0.08	8.01	-11.07	13.00	5230
KO	10.43	26.89	-0.04	6.98	-13.72	10.31	3305
MCD	7.12	24.25	0.01	7.50	-10.08	10.50	4214
MMM	5.82	29.95	-1.09	22.53	-31.17	12.25	80485
MRK	19.06	35.23	0.01	7.94	-16.96	17.87	5087
MSFT	10.01	29.62	-0.18	6.07	-11.82	9.69	1997
PFE	10.26	25.33	-2.78	68.38	-37.66	9.73	897033
PG	3.50	28.68	0.08	7.39	-13.54	15.08	4027
T	6.01	30.34	0.34	16.22	-20.07	22.76	36490
UTX	11.97	28.77	-1.13	28.55	-33.20	12.79	137065
VZ	1.90	27.61	0.17	7.64	-12.61	13.66	4503
WMT	11.39	29.21	0.13	5.83	-10.26	10.50	1681
XOM	8.91	24.83	0.09	11.92	-15.03	15.86	16591

Table IV. RMSE, MAE and HMSE for the Two Bivariate Systems

The table reports the RMSE, MAE and HMSE for each element of the conditional covariance matrix estimated using five multivariate conditional volatility models over the forecast period. The squares and cross-products of daily returns are used as proxies for the actual variances and covariances.

	EWMA	GARCH DCC	LM-EWMA	LM-EWMA DCC	CGARCH DCC
Panel A. Root Mean Square Error (RMSE)					
Variances					
Stock	4.7483	4.7953	4.7459	4.7459	4.7649
Bond	0.3964	0.3978	0.3957	0.3957	0.3988
S&P500	4.0336	4.0921	4.0434	4.0434	4.0646
DJIA	3.6876	3.7295	3.6900	3.6900	3.7076
Covariances					
Stock-Bond	0.7442	0.7593	0.7432	0.7536	0.7575
S&P500-DJIA	3.7951	3.8402	3.8015	3.8015	3.8166
Panel B. Mean Absolute Error (MAE)					
Variances					
Stock	1.5342	1.5372	1.5337	1.5337	1.5577
Bond	0.1803	0.1874	0.1799	0.1799	0.1880
S&P500	1.4088	1.4251	1.4089	1.4089	1.4407
DJIA	1.3079	1.3232	1.3077	1.3077	1.3357
Covariances					
Stock -Bond	0.3298	0.3335	0.3278	0.3295	0.3398
S&P500 -DJIA	1.3284	1.3435	1.3296	1.3306	1.3583
Panel C. Heteroskedasticity-adjusted Mean Square Error (HMSE)					
Variances					
Stock	13.5870	8.6617	11.3285	11.3285	8.9257
Bond	5.0709	4.1601	4.6114	4.6114	4.5520
S&P500	8.1334	5.5538	6.8703	6.8703	5.6967
DJIA	9.6358	5.9739	7.9015	7.9015	6.0413
Covariances					
S&P500-DJIA	9.7683	6.0861	8.0231	7.8463	6.1982

Table V. Mincer – Zarnowitz Regressions for the Two Bivariate Systems

The table reports the estimated coefficients of the Mincer-Zarnowitz regressions for the elements of the covariance matrix. The *p*-values are for the tests of the joint hypothesis H_0: $\alpha_{ij} = 0$ and $\beta_{ij} = 1$. The numbers in the parentheses are the *t*-statistics to test $\alpha_{ij} = 0$ and $\beta_{ij} = 1$, respectively.

	Intercept	**Slope**	**R^2**	***p*-value**	**Intercept**	**Slope**	**R^2**	***p*-value**
	Panel A. EWMA				***Panel B. GARCH-DCC***			
Stock	0.167	0.888	0.176	0.000	0.312	0.783	0.169	0.000
	(1.218)	(-0.798)			(3.750)	(-2.993)		
Bond	0.058	0.657	0.037	0.000	0.051	0.642	0.030	0.000
	(5.443)	(-4.687)			(4.248)	(-4.656)		
S&P500	0.132	0.906	0.207	0.001	0.257	0.801	0.193	0.000
	(1.110)	(-0.739)			(3.423)	(-2.748)		
DJIA	0.142	0.889	0.179	0.000	0.242	0.801	0.168	0.000
	(1.275)	(-0.876)			(3.204)	(-2.627)		
Stock-Bond	-0.010	0.698	0.046	0.000	-0.052	0.554	0.022	0.000
	(-1.263)	(-2.762)			(-3.593)	(-3.472)		
S&P500-DJIA	0.130	0.899	0.195	0.000	0.228	0.811	0.183	0.000
	(1.180)	(-0.789)			(3.129)	(-2.490)		
	Panel C. LM-EWMA				***Panel D. LM-EWMA-DCC***			
Stock	-0.006	1.011	0.174	0.154	-0.006	1.011	0.174	0.154
	(-0.044)	(0.074)			(-0.044)	(0.074)		
Bond	0.050	0.706	0.037	0.000	0.050	0.706	0.037	0.000
	(4.377)	(-3.853)			(4.377)	(-3.853)		
S&P500	0.000	1.011	0.201	0.224	0.000	1.011	0.201	0.224
	(-0.005)	(0.084)			(-0.005)	(0.084)		
DJIA	0.005	1.001	0.175	0.051	0.005	1.001	0.175	0.051
	(0.040)	(0.011)			(0.040)	(0.011)		
Stock-Bond	-0.010	0.735	0.046	0.000	-0.051	0.648	0.030	0.000
	(-1.278)	(-2.357)			(-3.765)	(-3.105)		
S&P500-DJIA	0.003	1.008	0.189	0.120	-0.006	1.012	0.189	0.202
	(0.025)	(0.057)			(-0.047)	(0.092)		
	Panel E. CGARCH-DCC							
Stock	0.248	0.802	0.178	0.000				
	(2.712)	(-2.329)						
Bond	0.059	0.606	0.028	0.000				
	(4.916)	(-5.097)						
S&P500	0.208	0.817	0.203	0.000				
	(2.457)	(-2.172)						
DJIA	0.200	0.814	0.177	0.000				
	(2.411)	(-1.823)						
Stock-bond	-0.067	0.625	0.025	0.000				
	(-4.375)	(-2.952)						
S&P500-DJIA	0.184	0.822	0.192	0.000				
	(2.074)	(-2.035)						

