

Concise stability conditions for systems with static nonlinear feedback expressed by a quadratic program ^{*}

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Abstract

We consider the stability of the feedback connection of a strictly proper linear time invariant (LTI) stable system with a static nonlinearity expressed by a convex quadratic program (QP). From the Karush-Kuhn-Tucker (KKT) conditions for the QP, we establish quadratic constraints that may be used with a quadratic Lyapunov function to construct a stability criterion via the S-procedure. The approach is based on existing results in the literature, but gives a more parsimonious linear matrix inequality (LMI) criterion and is much easier to implement. Our approach can be extended to model predictive control (MPC), and gives equivalent results to those in the literature but with a much lower dimension LMI criterion.

Key words: *Constrained system, quadratic program, LMI, S-Procedure, predictive control*

1 Introduction

The stability analysis of a closed loop system consisting of an LTI system in feedback with a static nonlinearity has been studied for a long time (see, e.g. [1–5] for general analysis; [6–8] for saturation analysis and [9–11] for anti-windup design). Primbs [12] and Primbs and Giannelli [13] observed that an important subclass of such nonlinearities can be represented as the solution of a convex QP — see Fig. 1. Examples include saturation, deadzone, combination of saturation and deadzone, as well as the piecewise affine mapping between state estimate and control input obtained with MPC (see, e.g., [14–16]). They developed a new approach to derive stability by showing that a candidate Lyapunov function is decreasing subject to the system dynamics and constraints determined by the KKT conditions for the QP. This approach is implemented by applying the S-procedure [17, 18], which leads to the stability criterion in terms of an LMI [19].

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One acknowledged drawback of the method is that “*a priori*, it is not clear how effective a constraint will be” [13]. The inclusion of redundant constraints leads to an LMI with large dimension, which both increases the computational burden and reduces the numerical accuracy. For example, in the case of a saturation, ten constraints are required to establish the stability criterion using Primbs’ method.

In this paper we are concerned with a strictly proper stable LTI system interconnected with a non-linearity, which is expressed by a convex QP with linear constraints

$$Lu + My \preceq b \tag{1}$$

for some fixed L, M and $b \succeq 0$ with appropriate dimensions — see Fig. 1. Here “ \preceq ” and “ \succeq ” signify term by term inequality. We first derive from the KKT conditions the essential linear and quadratic constraints that are useful to establish stability using Primbs’ method. Then we show that these constraints can be replaced by three constraints concisely. When we can set $M = LN$, for some fixed N with appropriate dimensions, the three constraints can be simplified further, and only involve terms in u, y, \dot{u} and \dot{y} (where \dot{u} exists) for continuous case.

Furthermore, we specifically focus on the QP with symmetric constraints. This case is important, because it involves almost all the nonlinearities presented by Primbs et al. and many MPC controllers. In this case, to guarantee global stability, the relation $M = LN$ in (1) is required as a necessary and sufficient condition for all $y \in \mathbb{R}^m$ to be feasible for the corresponding QP. It can be shown that we can use the three constraints proposed in this paper instead of the original ten by Primbs et al. to establish stability, which leads to a much lower dimension LMI criterion while not increasing the conservatism in stability.

All of these results have their counterparts for the discrete time case. In the numerical example investigated in this paper, we apply the results to the robustness and stability analysis of MPC. The reduction of the LMI dimension is particularly significant when the MPC has a long prediction horizon, which results in an LMI criterion with an immense dimension.

The paper is structured as follows. In section 2, the continuous time case is considered. In section 3 we derive results for the discrete time case that correspond to those of section 2. In section 4 we use three examples: saturation, deadzone and MPC [13] to illustrate the benefits of the new approach. Section 5 concludes the paper. Some of the proofs are given in the appendix.

We use the following notations of $M \in \mathbb{R}^{m \times n}$:

- M^\dagger : If $\text{rank}(M) = n$, then the pseudoinverse of M is defined as $M^\dagger = (M^T M)^{-1} M^T$.
- M_\perp : If $m < n$ and $\text{rank}(M) = m$, then $M_\perp \in \mathbb{R}^{n \times (n-m)}$ and the rows of M_\perp^T span the null space of the subspace spanned by the rows of M , such that $M M_\perp = 0$; if $m > n$ and $\text{rank}(M) = n$, then $M_\perp \in \mathbb{R}^{(m-n) \times m}$ and the columns of $(M_\perp)^T$ span the null space of the subspace spanned by the columns of M , such that $M_\perp M = 0$.
- $\text{He}(M)$: If M is a square matrix, then $\text{He}(M) := M + M^T$.

2 Continuous time case

2.1 Problem setup

Consider a stable continuous time multivariable system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}\tag{2}$$

with $x(t) \in \mathbb{R}^n$, $u(t), y(t) \in \mathbb{R}^m$. This system has a feedback connection with a nonlinearity expressed by a QP

$$u = \phi(y) = \arg \min_{\tilde{u}} \frac{1}{2} \tilde{u}^T H \tilde{u} + \tilde{u}^T F y\tag{3}$$

$$\text{s.t. } L\tilde{u} + My \preceq b\tag{4}$$

with Hessian matrix $H = H^T > 0$. Here the dimensions of the fixed terms are $H \in \mathbb{R}^{m \times m}$, $F \in \mathbb{R}^{m \times m}$, $L \in \mathbb{R}^{l \times m}$, $M \in \mathbb{R}^{l \times m}$ and $b \in \mathbb{R}^l$. We assume that $\phi(0) = 0$, hence the constant vector $b \succeq 0$.

