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An improved stability criterion for a class of Lur'e systems

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***Abstract*— We consider the stability of the feedback connection of a linear time invariant (LTI) plant with a static nonlinearity expressed by a certain class of quadratic program. By generalizing the class of candidate Lyapunov functions we improve on existing results in the literature. A Lyapunov function is constructed via the S-procedure from quadratic constraints established using the Karush-Kuhn-Tucker (KKT) conditions. The stability criterion can be expressed as a linear matrix inequality (LMI) condition. We discuss some simple examples that demonstrate the improved results.**

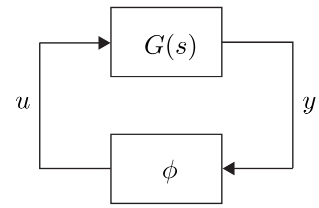

Fig. 1. A Lur'e system

I. Introduction

The stability analysis of Lur'e systems (see Fig. 1), namely an LTI plant connected with a memoryless nonlinearity in the feedback path satisfying a sector bound condition, is a classical problem in control theory [8]. There are many stability tests for this problem based on the circle criterion, the Popov criterion and the use of Zames-Falb multipliers, etc.. Recently a new LMI stability test was proposed for a subclass of Lur'e systems where the nonlinearity can be expressed by a quadratic program (QP) [14], [16]. This is a nontrivial subclass, which includes feedback systems that contain common nonlinearities, such as saturation, deadzone and combinations of saturation and deadzone. Furthermore, input constrained model predictive control (MPC) can also be cast into this subclass, which enables a simple test for the stability analysis of input constrained MPC. We will refer to the test as *Primbs' method* throughout this paper.

Primbs' method uses the S-procedure to represent the implication that a candidate Lyapunov function is decreasing subject to some constraints derived from the KKT conditions for the QP. When the Lyapunov function is quadratic in the state x and the nonlinearity corresponds to a saturation or a deadzone the test is equivalent to the circle criterion. More generally, [14] and [16] propose a Lyapunov function that is quadratic in both the state and input $[x^T, u^T]^T$. This subsumes the circle criterion for these cases and can also be shown to give a considerably less conservative stability criterion than the Popov criterion for some simple cases.

It is also claimed [16] that it may give a less conservative stability criterion than the use of Zames-Falb multipliers. The veracity of this claim is beyond the scope of this paper. Nevertheless we give a simple example where the Popov criterion is less conservative than Primb's method. That is to say we show that while for some examples Primbs' method gives a less conservative test, for other examples the Popov criterion gives a less conservative test. Indeed we show that for some nonlinearities Primbs' method may give worse results than the circle criterion.

The main contribution of this paper is to propose a new stability criterion that is based on Primbs' method but which further reduces the conservatism of the approach. The improvement is achieved by constructing a novel candidate Lyapunov function, that involves not only a quadratic term in $[x^T, u^T]^T$, but also an integral term with the nonlinearity. This construction is inspired by the Lyapunov function associated with the Popov criterion which is a combination of a quadratic term in state x and an integral term involving the nonlinearity [7], [12], [1]. Furthermore, to achieve a stability criterion in a concise LMI form, we use the three conditions developed in [10], [9] which were shown to be equivalent to a large number of conditions originally derived for Primbs' method [14], [16]. Thus we develop concise stability conditions which can never be worse than Primbs' method. We demonstrate numerical examples where the new test is less conservative than both Primbs' method and the Popov criterion.

II. Continuous time case

A. Problem setup

Consider a strictly proper stable continuous time MIMO plant $G(s)$ with equal number of inputs and outputs, which has a negative feedback connection with a nonlinearity $\phi(y) : \mathbb{R}^m \to \mathbb{R}^m$ expressed by a QP

$$\phi(y(t)) = \arg\min_{\tilde{u}} \frac{1}{2}\tilde{u}^T H \tilde{u} + \tilde{u}^T F y(t) \quad \text{subject to } Lu(t) + LNy(t) \preceq b \tag{1}$$

where $H = H^T > 0$ and each elements of the vector b is nonnegative, which is represented as $b \succeq 0$. Here $H, F, N \in \mathbb{R}^{m\times m}$, $L \in \mathbb{R}^{l\times m}$, $b \in \mathbb{R}^l$. We assume $\phi(0) = 0$.