Table VI. Comparison of Conditional Volatilities: Bivariate Portfolios

The table reports the average conditional volatilities for the two bivariate portfolios, constructed with the objective of minimizing variance subject to the target excess return of 1. Each row in the table shows the results for the pair of expected returns in the corresponding first two columns. The overall returns are the pair of weighted returns using the Bayesian prior probabilities. The lowest volatility in each row is normalised to 100.

Panel A. Stock-Bond Portfolio

μ_{Stock}	μ_{Bond}	EWMA	GARCH DCC	LM EWMA	LM EWMA DCC	CGARCH DCC	Const
0.00	1.00	100.247	102.156	100.000	101.909	104.206	105.938
0.16	0.99	100.243	102.330	100.000	101.983	104.557	105.148
0.31	0.95	100.402	102.580	100.000	102.077	105.025	104.422
0.45	0.89	100.505	102.587	100.000	101.956	105.457	103.754
0.59	0.81	100.580	102.465	100.000	101.827	105.655	103.074
0.71	0.71	100.521	102.632	100.000	101.746	105.472	102.840
0.81	0.59	100.390	102.705	100.000	102.017	104.974	103.507
0.89	0.45	100.317	102.673	100.000	102.040	104.832	105.564
0.95	0.31	100.237	101.994	100.000	101.301	104.207	108.904
0.99	0.16	100.465	101.949	100.000	100.509	103.752	111.038
1.00	0.00	100.385	103.277	100.000	102.335	105.200	106.434
Overall	*(weighted)*	*100.208*	*102.097*	*100.000*	*101.496*	*104.307*	*108.365*

Panel B. S&P500-DJIA Portfolio

μ_{SP500}	μ_{DJIA}	EWMA	GARCH DCC	LM EWMA	LM EWMA DCC	CGARCH DCC	Const
0.00	1.00	100.402	102.054	100.089	100.000	102.233	101.206
0.16	0.99	100.491	101.925	100.076	100.000	102.227	101.435
0.31	0.95	100.511	101.534	100.090	100.000	102.136	101.865
0.45	0.89	100.308	100.836	100.022	100.000	101.695	102.751
0.59	0.81	100.172	100.917	100.000	100.559	101.547	105.244
0.71	0.71	100.191	102.031	100.000	100.658	102.946	102.088
0.81	0.59	100.184	101.040	100.000	100.658	101.317	107.019
0.89	0.45	100.377	101.256	100.000	100.586	101.988	106.425
0.95	0.31	100.347	101.473	100.000	100.404	102.137	103.928
0.99	0.16	100.256	101.681	100.000	100.219	102.119	102.558
1.00	0.00	100.261	101.697	100.000	100.087	102.045	101.784
Overall	*(weighted)*	*100.204*	*102.057*	*100.000*	*100.719*	*103.022*	102.508

Table VII. Comparison of Out-of-Sample Volatilities: Bivariate Portfolios

The table reports out-of-sample volatilities for the two bivariate portfolios, constructed with the objective of minimizing variance subject to the target excess return of 1. Each row in the table shows the results for the pair of expected returns in the corresponding first two columns. The overall returns are the pair of weighted returns using the Bayesian prior probabilities. The lowest volatility in each row is normalised to 100.