We can also include equality constraints

$$K\tilde{u} + Gy = d$$

with $K \in \mathbb{R}^{g \times m}$, $G \in \mathbb{R}^{g \times m}$ and $d \in \mathbb{R}^g$ in the QP. Since the assumption $\phi(0) = 0$ requires $d = 0$, the equality constraints can be expressed by two inequality constraints

$$\begin{aligned}K\tilde{u} + Gy &\preceq 0 \\ -K\tilde{u} - Gy &\preceq 0\end{aligned}$$

which can be included in (4). Hence in this paper, we only consider the case when the objective function is subject to inequality constraints.

2.2 Stability establishment

From the KKT conditions for the QP describing the saturation or deadzone function, Primbs and Giannelli [13, 20] develop equality and inequality constraints that involve y , \dot{y} , u , \dot{u} , λ and $\dot{\lambda}$ (where λ is the Lagrangian multiplier corresponding to the constraints). Using the S-procedure, these constraints with a Lyapunov function are employed to establish the stability criterion. This method is also generalized to multivariable discrete time cases in [12]. However, exactly what kind of constraints are useful to establish the stability criterion is not pointed out [12]. In the following Proposition we summarize five constraints that are useful for establishing stability using the S-procedure, and show their equivalent three constraints.

Proposition 1 Consider the QP with objective function (3) subject to (4)

1) The following constraints can be derived from KKT conditions

$$Hu + Fy + L^T \lambda = 0 \quad (5)$$

$$H\dot{u} + F\dot{y} + L^T \dot{\lambda} = 0 \quad \text{where } \dot{u} \text{ and } \dot{\lambda} \text{ exist} \quad (6)$$

$$\lambda^T Lu + \lambda^T My \geq 0 \quad (7)$$

$$\lambda^T L\dot{u} + \lambda^T M\dot{y} = 0 \quad \text{where } \dot{u} \text{ exists} \quad (8)$$

$$\dot{\lambda}^T L\dot{u} + \dot{\lambda}^T M\dot{y} = 0 \quad \text{where } \dot{u} \text{ and } \dot{\lambda} \text{ exist} \quad (9)$$

where $\lambda \in \mathbb{R}^l$ is the Lagrangian multiplier corresponding to the inequality constraints in the KKT conditions.

2) When $\text{rank}(L) = m$, we have:

2.1) The following three constraints also hold (with the usual caveats about the existence of \dot{u} and $\dot{\lambda}$):

$$(u + L^\dagger My)^T (Hu + Fy) - ((I - LL^\dagger)My)^T \lambda \leq 0 \quad (10)$$

$$(\dot{u} + L^\dagger M\dot{y})^T (Hu + Fy) - ((I - LL^\dagger)M\dot{y})^T \lambda = 0 \quad (11)$$

$$(\dot{u} + L^\dagger M\dot{y})^T (H\dot{u} + F\dot{y}) - ((I - LL^\dagger)M\dot{y})^T \dot{\lambda} = 0 \quad (12)$$

2.2) Suppose that the Lyapunov function is $V(x, u) = [x^T, u^T]P[x^T, u^T]^T$. Then when establishing stability by the S-procedure, the LMI derived from the constraints (5)-(9) and the LMI from the three constraints (10)-(12) are equivalent.

Proof: See Appendix. □

In this proposition, we note two points: First, in the reduction in 2.2), it can also be shown that when the candidate Lyapunov function is chosen as $V(x) = x^T Px$, the LMI derived from the constraints (5)-(9) are equivalent to the LMI from (10). However, it has been shown that the stability criterion based on the quadratic Lyapunov function $V(x, u)$ is usually less conservative than the one based on $V(x) = x^T Px$ (see [20] and [21]). Hence in this paper we are mainly concerned with the Lyapunov function $V(x, u)$. Second, although we can use a less number of constraints to establish stability, the resulting LMI dimension is not reduced because the three constraints (10)-(12) have the same number of variables as the original five constraints (5)-(9).

Now we consider an important special case of the constraint (4): When $M = LN$ with $N \in \mathbb{R}^{m \times m}$ and $\text{rank}(L) = m$, (4) takes the form of

$$L\tilde{u} + LNy \preceq b \quad (13)$$

In this case, the variables λ and $\dot{\lambda}$ in the three constraints (10)-(12) are eliminated so that the three constraints can be further reduced:

Lemma 1 For the QP with objective function (3) subject to (13), the following three conditions hold (with the usual caveats about the existence of \dot{u} and $\dot{\lambda}$):

$$(u + Ny)^T(Hu + Fy) \leq 0 \quad (14)$$

$$(\dot{u} + N\dot{y})^T(Hu + Fy) = 0 \quad (15)$$

$$(\dot{u} + N\dot{y})^T(H\dot{u} + F\dot{y}) = 0 \quad (16)$$

Proof: (14)-(16) can be derived by substituting $M = LN$ into (10)-(12) and the fact that $L^\dagger L = I$. \square

Remark: When $N = 0$ or $F = 0$, the first condition corresponds to the sector bound condition long established for saturation and deadzone. More recently Heath, *et al.* [15,22,23] established such a sector bound in the context of MPC, and used it in a stability criterion. This lemma gives two extra slope rate conditions, which are expected to yield less conservative stability result.