From the KKT conditions for this QP, three properties can be derived:

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Lemma 1 (QP properties — continuous time case [9]): The constrained QP (1) has the following properties

$$(\phi + Ny)^T(H\phi + Fy) \le 0 \quad (2)$$
$$(\dot{\phi} + N\dot{y})^T(H\phi + Fy) = 0 \text{ where } \dot{u} \text{ exists} \quad (3)$$
$$(\dot{\phi} + N\dot{y})^T(H\dot{\phi} + F\dot{y}) = 0 \text{ where } \dot{u} \text{ exists} \quad (4)$$

Here we use ϕ and $\dot{\phi}$ to represent $\phi(y(t))$ and $d\phi(y(t))/dt$ respectively for conciseness.
Proof: See [9]. □

B. New stability criterion

Based on the three conditions (2)-(4) and a new Lur'e type Lyapunov function, a stability criterion can be established as follows.

Let G have state space representation $G(s) \sim (A, B, C, 0)$. Define Π_i for $i = 1, 2, 3, p$ as

$$\Pi_i = \begin{bmatrix} \Pi_i^{(11)} & {\Pi_i^{(21)}}^T & {\Pi_i^{(31)}}^T \\ \Pi_i^{(21)} & \Pi_i^{(22)} & {\Pi_i^{(32)}}^T \\ \Pi_i^{(31)} & \Pi_i^{(32)} & \Pi_i^{(33)} \end{bmatrix} \quad (5)$$

where for Π_1

$$\Pi_1^{(11)} = -(C^T F^T NC + C^T N^T FC)$$
$$\Pi_1^{(21)} = -(FC + HNC)$$
$$\Pi_1^{(22)} = -2H,\ \Pi_1^{(31)} = 0,\ \Pi_1^{(32)} = 0,\ \Pi_1^{(33)} = 0$$

for Π_2

$$\Pi_2^{(11)} = -(C^T F^T NCA + A^T C^T N^T FC)$$
$$\Pi_2^{(21)} = -(HNCA + B^T C^T N^T FC)$$
$$\Pi_2^{(22)} = -(HNCB + B^T C^T N^T H)$$
$$\Pi_2^{(31)} = -FC,\ \Pi_2^{(32)} = -H,\ \Pi_2^{(33)} = 0$$

for Π_3

$$\Pi_3^{(11)} = -A^T C^T (F^T N + N^T F)CA$$
$$\Pi_3^{(21)} = -B^T C^T (F^T N + N^T F)CA$$
$$\Pi_3^{(22)} = -B^T C^T (F^T N + N^T F)CB,\ \Pi_3^{(33)} = -2H$$
$$\Pi_3^{(31)} = -(HN + F)CA,\ \Pi_3^{(32)} = -(HN + F)CB$$

and for Π_p

$$\Pi_p^{(11)} = -C^T N^T FCA - A^T C^T F^T NC$$
$$\Pi_p^{(21)} = -FCA - HNCA - B^T C^T F^T NC$$
$$\Pi_p^{(22)} = -FCB - HNCB - B^T C^T F^T - B^T C^T N^T H$$
$$\Pi_p^{(31)} = 0,\ \Pi_p^{(33)} = 0,\ \Pi_p^{(33)} = 0$$

Result 1 The system is stable if there exists a positive definite matrix $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$ with dimension corresponding to $[x^T, u^T]^T$, such that the following LMI can be satisfied

$$\Pi_0 + \lambda \Pi_p + \sum_{i=1}^{3} r_i \Pi_i < 0 \quad (6)$$

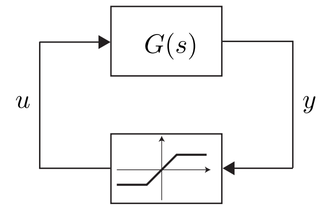

Fig. 2. LTI plant connected in feedback with a saturation function

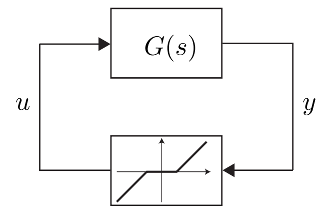

Fig. 3. LTI plant connected in feedback with a deadzone function

where

$$\Pi_0 = \begin{bmatrix} A^T P_{11} + P_{11} A & A^T P_{12} + P_{11} B & P_{12} \\ P_{12}^T A + B^T P_{11} & P_{12}^T B + B^T P_{12} & P_{22} \\ P_{12}^T & P_{22} & 0 \end{bmatrix} \quad (7)$$