Panel A. Stock-Bond Portfolio

μ_{Stock}	μ_{Bond}	EWMA	GARCH DCC	LM EWMA	LM EWMA DCC	CGARCH DCC	Const
0.00	1.00	100.540	102.133	100.000	101.876	104.291	105.164
0.16	0.99	100.532	102.228	100.000	101.950	104.659	104.432
0.31	0.95	100.537	102.320	100.000	102.051	105.031	103.736
0.45	0.89	100.530	102.372	100.000	102.119	105.435	103.109
0.59	0.81	100.446	102.462	100.000	102.208	105.816	102.696
0.71	0.71	100.401	102.638	100.000	102.332	105.791	102.695
0.81	0.59	100.370	102.808	100.000	102.673	105.027	103.413
0.89	0.45	100.303	103.046	100.000	102.642	105.631	105.039
0.95	0.31	100.328	102.932	100.000	102.117	105.474	107.664
0.99	0.16	100.386	103.133	100.000	100.936	104.811	109.317
1.00	0.00	100.428	105.932	100.000	103.005	107.182	104.709
Overall	*(weighted)*	*100.312*	*102.920*	*100.000*	*102.192*	*105.446*	*107.262*

Panel B. S&P500-DJIA Portfolio

μ_{SP500}	μ_{DJIA}	EWMA	GARCH DCC	LM EWMA	LM EWMA DCC	CGARCH DCC	Const
0.00	1.00	100.292	103.208	100.130	100.000	103.694	101.879
0.16	0.99	100.302	102.995	100.110	100.000	103.956	102.280
0.31	0.95	100.352	102.353	100.132	100.000	104.288	103.012
0.45	0.89	100.323	101.148	100.048	100.000	102.441	104.170
0.59	0.81	100.304	101.361	100.000	100.408	102.366	105.967
0.71	0.71	100.073	102.290	100.000	100.589	102.886	104.349
0.81	0.59	100.142	102.236	100.000	100.717	101.208	109.425
0.89	0.45	100.421	101.519	100.000	100.617	101.730	107.279
0.95	0.31	100.456	102.817	100.000	100.476	102.217	104.268
0.99	0.16	100.393	102.854	100.000	100.314	102.488	102.723
1.00	0.00	100.374	102.871	100.000	100.218	102.621	101.934
Overall	*(weighted)*	*100.096*	*102.368*	*100.000*	*100.691*	*102.936*	*104.723*

Table VIII. Diebold-Mariano Tests of the Stock-Bond Portfolio

The table reports the *t*-statistics of the Diebold-Mariano tests for the Stock-Bond portfolio using the improved version of the test described in Engle and Colacito (2006). Panels A and B correspond to [μ_{Stock}, μ_{Bond}] = [0.95; 0.31] and [0.99; 0.16], respectively. Panel C reports the joint tests of all the expected vectors of returns. The *t*-statistics for the standard version of the Diebold-Mariano test are reported in parentheses. A positive number indicates that the model in the row is better than that in the column, and vice-versa.

	EWMA	GARCH DCC	LM EWMA	LM-EWMA DCC	CGARCH DCC	Const
			Panel A. μ=[0.95; 0.31]			
EWMA		2.005*	-1.227	3.539***	2.763***	2.645***
		(2.746***)	(-2.151**)	(3.475***)	(4.757***)	(4.211***)
GARCH DCC	-2.005*		-2.302**	-0.952	1.132	1.632
	(-2.746***)		(-3.075**)	(-1.189)	(3.491***)	(2.469**)
LM-EWMA	1.227	2.302**		3.946***	3.175***	2.788***
	(2.151**)	(3.075***)		(4.042***)	(5.041***)	(4.391***)
LM-EWMA DCC	-3.539***	0.952	-3.946***		1.952**	2.102**
	(-3.475***)	(1.189)	(-4.042***)		(3.943***)	(3.094***)
CGARCH DCC	-2.763***	(-1.132)	-3.175***	-1.920*		1.065
	(-4.757***)	(-3.491***)	(-5.041***)	(-4.060***)		(1.382)
Constant	-2.645***	-1.632	-2.788***	-2.102**	-1.065	
	(-4.211***)	(-2.469**)	(-4.391***)	(-3.094***)	(-1.382)	
			Panel B. μ=[0.99; 0.16]			
EWMA		0.804	-0.208	-0.287	1.023	2.272**
		(1.895*)	(-1.985**)	(-1.147)	(3.136***)	(4.266***)
GARCH DCC	-0.804		-0.812	-1.455	0.807	1.526
	(-1.895*)		(-2.087**)	(-1.723*)	(3.780***)	(2.673***)
LM-EWMA	0.208	0.812		-0.112	1.006	2.461**
	(1.985**)	(2.087**)		(1.766*)	(3.285***)	(4.499***)
LM-EWMA DCC	0.287	1.455	0.112		2.101**	2.341**
	(-1.147)	(1.723*)	(-1.766*)		(3.090***)	(3.886***)
CGARCH DCC	-1.023	-0.807	-1.006	-2.017**		1.560
	(-3.136***)	(-3.780***)	(-3.285***)	(-3.179***)		(1.954*)
Constant	-2.272**	-1.526	-2.461**	-2.366**	-1.560	
	(-4.266***)	(-2.673***)	(-4.499***)	(-3.945***)	(-1.954*)	
			Panel C. Joint tests			
EWMA		2.001**	-0.422	1.741*	1.742*	3.229***
		(4.072***)	(-6.393***)	(5.806***)	(4.710***)	(7.849***)
GARCH DCC	-2.001**		-2.054**	-1.944*	0.171	0.044
	(-4.072***)		(-4.442***)	(-2.385**)	(4.719***)	(2.156**)
LM-EWMA	0.422	2.054**		1.807*	1.726*	3.629***
	(6.393***)	(4.442***)		(6.624***)	(4.957***)	(8.451***)
LM-EWMA DCC	-1.741*	1.944*	-1.807*		1.462	2.209**
	(-5.806***)	(2.385**)	(-6.624***)		(3.719***)	(5.003***)
CGARCH DCC	-1.742*	-0.171	-1.726*	-1.462		-0.322
	(-4.710***)	(-4.719***)	(-4.957***)	(-3.719***)		(-0.134)
Constant	-3.229***	-0.044	-3.629***	-2.209**	0.322	
	(-7.849***)	(-2.156**)	(-8.451***)	(-5.003***)	(0.134)	