The three quadratic constraints (14), (15) and (16) may be used in the manner proposed by Primbs and Giannelli [13,20] to establish stability by the S-procedure:

Theorem 1 (stability criterion): The system (2) with a nonlinear feedback expressed by a QP (3) subject to (13) is stable if there is a symmetric positive definite matrix

$$P = \begin{bmatrix} P_{11} & P_{21}^T \\ P_{21} & P_{22} \end{bmatrix} \quad (17)$$

such that the following LMI is satisfied:

$$\text{He}(\Pi_0 + \sum_{i=1}^3 r_i \Pi_i) < 0 \quad (18)$$

with scalars $r_1 \geq 0$, $r_2, r_3 \in \mathbb{R}$. Here Π_i with $i = 0, 1, 2, 3$ are

$$\Pi_0 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}^T P \begin{bmatrix} A & B & 0 \\ 0 & 0 & I \end{bmatrix} \quad (19)$$

$$\Pi_1 = - \begin{bmatrix} NC & I & 0 \end{bmatrix}^T \begin{bmatrix} FC & H & 0 \end{bmatrix} \quad (20)$$

$$\Pi_2 = \begin{bmatrix} NCA & NCB & I \end{bmatrix}^T \begin{bmatrix} FC & H & 0 \end{bmatrix} \quad (21)$$

$$\Pi_3 = \begin{bmatrix} NCA & NCB & I \end{bmatrix}^T \begin{bmatrix} FCA & FCB & H \end{bmatrix} \quad (22)$$

\square

Remark: From the LMI (18), it is obvious that the assumption A is Hurwitz must hold.

Remark: The LMI dimension reduction is obvious by comparing the LMI (18) established by (14) - (16) with the LMI (72) established by (10) - (12) (or equivalently (5) - (9)): the LMI (72) has the dimension corresponding to the vector $\varphi = [x^T, u^T, \dot{u}^T, \lambda^T, \dot{\lambda}^T]^T$; while the LMI (18) has the dimension corresponding to $\phi = [x^T, u^T, \dot{u}^T]^T$.

In the following subsection, the three conditions in Lemma 1 and the stability criterion in Theorem 1 will be used for the symmetric constraints.

2.3 Symmetric constraints

In this subsection we consider particularly the case when the QP is subject to symmetric constraints with guaranteed feasibility. The results in this subsection can be extended to the discrete time case. We show that the extra conditions introduced by the symmetry (in Primbs' method) do not take any effect in reducing the conservatism when establishing stability by the S-procedure. Hence all the constraints derived from (23) by following Primbs' method can be reduced to the three properties (14)- (16), and stability can be established as Theorem 1.

We consider the following constraint

$$\begin{bmatrix} L \\ -L \end{bmatrix} \tilde{u} + \begin{bmatrix} L \\ -L \end{bmatrix} Ny \preceq \begin{bmatrix} b \\ b \end{bmatrix} \quad (23)$$

where $L \in \mathbb{R}^{l \times m}$ with $l \geq m$, $N \in \mathbb{R}^{m \times m}$ and $b \succeq 0$ with $b \in \mathbb{R}^l$.

Remark: The relation $u(t) = -Ny(t)$ is always feasible for the QP with objective function (3) and subject to (23).

The following Lemma shows that to achieve global stability, the QP must have the constraint form as (23).

Lemma 2 Consider the QP with objective function (3) and subject to the general symmetric constraint

$$\begin{bmatrix} L \\ -L \end{bmatrix} \tilde{u} + \begin{bmatrix} M \\ -M \end{bmatrix} y \preceq \begin{bmatrix} b \\ b \end{bmatrix} \quad (24)$$

where $u, y \in \mathbb{R}^m$, $L \in \mathbb{R}^{l \times m}$ with $l \geq m$, $M \in \mathbb{R}^{l \times m}$ and $b \succeq 0$ with $b \in \mathbb{R}^l$. Then the only constraint form for the QP having guaranteed feasibility for all $y \in \mathbb{R}^m$ is (23). \square

Remark: Suppose the objective function (3) is expressed as $J(u, y)$, then it is obvious that $J(u, y) = J(-u, -y)$. Furthermore, if (u, y) is feasible, then $(-u, -y)$ must also be feasible. Therefore the function $u = \phi(y)$ is odd.

Proof: We can write the constraint (24) as

$$\begin{bmatrix} L \\ -L \end{bmatrix} \tilde{u} + \begin{bmatrix} L \\ -L \end{bmatrix} N_1 y + \begin{bmatrix} (L_\perp)^T \\ -(L_\perp)^T \end{bmatrix} N_2 y \preceq \begin{bmatrix} b \\ b \end{bmatrix} \quad (25)$$

where $N_1 \in \mathbb{R}^{m \times m}$ and $N_2 \in \mathbb{R}^{(l-m) \times m}$, then the function $u = \phi(y)$ is not feasible $\forall y \in \mathbb{R}^m$ if $(L_\perp)^T N_2 \neq 0$ ¹. This can be shown as follows: by row transformations we can make sure that at least one column of $(L_\perp)^T$, denoted as $(L_\perp)^T(:, j)$, has the property that $(L_\perp)^T(i, j) \geq 0$ and not all $(L_\perp)^T(i, j) = 0$ for $i = 1, \dots, l$. Premultiplying (25) by $\begin{bmatrix} L_\perp(j, :) & \mathbf{0}_{1 \times l} \end{bmatrix}$, we have

$$L_\perp(j, :)(L_\perp)^T N_2 y \leq L_\perp(j, :)b \quad (26)$$

Suppose $L_\perp(j, k) \neq 0$ with $1 \leq k \leq l$ and there is at least one entry at the k th row of $(L_\perp)^T N_2$ not equal to zero (otherwise the corresponding constraint would degenerate to (23)). Then (26) defines a half hyperplane, which can not guarantee $\forall y \in \mathbb{R}^m$ satisfy the constraint (25). \square

¹Here " \neq " denotes not all entries are equal to zeros.

Proposition 2 (reduction for symmetric constraints): Consider the QP with objective function (3) subject to (23).