Here the multipliers λ, $r_i \in \mathbb{R}$ with $i = 1, 2, 3$, $\lambda \ge 0$ and $r_1 \ge 0$.
Proof: See Appendix. □

Remark: When the sector bound condition (2) takes the simple form $\phi^T(\phi - Ky) \le 0$ with K a scalar or a positive diagonal matrix, the stability criterion (6) subsumes the Popov criterion in LMI form with nonnegative multiplier, which may be expressed as [2]:

$$\begin{bmatrix} A^T P + PA & PB - KC^T - KA^T C^T \lambda \\ B^T P - \lambda KCA - KC & -2K - \lambda CB - B^T C^T \lambda \end{bmatrix} < 0 \quad (8)$$

with $\lambda \ge 0$. This can be seen by letting $P_{12} = 0$, $P_{22} = 0$ and $r_2 = r_3 = 0$. However the multiplier λ in the Popov criterion can also be relaxed to any real number (see [13]).

C. Examples

Saturation nonlinearity (see Fig. 2):

A saturation given by $u(t) = \text{sat}\,(y(t)) = \frac{y(t)}{\max\{1, |y(t)|\}}$ can be expressed by an optimization problem as [16]

$$u(t) = \arg\min_{\tilde{u}} \frac{1}{2} (\tilde{u} - y(t))^2 \quad \text{s.t. } |u(t)| \le 1 \quad (9)$$

It is straightforward to see that this saturation function falls into the QP (1) when we set $H = 1$, $F = -1$, $L = 1$, $N = 0$ and $b = 1$. Its corresponding candidate Lyapunov function is

$$V(x, u) = [x^T, u^T]^T P [x^T, u^T] + 2\lambda \int_0^y \phi^T d\tau \quad (10)$$

Deadzone nonlinearity (see Fig. 3):

The deadzone

$$u = \begin{cases} y + 1 \text{ for } y < -1 \\ 0 \text{ for } -1 \le y \le 1 \\ y - 1 \text{ for } y > 1 \end{cases} \quad (11)$$

can be expressed as [15]

$$u = \arg\min_{\tilde{u}} \frac{1}{2}\tilde{u}^2 \quad \text{subject to } |u - y| \leq 1 \tag{12}$$

It is straightforward to see that this saturation function falls into the QP (1) when we set $H = 1$, $F = 0$, $L = 1$, $N = -1$ and $b = 1$. The corresponding candidate Lyapunov function take the same form as (10).

Combined deadzone and saturation nonlinearity (see Fig. 4):

The combined deadzone and saturation

$$u = \begin{cases} -1 \text{ for } y < -2 \\ y+1 \text{ for } -2 \leq y < -1 \\ 0 \text{ for } -1 \leq y \leq 1 \\ y-1 \text{ for } 1 < y \leq 2 \\ 1 \text{ for } y > 2 \end{cases} \tag{13}$$

can be expressed as

$$\begin{aligned} u &= \begin{bmatrix} 0 & 1 \end{bmatrix} v \\ v &= \arg\min_{\tilde{v}} \frac{1}{2}\tilde{v}^T H \tilde{v} + F\psi \\ &\quad \text{subject to } Lv + LN\psi \preceq b \\ \psi &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} y \end{aligned} \tag{14}$$

where

$$\begin{aligned} H &= \begin{bmatrix} 1+m & -1 \\ -1 & 1 \end{bmatrix} \text{ with any } m > 0 \\ F &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \\ L &= \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{aligned} \tag{15}$$

In this case, we must augment the LTI plant as $\tilde{G}(s) = G(s)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and consider v to be the input of the augmented plant and ψ its output.

Remark: This formulation is not unique. There are many other ways of parameterizing a combined deadzone and saturation in terms of a quadratic program.

Observation 1: It is possible to show analytically that Primbs' method using only Π_1 for a combined saturation and deadzone modelled as the quadratic program (14) with coefficients (15) gives a maximum gain $K = 1/\|G\|_\infty$ for guaranteed stability.
Proof: See Appendix. □

D. Numerical example

To make a comparison of how conservative the different approaches are, we consider a strictly proper stable LTI plant with a nonlinearity in the negative feedback path and an additional scalar gain $K \geq 0$ between the output of the plant

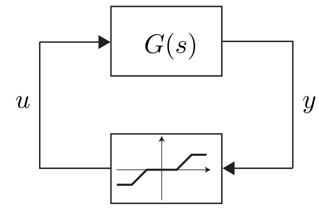

Fig. 4. LTI plant connected in feedback with a combined deadzone and saturation function