Table IX. Diebold-Mariano Tests of the S&P500-DJIA Portfolio

The table reports the *t*-statistics of the Diebold-Mariano tests for the S&P500-DJIA portfolio using the improved version of the test described in Engle and Colacito (2006). The *t*-statistics for the standard version of the Diebold-Mariano test are reported in parentheses. A positive number indicates that the model in the row is better than that in the column, and vice-versa.

	EWMA	**GARCH DCC**	**LM EWMA**	**LM EWMA DCC**	**CGARCH DCC**	**Const**
EWMA		2.671***	0.338	1.246	3.179***	2.598***
		(5.166***)	(-3.329***)	(1.469)	(6.374***)	(5.222***)
GARCH DCC	-2.671***		-3.033***	-3.296***	-0.368	2.130**
	(-5.166***)		(-6.004***)	(-5.545***)	(1.307)	(3.795***)
LM EWMA	-0.338	3.033***		1.412	3.674***	2.769***
	(3.329***)	(6.004***)		(2.813***)	(7.161***)	(5.455***)
LM EWMA DCC	-1.246	3.296***	-1.412		2.611***	2.848***
	(-1.469)	(5.545***)	(-2.813***)		(5.939***)	(5.438***)
CGARCH DCC	-3.179***	0.368	-3.674***	-2.611***		2.004**
	(-6.374***)	(-1.307)	(-7.161***)	(-5.939***)		(3.161***)
Constant	-2.598***	-2.130**	-2.769***	-2.848***	-2.004**	
	(-5.222***)	(-3.795***)	(-5.455***)	(-5.438***)	(-3.161***)	

Table X. Comparison of Volatilities: Multivariate Portfolios

The table reports the volatilities of the global minimum variance portfolios. The lowest volatility in each row is normalised to 100.

	EWMA	**GARCH DCC**	**LM EWMA**	**LM EWMA DCC**	**CGARCH DCC**
Panel A. Conditional Volatilities					
International stock and bond portfolio	103.972	113.069	100.000	105.886	106.256
DJIA portfolio	126.605	102.881	106.554	100.000	102.478
Panel B. Out-of-Sample Volatilities					
International stock and bond portfolio	102.734	112.645	100.000	103.551	102.853
DJIA portfolio	125.268	104.119	105.393	100.000	103.663

Table XI. Diebold–Mariano Joint Tests: Hedging Multivariate Portfolios

The table reports the *t*-statistics of the Diebold–Mariano joint tests for the hedging multivariate portfolios, using the improved test of Engle and Colacito (2006). Panel A corresponds to the international stock and bond portfolio, while Panel B corresponds to the DJIA portfolio. The *t*-statistics for the standard test are reported in parentheses. A positive number indicates that the model in the row is better than the model in the column, and vice-versa.

	EWMA	**GARCH DCC**	**LM EWMA**	**LM EWMA DCC**	**CGARCH DCC**	**Const**
Panel A. International Stock and Bond Portfolio						
EWMA		-3.44***	-3.62***	-4.63***	-3.49***	-3.04***
		(-7.12***)	(-9.13***)	(-10.64***)	(-7.39***)	(-2.61***)
GARCH	3.44***		2.44***	-2.78***	1.28	0.59
DCC	(7.12***)		(4.37***)	(-4.88***)	0.99	(5.06***)
LM-EWMA	3.62***	-2.44***		-3.74***	-2.58***	-1.50
	(9.13***)	(-4.37***)		(-7.51***)	(-4.53***)	(-0.37)
LM-EWMA	4.63***	2.78***	3.74***		2.66***	4.39***
DCC	(10.64***)	(4.88***)	(7.51***)	(4.48***)	(9.57***)	
CGARCH	3.49***	-1.28	2.58***	-2.66***		0.45
DCC	(7.39***)	(-0.99)	(4.53***)	(-4.48***)		(3.97***)
Constant	3.04***	-0.59	1.50	-4.39***	-0.45	
	(2.61***)	(-5.06***)	(0.37)	(-9.57***)	(-3.97***)	
Panel B. DJIA Portfolio						
EWMA		-7.02***	-9.81***	-9.79***	-9.56***	-13.68***
		(-37.75***)	(-40.43***)	(-36.79***)	(-38.16***)	(-32.09***)
GARCH	7.02***		1.77*	-1.52	-1.00	0.86
DCC	(37.75***)		(26.10***)	(-4.10***)	(1.28)	(11.10***)
LM-EWMA	9.81***	-1.77*		-8.79***	-7.58***	-6.57***
	(40.43***)	(-26.10***)		(-27.93***)	(-28.70***)	(-12.43***)
LM-EWMA	9.79***	1.52	8.79***		2.54**	7.18***
DCC	(36.79***)	(4.10***)	(27.93***)		(6.32***)	(13.77***)
CGARCH	9.56***	1.00	7.58***	-2.54**		3.82***
DCC	(38.16***)	(-1.28)	(28.70***)	(-6.32***)		(11.92***)
Constant	13.68***	-0.86	6.57***	-7.18***	-3.82***	
	(32.09***)	(-11.10***)	(12.43***)	(-13.77***)	(-11.92***)	