1) The following constraints can be derived from KKT conditions (with the usual caveats about the existence of \dot{u} , $\dot{\lambda}^+$ and $\dot{\lambda}^-$):

$$Hu + Fy + L^T \lambda^+ - L^T \lambda^- = 0 \quad (27)$$

$$H\dot{u} + F\dot{y} + L^T \dot{\lambda}^+ - L^T \dot{\lambda}^- = 0 \quad (28)$$

$$(\lambda^+)^T Lu + (\lambda^+)^T LNy \geq 0 \quad (29)$$

$$-(\lambda^-)^T Lu - (\lambda^-)^T LNy \geq 0 \quad (30)$$

$$(\lambda^+)^T L\dot{u} + (\lambda^+)^T LN\dot{y} = 0 \quad (31)$$

$$-(\lambda^-)^T L\dot{u} - (\lambda^-)^T LN\dot{y} = 0 \quad (32)$$

$$(\dot{\lambda}^+)^T L\dot{u} + (\dot{\lambda}^+)^T LN\dot{y} = 0 \quad (33)$$

$$-(\dot{\lambda}^-)^T L\dot{u} - (\dot{\lambda}^-)^T LN\dot{y} = 0 \quad (34)$$

$$(\lambda^+)^T \lambda^- = 0 \quad (35)$$

$$(\dot{\lambda}^+)^T \dot{\lambda}^- = 0 \quad (36)$$

with λ^+ and λ^- corresponding to the constraints $Lu + LNy \preceq b$ and $-Lu - LNy \preceq b$ respectively.

2) Suppose that the Lyapunov function is $V(x, u) = [x^T, u^T]P[x^T, u^T]^T$. Then when establishing stability by the S-procedure, the LMI derived from the ten constraints (27)-(36) and the LMI derived from the three constraints (14)-(16) under the condition that $\text{rank}(L) = m$ are equivalent.

Proof: The proof of 1) is a straightforward generalization of that in Primbs and Giannelli [13, 20]; the proof of 2) is similar to that of Proposition 1; see Li [24] for details. \square

3 Discrete time case

In this section, we consider the discrete time case. The stability criteria and the reduction results can be derived in a similar manner as those proposed in the continuous time case. We only present the results corresponding to $Lu_k + LNy_k \preceq b$ for conciseness, since these results can be extended for the symmetric constraint case easily.

3.1 Problem setup

Given a discrete time multivariable system

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{aligned} \quad (37)$$

with $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$ and $y_k \in \mathbb{R}^m$. Suppose this system has a feedback connection with a nonlinearity or controller expressed by a discrete quadratic program, which is represented as

$$u_k = \phi(y_k) = \arg \min_{\tilde{u}} \frac{1}{2} \tilde{u}^T H \tilde{u} + \tilde{u}^T F y_k$$

$$\text{subject to } L \tilde{u} + L N y_k \preceq b \quad (38)$$

with Hessian matrix $H = H^T > 0$, the constant vector $b \succeq 0$. The dimensions for the fixed terms are $H \in \mathbb{R}^{m \times m}$, $F \in \mathbb{R}^{m \times m}$, $L \in \mathbb{R}^{l \times m}$ with $l \geq m$, $N \in \mathbb{R}^{m \times m}$ and $b \in \mathbb{R}^l$.

3.2 Main results

Consider a discrete time system (37) in feedback with a nonlinearity expressed as a QP (38). We derive the three constraints directly from KKT conditions in Lemma 3.

Lemma 3 (QP properties — discrete time case):

The QP (38) satisfies the following properties

$$(u_k + N y_k)^T (H u_k + F y_k) \leq 0 \quad (39)$$

$$(\Delta u_{k+1} + N \Delta y_{k+1})^T (H u_k + F y_k) \geq 0 \quad (40)$$

$$(\Delta u_{k+1} + N \Delta y_{k+1})^T (H u_{k+1} + F y_{k+1}) \leq 0 \quad (41)$$

with $\Delta u_{k+1} = u_{k+1} - u_k$ and $\Delta y_{k+1} = y_{k+1} - y_k$.

Proof: See Appendix A. □

Since the derivations of the reduction result and stability criterion for discrete time case are similar to those of continuous time case, they are given below without proofs (see Li [24] for details).

Theorem 2 (Stability criterion — discrete time case):

The system is stable if there is a symmetric positive definite matrix P with the same structure as (17), such that the LMI (42) is satisfied:

$$\Pi_0 + \text{He} \left(\sum_{i=1}^3 r_i \Pi_i \right) \leq 0 \quad (42)$$

with $r_1 \geq 0$, $r_2 \geq 0$ and $r_3 \geq 0$. Here Π_i with $i = 0, \dots, 3$ are

$$\Pi_0 = \begin{bmatrix} A & B & 0 \\ 0 & I & I \end{bmatrix}^T P \begin{bmatrix} A & B & 0 \\ 0 & I & I \end{bmatrix} - \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \quad (43)$$

$$\Pi_1 = - \begin{bmatrix} N C & I & 0 \end{bmatrix}^T \begin{bmatrix} F C & H & 0 \end{bmatrix} \quad (44)$$

$$\Pi_2 = \begin{bmatrix} N(CA - C) & N C B & I \end{bmatrix}^T \begin{bmatrix} F C & H & 0 \end{bmatrix} \quad (45)$$

$$\Pi_3 = - \begin{bmatrix} N(CA - C) & N C B & I \end{bmatrix}^T \times \begin{bmatrix} F C A & F C B + H & H \end{bmatrix} \quad (46)$$

□

Note that the LMI (42) corresponds to the vector $\varphi_k = [x_k^T, u_k^T, \Delta u_{k+1}^T]$.