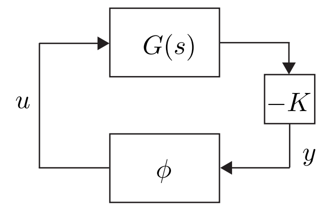

Fig. 5. LTI system connected in feedback with a nonlinearity and an additional gain K.

and the input of the nonlinearity (see Fig. 5). The numerical tests were performed in Matlab with the software packages SEDUMI [18] and YALMIP [11]. The tests were performed with two different plants:

$$G_1(s) = \frac{1}{s^4 + s^3 + 8s^2 + 2s + 1} \tag{16a}$$

$$G_2(s) = \frac{1}{s^4 + s^3 + 8s^2 + 3s + 1} \tag{16b}$$

Table 1 shows maximum values of K for which the various criteria guarantee stability. The tests used were the classical circle and Popov criteria, Primbs' method using only Π_1, Primbs' method using all conditions (i.e. using Π_1 to Π_3), the new method using only Π_1 and Π_p, and the new method using all conditions (i.e. using Π_1 to Π_3 and Π_p). The nonlinearities ϕ were saturation (9), deadzone (12) and the combined deadzone and saturation (14). For each case the larger the value of K, the less conservative the test. We may observe the following:

- Both the saturation (9) and the deadzone (12) belong to the same sector, so the circle criterion gives the same result for both these cases, and similarly the Popov criterion gives the same result. On the other hand Primbs' method gives a different result for the two cases, as discussed in [14], [16].
- For the saturation with G_1, Primbs' method is more conservative than the Popov criterion. For this case the new method (using all conditions) is less conservative than all other tests considered.
- For the saturation with G_2, Primbs' method (using all conditions) is less conservative than the Popov criterion. Again for this case the new method (using all conditions) is less conservative than all other tests considered.
- The new method (using all conditions) is never more conservative than Primbs' method. However for the deadzone it is no better than Primbs' method, while for the combined deadzone and saturation it is no better

TABLE I

A COMPARISON OF THE EXISTING APPROACHES AND THE NEW APPROACH: CONTINUOUS CASE. IN EACH CASE THE MAXIMUM VALUE OF K FOR WHICH THE SYSTEM IS GUARANTEED STABLE IS SHOWN.

ϕ		Circle	Popov	Primbs Π_1	Primbs all	New Π_1, Π_p	New all
(9)	G_1	1.69	4.03	1.69	3.98	4.03	9.63
(9)	G_2	2.94	5.41	2.94	6.93	5.41	11.56
(12)	G_1	1.69	4.03	1.69	10.16	4.03	10.16
(14)	G_1	3.38	8.05	0.64	0.64	0.64	0.64

than Primbs' method based on Π_1 alone.

- For the deadzone and saturation the circle criterion is less conservative than both Primbs' method and the new method. Note that this nonlinearity is in a smaller sector than either the deadzone or the saturation alone. The value for both Primbs' method and the new method is found by optimizing over m in (15). Nevertheless there may be differently structured quadratic programs that give the same nonlinearity but less conservative results that we have not tested.

  Note that $1/\|G_1\|_\infty = 0.64$ (c.f. Observation 1).

III. DISCRETE TIME CASE

A. Problem setup

Similar results can be obtained for discrete time systems. We will consider only a single-input single-output strictly proper stable discrete time plant $G(z)$, which has a negative feedback connection with a nonlinearity $\phi(y) : \mathbb{R} \to \mathbb{R}$ expressed by the QP

$$\phi(y(k)) = \arg\min_u \frac{1}{2}u^T u - u^T K y(k) \quad \text{subject to } Lu(k) \preceq b \tag{17}$$

where $b \succeq 0$. Here K is scalar and $L \in \mathbb{R}^l$, $b \in \mathbb{R}^l$.

From the KKT conditions for this QP, three properties can be derived:

Lemma 3 (QP properties—discrete time case): The constrained QP proposed above has the following properties

$$\phi(k)^T(\phi(k) - Ky(k)) \leq 0 \tag{18}$$

$$\Delta\phi(k+1)^T(\phi(k) - Ky(k)) \geq 0 \tag{19}$$

$$\Delta\phi(k+1)^T(\phi(k+1) - Ky(k+1)) \leq 0 \tag{20}$$

with $\Delta\phi(k+1) = \phi(k+1) - \phi(k)$. Here we use $\phi(k)$ and $\phi(k+1)$ to represent $\phi(y(k))$ and $\phi(y(k+1))$ respectively for conciseness.