Table XII. RMSE for Longer Horizon Forecasts: Bivariate Systems

The table reports the RMSE for each element of the forecast conditional covariance matrix over the forecast period. The benchmarks are the realised variances and covariances, proxied by the sum of squares and cross products of returns over the forecast horizons, respectively.

	EWMA	**GARCH DCC**	**LM EWMA**	**LM-EWMA DCC**	**CGARCH DCC**
Panel A. One Week (5-Step) ahead Forecasts					
Variances					
Stock	12.675	13.207	12.668	12.668	12.330
Bond	0.918	0.938	0.901	0.901	0.952
S&P500	10.308	10.676	10.411	10.411	10.039
DJIA	9.611	9.892	9.638	9.638	9.286
Covariances					
Stock-Bond	1.814	1.892	1.788	1.811	1.880
S&P500-DJIA	9.783	10.062	9.851	9.858	9.460
Panel B. One Month (21-Step) ahead Forecasts					
Variances					
Stock	49.667	54.759	47.789	47.789	51.491
Bond	2.230	2.348	2.128	2.128	2.487
S&P500	42.384	47.684	41.015	41.015	44.696
DJIA	38.029	40.468	36.789	36.789	38.583
Covariances					
Stock-Bond	4.523	4.875	4.737	4.691	4.536
S&P500-DJIA	39.655	43.093	38.330	38.350	40.755
Panel C. One Quarter (63-Step) ahead Forecasts					
Variances					
Stock	151.499	168.737	146.514	146.514	165.768
Bond	5.348	5.630	4.983	4.983	5.974
S&P500	133.140	151.082	129.748	129.748	142.108
DJIA	113.798	128.904	111.416	111.416	120.125
Covariances					
Stock-Bond	7.729	10.210	7.893	10.470	9.377
S&P500-DJIA	121.775	137.591	118.879	118.890	128.746

Table XIII. Mincer – Zarnowitz Regressions for Longer Horizons: Bivariate Systems

The table reports the results of the Mincer-Zarnowitz regressions for longer horizon forecasts of each element of the covariance matrix. The *p*-values are for the tests of the joint hypothesis: H_0: $\alpha_{ij} = 0$ and $\beta_{ij} = 1$.

	EWMA				GARCH-DCC				LM-EWMA			
	Intercept	**Slope**	R^2	***p*-value**	**Intercept**	**Slope**	R^2	***p*-value**	**Intercept**	**Slope**	R^2	***p*-value**
Panel A. One Week (5-Step) ahead Forecasts												
Stock	1.042	0.860	0.412	0.000	1.751	0.750	0.401	0.000	0.074	0.997	0.408	0.025
Bond	0.282	0.680	0.161	0.000	0.251	0.664	0.120	0.000	0.199	0.782	0.170	0.000
S&P500	0.759	0.895	0.497	0.002	1.239	0.806	0.480	0.000	-0.051	1.019	0.480	0.341
DJIA	0.839	0.870	0.437	0.000	1.237	0.796	0.423	0.000	0.049	0.999	0.426	0.021
Stock-Bond	-0.066	0.686	0.176	0.000	-0.184	0.610	0.122	0.000	-0.061	0.752	0.183	0.000
S&P500-DJIA	0.758	0.884	0.472	0.000	1.134	0.811	0.457	0.000	-0.011	1.011	0.456	0.133
Panel B. One Month (21-Step) ahead Forecasts												
Stock	8.454	0.730	0.322	0.000	13.066	0.564	0.305	0.000	5.739	0.825	0.348	0.103
Bond	0.931	0.764	0.350	0.003	0.819	0.747	0.274	0.008	0.615	0.871	0.389	0.148
S&P500	6.408	0.784	0.394	0.005	11.133	0.604	0.342	0.000	3.996	0.875	0.413	0.325
DJIA	6.708	0.748	0.346	0.001	9.431	0.629	0.330	0.000	4.459	0.836	0.365	0.129
Stock-Bond	-0.297	0.776	0.379	0.034	-0.543	0.662	0.335	0.000	-0.344	0.739	0.326	0.001
S&P500-DJIA	6.259	0.768	0.371	0.003	9.708	0.625	0.339	0.000	3.968	0.860	0.391	0.236
Panel C. One Quarter (63-Step) ahead Forecasts												
Stock	44.494	0.536	0.172	0.005	60.382	0.336	0.065	0.000	37.168	0.623	0.167	0.104
Bond	3.462	0.660	0.468	0.000	2.910	0.687	0.353	0.016	2.641	0.754	0.482	0.041
S&P500	34.961	0.614	0.244	0.014	50.654	0.414	0.098	0.000	28.389	0.697	0.206	0.192
DJIA	34.137	0.583	0.221	0.008	49.335	0.379	0.086	0.000	29.114	0.647	0.196	0.094
Stock-Bond	-0.714	0.886	0.669	0.422	-1.326	0.721	0.479	0.006	-0.756	0.919	0.649	0.840
S&P500-DJIA	33.000	0.600	0.232	0.011	47.863	0.401	0.091	0.000	27.280	0.677	0.221	0.150