We also have the reduction result:

Proposition 3: Consider the QP (38). We have

1) The following constraints can be derived from KKT conditions:

$$Hu_k + Fy_k + L^T \lambda_k = 0 \quad (47)$$

$$Hu_{k+1} + Fy_{k+1} + L^T \lambda_{k+1} = 0 \quad (48)$$

$$\lambda_k^T L u_k + \lambda_k^T L N y_k \geq 0 \quad (49)$$

$$\lambda_k^T L \Delta u_{k+1} + \lambda_k^T L N \Delta y_{k+1} \leq 0 \quad (50)$$

$$\lambda_{k+1}^T L \Delta u_{k+1} + \lambda_{k+1}^T L N \Delta y_{k+1} \geq 0 \quad (51)$$

2) Suppose that the candidate Lyapunov function is $V(x_k, u_k) = [x_k^T, u_k^T]^T P [x_k^T, u_k^T]$. Then when establishing stability by the S-procedure, the LMI derived from the five constraints (47)-(51) and the LMI from the three constraints (39)-(41) under the condition that $\text{rank}(L) = m$ are equivalent.

Remark: For the symmetric constraint corresponding to (23) in discrete time case, ten conditions corresponding to (27)-(36) can be derived and further reduced to the same stability criterion in Theorem 2 established by the three conditions (39)-(41) in a similar way as the continuous case.

4 Examples

4.1 Examples of Symmetric Constraints

The benefits of the reduction for symmetric constraints are best illustrated by two examples.

Example 1-Saturation nonlinearities:

Primbs and Giannelli [13] consider the stability analysis of a strictly proper SISO system interconnected with a saturation nonlinearity (see Fig. 2):

$$u = \begin{cases} -1 & \text{for } y < -1 \\ y & \text{for } -1 \leq y \leq 1 \\ 1 & \text{for } y > 1 \end{cases} \quad (52)$$

which can be expressed by an optimization problem as

$$\begin{aligned} u &= \arg \min_{\tilde{u}} \frac{1}{2} (\tilde{u} - y)^2 \\ \text{s.t. } & |\tilde{u}| \leq 1 \end{aligned}$$

It is straightforward to see that this saturation function falls into the QP with objective function (3) subject to (23) when we set $H = 1$, $F = -1$, $L = 1$, $N = 0$ and $b = 1$. With these values, the conditions (27)-(36) correspond to the conditions derived by Primbs and Giannelli [13].

For this particular case, we have the following corollary by applying Proposition 2.

Corollary 1 (reduction for a saturation): Given a SISO system interconnected with a saturation function (52). Suppose that the candidate Lyapunov function is $V(x, u) = [x^T, u]P[x^T, u]^T$. Then when using the S-procedure to establish stability, the LMI derived from the ten conditions by Primbs [13] is equivalent to the LMI from the following three conditions:

$$\begin{aligned} u(u - y) &\leq 0 \\ \dot{u}(u - y) &= 0 \text{ where } \dot{u} \text{ exists} \\ \dot{u}(\dot{u} - \dot{y}) &= 0 \text{ where } \dot{u} \text{ exists} \end{aligned} \tag{53}$$

□

Example 2-Deadzone nonlinearities:

Primbs and Giannelli [20] also consider a system connected with a deadzone (see Fig. 3)

$$u = \begin{cases} y + 1 & \text{for } y < -1 \\ 0 & \text{for } -1 \leq y \leq 1 \\ y - 1 & \text{for } y > 1 \end{cases} \tag{54}$$

This may be expressed as

$$\begin{aligned} u &= \arg \min_{\tilde{u}} \frac{1}{2} \tilde{u}^2 \\ &\text{subject to } |\tilde{u} - y| \leq 1 \end{aligned}$$

It is straightforward to see that this deadzone function falls into the general case when we set $H = 1$, $F = 0$, $L = 1$, $N = -1$ and $b = 1$ in (3). The conditions (27)-(36) correspond to the conditions derived by Primbs and Giannelli [20]. In parallel with Corollary 1, we have the following corollary by applying Proposition 2.

Corollary 2 (reduction for a deadzone): Given a SISO system interconnected with a deadzone function (54). Suppose that the candidate Lyapunov function is $V(x, u) = [x^T, u]P[x^T, u]^T$. Then when using the S-procedure to establish stability, the LMI derived from the ten conditions by Primbs [20] is equivalent to the LMI from the following three conditions:

$$\begin{aligned} u(u - y) &\leq 0 \\ u(\dot{u} - \dot{y}) &= 0 \text{ where } \dot{u} \text{ exists} \\ \dot{u}(\dot{u} - \dot{y}) &= 0 \text{ where } \dot{u} \text{ exists} \end{aligned} \tag{55}$$

□

Remark: Using (53) or (55) to establish the stability criterion requires an LMI with the same dimension as the vector $[x^T, u, \dot{u}]^T$ and with three multipliers; using the conditions proposed by Primbs requires an LMI with the same dimension as the vector $\varphi = [x^T, u, \dot{u}, \lambda^+, \lambda^-, \dot{\lambda}^+, \dot{\lambda}^-]^T$ and with ten multipliers.

4.2 MPC example

Since MPC controller can be expressed by a QP [25], a branch of studying stability of MPC is conducted by investigating the QP properties, which always leads to direct approaches to stability analysis (see, e.g., [15,16,22,26] by Heath et al.) and synthesis (see, e.g., [14] by Bemporad et al.). Primbs' approach for its MPC extension [12] also fall into this branch. The approach developed in this paper greatly extends the implementability of Primbs' by reducing redundant constraints used in Primbs' approach.