Proof: See [10]. □

B. New stability criterion

We write $G(z) \sim (A, B, C, 0)$. Define Π_1, Π_2, Π_3 and Π_p as

$$\Pi_1 = \begin{bmatrix} 0 & -KC^T & 0 \\ -KC & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} 0 & 0 & KC^T \\ 0 & 0 & 1 \\ KC & 1 & 0 \end{bmatrix}$$

$$\Pi_3 = \begin{bmatrix} 0 & 0 & -K(CA)^T \\ 0 & 0 & -(KCB+1)^T \\ -KCA & -(KCB+1) & -2 \end{bmatrix}$$
$$\Pi_p = \begin{bmatrix} \Pi_p^{(11)} & {\Pi_p^{(21)}}^T & 0 \\ \Pi_p^{(21)} & \Pi_p^{(22)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{21}$$

with

$$\Pi_p^{(11)} = K^2(A - I_n)^T C^T C(A - I_n)$$
$$\Pi_p^{(21)} = (KB^T C^T - 1)KC(A - I_n) \tag{22}$$
$$\Pi_p^{(22)} = KB^T C^T (KCB - 1) - KCB$$

where the identity matrix $I_n \in \mathbb{R}^{n \times n}$ with n the number of states.

Result 2 (stability criterion — discrete time case): Consider a SISO strictly proper discrete time system $G(z)$ connected with a nonlinearity expressed as a QP which satisfies three conditions (18)-(20). Then the system is stable if there is a symmetric positive definite matrix $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$ such that the following LMI is satisfied:

$$\Pi_0 + \sum_{i=1}^{3} r_i \Pi_i < 0 \tag{23}$$

where

$$\Pi_0 = \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} & \tilde{P}_{13} \\ \tilde{P}_{21} & \tilde{P}_{22} & \tilde{P}_{23} \\ \tilde{P}_{31} & \tilde{P}_{32} & \tilde{P}_{33} \end{bmatrix} \tag{24}$$

with

$$\tilde{P}_{11} = A^T P_{11} A - P_{11}$$
$$\tilde{P}_{21} = \tilde{P}_{12}^T = B^T P_{11} A + P_{12}^T A - P_{12}^T$$
$$\tilde{P}_{22} = B^T P_{11} B + P_{12}^T B + B^T P_{12} \tag{25}$$
$$\tilde{P}_{31} = \tilde{P}_{13}^T = P_{12}^T A$$
$$\tilde{P}_{32} = \tilde{P}_{23}^T = P_{12}^T B + P_{22}, \tilde{P}_{33} = P_{22}$$

Proof: See Appendix. □

Remark: Haddad and Bernstein [3] constructed the Lyapunov function for the discrete time Popov criterion developed by Szegö and Pearson [19] and Jury and Lee [6], [5]. By using the KYP lemma and straightforward manipulations, the corresponding result for SISO case can be transformed into an LMI

$$\begin{bmatrix} A^T PA - P & A^T PB \\ B^T PA & B^T PB \end{bmatrix} + \lambda \begin{bmatrix} \Pi_p^{(11)} & {\Pi_p^{(21)}}^T \\ \Pi_p^{(21)} & \Pi_p^{(22)} \end{bmatrix} < 0 \tag{26}$$

with $\Pi_p^{(11)}$, $\Pi_p^{(21)}$ and $\Pi_p^{(22)}$ given by (22). For this case (23) subsumes the Popov criterion (26) by setting $P_{12} = 0$, $P_{22} = 0$ and $r_2 = r_3 = 0$.

C. Numerical example

As with the continuous case we consider a strictly proper stable LTI plant with a saturation (9) in the negative feedback and with an additional scalar gain $K \geq 0$ between the output

TABLE II

A COMPARISON OF THE EXISTING APPROACHES AND THE NEW APPROACH: DISCRETE CASE.

	Circle	Popov	Primbs	New
$\tilde{G}_1(z)$	0.2080	0.4346	0.3877	0.5498
$\tilde{G}_2(z)$	0.3584	0.6045	0.6943	0.9111

of the plant and the input of the saturation. Once again tests were performed with two different plants.