Table XIV. Comparison of Out-of-Sample Volatilities: Bivariate Portfolios with Weekly Rebalancing

The table reports out-of-sample volatilities for the weekly rebalanced bivariate portfolios, constructed with the objective of minimizing variance subject to the target excess return of 1. Each row in the table reports the results for the pair of expected returns in the corresponding first two columns. The overall returns are the pair of weighted returns using the Bayesian prior probabilities. The lowest volatility in each row is normalised to 100.

Panel A. Stock-Bond Portfolio

μ_{Stock}	μ_{Bond}	EWMA	GARCH DCC	LM EWMA	LM EWMA DCC	CGARCH DCC	Const
0.00	1.00	100.472	100.590	100.000	100.000	102.243	105.903
0.16	0.99	100.524	100.524	100.058	100.000	102.446	105.533
0.31	0.95	100.562	100.449	100.169	100.000	102.528	105.056
0.45	0.89	100.635	100.424	100.265	100.000	102.594	104.553
0.59	0.81	100.730	100.389	100.292	100.000	102.530	103.990
0.71	0.71	100.652	100.478	100.217	100.000	102.262	103.306
0.81	0.59	100.528	100.642	100.000	100.000	101.925	102.491
0.89	0.45	100.447	100.767	100.000	100.032	101.374	102.204
0.95	0.31	100.376	101.207	100.000	100.107	102.468	103.165
0.99	0.16	100.670	101.547	100.000	100.485	103.486	105.149
1.00	0.00	100.921	101.800	100.000	100.879	102.550	105.271
Overall	*(weighted)*	*100.360*	*101.099*	*100.000*	*100.049*	*102.178*	*102.913*

Panel B. S&P500-DJIA Portfolio

μ_{SP500}	μ_{DJIA}	EWMA	GARCH DCC	LM EWMA	LM EWMA DCC	CGARCH DCC	Const
0.00	1.00	100.621	100.966	100.207	100.000	100.897	101.655
0.16	0.99	100.762	100.645	100.352	100.000	100.821	102.345
0.31	0.95	100.941	100.141	100.612	100.000	100.612	103.388
0.45	0.89	101.749	100.000	101.435	100.700	100.770	105.493
0.59	0.81	100.842	100.000	100.591	100.364	100.682	105.755
0.71	0.71	100.184	100.811	100.000	100.199	100.719	101.560
0.81	0.59	100.431	101.940	100.000	100.367	101.574	107.827
0.89	0.45	100.396	100.759	100.000	100.231	100.660	105.805
0.95	0.31	100.400	101.422	100.044	100.000	101.244	102.222
0.99	0.16	100.278	101.893	100.111	100.000	101.336	100.557
1.00	0.00	100.331	102.118	100.199	100.132	101.390	100.000
Overall	*(weighted)*	*100.225*	*100.856*	*100.000*	*100.187*	*100.677*	*101.613*

Table XV. Comparison of Out-of-Sample Volatilities: Bivariate Portfolios with Quarterly Rebalancing

The table reports out-of-sample volatilities for the quarterly rebalanced bivariate portfolios, constructed with the objective of minimizing variance subject to the target excess return of 1. Each row in the table reports the results for the pair of expected returns of the corresponding first two columns. The overall returns are the pair of weighted returns using the Bayesian prior probabilities. The lowest volatility in each row is normalised to 100.