For a comparison, we consider the MPC example used by Primbs [12] with 2-norm bounded unstructured uncertainty. The extension to the structured uncertainty is obvious and the extension to two-stage MPC can be found in [24].

4.2.1 Extension to MPC

The plant with the unstructured uncertainty is expressed as

$$\begin{aligned}x_{k+1} &= Ax_k + B_u u_k + B_w w_k \\p_k &= Cx_k + D_u u_k + D_w w_k \\w_k &= \Delta_k p_k\end{aligned}$$

with $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$, $p_k \in \mathbb{R}^{n_p}$ and $w_k \in \mathbb{R}^{n_w}$. Here the memoryless operator Δ_k satisfies

$$\|\Delta_k\|_2 \leq 1 \tag{56}$$

so that $w_k^T w_k \leq p_k^T p_k$ holds.

Suppose the MPC controller is

$$\begin{aligned}J_k &= x_{k+N_p}^T P x_{k+N_p} + \sum_{i=0}^{N_p-1} \left[x_{k+i}^T Q x_{k+i} + u_{k+i}^T R u_{k+i} \right] \\&\text{subject to } L_k u_k \preceq b_k \text{ with } b_k \succeq 0\end{aligned} \tag{57}$$

with $L_k \in \mathbb{R}^{n_c \times n_u}$ and $b_k \in \mathbb{R}^{n_c}$. In a standard way, the MPC controller (57) can be converted to a QP, so that the results proposed in this paper can be applied. We consider two cases: 1) the candidate Lyapunov function is $V_k = x_k^T P x_k$, and the sector bound constraint (39) together with the disturbance constraint (56) are used to establish stability; 2) the candidate Lyapunov function is $V_k = [x_k^T, U_k^T] P [x_k^T, U_k^T]^T$ with $U_k = [u_k^T, u_{k+1}^T, \dots, u_{k+N_p-1}^T]$, and the three constraints (39)-(41) together with the disturbance constraint (56) are used to establish stability. We mainly focus on case 2) for the discussion of LMI dimension reduction.

In the second case, the LMI stability criterion corresponds to the vector $[x_k^T, U_k^T, u_{k+N_p}^T, w_k^T]^T$, while the LMI of Primbs corresponds to the vector $[x_k^T, U_k^T, \lambda_k^T, w_k^T, U_{k+1}^T, \lambda_{k+1}^T, w_{k+1}^T, 1]^T$, where $\lambda_k, \lambda_{k+1} \in \mathbb{R}^{N_p n_c}$ are Lagrangian Multipliers associated with the input constraints for the whole prediction horizon. The dimensions of the resulting LMIs from our approach and Primbs' approach are $n_x + (N_p + 1)n_u + n_w$ and $n_x + (2N_p + 1)n_u + 2n_w + 2N_p n_c + 1$ respectively.

4.2.2 Primbs' example

The plant's state space matrices are

$$\begin{aligned}
 A &= \begin{bmatrix} 4/3 & -2/3 \\ 1 & 0 \end{bmatrix} & B_u &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} & B_w &= \begin{bmatrix} \theta \\ 0 \end{bmatrix} \\
 C &= \begin{bmatrix} 1 & 0 \end{bmatrix} & D_u &= 0 & D_w &= 0
 \end{aligned}$$

where θ is a fixed value for the size of uncertainty. This system is subject to $|u| \leq 1$. The MPC controller is (57) with the horizon $N_p = 3$ and the parameters Q and R as

$$Q = \begin{bmatrix} 1 & -2/3 \\ -2/3 & 3/2 \end{bmatrix} \qquad R = 1$$

In case 1), the sufficient condition of θ for the system to be stable is $0 \leq \theta \leq 0.03$. In case 2), the range of θ for the system to be stable is $0 \leq \theta \leq 0.19$, which is the same result achieved by Primbs [12]; the dimensions of the two LMIs derived from our approach and Primbs' approach are 7 and 24 respectively, with $n_x = 2$, $n_u = 1$, $n_w = 1$, $n_c = 2$. In Fig. 4, a comparison for case 2) is made between the dimensions (denoted by square markers) of the LMIs derived from the Primbs' approach and our new approach at various lengths of prediction horizon. This figure shows that as the length of prediction horizon grows, the dimension difference between the two approaches is enlarged. When the prediction horizon is 30, the LMI dimension from Primbs' approach is about 5.5 times of the one from the new approach.

From this numerical example we can see that our result is no worse than Primbs', but our reduction is much easier to implement and the LMI criterion has a much lower dimension compared with that of Primbs. The benefits of such a reduction become especially important when a long prediction horizon is required.

5 Conclusion

We have considered Primbs' method for assessing the stability of a closed-loop system with a static nonlinearity that may be expressed as the solution of a class of QP. This includes simple nonlinearities, such as saturation functions and deadzone functions; the method can also be extended to MPC applications. We proposed three conditions that lead to a concise and parsimonious application of the S-procedure. For continuous time systems we have shown analytically that the results are no worse than those of Primbs for a fairly broad class of nonlinearities, and considered saturation and deadzone nonlinearities as examples. All the results have their discrete time counterparts and we have demonstrated the result using a numerical example.

Appendix – Proofs of Results

Proof of Proposition 1:

Proof of 1): The KKT conditions [27] for the QP are

$$Hu + Fy + L^T \lambda = 0 \quad (58)$$

$$Lu + My + s = b \quad (59)$$

$$\lambda^T s = 0 \quad (60)$$

$$\lambda \succeq 0 \quad (61)$$

$$s \succeq 0 \quad (62)$$

Note that (5) and (6) are just (58) and its derivative. From (60) - (62), it can be shown that $\lambda^T \dot{s} = 0$ and $\dot{\lambda}^T s = 0$ also hold (see [20,24] for details). Then premultiplying (59) and its derivative by λ^T respectively leads to (7) and (8); premultiplying the derivative of (59) by $\dot{\lambda}^T$ leads to (9).