1) Suppose the plant is
$$\tilde{G}_1(z) = \frac{z-0.4}{(z-0.9)(z-0.1)(z-0.8)} \tag{27}$$
Table 2 shows maximum values of K for which the circle criterion, the Popov criterion, Primbs' method and the new approach guarantee stability. For this case the Popov criterion is less conservative than Primbs' method, but the new approach is less conservative than both.
2) Suppose the plant is
$$\tilde{G}_2(z) = \frac{z-0.5}{(z-0.9)(z+0.1)(z-0.8)} \tag{28}$$
Once again Table 2 shows maximum values of K for each method. This time Primbs' method is less conservative than the Popov criterion, and once again both methods are more conservative than the new approach.

IV. CONCLUSION AND DISCUSSION

We have presented a new stability test for linear plants connected in feedback with a static nonlinearity that may be represented by a quadratic program. The test is based on Primbs' method [14], [16], but with the candidate Lyapunov function augmented with a term corresponding to that for the Popov criterion. The test is guaranteed to be no more conservative than Primbs' method, and no more conservative than the Popov criterion when constraint (2) takes the simple form $\phi^T(\phi - Ky) \leq 0$ and the corresponding multiplier is positive. We have given numerical examples where the test is less conservative than both Primbs' method and the Popov criterion.

The following questions remain open:

1) Is it possible to show that the candidate Lyapunov function remains positive even when the multiplier corresponding to the Popov term is negative? If this can be shown, then the test would also be shown to subsume the Popov criterion when constraint (2) takes the simple form $\phi^T(\phi - Ky) \leq 0$.
2) It is possible to represent nonlinearities such as the combined deadzone and saturation (14) with many different quadratic programs. Is it possible to specify the quadratic program that gives the least conservative test?
3) How should the discrete case be generalized to more general sectors and multivariable nonlinearities? It is straightforward to handle different single-input single-output nonlinearities via loop transformations. Furthermore it is straightforward to handle repeated nonlinearities for the multivariable case. Thus test can be applied to model predictive control with only simple bounds on the constraints (the corresponding quadratic program may be replaced by a feedback loop with repeated saturation functions [17],[4]).
4) How does the test compare with the use of Zames-Falb multipliers? One contribution of this paper is to show that Primbs' method with a piecewise quadratic cost function complements the Popov criterion, in the sense that one or the other may be less conservative depending on the plant.

V. APPENDIX

Proof of Result 1:

By defining $u := -\phi$, the three constraints (2)-(4) can be written in quadratic forms in terms of $v := [x^T, u^T, \dot{u}^T]^T$ as
$$\sigma_1 := v^T \Pi_1 v \geq 0 \quad \sigma_2 := v^T \Pi_2 v = 0 \quad \sigma_3 := v^T \Pi_3 v = 0 \tag{29}$$
with Π_1, Π_2 and Π_3 as (5).

Suppose the candidate Lyapunov function has the form
$$V(x,u) = [x^T, u^T]^T P [x^T, u^T] - 2\lambda \int_0^y [\phi^T F \dot{\tau}(t) + \phi^T H N \dot{\tau}(t) + y^T N^T F \dot{\tau}(t)] dt \tag{30}$$
with $P = P^T > 0$ and $\lambda \geq 0$. From (2), we have
$$-(\phi^T H N y + \phi^T F y + y^T N^T F y) \geq \phi^T H \phi \geq 0 \tag{31}$$
which validates the positivity of this Lyapunov function (30).

The system is stable if under the conditions (29), the first derivative of Lyapunov function (30) is less than zero, which can be expressed by a quadratic inequality $\sigma_0 := v^T \tilde{\Pi}_p v < 0$ where $\tilde{\Pi}_p = \Pi_0 + \Pi_p$ with Π_0 and Π_p as (7) and (5). Using the S-procedure, this implication can be represented by $\sigma_0 + \sum_{i=1}^{3} r_i \sigma_i < 0$ with $r_1 \geq 0$ and $r_2, r_3 \in \mathbb{R}$, which can be expressed in one LMI as (6). □