Panel A. Stock-Bond Portfolio

μ_{Stock}	μ_{Bond}	EWMA	GARCH DCC	LM EWMA	LM EWMA DCC	CGARCH DCC	Const
0.00	1.00	100.622	100.000	100.000	100.000	100.415	110.581
0.16	0.99	101.031	100.000	100.412	100.000	100.412	109.691
0.31	0.95	101.603	100.000	100.601	100.000	100.200	108.417
0.45	0.89	101.908	100.000	100.954	100.191	100.000	107.252
0.59	0.81	102.135	100.000	101.246	100.534	100.000	106.228
0.71	0.71	101.789	100.650	101.138	100.813	100.000	105.528
0.81	0.59	100.728	101.019	100.146	100.146	100.000	104.803
0.89	0.45	100.783	102.611	100.000	100.261	101.044	106.658
0.95	0.31	100.818	103.855	100.467	100.000	102.336	109.813
0.99	0.16	100.526	103.891	100.315	100.000	103.260	114.826
1.00	0.00	100.000	102.844	100.267	101.422	101.778	113.867
Overall	*(weighted)*	*100.727*	*103.749*	*100.336*	*100.000*	*102.098*	*109.219*

Panel B. S&P500-DJIA Portfolio

μ_{SP500}	μ_{DJIA}	EWMA	GARCH DCC	LM EWMA	LM EWMA DCC	CGARCH DCC	Const
0.00	1.00	100.000	102.597	100.519	100.519	103.377	101.818
0.16	0.99	100.000	102.649	100.442	100.442	103.311	101.987
0.31	0.95	100.000	102.482	100.355	100.177	102.837	102.128
0.45	0.89	100.000	101.713	100.000	100.000	101.976	102.635
0.59	0.81	100.629	101.258	100.449	100.000	101.797	105.121
0.71	0.71	101.020	100.960	100.000	101.621	106.363	102.461
0.81	0.59	100.000	106.854	101.168	102.103	106.464	109.891
0.89	0.45	100.000	105.181	100.361	101.084	105.301	108.434
0.95	0.31	100.163	103.431	100.000	100.490	104.739	104.248
0.99	0.16	100.206	102.263	100.000	100.206	103.909	102.469
1.00	0.00	100.000	101.716	100.000	100.000	103.676	101.471
Overall	*(weighted)*	*100.707*	*100.865*	*100.000*	*101.623*	*106.049*	*102.937*

Table XVI. Comparison of Volatilities: Multivariate Portfolios with Different Rebalancing Frequencies

The table reports out-of-sample volatilities of the global minimum variance multivariate portfolios. Conditional volatilities are reported in parentheses. The lowest volatility in each row is normalised to 100.

	EWMA	GARCH DCC	LM EWMA	LM-EWMA DCC	CGARCH DCC
Panel A. International Stock and Bond Portfolio					
Monthly rebalancing	102.884 (102.502)	123.706 (113.730)	100.000 (100.000)	104.699 (105.481)	110.065 (111.344)
Quarterly rebalancing	106.339 (103.374)	128.935 (116.535)	100.000 (100.000)	105.900 (107.679)	113.939 (110.481)
Panel B. DJIA Portfolio					
Weekly rebalancing	107.748 (106.180)	102.985 (100.444)	104.699 (102.825)	100.000 (100.000)	102.053 (100.597)
Monthly rebalancing	105.261 (107.272)	103.892 (104.615)	103.612 (104.311)	100.000 (100.000)	102.227 (104.273)
Quarterly rebalancing	115.392 (114.665)	107.737 (108.247)	108.621 (107.377)	100.000 (100.000)	104.939 (103.760)

Table XVII. Comparison of Out-of-Sample Volatilities: Bivariate Portfolios with 5-Year Estimation Window

The table reports out-of-sample volatilities for the minimum variance bivariate portfolios, constructed using 5-year estimation window and subject to the excess target return of 1. Each row in the table reports the results for the pair of expected returns in the corresponding first two columns. The overall returns are the pair of weighted returns using the Bayesian prior probabilities. The lowest volatility in each row is normalised to 100.

Panel A. Stock-Bond Portfolio

μ_{Stock}	μ_{Bond}	EWMA	GARCH DCC	LM EWMA	LM EWMA DCC	CGARCH DCC	FIGARCH DCC	Const
0.00	1.00	100.564	102.177	100.000	102.075	103.637	106.967	105.610
0.16	0.99	100.608	102.128	100.000	102.103	103.572	108.715	104.890
0.31	0.95	100.588	102.009	100.000	102.058	103.552	110.828	104.140
0.45	0.89	100.556	101.922	100.000	101.992	103.659	112.969	103.474
0.59	0.81	100.471	101.883	100.000	101.904	103.808	114.955	102.974
0.71	0.71	100.387	101.971	100.000	101.817	103.923	116.293	103.015
0.81	0.59	100.290	102.170	100.000	101.811	103.861	116.761	104.032
0.89	0.45	100.162	102.184	100.000	101.697	103.424	116.293	106.272
0.95	0.31	100.236	101.972	100.000	101.191	103.523	114.823	109.278
0.99	0.16	100.395	101.376	100.000	100.617	103.177	111.655	110.229
1.00	0.00	100.427	102.484	100.000	102.902	104.224	104.115	104.567
Overall (weighted)		*100.207*	*102.010*	*100.000*	*101.296*	*103.446*	*115.106*	*108.884*