Proof of 2.1): Premultiplying (58) by u^T and using (7) yields

$$u^T(Hu + Fy) - y^T M^T \lambda \leq 0 \quad (63)$$

Premultiplying (58) by $y^T M^T (L^\dagger)^T$ yields

$$y^T M^T (L^\dagger)^T (Hu + Fy) + y^T M^T (L^\dagger)^T L^T \lambda = 0 \quad (64)$$

Then the sum of (64) and (63) yields (10). (11) and (12) can be derived in a similar way.

Proof of 2.2): Introducing a vector $\varphi = [x^T, u^T, \dot{u}^T, \lambda^T, \dot{\lambda}^T]^T$, the linear equality constraints from (5) and (6) can be expressed as

$$E\varphi = 0 \quad (65)$$

with E as (66)

$$E = \begin{bmatrix} FC & H & 0 & L^T & 0 \\ FCA & FCB & H & 0 & L^T \end{bmatrix} \quad (66)$$

The candidate Lyapunov function is chosen as $V(x) = [x^T, u^T]P[x^T, u^T]^T$, and $\dot{V} \leq 0$ takes a quadratic form as

$$\varphi^T \tilde{\Pi}_0 \varphi \leq 0 \quad (67)$$

with

$$\tilde{\Pi}_0 = \text{He} \left\{ \begin{bmatrix} I_n & 0 & 0_{n \times (m+2l)} \\ 0 & I_m & 0_{m \times (m+2l)} \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{21}^T \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} A & B & 0_{n \times m} & 0_{n \times 2l} \\ 0 & 0 & I_m & 0_{m \times 2l} \end{bmatrix} \right\}$$

Here $0_{r \times c}$ denotes $r \times c$ zero matrix; 0_r and I_r denote $r \times r$ zero matrix and $r \times r$ identity matrix respectively.

The quadratic constraints (7), (8) and (9) can be expressed in quadratic forms as

$$\varphi^T \tilde{\Pi}_1 \varphi \geq 0 \quad \varphi^T \tilde{\Pi}_2 \varphi = 0 \quad \varphi^T \tilde{\Pi}_3 \varphi = 0 \quad (68)$$

with

$$\begin{aligned}\tilde{\Pi}_1 &= \text{He} \left\{ \begin{bmatrix} 0_{l \times (n+2m)} & I_l & 0_l \end{bmatrix}^T \begin{bmatrix} MC & L & 0_{l \times (m+2l)} \end{bmatrix} \right\} \\ \tilde{\Pi}_2 &= \text{He} \left\{ \begin{bmatrix} 0_{l \times (n+2m)} & I_l & 0_l \end{bmatrix}^T \begin{bmatrix} MCA & MCB & L & 0_{l \times (2l)} \end{bmatrix} \right\} \\ \tilde{\Pi}_3 &= \text{He} \left\{ \begin{bmatrix} 0_{l \times (n+2m+l)} & I_l \end{bmatrix}^T \begin{bmatrix} MCA & MCB & L & 0_{l \times (2l)} \end{bmatrix} \right\}\end{aligned}$$

The system is stable if we can show that under the constraints (65) and (68), the inequality (67) holds. Using the S-procedure, this relation can be expressed in one LMI as

$$E_{\perp}^T \Omega E_{\perp} \leq 0 \quad (69)$$

with E_{\perp}^T in row echelon form as

$$E_{\perp}^T = \begin{bmatrix} I & 0 & 0 & -(FC)^T L^{\dagger} & -(FCA)^T L^{\dagger} \\ 0 & I & 0 & -HL^{\dagger} & -(FCB)^T L^{\dagger} \\ 0 & 0 & I & 0 & -HL^{\dagger} \\ 0 & 0 & 0 & I - LL^{\dagger} & 0 \\ 0 & 0 & 0 & 0 & I - LL^{\dagger} \end{bmatrix} \quad (70)$$

such that $EE_{\perp} = 0$ (Note that $L^T(I - LL^{\dagger})^T = 0$ and $\text{rank}(I - LL^{\dagger}) = l - m.$), and

$$\Omega = \tilde{\Pi}_0 + \sum_{i=1}^3 r_i \tilde{\Pi}_i = \begin{bmatrix} \Omega_{11} & \Omega_{21}^T \\ \Omega_{21} & 0 \end{bmatrix} \quad (71)$$

where $\Omega_{11} = \text{He}(\Pi_0)$ with Π_0 as (19), and

$$\Omega_{21} = \begin{bmatrix} r_1 MC + r_2 MCA & r_1 L + r_2 MCB & r_2 L \\ r_3 MCA & r_3 MCB & r_3 L \end{bmatrix}$$

where $r_1 \geq 0$, $r_2 \in \mathbb{R}$ and $r_3 \in \mathbb{R}$ are the multipliers corresponding to the constraints (68) respectively.