Proof of Observation 1: The LMI stability condition for stability is
$$\begin{bmatrix} A^T P + PA & \lambda K C^T & PB - KC^T \\ \lambda K C & -2\lambda & 2 \\ B^T P - KC & 2 & -2 \end{bmatrix} < 0 \tag{32}$$
where $\lambda = 1 + m \geq 1$. By the KYP lemma this is equivalent to requiring
$$\begin{bmatrix} 0 & (j\omega I - A)^{-1} B \\ 1 & 0 \\ 0 & 1 \end{bmatrix}^* \begin{bmatrix} 0 & \lambda K C^T & -KC^T \\ \lambda K C & -2\lambda & 2 \\ -KC & 2 & -2 \end{bmatrix} \times \begin{bmatrix} 0 & (j\omega I - A)^{-1} B \\ 1 & 0 \\ 0 & 1 \end{bmatrix} < 0 \text{ for all } \omega$$

and hence for all ω

$$\left[\begin{array}{cc} KG(j\omega)+KG(j\omega)^*+2 & -2-\lambda KG(j\omega) \\ -2-\lambda KG(j\omega)^* & 2\lambda \end{array}\right] > 0$$

Taking Schur complements gives

$$\lambda^2K^2G(j\omega)^*G(j\omega)-4\lambda+4<0 \text{ for all } \omega \tag{33}$$

There exists a real $\lambda \geq 1$ for which the inequality is satisfied if and only if $K\|G\|_\infty \leq 1$. $\square$

Proof of Result 2:

We firstly derive a relation which will be used later. Subtracting (19) from (20) gives $\Delta\varphi(k+1)[\Delta\varphi(k+1)-K\Delta y(k+1)] \leq 0$, i.e. $\Delta\varphi(k)[\Delta\varphi(k)-K\Delta y(k)] \leq 0$, which is equivalent to $0 \leq \frac{\Delta\varphi(k)}{\Delta y(k)} \leq K$. Hence we have

$$0 \leq \frac{\varphi(y)-\varphi(\hat{y})}{y-\hat{y}} \leq K \tag{34}$$

In the following, we establish stability based on the three conditions (18)-(20) and the candidate Lyapunov function

$$V(k) := V(x(k),u(k)) = \xi(k)^TP\xi(k)+2\lambda\int_0^{y(k)}\varphi(\sigma)d\sigma$$

with $\lambda \geq 0$ and $\xi(k) := [x(k)^T, u(k)^T]^T$. Then the corresponding Lyapunov function difference is

$$\begin{aligned}&\Delta V(k) := V(k+1)-V(k)\\ =&\xi(k+1)^TP\xi(k+1)-\xi(k)^TP\xi(k)+2\lambda\int_{y(k)}^{y(k+1)}\varphi(\sigma)d\sigma\end{aligned} \tag{35}$$

Using the relation (34), we have

$$\begin{aligned}&\int_{y(k)}^{y(k+1)}[\varphi(\sigma)-\varphi(y(k))]d\sigma\\ \leq& K\int_{y(k)}^{y(k+1)}[\sigma-y(k)]d\sigma = \frac{K}{2}[y(k+1)-y(k)]^2\end{aligned}$$

which is

$$\begin{aligned}&2\int_{y(k)}^{y(k+1)}\varphi(\sigma)d\sigma\\ \leq& 2\varphi(y(k))[y(k+1)-y(k)]+K[y(k+1)-y(k)]^2\end{aligned}$$

Therefore using $u=-\varphi$, (35) is

$$\begin{aligned}&\Delta V(x)\\ =&x^T(A^TPA-P)x+x^TA^TPBu+u^TB^TPAx\\ &+\varphi^TB^TPB\varphi+2\lambda\varphi^T[y(k+1)-y(k)]^2\\ =&x^T(A^TPA-P)x+x^TA^TPBu+u^TB^TPAx\\ &+u^TB^TPBu-2\lambda u^T[(CA-C)x+CBu]\\ &+\lambda[(CA-C)x+CBu]^T[(CA-C)x+CBu)]\end{aligned} \tag{36}$$

Then $\Delta V(k)<0$ can be represented in a quadratic form as $v^T(\Pi_0+\Pi_p)v<0$ where $v := [x(k)^T, u(k)^T, \Delta u(k)^T]^T$ and Π_0, Π_p as (24) and (21) respectively. Furthermore, (18)-(20) can be represented in quadratic forms as

$$v^T\Pi_1v \leq 0 \qquad v^T\Pi_2v=0 \qquad v^T\Pi_3v=0 \tag{37}$$

with Π_1, Π_2 and Π_3 as (21).

Therefore, the system is stable if under the conditions (37), the Lyapunov function difference $\Delta V(x) \leq 0$ can be satisfied. By the S-procedure, this implication can be represented by an LMI as (6). $\square$

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