Panel B. S&P500-DJIA Portfolio

μ_{SP500}	μ_{DJIA}	EWMA	GARCH DCC	LM EWMA	LM EWMA DCC	CGARCH DCC	FIGARCH DCC	Const
0.00	1.00	100.217	101.485	100.031	100.000	101.671	158.014	101.702
0.16	0.99	100.210	101.446	100.000	100.000	101.683	165.843	102.182
0.31	0.95	100.254	101.458	100.021	100.000	101.733	170.716	103.042
0.45	0.89	100.282	101.408	100.000	100.047	101.847	145.211	104.570
0.59	0.81	100.226	101.068	100.000	100.411	100.503	139.945	107.094
0.71	0.71	100.102	101.108	100.000	100.382	101.439	111.423	104.680
0.81	0.59	100.098	101.238	100.000	100.543	100.285	117.528	110.305
0.89	0.45	100.469	101.278	100.000	100.540	101.293	118.835	107.841
0.95	0.31	100.528	101.468	100.000	100.372	101.566	129.928	104.463
0.99	0.16	100.470	101.510	100.000	100.198	101.659	136.940	102.748
1.00	0.00	100.413	101.535	100.000	100.118	101.712	139.327	101.860
Overall (weighted)		*100.123*	*101.209*	*100.000*	*100.448*	*101.403*	*112.275*	*105.138*

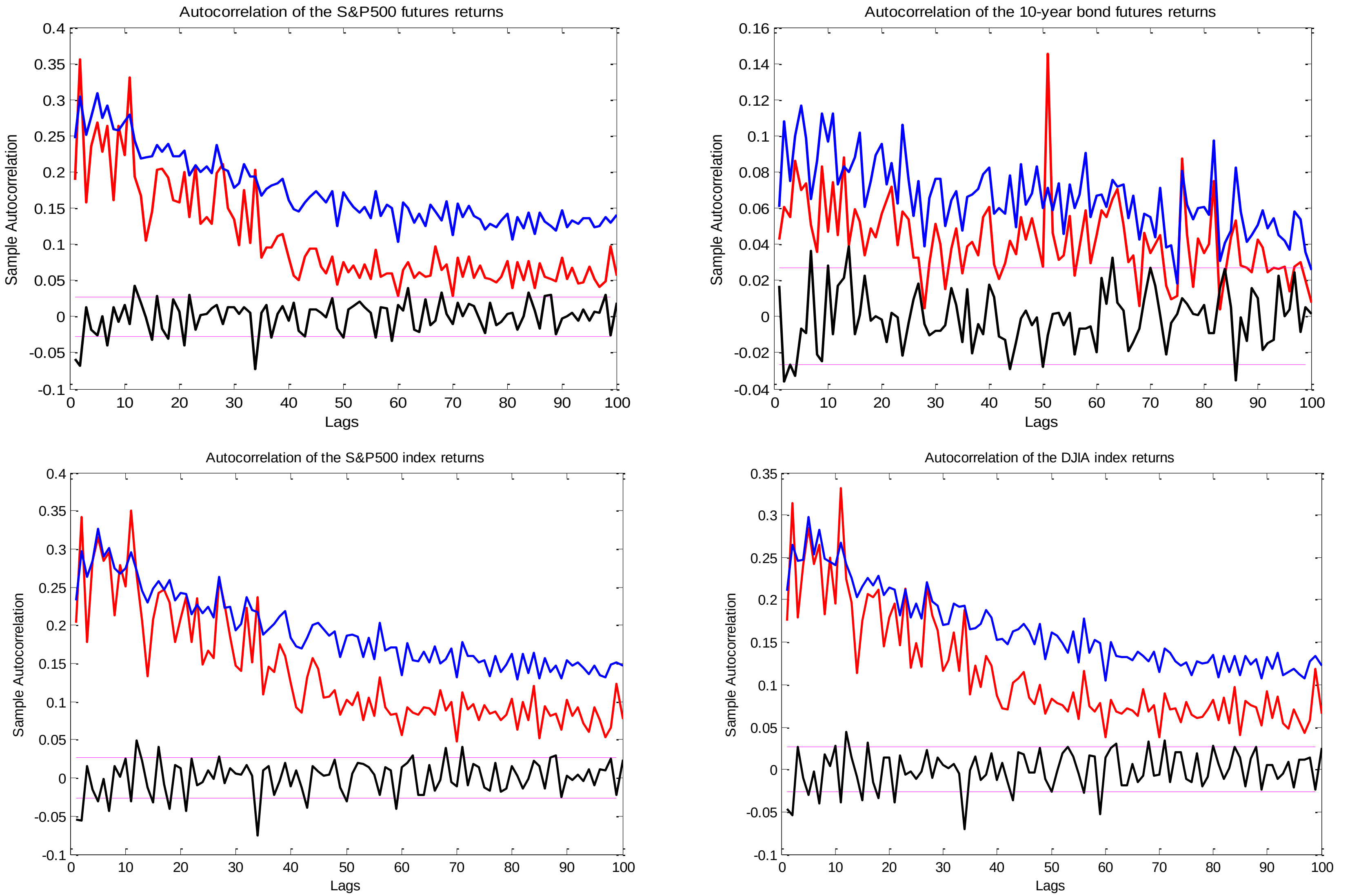

Figure 1. *Autocorrelations of returns (black line), absolute returns (blue line,) and squared returns (red line).*