After multiplication the LMI (69) can be expressed as

$$E_{\perp}^T \Omega E_{\perp} = \bar{\Pi}_0 + \text{He} \left(\sum_{i=1}^3 r_i \bar{\Pi}_i \right) \quad (72)$$

with $\bar{\Pi}_0 = \tilde{\Pi}_0$ and

$$\begin{aligned}\bar{\Pi}_1 &= - \begin{bmatrix} L^{\dagger} MC & I_m & 0_{m \times 3m} \end{bmatrix}^T \begin{bmatrix} FC & H & 0_{m \times 3m} \end{bmatrix} \\ &\quad + \begin{bmatrix} (I - LL^{\dagger}) MC & 0_{m \times 4m} \end{bmatrix}^T \begin{bmatrix} 0_{m \times (n+2m)} & I_m & 0_m \end{bmatrix} \\ \bar{\Pi}_2 &= \begin{bmatrix} L^{\dagger} MCA & L^{\dagger} MCB & I_m & 0_{m \times 2m} \end{bmatrix}^T \begin{bmatrix} FC & H & 0_{m \times 3m} \end{bmatrix} \\ &\quad + \begin{bmatrix} (I - LL^{\dagger}) MCA & (I - LL^{\dagger}) MCB & 0_{m \times 3m} \end{bmatrix}^T \begin{bmatrix} m \times n + 2m & I_m & 0_m \end{bmatrix} \\ \bar{\Pi}_3 &= \begin{bmatrix} L^{\dagger} MCA & L^{\dagger} MCB & I_m & 0_{m \times 2m} \end{bmatrix}^T \times \begin{bmatrix} FCA & FCB & H & 0_{m \times 2m} \end{bmatrix} \\ &\quad + \begin{bmatrix} (I - LL^{\dagger}) MCA & (I - LL^{\dagger}) MCB & 0_{m \times 3m} \end{bmatrix}^T \begin{bmatrix} 0_{m \times (n+3m)} & I_m \end{bmatrix}\end{aligned}$$

It is straightforward to show that the same LMI with (72) can be derived from the three constraints (10), (11) and (12) directly using the S-procedure. \square

Proof of Lemma 3:

The KKT conditions [27] for the discrete QP at time instant k are

$$Hu_k + Fy_k + L^T \lambda_k = 0 \quad (73)$$

$$Lu_k + LNy_k + s_k = b \quad (74)$$

$$\lambda_k^T s_k = 0 \quad (75)$$

$$\lambda_k \succeq 0 \quad (76)$$

$$s_k \succeq 0 \quad (77)$$

The KKT conditions at $k + 1$ follow in a similar rule. The first inequality (39) follows immediately from (14).

Premultiplying (73) at time k by Δu_{k+1}^T yields

$$\Delta u_{k+1}^T (Hu_k + y_k) + \Delta u_{k+1}^T L^T \lambda_k = 0$$

This yields

$$\begin{aligned} & \Delta u_{k+1}^T (Hu_k + y_k) \\ &= -\Delta u_{k+1}^T L^T \lambda_k \\ &= (y_{k+1} - y_k)^T N^T L^T \lambda_k + (s_{k+1} - s_k)^T \lambda_k \\ & \quad \text{from (74) at times } k \text{ and } k + 1 \\ &\geq \Delta y_{k+1}^T N^T L^T \lambda_k \\ & \quad \text{from (76) at time } k, \text{ (77) at time } k + 1 \text{ and (75) at time } k \\ &= -\Delta y_{k+1}^T N^T (Hu_k + Fy_k) \text{ from (73) at time } k \end{aligned}$$

Hence (40).

Similarly premultiplying (73) at time $k + 1$ by Δu_{k+1}^T yields

$$\Delta u_{k+1}^T (Hu_{k+1} + y_{k+1}) + \Delta u_{k+1}^T L^T \lambda_{k+1} = 0$$

This yields

$$\begin{aligned} & \Delta u_{k+1}^T (Hu_{k+1} + y_{k+1}) \\ &= -\Delta u_{k+1}^T L^T \lambda_{k+1} \\ &= (y_{k+1} - y_k)^T N^T L^T \lambda_{k+1} + (s_{k+1} - s_k)^T \lambda_{k+1} \\ &\leq -\Delta y_{k+1}^T N^T (Hu_{k+1} + Fy_{k+1}) \end{aligned}$$

Hence (41). \square

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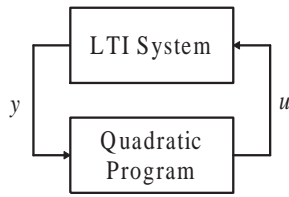


Figure 1: The system connected with a nonlinearity expressed as a QP

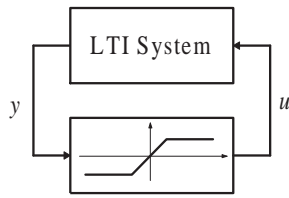


Figure 2: The system connected with a saturation function

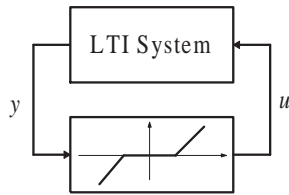


Figure 3: The system connected with a deadzone function

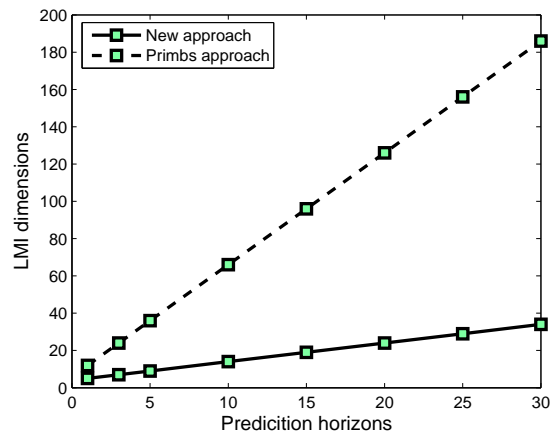


Figure 4: A comparison of the dimensions of the LMIs derived respectively from the new and Primbs' approach at different prediction horizons.